Sources of Emittance Growth

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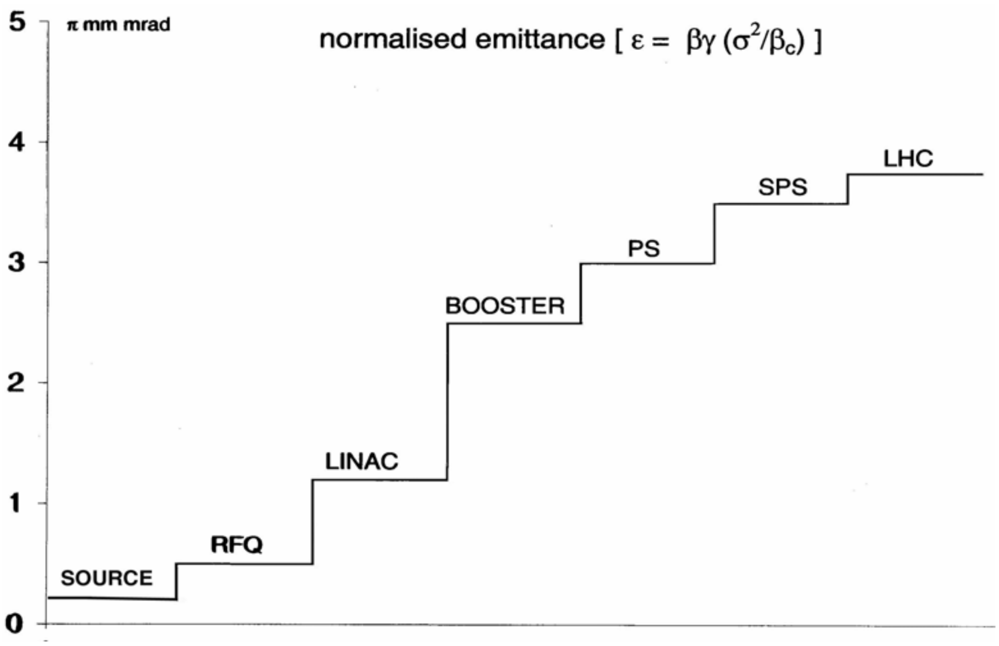
Menu

- Overview
- Definition of emittance , filamentation
- Mismatch at transfer
- Scattering on a foil
- Scattering on the residual gas
- Crossing resonances, instabilities
- Power supply noise and ripple
- Intra-beam scattering
- Conclusions

Emittance History (LHC-Design)



Emittance Definition

Betatron equation (linear!): $x_{(s)}'' + K_{(s)}\, x_{(s)} = 0$

Solution for particle "i"

$$x = A_i\sqrt{\beta}\,\cos(\psi+\delta_i)$$
$$x' = -A_i\sqrt{1/\beta}\,\{\alpha\cos(\psi+\delta_i) + \sin(\psi+\delta_i)\}$$

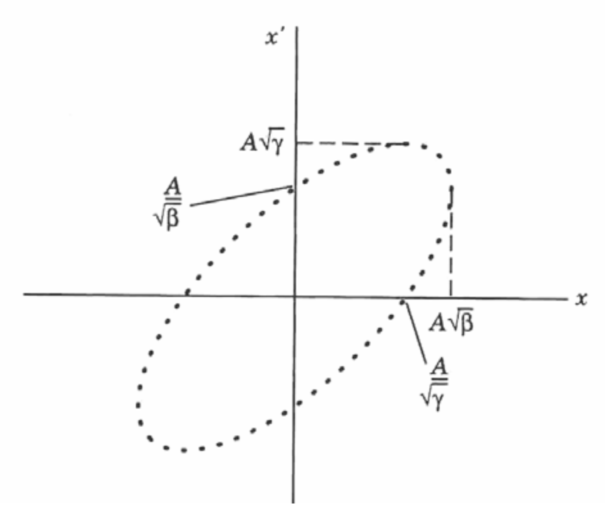
or :

$$x = A_i\sqrt{\beta}\,\cos(\psi+\delta_i)$$
$$p_x = \alpha x + \beta x' = -A_i\sqrt{\beta}\,\sin(\psi+\delta_i)$$

The motion of each particle describes a circle in (x, p_x) space! The Courant&Snyder invariant ("single particle emittance") is:

$$\varepsilon_i \equiv \pi\frac{1}{\beta}\left(x_i^2 + p_{xi}^2\right) = \pi A_i^2 = \pi\, 2\sigma_x^2/\beta$$

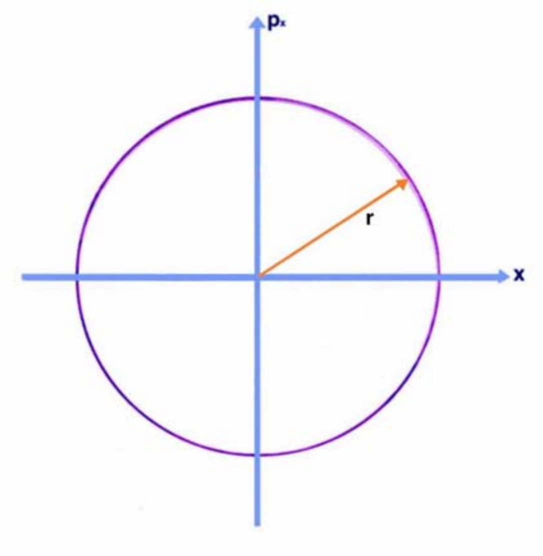
Single Particle (x x')-Phase-Space Trajectory



Area of ellipse: $\pi\, x_{max}^2/\beta = \pi\, x'^2_{max}\,\gamma$: **'single particle emittance'**

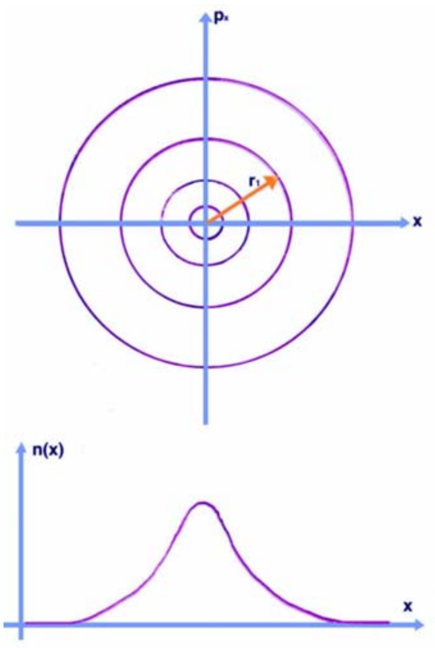
(also called : Courant&Snyder –invariant)

Single Particle Trajectory in Normalized Phase-Space



Area of circle: ‘single particle emittance’

Many Particle Trajectories and Projected Density



Two common beam emittance definitions

--- Beam emittance = " average" of ε_i over all beam particles---

Definition referring to the a fixed fraction of particles:

The beam emittance ($\varepsilon_{\%}$) is the area of the circle in (x, p_x) space that contains the motion of a given fraction (F) of the particles. Usually one refers to F = 39% or 86% or 95% of the beam.

