# INSTABILITIES IN LINACS

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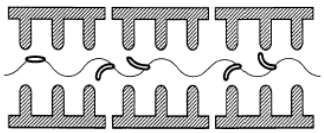


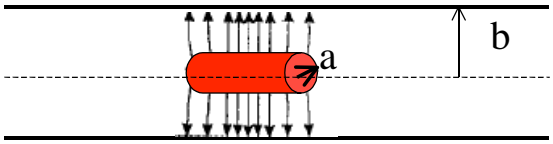
CAS THE CERN ACCELERATOR SCHOOL

**Zeuthen, 15-26 September 2003**

- **Introduction to Wake fields**
- **Instability in Linacs:Beam Beak Up**

# Circular Perfectly Conducting Pipe

## - Beam at Center- Static Approximation $\gamma \to \infty$




$$\varphi(b) = 0$$

$$E_r = \frac{I}{2\pi\varepsilon_o a^2 v} r \quad \textit{for} \quad r \le a$$

$$E_r = \frac{I}{2\pi\varepsilon_o v}\frac{1}{r} \quad \textit{for} \quad r > a$$

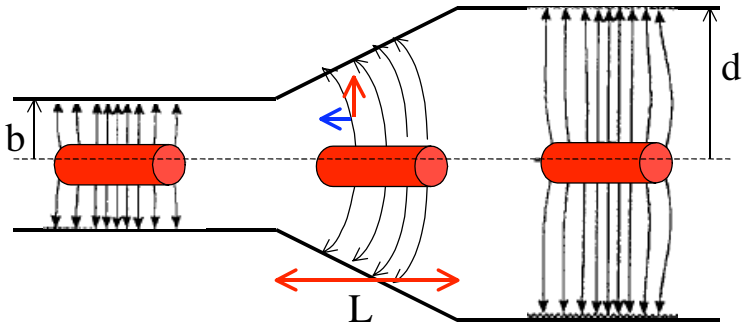
$$B_\vartheta = \frac{\beta}{c} E_r$$

$$\varphi(r) = \int_r^b E_r(r')dr'$$

$$= \frac{I}{4\pi\varepsilon_o v}\left(1 + 2\ln\frac{b}{a} - \frac{r^2}{a^2}\right) \quad \textit{for} \quad r \le a$$

$$= \frac{I}{2\pi\varepsilon_o v}\ln\frac{b}{r} \quad \textit{for} \quad a \le r \le b$$

# Circular Perfectly Conducting Pipe with Transition




There is a longitudinal $E_z(r,z)$ field in the transition and a test particle experience a voltage given by:

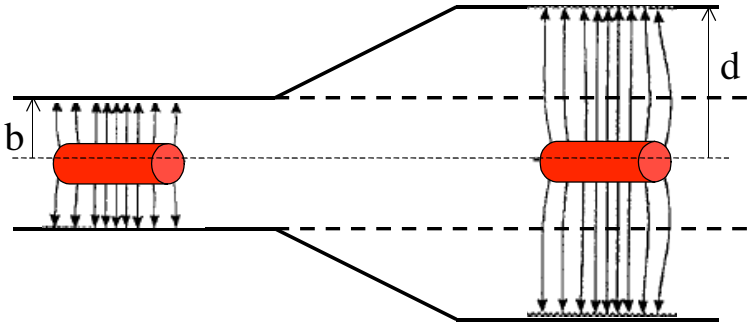
$$V = -\int_0^L E_z(r,z)dz = \varphi(r,L) - \varphi(r,0) = \frac{I}{2\pi\varepsilon_o v}\ln\frac{d}{b}$$

decelerating if d > b

$$P_b = VI = \frac{I^2}{2\pi\varepsilon_o v}\ln\frac{d}{b}$$

Power lost by the beam

**For d > b the power is deposited to the energy of the fields: moving from left to right the beam induces the fields in the additional space available**




The additional power passing through the right part of the beam pipe is:

$$P_{em} = \int_b^d \frac{E_r B_\vartheta}{\mu} 2\pi r dr = \frac{I^2}{2\pi\varepsilon_o v} ln\frac{d}{b}$$

The image charges travel with the same particle velocity v.
Since both the particles and the image charges move on parallel paths, in the limit **v = c** they do not interact with each other, no matter how close to the wall the particles are.

