Non−Linear

Imperfections

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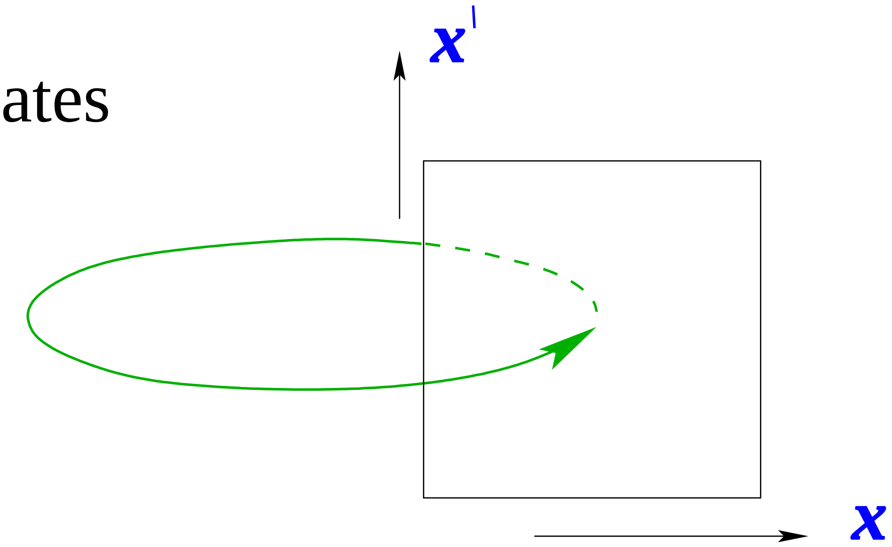
Non−Linear Imperfections

- equation of motion
  - → Hills equation
  - → sine and cosine like solutions + one turn map
- Poincare section
  - → normalized coordinates
- resonances
  - → tune diagram and fixed points
- non−linear resonances
  - → driving terms
- perturbation treatment of non−linear resonances
  - → amplitude growth and detuning — guadrupole
  - → fixed points and slow extraction — sextupole
  - → pendulum model and resonance overlap — octupole
- Hamiltonian dynamics and variable transformations
  - → Hamilton function
  - → generating functions
  - → Equations of motion for action angel variables

Poincare Section I

- Display coordinates after each turn:



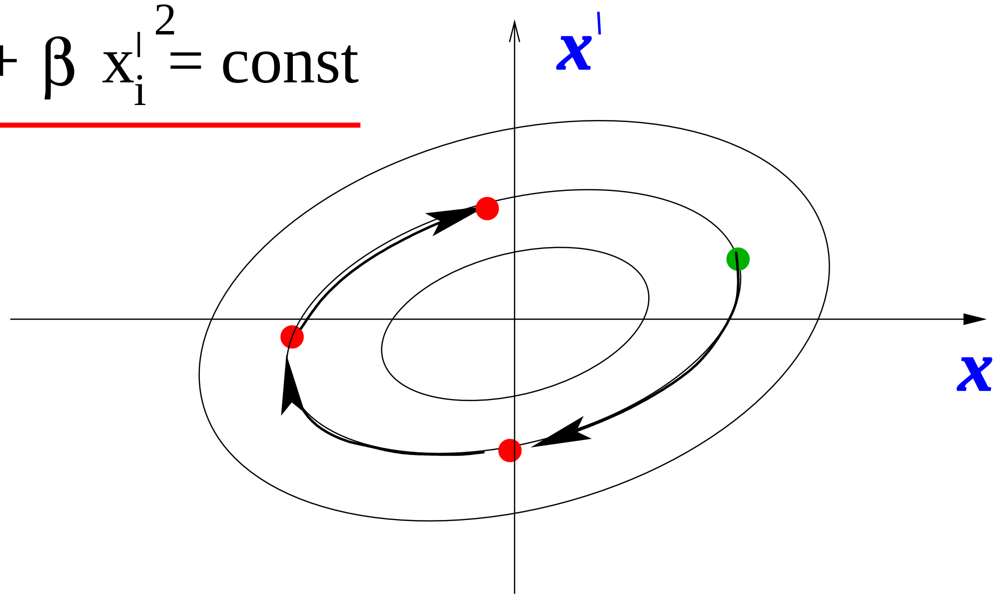
- Linear β − motion:

$$x_i = \sqrt{\beta(s)} \cdot \sin(2\pi Q i + \phi_0)$$

$$x'_i = [\cos(2\pi Q i + \phi_0) + \alpha(s) \cdot \sin(2\pi Q i + \phi_0)] / \sqrt{\beta(s)}$$