$\varepsilon_{\%}$ is (sometimes) called: "geometrical emittance"

Definition referring to the standard deviation of the projected distribution:

Let σ_x be the standard deviation of the particle density projected on the x-axis (i.e. the "rms beam size" as measured e.g. on a profile detector). Then the emittance ($\varepsilon_{k\sigma}$) is defined as the area in (x, p_x) space with radius $k\sigma_x$. Usually one choses k=1 or 2 or 2.5.

$\varepsilon_{k\sigma} = (k\sigma)^2/\beta$ is (sometimes) called " 'k'-rms emittance "

$$\int_0^{\varepsilon_{F\%}} n(\varepsilon_i)\, d\varepsilon_i = F$$

$$\varepsilon_{k\sigma} = \pi\,(k\sigma_x)^2/\beta$$

Emittance definition, cont.

Usually one can assume stationary conditions (distribution is time independend i.e. uniformly distributed with resp. to betatron phase: $\psi = 0$ to 2π)

The fraction of particles within a given element of phase space with area dx dp_x is
$dF = n(x, p_x)dxdp_x$. Switching to polar coordinates $(r^2 = x^2 + p_x^2,\ \psi)$: $dF = n(r,\psi)\,d\psi\, rdr$

The fraction of particles that have their motion contained in a circlea of radius "a" (emittance $\varepsilon = \pi a^2/\beta$) is

$$F = \int_0^{2\pi}\int^{a} n(r,\psi)d\psi\, rdr = \int_0^{2\pi}\int^{a} n(r)\, n(\psi)d\psi\, rdr = \int_o^{a} n(r)\, rdr \qquad (\ \psi\text{'s uniformly distributed!})$$

$$F = \int_o^{a^2} n(r^2)\, d(r^2) \ = \int_o^{\varepsilon_a} n(\varepsilon)\, d\varepsilon$$

Gaussian distribution

Suppose that the distribution in transverse coordinate (x) is Gaussian (independend of time)

$$n(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

The distribution in p_x is also Gaussian (with the same standard deviation) and the density in x, p_x space is the product

$$n(p_x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{p_x^2}{2\sigma_x^2}}, \quad n(x, p_x) = \frac{1}{2\pi\sigma_x^2} e^{-\frac{x^2+p_x^2}{2\sigma_x^2}}, \quad n(r) = \frac{r}{\sigma_x^2} e^{-\frac{r^2}{2\sigma_x^2}}$$

Gaussian distribution, cont.

Then the fraction of particles that have their motion contained in a circle of radius "a" (emittance $\varepsilon = \pi a^2/\beta$) is

$$F_{Gauss} = \int_0^a \frac{1}{\sigma_x^2} e^{-\frac{r^2}{2\sigma_x^2}} r dr = 1 - e^{-\frac{a^2}{2\sigma_x^2}}$$

$k = a/\sigma_x$	$\varepsilon_{k\sigma}$	F_{Gauss}	F_{unif} *)
1	$\pi\,(1\,\sigma_x)^2/\beta$	39.3 %	50%
2	$\pi\,(2\,\sigma_x)^2/\beta$	86.4 %	100%
2.5	$\pi\,(2.5\,\sigma_x)^2/\beta$	95.6 %	(100 %)

*) uniform density in (x, p_x) space

Gaussian distribution, cont., cont.

Average and root mean square of (the single particle) emittances

$$<\varepsilon> = \int_0^\infty \varepsilon\, n(\varepsilon)\, d\varepsilon = \int_0^\infty \frac{\varepsilon}{\varepsilon_o} e^{-\frac{\varepsilon}{\varepsilon_o}} d\varepsilon = \varepsilon_o = \pi\, 2\sigma_x^2/\beta$$

$$<\varepsilon^2> = \int_0^\infty \varepsilon^2\, n(\varepsilon)\, d\varepsilon = \int_0^\infty \frac{\varepsilon^2}{\varepsilon_o} e^{-\frac{\varepsilon}{\varepsilon_o}} d\varepsilon = 2\varepsilon_o^2 \quad \Rightarrow \quad \sqrt{<\varepsilon^2>} = 2\sqrt{2}\,\sigma_x^2/\beta$$

Betatron amplitude distribution:

$$x_i = r_i \cos(\psi_i)$$
$$p_{xi} = -r_i \sin(\psi_i)$$

Let $g(r_i, \psi_i)$ be the distribution of betatron amplitudes and phases
Then the probability to find particle "i" in a range dx near x is

$$n(x) = \frac{1}{\pi} \int_0^{2\pi} \int_{|x|}^{\infty} \frac{g(r_i, \psi_i)}{\sqrt{r_i^2 - x^2}} r_i \, dr_i \, d\psi_i$$

for uniform ψ_i

$$n(x) = 2 \int_{|x|}^{\infty} \frac{g(r_i)}{\sqrt{r_i^2 - x^2}} r_i \, dr_i$$

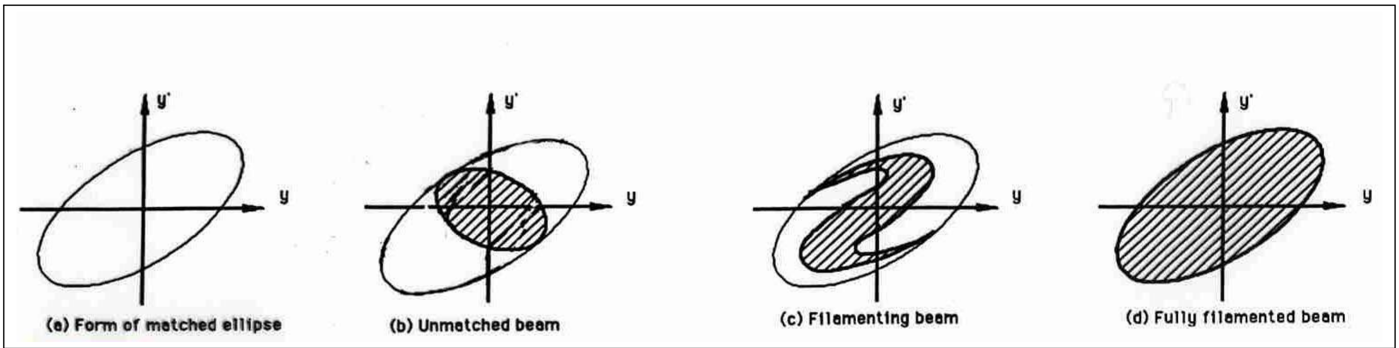
Let $n(x)$ be Gaussian (with variance σ_x^2) The corresponding amplitude distribution

$$g(r_i) = \frac{r_i}{\sigma_x^2} e^{-r_i^2 / (2\sigma_x^2)}$$ is a 'Rayleigh distribution' with variance $\langle r_i^2 \rangle \equiv \sigma_r^2 = 2\sigma_x^2$