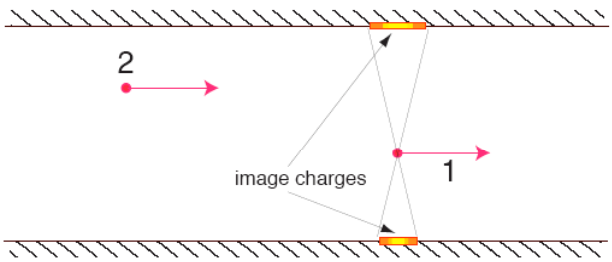



**FIGURE 2.** Particles traveling inside a perfectly conducting pipe of arbitrary cross section. Shown are the image charges on the wall generated by the leading charge.

If a beam particle moves along a straight line with the speed of light, the electromagnetic field of this particle scattered off the boundary discontinuities will not overtake it and, furthermore, will not affect the charges that travel ahead of it.

The field can interact only with the trailing charges in the beam that move behind it. This constitutes the principle of causality in the theory of wake fields

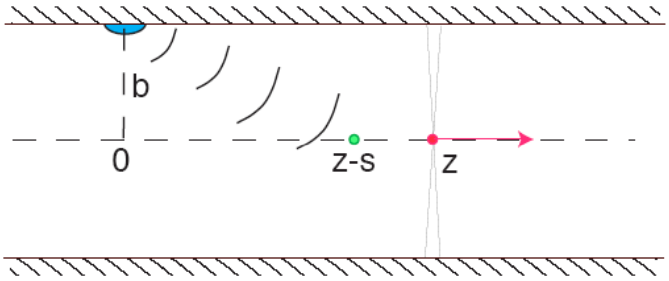
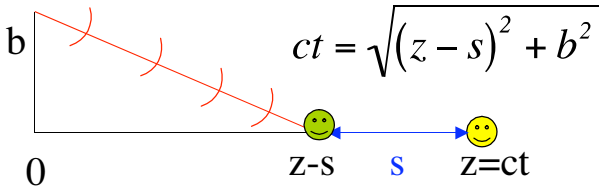



**FIGURE 3.** A wall discontinuity located at $z = 0$ scatters the electromagnetic field of an ultrarelativistic particle. When the particle moves to location $z$, the scattered field arrives to point $z - s$ .

We can estimate the distance at which the electromagnetic field produced by a leading charge reaches a trailing particles traveling at a distance s behind.




$$z^2 = (z-s)^2 + b^2 \quad \Longrightarrow \quad z_{catch-up} \approx \frac{b^2}{2s} \quad \textit{for } s \ll b$$

Only after the leading charge has traveled $z_{catch-up}$ away from the discontinuity, can a particle at point **s behind it** feel the field generated by the discontinuity.

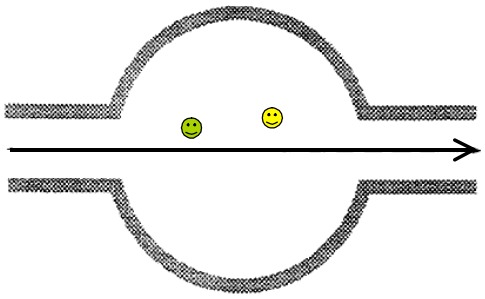
## Numerical Analysis

The study of the fields requires to solve the Maxwell equations in a given structure taking the beam current as source of fields. This is a quite complicated task for which it has been necessary to develop dedicated computer codes, which solve the e.m. problem in the frequency or in the time domain. There are several useful codes for the design of accelerator devices: **MAFIA, ABCI, URMEL,** etc...

## Theoretical Analysis

The parasitic fields depends on the particular charge distribution of the beam. It is therefore desirable to know what is the effect of a single charge, i.e. **find the Green function**, in order to reconstruct the fields produced by any charge distribution.

# Wake Potentials



$$\boldsymbol{F} = q\left[E_z\hat{z} + \left(E_x - vB_y\right)\hat{x} + \left(E_y + vB_x\right)\hat{y}\right] \equiv \boldsymbol{F}_{//} + \boldsymbol{F}_{\perp}$$

there can be two effects on the **test charge** :

1) a longitudinal force which changes its energy,

2) a transverse force which deflects its trajectory.

If we consider a device of length L:

the Energy Gain (Joule) is:

$$U = \int_0^L F_z ds$$

the Transverse Deflecting Kick (Newton · meter) is:

$$\boldsymbol{M} = \int_0^L \mathbf{F}_\perp ds$$

These quantities, normalised to the charges, are called ***wake-potentials*** (volt/coulomb) and are both function of the distance z.

Note that the integration is performed over a given path of the trajectory.