$$\gamma x_i^2 + 2\alpha x_i x'_i + \beta x'^2_i = \text{const}$$

ellipse



- the ellipse orientation and the half axis length vary along the machine

Poincare Section II

- for the sake of simplicity assume $\alpha = 0$

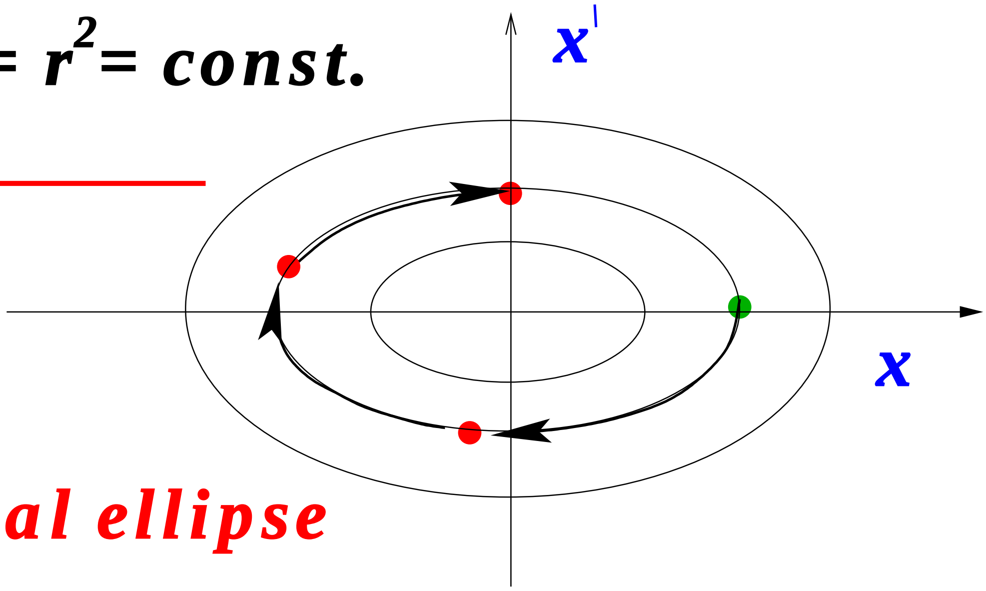
  at the location of the Poincare Section

$\longrightarrow$

$$x = \sqrt{\beta}\, r \cdot \cos(2\pi Q\, i + \phi_0)$$

$$x' = r \cdot \sin(2\pi\, Q\, i + \phi_0)/\sqrt{\beta}$$

$$\frac{x^2}{a^2} + \frac{x'^2}{b^2} = r^2 = const.$$



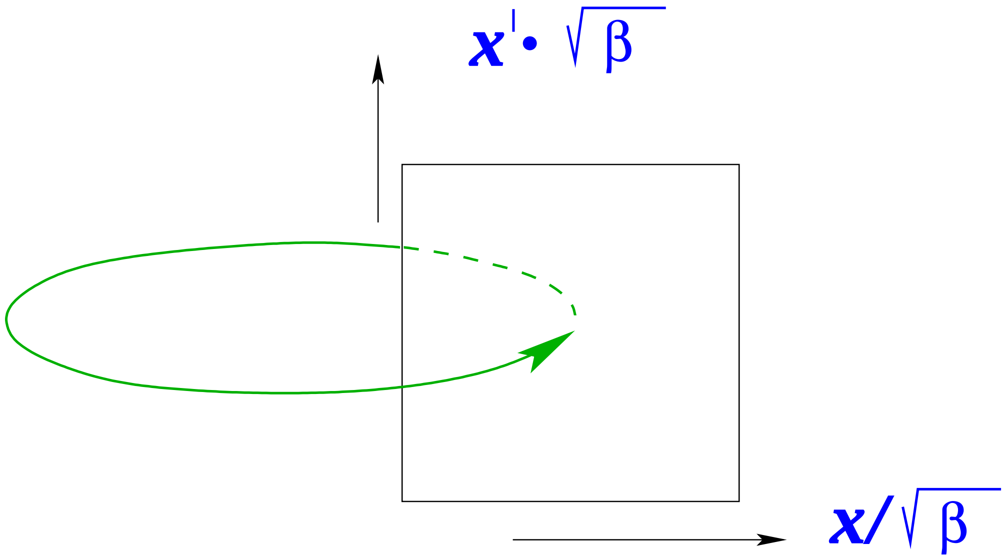
$\longrightarrow$ ***horizontal ellipse***