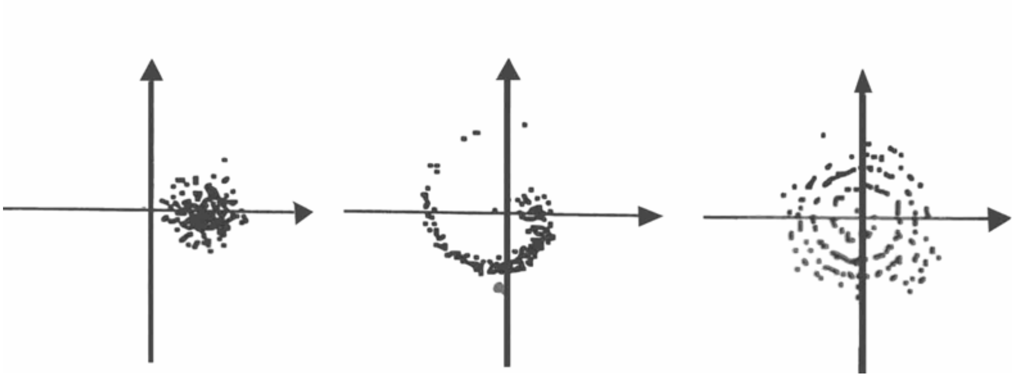
Filamentation

Nonlinearity of betatron oscillation causes a dependence of betatron frequency on amplitude ---> particles go around in phase space at (slightly) different speed. Over sufficiently long time ($\Delta\omega t >> 1$) a mismatched beam 'smears out' and the larger phase space becomes filled out

Filamentation = Randomisation of betatron phases (in a mismatched beam)
---> emittance dilution (apparent blow up)



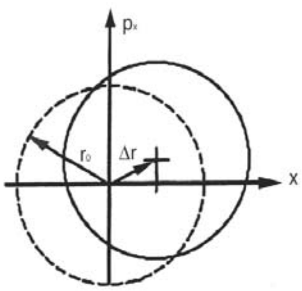
Simulation of filamentation



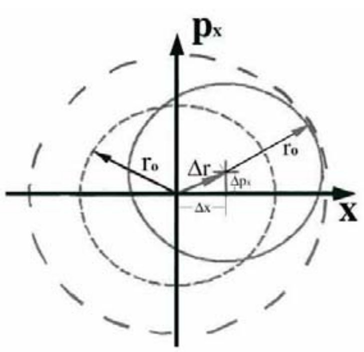
injection after some time after long time

Injection error

Assume the beam is injected with displacemant Δx and angular error $\Delta x'$ into an otherwise matched phase space (Fig.)



$$\Delta r^2 = \Delta x^2 + \Delta p_x^2$$



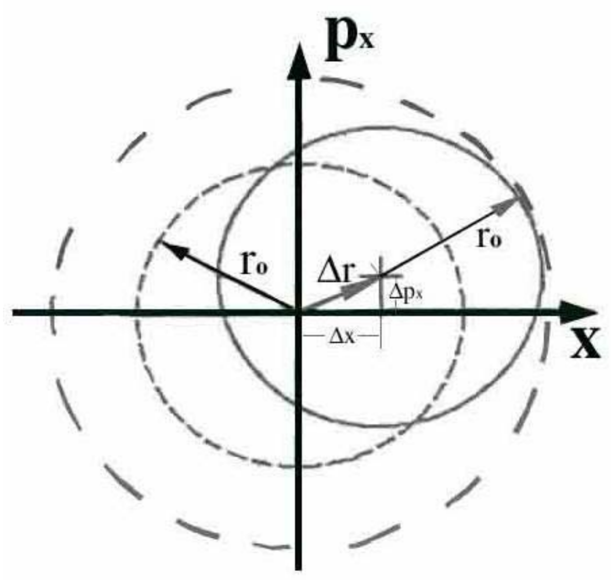
The radius enclosing the motion of the given fraction of particles is

$$r_{\%} \dashrightarrow r_{\%} + \Delta r = r_{\%} + \{ \Delta x^2 + (\beta \Delta x' + \alpha \Delta x)^2 \}^{1/2}$$

The emittance containing the given fraction of particles (after filamentation) is

$$\boxed{\varepsilon_{\%} \dashrightarrow \varepsilon_{\%} (1 + \Delta r / r_0)^2}$$

Injection error, increase of $\varepsilon_{\%}$,cont



$$\varepsilon_{\%} \dashrightarrow \varepsilon_{\%} \,(1 + \Delta r / r_0)^2$$

Injection error, increase of $\varepsilon_{\%}$, special cases

Position or angular error only:

Δx only: $(\Delta r = \Delta x\,(1+\alpha^2)^{1/2}$ ::: $\varepsilon_{\%} \dashrightarrow \varepsilon_{\%} \{ 1 + \Delta x\,(1+\alpha^2)^{1/2} / r_{\%o} \}^2$

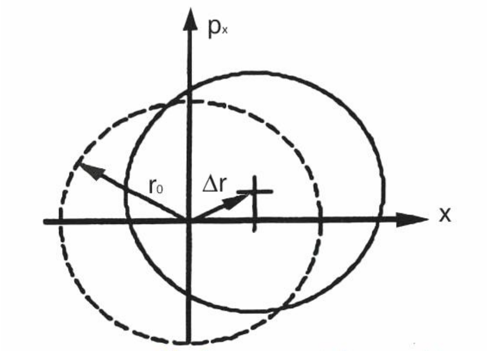
$\Delta x'$ only: $(\Delta r = \Delta x'\,\beta)$ ::: $\varepsilon_{\%} \dashrightarrow \varepsilon_{\%} \{ 1 + \Delta x'\,\beta / r_{\%o} \}^2$

For small $\Delta r \ll r_{\%o}$:

$$\varepsilon_{\%} \dashrightarrow \varepsilon_{\%} \,(1 + 2\,\Delta r / r_{\%o}) = \varepsilon_{\%} \,[\,1 + 2\,\{ \Delta x^2 + (\beta\Delta x' + \alpha\,\Delta x)^2 \}^{1/2} / r_{\%o}\,]$$

Injection error, increase of rms-emittance

Assume (again) the beam is injected with an error { Δx , Δp_x }



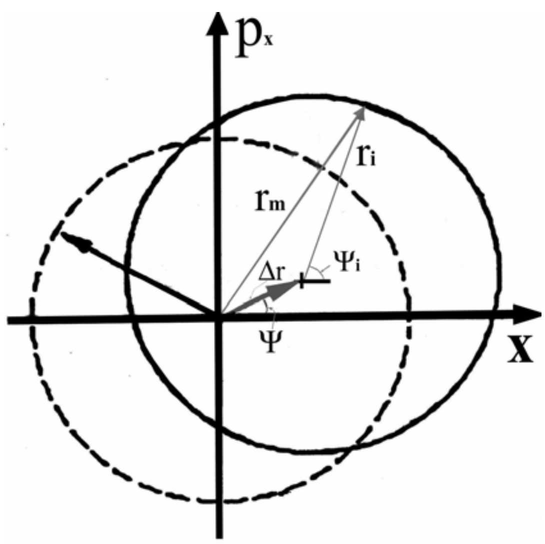
By (simple) geometrical arguments one finds that after filamentation the beam has the new

$$\sigma_x^2 = \sigma_{xo}^2 + ½\, \Delta r^2 \;;$$

$$\boxed{\varepsilon_{k\sigma} \dashrightarrow \varepsilon_{k\sigma} \{ 1 + ½\, (\Delta r^2 / \sigma_{xo}^2) \}}$$

This is *independent of the initial (and final) distribution*. Rem.: $\Delta r^2 = \{ \Delta x^2 + (\beta \Delta x' + \alpha\, \Delta x)^2 \}$

Increase of rms emittance, cont.



index m: machine phase space (origin 0 , Fig.)
index i ; injected beam (displaced by Δr) space

A test particle (cf. Figure)has

$$\vec{r}_m = \vec{r}_i + \Delta \vec{r} \quad \text{i.e.}$$

$$x_m = r_i \cos \psi_i + \Delta r \cos \psi$$

$$p_m = r_i \sin \psi_i + \Delta r \cos \psi \qquad (*)$$

The injection error (Δr and Ψ) is fixed but the phases Ψ_i of the injected particles are uniformly distributed from 0 --> 2π .