***Longitudinal wake potential***
*(Volt/Coulomb)*

$$w_{//} = -\frac{U}{q_o q}$$

***Transverse wake potential***
*(Volt/Coulomb meter)*

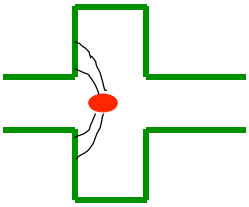
$$\boldsymbol{w}_\perp = \frac{1}{r_o}\frac{\boldsymbol{M}}{q_o q}$$

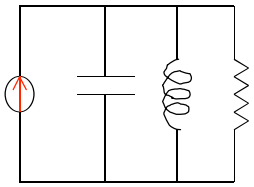
The sign minus in the longitudinal wake-potential means that the test charge loses (gains) energy when the wake is positive.

Positive transverse wake means that the transverse force is defocusing.

# Longitudinal wake potential of a resonant HOM

When a charge crosses a resonant structure, it excites the fundamental mode and high order modes (HOM). Each mode can be treated as an electric RLC circuit loaded by an impulsive current.





Just after the charge passage, the capacitor is charged with a voltage $V_o = Cq_o$ and the electric field is $E_{so} = V_o / l_o$.

The time evolution of the electric field is governed by the same differential equation of the voltage

$$\ddot{V} + \frac{1}{RC}\dot{V} + \frac{1}{LC}V = \frac{1}{C}\dot{I}$$

The passage of the impulsive current charges only the capacitor, which changes its potential by an amount $V_c(0)$.

This potential will oscillate and decay producing a current flow in the resistor and inductance.

For t>0 the potential satisfy the following equation and boundary conditions:

$$\ddot{V} + \frac{1}{RC}\dot{V} + \frac{1}{LC}V = 0$$

$$V(t = 0^+) = \frac{q}{C} \equiv V_0$$

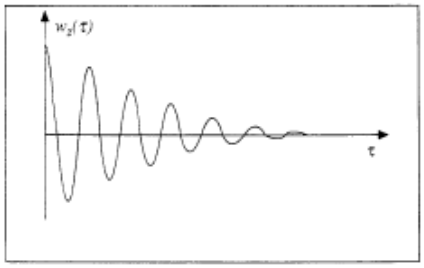
$$\dot{V}(t = 0^+) = \frac{\dot{q}}{C} = \frac{I(0^+)}{C} = \frac{V_0}{RC}$$

$$V(t) = V_0 e^{-\gamma t}\left[\cos(\bar{\omega} z / c) - \frac{\gamma}{\bar{\omega}}\sin(\bar{\omega} t)\right]$$

$$\bar{\omega}^2 = \omega_r^2 - \gamma^2$$

putting z = -ct (z is negative behind the charge),

$$w_{//}(z) = \frac{-V(z)}{q_o} = w_o e^{\gamma z / c}\left[\cos(\bar{\omega} z / c) + \frac{\gamma}{\bar{\omega}}\sin(\bar{\omega} z / c)\right]$$



It is also useful to define the ***loss factor*** as the normalised energy lost by the **source charge $q_o$**

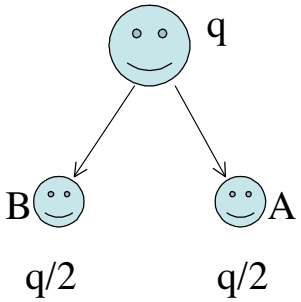
$$k = -\frac{U(z=0)}{q_o^2}$$

Although in general the loss factor is given by the longitudinal wake at z=0, for charges travelling with the light velocity the longitudinal wake potential is discontinuous at z=0

The exact relationship between *k* and *w(z→0)* is given by the ***beam loading theorem*:**

$$k = \frac{w_{//}(z \to 0)}{2}$$

Causality requires that the longitudinal wake potential of a charge travelling with the velocity of light is discontinuous at the origin.