- for $\alpha \neq 0$

  one can define a new set of coordinates via linear combination of x and x' such that one axis of the ellipse is parallel to x−axis

Poincare Section III

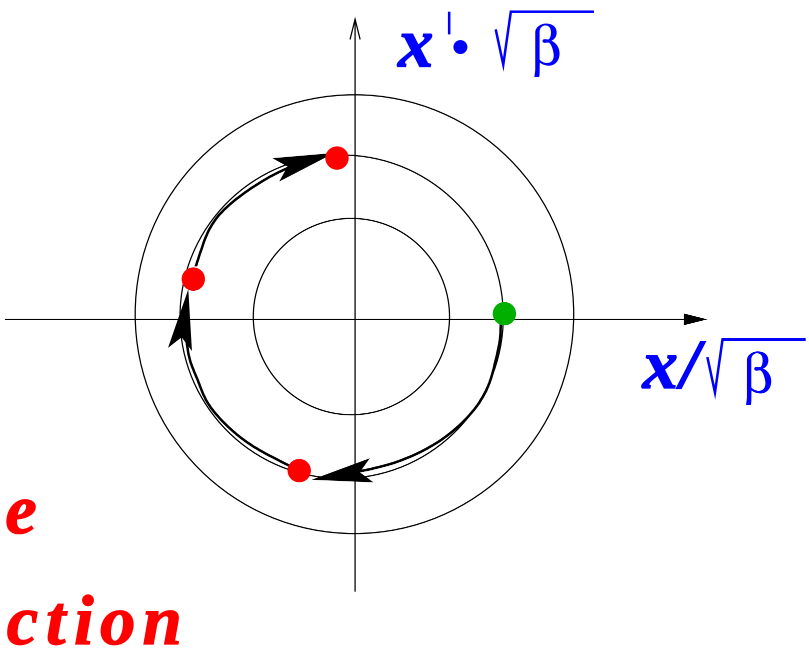
Display normalized coordinates:



normalized coordinates:

$$x/\sqrt{\beta} = r \cdot \cos(2\pi Q\, i + \phi_0)$$

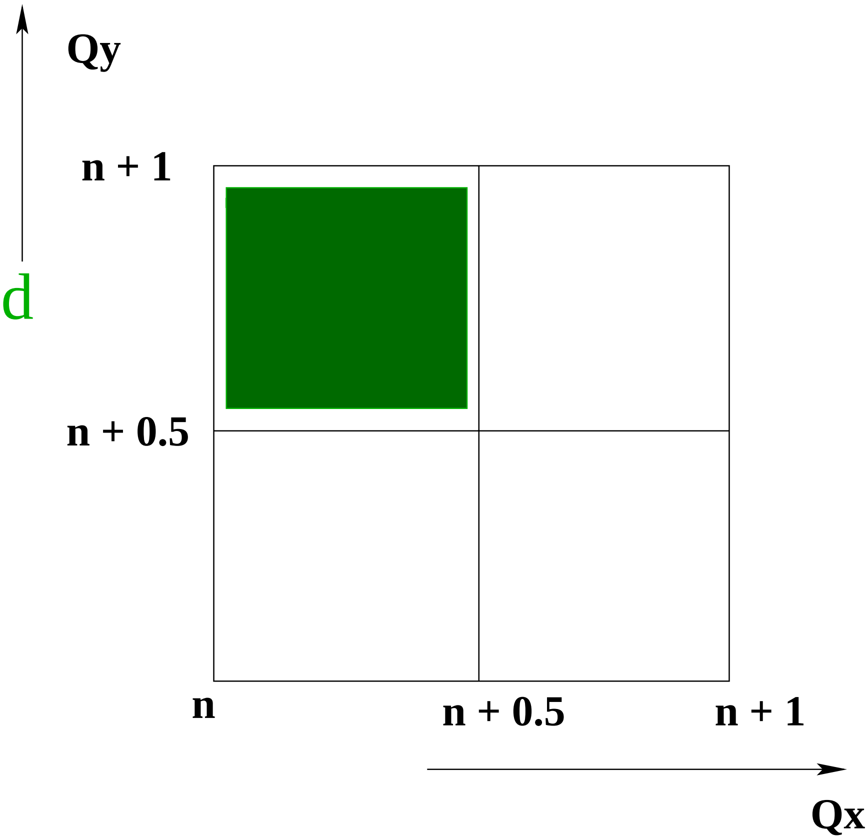
$$\sqrt{\beta} \cdot x' = -r \cdot \sin(2\pi Q\, i + \phi_0)$$