Averaging

$$< r_m^2 > \equiv < x_m^2 + p_m^2 > .$$

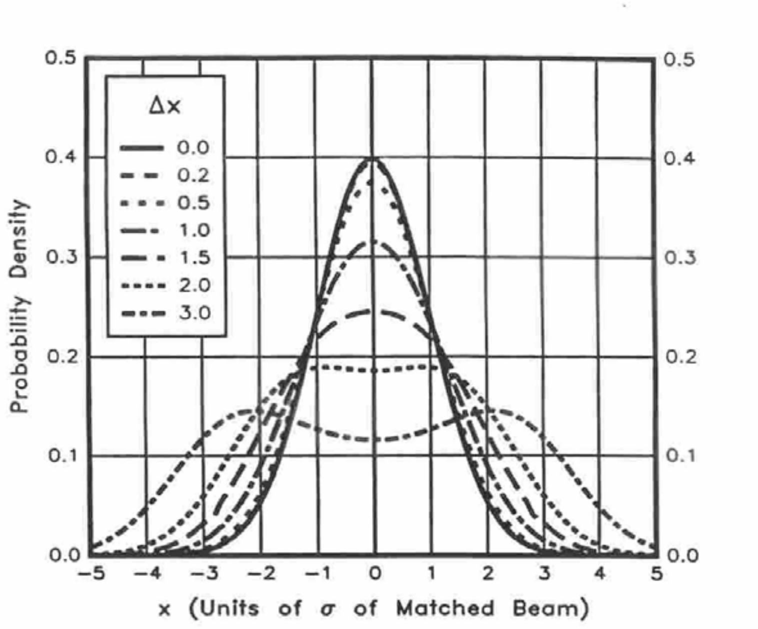
$$< r_m^2 > = < r_i^2 > + \Delta r^2 \quad \text{by virtue of } (*)$$

Now after filamentation

$$< x^2 > \equiv \tfrac{1}{2} < r_m^2 > = \tfrac{1}{2} < r_i^2 > + \tfrac{1}{2} \Delta r$$

$$\boxed{\sigma^2 = \sigma_i^2 + \tfrac{1}{2} \Delta r^2}$$

Distribution after filamentation of an injection error



Momentum error, dispersion error

Assume a beam is injected with a **momentum error** $\delta p/p$**. If** the dispersion D of the transfer **line matches the dispersion of the ring** ***and*** **the beam is centered around its displaced** ($\delta x = D\,\delta p/p$, $\delta x' = D'\delta p/p$) off-momentum **orbit, then perfect matching** will prevail. If however the beam is injected onto the center of the aperture, it has a position error

$$\mathbf{\Delta r = \{ D^2 + (\beta D' + \alpha D)^2 \}^{1/2}\, \delta p/p,}$$

and our previous formulae can be used

Next: **if the dispersion is mismatched** (without loss of generality assume that the machine has zero and the line has finite D ; in the general case we just can substitute D --> ΔD). Then for any particle

$$x = r\cos(\Psi) + D\,\Delta p/p\,, \quad p_x = r\sin(\Psi) + (\beta D' + \alpha D)\,\Delta p/p$$

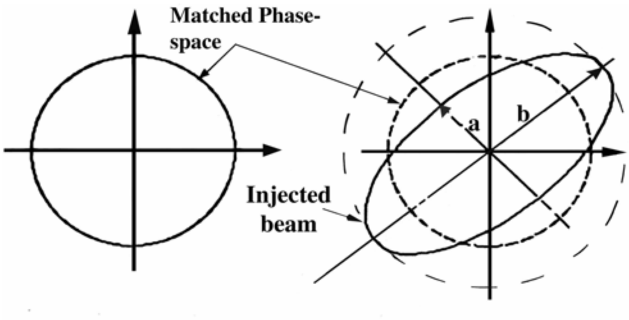
$$2\sigma^2 = \langle x^2 + p_x^2 \rangle = \langle r^2 + \{ D^2 + (\beta D' + \alpha D)^2\} (\Delta p/p)^2 \rangle ,$$

$$\sigma^2 = \sigma_0^2 + \{ D^2 + ½\,(\beta D' + \alpha D)^2\}(\sigma_p/p)^2$$

$$\boxed{\boldsymbol{\varepsilon_{k\sigma} \dashrightarrow \varepsilon_{k\sigma}\{ 1 + ½\,\{ D^2 + (\beta D' + \alpha D)^2\}(\sigma_p/p)^2 / \sigma_o^2) \}}}$$

Focussing error (mismatch) at transfer

Suppose the twiss functions at the "hand over point" (exit of the transfer line --> entrance of the ring) differ from the ring ones . Normalising the ring phase space trajectories to circles, the trajectories of the mismatched injected beam are ellipses. Particles contained in a given ellipse fill, after filamentation, the circumscribed circle of the ring phase space with an area encrease $\varepsilon_{\%}$ --> $\varepsilon_{\%}$ (b/a) (major axis/minor axis) fig



Calculation shows that this can be expressed as

$$\varepsilon_{\%} \rightarrow \varepsilon_{\%}\left(b/a\right) = \varepsilon_{\%}\left(F + \sqrt{F^2 - 1}\right) \qquad \text{where} \quad F = \tfrac{1}{2}\left(\beta_l\,\gamma_r + \beta_r\,\gamma_l - 2\,\alpha_l\,\alpha_r\right)$$

$$\text{or equivalently} \quad F = \frac{1}{2}\left(\frac{\beta_l}{\beta_r} + \frac{\beta_r}{\beta_l} + \left(\frac{\alpha_r}{\beta_r} - \frac{\alpha_l}{\beta_l}\right)^2 \beta_r \beta_l\right)$$

Mismatch at transfer, continued

To calculate the increase of the rms-emittance represent the particles motion at the end of the beam line by