$$U_A = q_A^2 k = \frac{q^2}{4} k$$

$$U_B = q_B^2 k + q_A q_B W_{//}(s)$$

$$= \frac{q^2}{4} k + \frac{q^2}{4} W_{//}(s)$$

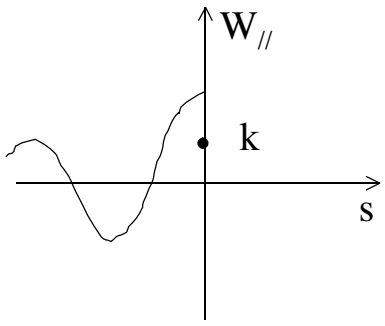
$$U_A + U_B = \frac{q^2}{2} k + \frac{q^2}{4} W_{//}(s)$$

$$s \to 0 \quad U_A + U_B = q^2 k$$

$$\frac{q^2}{2} k + \frac{q^2}{4} W_{//}(0) = q^2 k$$

$$\frac{W_{//}(0)}{4} = \frac{k}{2}$$

$$k = \frac{W_{//}(0)}{2}$$



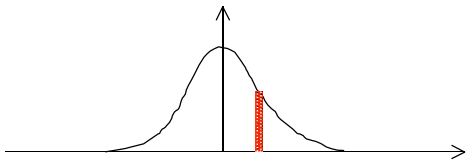
Relationship between transverse and longitudinal forces :
"**Panofsky-Wenzel theorem**".

$$\nabla_{\perp} F_{//} = \frac{\partial}{\partial z} \boldsymbol{F}_{\perp}$$

$$\nabla_{\perp} w_{//} = \frac{\partial}{\partial z} \boldsymbol{w}_{\perp}$$

## Wake potentials and energy loss of a bunched distribution

When we have a bunch with density λ(z), we may wander what is the amount of energy lost or gained by a single charge $e$ in the beam



To this end we calculate the effect on the charge from the whole bunch by means of the convolution integral:

$$U(z) = -e \int_{-\infty}^{\infty} w_{//}(z - z') \lambda(z') dz'$$

Which permits to define the ***wake potential of a distribution***

$$W_{//}(z) = -\frac{U(z)}{q q_o}$$

The total energy lost by the bunch is computed summing up the loss of all particles:

$$U_{bunch} = -\frac{1}{e} \int_{-\infty}^{\infty} U(z') \lambda(z') dz'$$

# Wake Fields Effects In Linear Accelerators: Beam Break Up

A beam injected off-center in a LINAC, because of the focusing quadrupoles, execute betatron oscillations. The displacement produces a transverse wake field in all the devices crossed during the flight, which deflects the trailing charges.

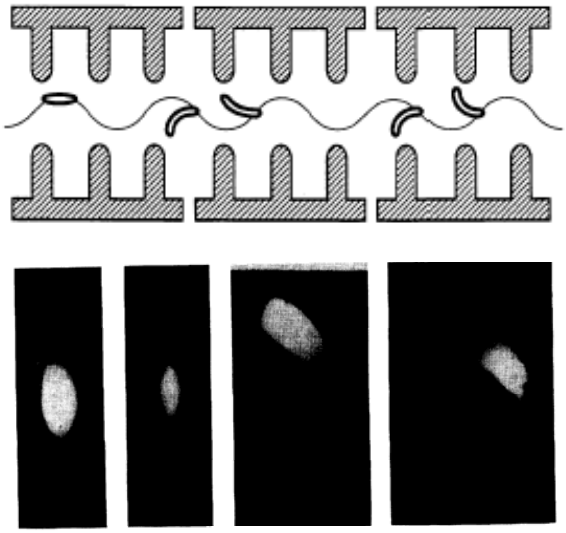
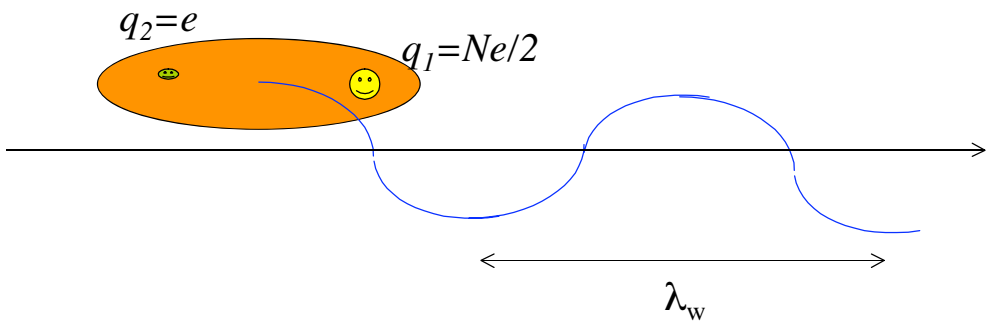


**Figure 3.4.** Four transverse beam profiles observed at the end of the SLAC linac are shown when the beam was carefully injected and injected with 0.2, 0.5, and 1 mm offsets. The beam sizes $\sigma_x$ and $\sigma_y$ are about 120 $\mu$m. (Courtesy John Seeman, 1991.)