→ circles in the Poincare Section

Resonances I

tune diagram with linear resonances:



stability:

avoid integer and

half integer

resonances!

higher order resonances:

$$n\,Q_x + m\,Q_y = r$$

the rational numbers

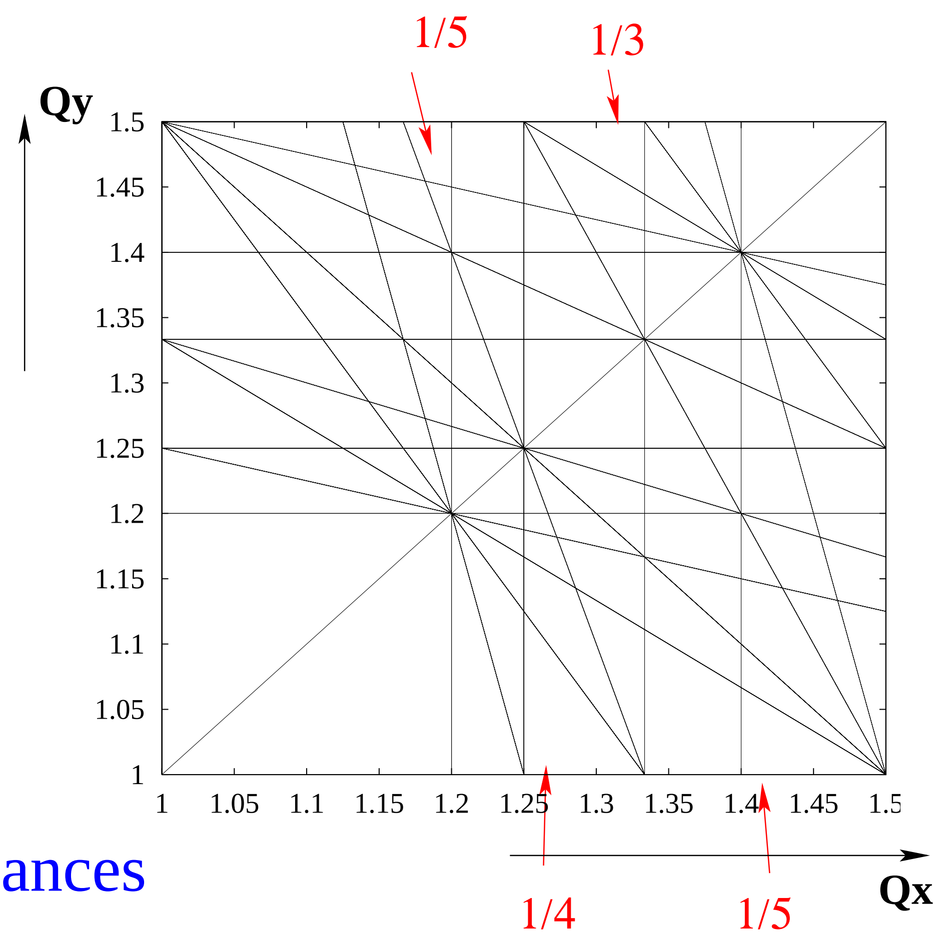
lie 'dense' in the

real numbers

there are resonances

everywhere!

avoid low order resonances

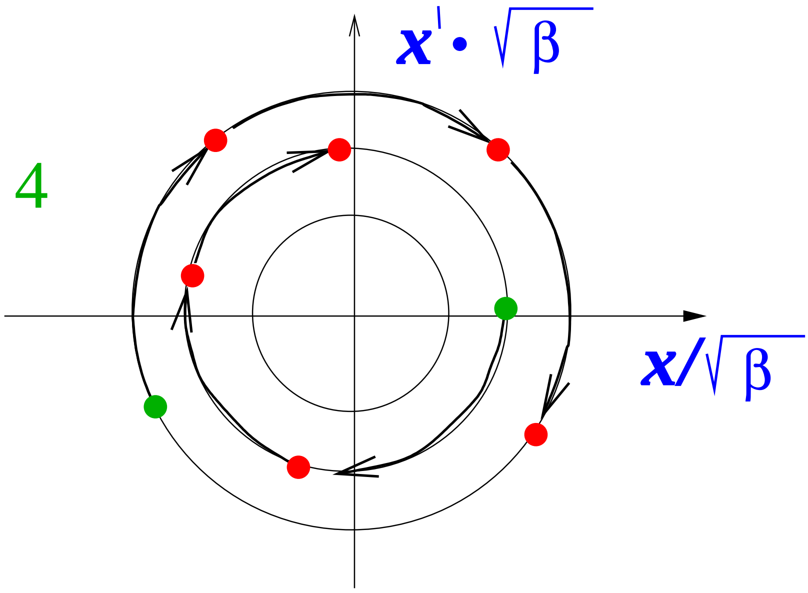


Resonances II

fixed points in the Poincare section:

$Q = N + 1 / n$

example: n = 4



→ *every point is mapped on itself after n turns!*

→ *every point is a 'fixed point'*

→ *motion remains stable if the resonances are not driven*

→ *sources for resonance driving terms?*

Non−Linear Resonances I

- Sextupoles + octupoles

- Magnet errors:

  pole face accuracy

  geometry errors

  eddy currents

  edge effects

- Vacuum chamber:

  LEP I welding

- Beam−beam interaction

→ careful analysis of all components

Non−Linear Resonances II

Taylor expansion for upright multipoles:

$$\boldsymbol{B_y + i \cdot B_x} = \sum_{n=0} \frac{1}{n!} \cdot f_n \cdot \left(x + i\,y\right)^n$$

with: $$f_n = \frac{\partial^{n+1} \mathbf{B}_y}{\partial\, \mathbf{x}^{n+1}}$$

multipole	order	B_x	B_y
dipole	0	0	B_0
quadrupole	1	$f_1\, y$	$f_1\, x$
sextupole	2	$f_2\, x\, y$	$\frac{1}{2} f_2 \cdot (x^2 - y^2)$
octupole	3	$\frac{1}{6} f_3 \cdot (3y\, x^2 - y^3)$	$\frac{1}{6} f_3 \cdot (x^3 - 3x\, y^2)$

skew multipoles:

rotation of the magnetic field by 1/2 of the azimuthal magnet symmetry: 90^o for dipole

45^o for quadrupole

30^o for sextupole; etc

Perturbation I

- perturbed equation of motion:

$$\frac{d^2x}{d\,s^2}+\left(\frac{2\pi}{L}\cdot\,Q_x\right)^2\cdot x=\frac{F_x\,(x,y)}{v\,\cdot p}$$

$$\frac{d^2y}{d\,s^2}+\left(\frac{2\pi}{L}\cdot\,Q_y\right)^2\cdot y=\frac{F_y\,(x,y)}{v\,\cdot p}$$

- assume motion in one degree only:

$y \equiv 0$ is a solution of the vertical equation of motion

$$\longrightarrow \quad B_x \equiv 0; \qquad B_y = \frac{1}{n!}\cdot f_n \cdot x^n \qquad F_x = -v_s \cdot B_y$$

- perturbed horizontal equation of motion:

$$\frac{d^2x}{d\,s^2}+\left(\frac{2\pi}{L}\cdot Q_x\right)^2\cdot x=\frac{-1}{n!}\cdot k_n(s)\cdot x^n$$

- normalized strength:

$$k_n = 0.3\ \cdot\ \frac{f_n\,[T/m^n]}{p\,[GeV/c]}\ ;\quad [k_n] = 1\,/\,m^{n+1}$$