$x_l = A_i\,\sqrt{b/a}\ \sin\psi_i$, $p_l = A_i\,\sqrt{a/b}\ \cos\psi_i$ where πA_i^2 is the single particle emittance and the average $\pi < A_i^2 > = \pi\, 2\,\sigma_x^2 / \beta$ is twice the rms- emittance in the line. The x_l, p_l set the initial conditions for the circular phase space motion with an amplitude $r_i^2 = x_l^2 + p_l^2$ in the ring. Averaging over all particles we find

$$< r_i^2 > = < A_i^2 > \frac{(b/a \;+\; a/b)}{2}$$ this is twice the rms emittance after filamentation in the ring. Thus

the blow up of the rms emittance is $\varepsilon_{k\sigma} \rightarrow \varepsilon_{k\sigma} \dfrac{(b/a \;+\; a/b)}{2}$ Noting that

$$\left(b/a\right) \equiv \left(F + \sqrt{F^2 - 1}\right) \quad , \quad \left(a/b\right) = \left(F - \sqrt{F^2 - 1}\right)$$ this may also be written

$$\boxed{\varepsilon_{k\sigma} \longrightarrow \varepsilon_{k\sigma}\, \mathrm{F}}$$

Recapitulation

	Steering error	Mismatch
Blow up $\varepsilon_{\%} / \varepsilon_{\%,0}$ of emittance containing given fraction of particles	$\left(1+\sqrt{\dfrac{\Delta x^2+(\beta\Delta x'+\alpha\Delta x)^2}{\beta\ \varepsilon_{\%}/\pi}}\right)^2$	$\left(F+\sqrt{F^2-1}\right)$
Blow up $\varepsilon_{rms} / \varepsilon_{rms,o}$ of rms-emittance	$\left(1+\frac{1}{2}\dfrac{\Delta x^2+(\beta\Delta x'+\alpha\Delta x)^2}{\beta\ \varepsilon_{rms}/\pi}\right)$	$F=\frac{1}{2}\left(\dfrac{\beta_l}{\beta_r}+\dfrac{\beta_r}{\beta_l}+\left(\dfrac{\alpha_r}{\beta_r}-\dfrac{\alpha_l}{\beta_l}\right)^2\beta_r\beta_l\right)$

An example

	Relative steering error	**Mismatch factor**
	$\dfrac{\Delta x^2+(\beta\Delta x'+\alpha\Delta x)^2}{\beta\ \varepsilon/\pi}=0.1$	$F=\frac{1}{2}\left(\dfrac{\beta_l}{\beta_r}+\dfrac{\beta_r}{\beta_l}+\left(\dfrac{\alpha_r}{\beta_r}-\dfrac{\alpha_l}{\beta_l}\right)^2\beta_r\beta_l\right)=1.1$
Dilution $\varepsilon_{\%} / \varepsilon_{\%,0}$ of geometrical emittance	**1.73**	**1.56**
Dilution $\varepsilon_{rms} / \varepsilon_{rms,o}$ of rms emittance	**1.05**	**1.1**

Scattering in a foil or a window

A particle (of charge number q_p (=1 for proton), momentum p [MeV/c], velocity $\beta_p = v_p/c$) traversing a foil (thickness L, material of 'radiation length' L_{rad} undergoes multiple Coulomb scattering. The rms scattering angle in each of the transverse planes is given by (ε_{corr} is a correction factor neglected later on)

$$\boxed{\theta_{rms} = \frac{14\ MeV/c}{p\beta_p}\, q_p \sqrt{\frac{L}{L_{rad}}}\,(1+\varepsilon_{corr})}$$

The (normalised) circular phase space trajectories of a matched beam at the entrance are converted into elliptical trajectories at the exit (Fig.). In fact for any particle scattered by an angle θ_i

$$x_i \rightarrow x_i = r_{io}\,\sin(\psi_i)$$

$$p_{xi} \rightarrow p_{xi} + \Delta p = r_{io}\,\cos(\psi_i) + \beta\,\theta_i \qquad (\text{ remember : } p_x = \alpha\, x + \beta\, x')$$

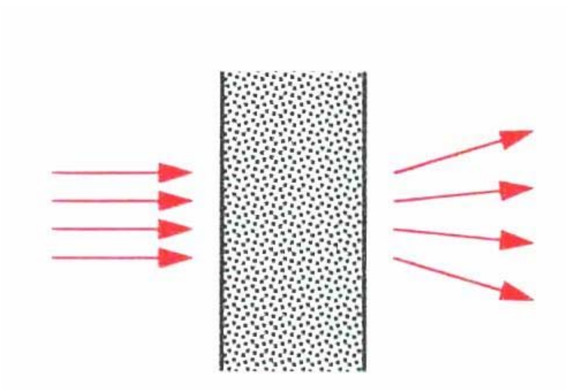
These are the initial conditions for the new betatron oscillation after the scattering event.Averaging over the beam we take the phases ψ_i of the original oscillation uniformly distributed and uncorrelatde with the scattering angle. After filamentation in the line and/or the subsequent ring the new betatron phase are also uniformly distributed and one obtains

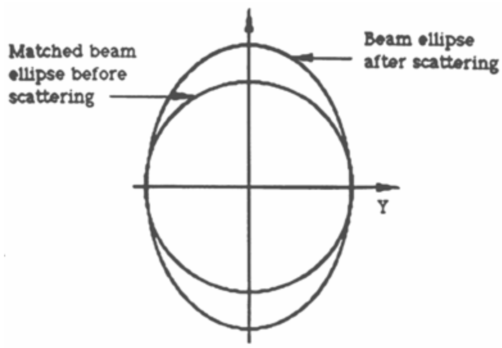
$$< r_i^2 > = < x_i^2 + p_{xi}^2 > = < r_{io}^2 > + < \beta^2\, \theta_i^2 >$$

$$\rightarrow 2\sigma_x^2 = 2\sigma_{xo}^2 + \beta^2\, \theta_{rms}^2 \qquad \text{or} \qquad \varepsilon_{k\sigma} = \pi(k\sigma_x)^2/\beta = \varepsilon_{k\sigma o} + \tfrac{1}{2}(k\theta_{rms})^2\beta$$

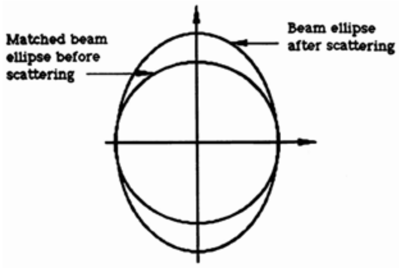
Thus blow up of $k\sigma$ emittance by $\boxed{\Delta\varepsilon_{k\sigma} = \pi\tfrac{1}{2}(k\theta_{rms})^2\ \beta}$

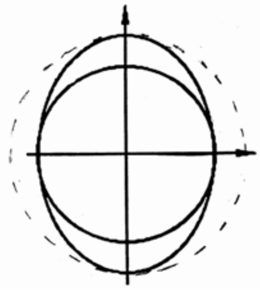
Transition of the beam through a foil





Phase space before and after scattering at the foil





Scattering in foil or thin window, cont.