In order to understand the effect, we consider a simple model with only two charges $q_1=Ne/2$ (leading = half bunch) and $q_2=e$ (trailing = single charge).




the leading charge executes free betatron oscillations: $y_1(s) = \hat{y}_1 \cos\left(\frac{\omega_y}{c} s\right)$

the trailing charge, at a distance $z$ behind, over a length $L_w$ experiences a deflecting force proportional to the displacement $y_1$, and dependent on the distance $z$:

$$\left\langle F_y^{self}(z, y_1) \right\rangle = \frac{Ne^2}{2L_w} w_\perp(z) y_1(s)$$

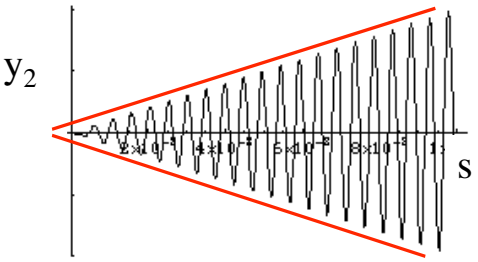
This force drives the motion of the trailing charge:

$$y_2'' + \left(\frac{\omega_y}{c}\right)^2 y_2 = \frac{Ne^2 w_\perp(z)}{\beta^2 E_o L_w}\hat{y}_1 \cos\left(\frac{\omega_y}{c}s\right)$$

This is the typical equation of a resonator driven at the resonant frequency. The solution is given by the superposition of the "free" oscillation and a "forced" oscillation which, being driven at the resonant frequency, grows linearly with *s*.

$$y_2(s) = \hat{y}_2 \cos\left(\frac{\omega_y}{c}s\right) + y_2^{forced}$$

$$y_2^{forced} = \frac{cNe^2 w_\perp(z)s}{4\omega_y E_o L_w}\hat{y}_1 \sin\left(\frac{\omega_y}{c}s\right)$$




At the end of the LINAC of length $L_L$, the oscillation amplitude is grown by (y1=y2):

$$\left(\frac{\Delta\hat{y}_2}{\hat{y}_2}\right)_{max} = \frac{cNew_\perp(z)L_L}{4\omega_y(E_o/e)L_w}$$

# Balakin-Novokhatsky-Smirnov Damping

The BBU instability is quite harmful and hard to take under control even at high energy with a strong focusing, and after a careful injection and steering.

A simple method to cure it has been proposed observing that the strong oscillation amplitude of the bunch tail is mainly due to the **"resonant" driving**.

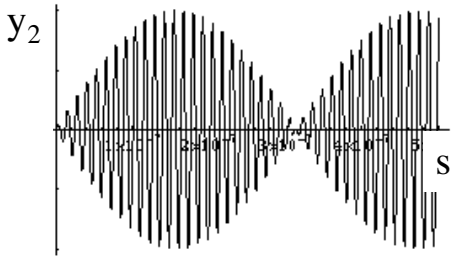
**If the tail and the head move with a different frequency, this effect can be significantly removed.**

Let us assume that the tail oscillate with a frequency $\omega_y+\Delta\omega_y$ , the equation of motion reads:

$$y_2'' + \left(\frac{\omega_y + \Delta\omega_y}{c}\right)^2 y_2 = \frac{Ne^2 w_\perp(z)}{2\beta^2 E_o L_w}\hat{y}_1 \cos\left(\frac{\omega_y}{c}s\right)$$

the solution of which is:

$$y_2(s) = \frac{c^2 Ne^2 w_\perp(z)}{4\omega_y \Delta\omega_y E_o L_w}\hat{y}_1\left[\cos\left(\frac{\omega_y + \Delta\omega_y}{c}s\right) - \cos\left(\frac{\omega_y}{c}s\right)\right]$$



by a suitable choice of $\Delta\omega_y$, it is possible to fully depress the oscillations of the tail.

$$\hat{y}_2 \cos\left(\frac{\omega_y + \Delta\omega_y}{c} s\right) + y_2(s) = \hat{y}_1 \cos\left(\frac{\omega_y}{c} s\right)$$

$$\Delta\omega_y = \frac{c^2 N e^2 w_\perp(z)}{4\omega_y E_o L_w}$$

The extra focusing at the tail can be obtained by:

Using a RFQ, where head and tail see a different focusing strength,

Exploit the energy spread across the bunch which, because of the chromaticity, induces a spread in the betatron frequency. An energy spread correlated with the position is attainable with the external accelerating voltage, or with the wake fields.

## References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