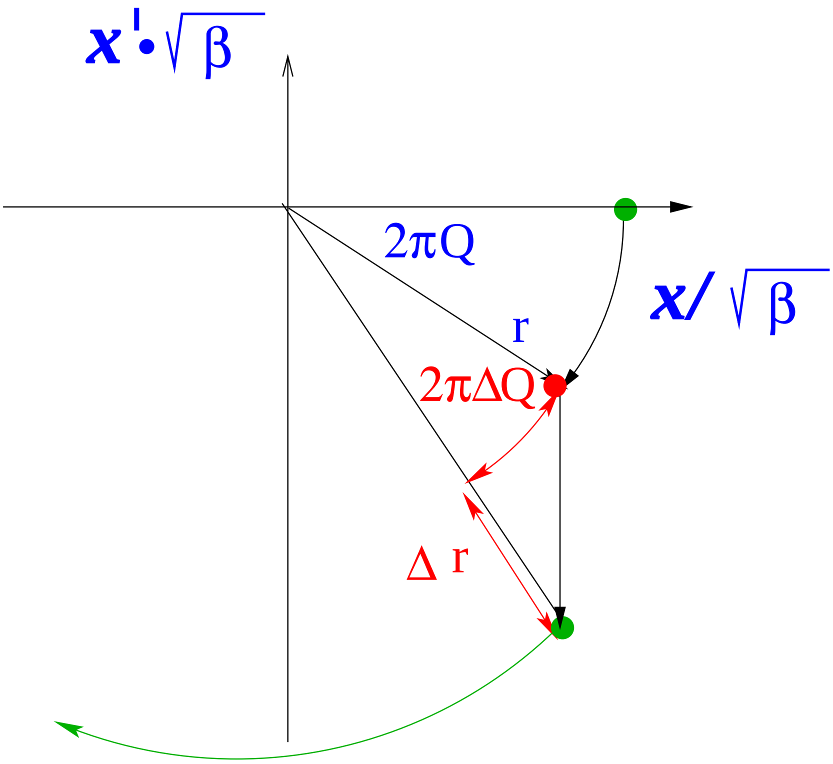
Perturbation II

- perturbation just infront of Poincare Section:

$$\Delta x' = \int \frac{F_y}{v \cdot p}\, ds \longrightarrow = \frac{-l}{2} \cdot k_n \cdot x^n$$

where 'l' is the length of the perturbation

- perturbed Poincare Map:



$k_n > 0$ $\longrightarrow$ increase in 'r'

increase in 'Q'

- stability of particle motion over many turns?

Perturbation III

- coordinates after 'i' itteration and before kick:

(1) $x_i/\sqrt{\beta} = r \cdot \cos(\phi_i) \qquad x'_i \cdot \sqrt{\beta} = -r \cdot \sin(\phi_i)$

(2) with: $\phi_i = \phi_{i-1} + 2\pi Q$

- coordinates after the perturbation kick:

(3) $x_{i+kick}/\sqrt{\beta} = x_i/\sqrt{\beta}$

(4) $x'_{i+kick} \cdot \sqrt{\beta} = x'_i \cdot \sqrt{\beta} + \dfrac{l}{n} \cdot k_n \cdot x_i^n \cdot \sqrt{\beta}$

- write new coordinates in circular coordinates

(5) $x_{i+kick}/\sqrt{\beta} = (r + \Delta r_i) \cdot \cos(\phi_i + \Delta\phi_i)$

(6) $x'_{i+kick} \cdot \sqrt{\beta} = -(r + \Delta r_i) \cdot \sin(\phi_i + \Delta\phi_i)$

Perturbation IV

- solve for 'Δr_i' and '$\Delta\phi_i$':

  → substitute (1) and (2) into (3) and (4)

  → set new expression equal to (5) and (6)

  → use: $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$

  $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

  and: $\sin(\Delta\phi) = \Delta\phi$; $\cos(\Delta\phi) = 1$

  to solve for 'Δr_i' and '$\Delta\phi_i$':

  → $$\Delta r_i = -\Delta x_i' \cdot \sqrt{\beta} \cdot \sin(\phi_i)$$

  $$\Delta\phi_i = \frac{-\Delta x_i' \cdot \sqrt{\beta} \cdot \cos(\phi_i)}{[r + \Delta x_i' \cdot \sqrt{\beta} \cdot \sin(\phi_i)]}$$

- substitute the kick expression:

$$\Delta r_i = \frac{l}{n!} \cdot k_n \cdot x_i^n \cdot \sqrt{\beta} \cdot \sin(\phi_i) \tag{7}$$

$$\Delta\phi_i = \frac{\frac{l}{n!} \cdot k_n \cdot x_i^n \cdot \sqrt{\beta} \cdot \cos(\phi_i)}{[r + \Delta r_i]} \tag{8}$$

Perturbation V

- quadrupole perturbation:

$$\Delta r_i = \boldsymbol{l} \cdot k_1 \cdot x_i \cdot \sqrt{\beta} \cdot \sin(\phi_{\boldsymbol{i}})$$

with: $x_i = \sqrt{\beta} \cdot r \cdot \cos(\phi_{\boldsymbol{i}})$

$$\Delta r_i = \boldsymbol{l} \cdot k_1 \cdot r \cdot \beta \cdot \sin(2\phi_{\boldsymbol{i}})$$

sum over many turns with: $\phi_{\boldsymbol{i}} = 2\pi Q \cdot i$

$$\longrightarrow \boxed{\sum_i \Delta r_i = 0 \quad \text{unless:} \quad Q = p/2}$$

(half integer resonance)

- tune change (first order in the perturbation):

$$\Delta\phi_i = \boldsymbol{l} \cdot k_1 \cdot \beta \cdot [1+\cos(2\phi_{\boldsymbol{i}})]/2$$

average change per turn: $\phi_{\boldsymbol{i}} = 2\pi Q \cdot i$

$$\boxed{\langle \Delta Q_i \rangle = \boldsymbol{l} \cdot k_1 \cdot \beta / 4\pi} \longrightarrow Q = Q_0 + \langle \Delta Q \rangle$$