Above we have tacitly assumed that the beta function is the same at the entrance and exit so that the beam remains matched in the *absence* of the scatterer. By re-adjusting the optics in the downstream part of the line the blow-up can in fact be reduced (approx. halved). The idea is to provide different β-functions ($\beta_0 > \beta_1$) at entrance (o) and exit (1). For simplicity, we perform the calculation only for $\alpha = 0$. Then, for any particle

$$x = A_o\sqrt{\beta_o}\cos(\psi_o) = A_1\sqrt{\beta_1}\cos(\psi_1)$$

$$x' = -A_o\sqrt{1/\beta_o}\sin(\psi_o) + \theta = -A_1\sqrt{1/\beta_1}\sin(\psi_1)$$

Taking the squares and averaging over the beam with the assumption that the beam is - and remains matched before and after the foil (so that all the $\cos^2$ and $\sin^2$ terms average to ½)

$$<x^2> = \frac{<A_o^2>\beta_o}{2} = \frac{<A_1^2>\beta_1}{2}$$

$$<x'^2> = \frac{<A_o^2>}{2\beta_o} + \theta_{rms}^2 = \frac{<A_o^2>}{2\beta_o}$$

The solution of this system is

$$\frac{<A_1^2>^2 - <A_o^2>^2}{<A_o^2>} = 2\beta_o\theta_{rms}^2$$

$$\frac{\beta_1}{\beta_o} = \frac{<A_o^2>}{<A_1^2>}$$

Remember that the 1σ-emittance is $\varepsilon_\sigma = \pi A^2/2$. Then, for small $\Delta\varepsilon/\varepsilon$, one obtains

$$\Delta\varepsilon_\sigma = \pi/4\ \theta^2_{rms} \qquad \text{(and hence } \Delta\varepsilon_{k\sigma} = \pi/4\ k^2\ \theta^2_{rms}\text{)}$$
$$\beta_1 = \beta_o\ [\,1 - (\pi/4)\,\theta^2_{rms}/\varepsilon_\sigma]$$

This is half the emittance increase than for constant $\beta_1 = \beta_o$ (after filamentation)

Multiple Coulomb scattering on the residual gas

This is treated in many papers of which I find the one by W. Hardt (CERN ISR-300/GS/68-11) especially instructive. Here we can use our previous results immediately by taking the resdal gas atmosphere as a "thin scatterer". For pure Nitrogen (N_2) at pressure P the radiation length is $L_{radN2} \approx 305\ m/(P/760\ torr)$ and the thickness traversed by the beam in time t is $L=\beta_p ct$. Then from (***) we get the blow-up of the kσ –emittance as

$$\Delta\varepsilon_{k\sigma} = \frac{\pi}{2}k^2 q_p^2 \left(\frac{14\ MeV/c}{p\beta_p}\right)^2 \bar{\beta}\ \frac{\beta_p c\, t}{305/(P/760\ torr)}$$

Here $\bar{\beta}$ is the average beta function as scattering occurs everywhere around the ring. With $p=938\ MeV/c{*}A_p{*}\beta_p\gamma_p$, A_p = mass number of the ion (1 for proton), one obtains Hardt's formula (except for a slight difference in the numerical factor as he takes 15 MeV/c in the scattering angle formula)

$$\boxed{\Delta\varepsilon_{k\sigma} \approx \frac{\pi}{2}k^2\ \frac{q_p^2}{A_p^2}\ 0.3\,\bar{\beta}\,\frac{P\,t}{\beta_p^3\gamma_p^2}} \qquad (\ P[torr]\quad t[sec]\ )$$

This relation is widely used to determine the vacuum requirement in a storage ring. For a synchrotron one has to integrate $\frac{dt}{\beta_p^2\gamma_p}$ over the acceleration cycle to get the blow up of the normalised emittance . For an atmosphere with different gases of partial pressures P_i we can define the N_2 equivalent P for multiple Coulomb scattering as $P_{N2equ} = \sum P_i\,(\,L_{rad,N2}/L_{rad,i})$

Hardt's classical internal paper (ISR-300/GS/68-11)

EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

ISR-300/GS/68-11

A FEW SIMPLE EXPRESSIONS FOR CHECKING VACUUM REQUIREMENTS IN A PROTON SYNCHROTRON

by

W. Hardt

Geneva - 14 March, 1968

PS/6445

Multiple Coulomb scattering, orders of magnitude

In LEAR the vacuum pressure (N2 eqivalent for scattering) is of the order of $P=10^{-12}$ torr.
Then: $L_{rad} = 2.3\ 10^{17}$ m (about 25 light years).

At p=100MeV/c ($\beta_p \approx 0.1$) an rms scattering angle of $\theta_{rms}=5\ 10^{-3}$ rad (which is about the acceptance limit of LEAR, s. below) is reached after a path length $L= 1.3\ 10^{-5} \times L_{rad} = 3\ 10^{12}$ m (about 2.7 light hours).

With the speed $\beta_p\, c = 3\ 10^{7}$ m/s the beam traverses this distance in ≈ 27 h ($\approx$ one day).
With the circumference of $C \approx 80$ m this corresponds to about $4\ 10^{10}$ revolutions.

For an average beta function of 10m, the rms scattering angle of $5\ 10^{-3}$ rad corresponds to an encrease of the 1σ -emmittance by $\Delta\varepsilon_{1\sigma} = 125\ \pi$ mm mrad .

With an acceptance of 125 π mm mrad 60 % of a Gaussian beam would be lost in 27 h.

Resonance crossing

Dangerous, low order resonances are usually avoided by choosing an appropriate working point (Qx, Qy). However high order resonances may be touched and traversed due to small, unavoidable or programmed tune changes. For a rapid traversal of a resonance pQ=integer the amplitude increase (for small $\Delta a/a$) is given by

$$\Delta a / a \approx \frac{\pi \, \Delta e}{p \sqrt{\Delta Q_t}} \approx \frac{10^{-3}}{p \sqrt{\Delta Q_t}} \quad \text{typically}$$

(p: order, Δe : width of the resonance, ΔQ_t: tune change per turn). The emittance growth after filamentation ($\Delta\varepsilon/\varepsilon = \frac{1}{2}\Delta a^2/a^2 \approx \Delta a/a$) is given by the same expression. Hence only few transitions can be tolerated even of high order resonances. For repeated random crossings the amplitude growth is multiplied by the square root of the number crossings.

For slow tune variation, particles can be trapped in resonance 'bands' which move them outwards, eventually even to the aperture limit. This can happen through a momentum diffusion (e.g. due to residual gas scattering s. below) leading to a tune diffusion via the chromaticity ξ: $\Delta Q/Q=\xi \, \Delta p/p$.

------> A very high 'stability' of the tune is essential

Power supply ripple

Could be treated by : $x'' + K(s)x = \varepsilon \left(d(s) + g(s)\, x \right)$ where $\varepsilon\, d(s)$ and $\varepsilon\, g(s)$ are small random functions representing bending- and focussing-ripple. Instead we use previous assuming that the particle receives small random kicks.