Perturbation VI

- resonance stop band: $Q \neq p/2$

  the map perturbation generates a tune oscillation

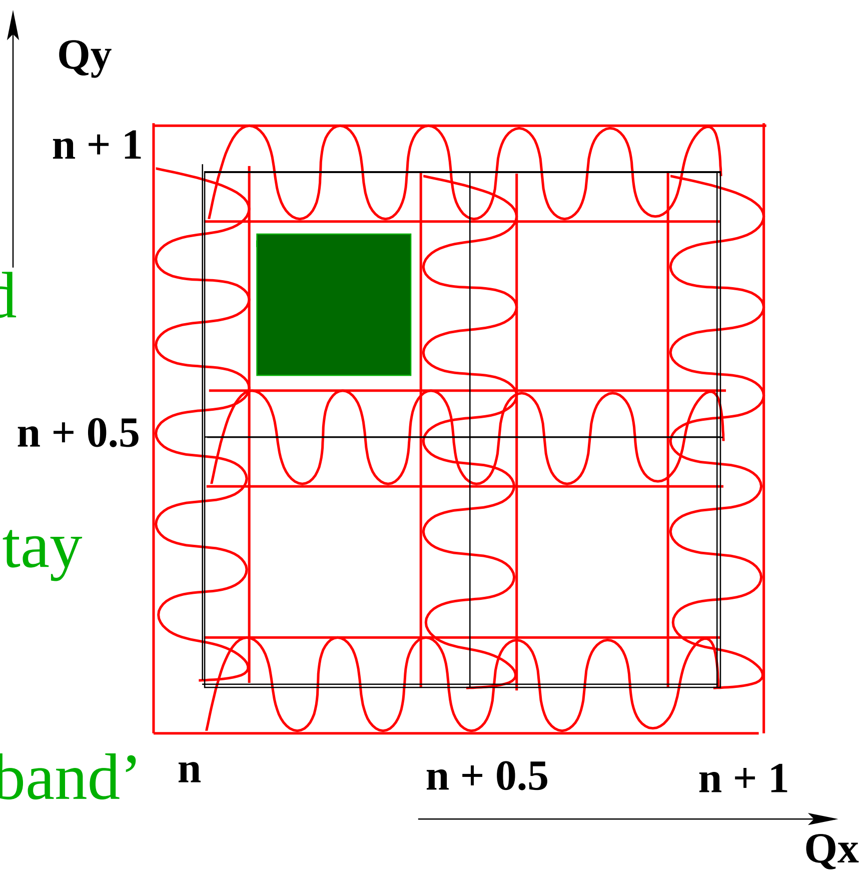
$$\delta Q_i = \boldsymbol{l} \cdot k_1 \cdot \beta \cdot \cos(4\pi \cdot Q \cdot i + 2\phi_0)/4\pi$$

→ particles will experience the half integer resonance if their tune satisfies:

$$\boxed{(p/2 - \langle\Delta Q\rangle) < (Q_0 + \langle\Delta Q\rangle) < (p/2 + \langle\Delta Q\rangle)}$$

- tune diagram:

  avoid integer and half integer resonances and stay away from the resonance 'stop band'



Perturbation VII

- sextupole perturbation:

$$\Delta r_i = l \cdot k_2 \cdot x_i^2 \cdot \sqrt{\beta} \cdot \sin(\phi_i)/2$$

with: $x_i = \sqrt{\beta} \cdot r \cdot \cos(\phi_i)$

$$\Delta r_i = l \cdot k_2 \cdot r_i^2 \cdot \beta^{3/2} \cdot [3 \sin(\phi_i) + \sin(3\phi_i)]/8$$

sum over many turns: $\phi_i = 2\pi Q \cdot i$

$\longrightarrow$ $\boxed{r = 0 \quad \text{unless:} \quad Q = p \text{ or } Q = p/3}$

- tune change (first order in the perturbation):

$$2\pi \Delta Q_i = l \cdot k_2 \cdot r_i \cdot \beta^{3/2} \cdot [3 \cos(2\pi Q i + \phi_0) + \cos(6\pi Q i + 3\phi_0)]/8$$

sum over many turns:
(unless: Q = p or Q = p/3)

$\boxed{\langle \Delta Q \rangle = 0}$

$\longrightarrow$ stop band increases with amplitude!