Dipole errors: Let $\theta(n)$ be the random deflection error *per turn* with variance $<\theta(n)^2> = \theta^2_{n,rms}$. Each kick introduces an increase in the kσ-emittance of $\frac{d\varepsilon_{k\sigma}}{dn} = \frac{\pi}{2}\beta(k\theta_{(n)})^2$ according to the previous results. For true (white) noise, the effect of the multiple kicks adds up statistically (i.e. "in square") . After $n = f_{rev}{*}t$ turns one therefore has:

$$\boxed{\Delta\varepsilon_{k\sigma} = \frac{\pi}{2}\beta(k\theta_{n,rms})^2 \; f_{rev}t}$$

Orders of magnitude: storage for 1 day: n= $f_{rev}{*}t = 10^{10}$ turns (more revolutions than earth around sun!!)
typical beta function: β = 20 m ,
assume one magnet giving a nominal angle of θ_o =10 mrad
with field jitter $(\Delta B/B)_{rms} = 10^{-6}$ ---> $\theta^2_{n,rms}$ =(10 mrad $(\Delta B/B)_{rms}$)^2 =$(10^{-5}$ mrad$)^2$
---> $\Delta\varepsilon_\sigma$= 10 π mm mrad.

This might suggest a (ΔI/I) ? =? (ΔB/B) power supply stability requirement of $1/10^6$
However .... the frequency content of the noise is very important!

Power supply ripple, importance of frequency content.

Assuming one localised kick per turn, calculate effect in the normalised variables $(x,\ p_x = \alpha x + \beta x')$.
The compound effect (at the observation time o) is: --the kick Δp_o at t=o, -- plus the kick Δp_1 at the previous turn *transposed by the 1 turn matrix* --plus....-- plus the kick n turns ago *transformed by the n-turn matrix*:

$$\begin{pmatrix} \Delta x \\ \Delta p_x \end{pmatrix} = \sum_{l=o}^{n} M_l \begin{pmatrix} 0 \\ \Delta p_l \end{pmatrix} \qquad (**)$$

where the ℓ-turn matrix is given by the expression $M_\ell = \begin{pmatrix} \cos \ell\mu & \sin \ell\mu \\ -\sin \ell\mu & \cos \ell\mu \end{pmatrix}$ with $\mu = 2\pi Q$

May also write: (**) $\Delta x = \sum_{\ell=0}^{n} \Delta p_\ell \sin \ell\mu \ , \quad \Delta p_x = \sum_{\ell=0}^{n} \Delta p_\ell \cos \ell\mu .$

Now assume purely sinusoidal kicks
$\Delta p_\ell = \Delta \sin \ell\mu_k$ with frequency f_k i.e. $\mu_k = 2\pi f_k / f_{rev}$ and amplitude Δ. Then the sum (**) gives

$$\Delta x = \frac{\Delta}{2}\left[\frac{\sin\{(n+1)\mu_-\}\cos\{n\mu_-\}}{\sin\{\mu_-\}} - \frac{\sin\{(n+1)\mu_+\}\cos\{n\mu_+\}}{\sin\{\mu_+\}}\right] \quad \text{where: } \mu_\pm = \tfrac{1}{2}(\mu_k \pm \mu) = \pi((f_k/f_{rev}) \pm Q)$$

This is bounded for $n \to \infty$ except when $\mu_\pm = m\pi$ i.e.

$\boxed{\text{for } (f_k/f_{rev}) \pm Q = m \quad \text{In that case: } \Delta x = n\Delta/2}$

--> ***The excitation is by those noise components that coincide in frequency f_k with the betatron sidebands***

$$f_k = (m \pm Q) f_{rev}$$

Power supply noise by differential equation

Use the simplified 'linear oscillator' eq. $\ddot{x} + \omega_\beta^2 x = \omega_\beta^2\, u(t)$ where $u(t)$ is the 'noisy deflection'.
The statistical properties of $u(t)$ are supposed to be independent of time ("$u(t)$ is stationary") and represented by the *autocorrelation function* :

$\varphi(\tau_1 - \tau_2) = \overline{u(\tau_1)u(\tau_2)}$. The *spectral power density function* $\phi(\omega)$ is defined by the Fourier transform of the autocorrelation function $\phi(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(\tau) e^{-i\omega\tau}\, d\tau \qquad \varphi(\tau) = \int_{-\infty}^{\infty} \phi(\omega) e^{i\omega\tau}\, d\omega$.

The response of the oscillator can be shown (CERN report by Hereward and Johnsen or book e.g. by Oksendahl) to be a noisy waveform $x(t)$ with a mean square amplitude $\overline{A^2} = 2\pi\, \phi(\omega_\beta)\, t$ that *increases linear with time and is proportional to the power density of the noise at the resonant frequency of the oscillator*. For white noise the autocorrelation is a δ-function

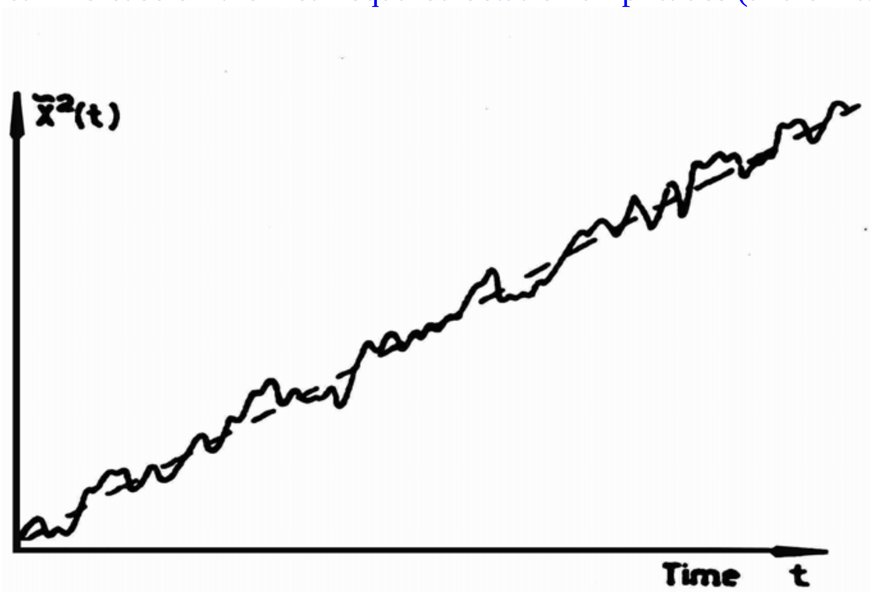
$\varphi(\tau_1 - \tau_2) = \overline{u_1^2(\tau_1)}\,\delta(\tau_1 - \tau_2) = u_{1,rms}^2\, \delta(\tau_1 - \tau_2)$ and the power density is constant $\phi(\omega) = \frac{1}{2\pi} u_{1,rms}^2$.

Then the mean square amplitude is $\overline{A^2} = u_{1rms}^2\, t$. In reality the noise on the bending magnet is filtered by the large inductance of the magnet. For any system with a transfer function **$H(\omega)$** and white noise with $\phi(\omega) = \phi_w$ at the input, the noise at the output is $\phi(\omega) = |H(\omega)|^2 \phi_w$.

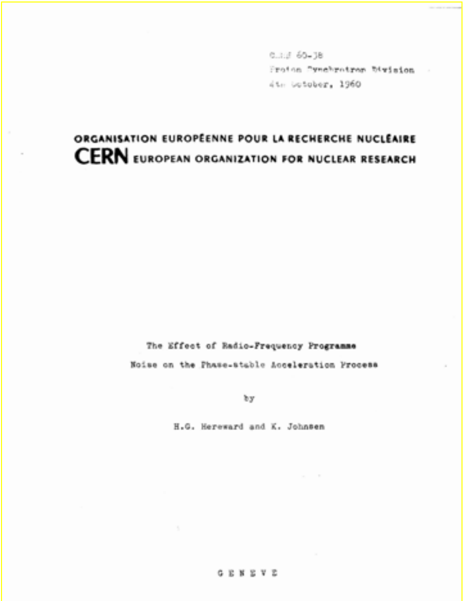
Thus for $H(\omega) = (i\,\omega L)^{-1}$ the mean squared amplitude response is $\overline{A^2} = \frac{u^2_{1rms}\; t}{(\omega_\beta L)^2}$ or by comparison with the previous result $\boxed{\Delta\varepsilon_{k\sigma} = \frac{\pi}{2}\beta \frac{(k\theta_{n,rms})^2}{(\omega_\beta L)^2} f_{rev} t}$

If the magnet excitation current has a noise density leading to a ripple of 10^{-3} at 50 Hz the effective density at the betatron frequency (which is several 100 KHz) is smaller than 10^{-9}.
These qualitative considerations show the way towards explaining the puzzle of the insensitivity to power supply ripple. However to deal with betatron excitation by a short kicker we have to introduce (at least) one more element.
Particles 'sample' the noisy kick once per turn. This can be expressed by multiplying the **$u(t)$** by a periodic δ-function **$\delta(t - m\,T)$** which we expand as $\delta(t - mT) = \sum_{m=-\infty}^{\infty} e^{i\,\omega_{rev} t}$. Then the betatron equation to be treated is $\ddot{x} + \omega_\beta^2 x = \omega_\beta^2\, u(t) \sum_{m=-\infty}^{\infty} e^{i\,m\,\omega_{rev} t}$ with the result $\overline{A^2} = \sum_{-\infty}^{\infty} \frac{u^2_{1rms}\; t}{\{(im\omega_{rev} + \omega_\beta)L\}^2}$

---> noise components with frequencies near the betatron sidebands $(n\pm Q)\omega_{rev}$ (only) lead to a linear increase of the mean squared betatron amplitudes (and emittance)



Hereward and Johnson's (yellow) report on noise (CERN/60-38)


60-38
Proton Synchrotron Division
4th October, 1960

ORGANISATION EUROPÉENNE POUR LA RECHERCHE NUCLÉAIRE
CERN EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

The Effect of Radio-Frequency Programme
Noise on the Phase-stable Acceleration Process

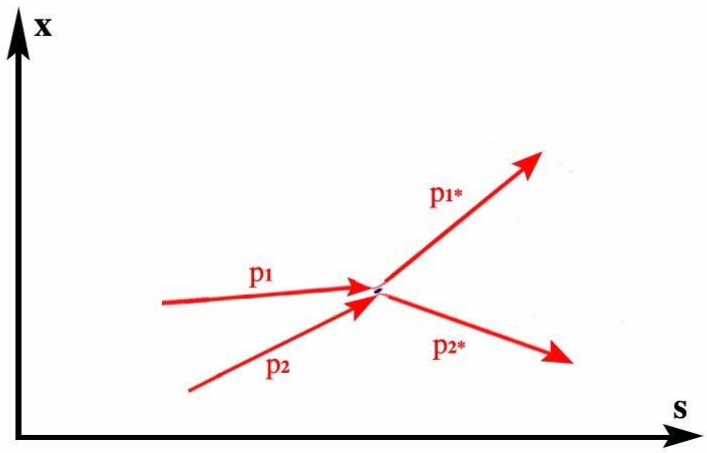
by

H.G. Hereward and K. Johnsen

GENEVE

Intra-beam scattering

Small angle (multiple) Coulomb scattering between particles of the beam.
In the collisions, energy transfer: longitudinal <----> horizontal <---->vertical occurs



Outline of the calculation

(due to A.Piwinski, 1974)

- Transform the momenta of the two colliding particle into their centre of mass system.
- Calculate the change of the momenta using the Rutherford cross-section.
- Transform the changed momenta back into the laboratory system.
- Calculate the change of the emittances due to the change of momenta at the given location of the collision.
- Take the average over all possible scattering angles (impact parameters from the size of the nucleus to the beam radius).
- Assume a 'Gaussian beam' (in all three planes). Take the average over momenta and transverse position of the particles at the given location on the ring circumference.
- Finally calculate the average around the circumference (taking the lattice function of the ring into account) to determine the change per turn.

Particularities of IBS.

The sum of the three emittances

- For constant lattice functions and below transition energy, the sum of the three emittances is constant (the beam behaves like a 'gas. in a box').
- Above transition the sum of the emittances always grows (due to the negative mass effect, i.e. particles 'being pushed go around slower').
- In any strong focussing lattice the sum of the emittances always grows (also below transition because of the 'friction' due to the derivatives of the lattice functions).
- The increase of the 6-dimentional phase space volume can be explained by transfer of energy from the common longitudinal motion into transverse energy spread.
- Although the sum grows there can be strong transfer of emittance and theoretically even reduction in one at the expense of fast growth in another plane (in practise the reduction in one plane has not been observed).

Particularities of IBS, scaling.

The exact IBS growth rates have to be calculated by computer codes. One determines "form factors" F giving $1/\tau_{x,y,l} = 1/\tau_0 * F_{x,y,l}$. The basic scaling is given by

$$\boxed{1/\tau_0 = \frac{N_b r_0^2 \left(\frac{q^2}{A}\right)^2}{(4/\pi^2)\,\gamma\,\varepsilon_x^*\varepsilon_y^*\varepsilon_l^* / E_0} \quad \propto \quad \frac{N_b \left(\frac{q^2}{A}\right)^2}{\gamma\,\varepsilon_x^*\varepsilon_y^*\varepsilon_l^*}}$$

N_b : number of particles per bunch, **r_0** : classical proton radius,

$\varepsilon_{x,y}^* = \pi\beta\gamma\sigma_{x,y}^2 / \beta_{x,y}$, $\varepsilon_l^* = \pi\beta\gamma\sigma_l\sigma_{\Delta p/p}(E_0/c)$ **:** normalised 1 σ emittances of bunch

β, γ **:** relativistic factors, **E_0** : proton rest mass

One notes:

- **Strong dependence on ion charge (q^4/A^2)**
- **Linear dependence on (normalised) phase space density ($N_b/(\varepsilon_x^* \varepsilon_y^* \varepsilon_l^*)$)**
- **Weak dependence on energy ($1/\gamma$)**

Conclusions.

- **Emittance (i.e. beam density) preservation is a mayor concern in the design of modern accelerators and colliders.**

- **There is a great number of 'single particle' and 'collective' effects which have to be perfectly controlled to avoid excessive 'beam heating'.**

- **Beam cooling (not treated in this talk) can –to some extent-- be used to fight emittance growth and even lead to very small equilibrium emittances. These are the result of the balance between the cooling and (many of) the heating mechanisms mentioned.**

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