Perturbation VIII

- what happens for Q = p; p/3 ?

$$\Delta r_i = \boldsymbol{l} \cdot k_2 \cdot r_i^2 \cdot \beta^{3/2} \cdot \boxed{[3\ \sin(2\pi\, Q\, i + \phi_0) + \sin(6\pi\, Q\, i + 3\,\phi_0)]/8}$$

constant for each kick

$$2\pi\, \Delta Q_i = \boldsymbol{l} \cdot k_2 \cdot r_i \cdot \beta^{3/2} \cdot \boxed{[3 \cos(2\pi\, Q\, i + \phi_0) + \cos(6\pi\, Q\, i + 3\phi_0)]/8}$$

amplitude 'r' increases every turn → instability

→ dephasing and tune change

→ motion moves off resonance

→ stop of the instability

→ what happens in the long run?

Perturbation IX

- let us assume: $Q = p/3$

$$\Delta r_i = \boldsymbol{l} \cdot k_2 \cdot r_i^2 \cdot \beta^{3/2} \left[\boxed{3 \sin(\phi_{\boldsymbol{i}})} + \sin(3\phi_{\boldsymbol{i}})\right] / 8$$

$$\Delta\phi_i = \boldsymbol{l} \cdot k_2 \cdot r_i \cdot \beta^{3/2} \left[\boxed{3 \cos(\phi_{\boldsymbol{i}})} + \cos(3\phi_{\boldsymbol{i}})\right] / 8 + 2\pi Q$$

the first terms change rapidly for each turn

→ the contribution of these terms are small and we omit these terms in the following (method of averaging)

$$\rightarrow \quad \Delta r_i = \boldsymbol{l} \cdot k_2 \cdot r_i^2 \cdot \beta^{3/2} \sin(3\phi_{\boldsymbol{i}}) / 8$$

$$\Delta\phi_i = \boldsymbol{l} \cdot k_2 \cdot r_i \cdot \beta^{3/2} \cos(3\phi_{\boldsymbol{i}}) / 8 + 2\pi Q$$

Perturbation X

- fixed point conditions: $Q_0 \gtrsim p/3;\ k_2 > 0$

Δr / turn = 0 and $\Delta\phi$ / turn = $2\pi p / 3$

with: $$\Delta r_i = \boldsymbol{l} \bullet k_2 \bullet r_i^2 \bullet \beta^{3/2} \bullet \sin(3\phi_i) / 8$$

$$\Delta\phi_i = 2\pi Q_0 + \boldsymbol{l} \bullet k_2 \bullet r_i \bullet \beta^{3/2} \bullet \cos(3\phi_i) / 8$$

→ $$\phi_{\text{fixed point}} = \pi/3;\ \pi;\ 5\pi/3;$$

$$r_{\text{fixed point}} = \frac{16\pi\,(Q_0 - p/3)}{\boldsymbol{l}\ k_2\ \beta^{3/2}}$$

→ r = 0 also provides a fixed point in the

x; x' (infinit set in the r, ϕ plane)

Perturbation XI

- fixed point stability:

linearize the equation of motion around the fixed points:

Poincare map: $$r_{i+1} = r_i + f(r_i, \phi_i)$$

$$\phi_{i+1} = \phi_i + g(r_i, \phi_i)$$

single sextupole kick:

→ $$f = \boldsymbol{l} \cdot k_2 \cdot r_i^2 \cdot \beta^{3/2} \sin(3\phi_{\boldsymbol{i}}) / 8$$

$$g = \boldsymbol{l} \cdot k_2 \cdot r_i \cdot \beta^{3/2} \cos(3\phi_{\boldsymbol{i}}) / 8$$

→ linearized map around fixed points:

$$\begin{pmatrix} r_{i+1} \\ \phi_{i+1} \end{pmatrix} = \left. \begin{pmatrix} \frac{\partial r_{i+1}}{\partial r_i} & \frac{\partial r_{i+1}}{\partial \phi_i} \\ \frac{\partial \phi_{i+1}}{\partial r_i} & \frac{\partial \phi_{i+1}}{\partial \phi_i} \end{pmatrix} \right|_{\text{fixed point}} \circ \begin{pmatrix} r_i \\ \phi_i \end{pmatrix}$$

Perturbation XII

- Jacobin matrix for single sextupole kick:

Jacobian matrix

$$\frac{\partial r_{i+1}}{\partial r_i} = 1; \qquad \frac{\partial r_{i+1}}{\partial \phi_i} = -3\boldsymbol{l} \bullet k_2 \bullet \beta^{3/2} \bullet r^2_{\text{fixed point}} / 8$$

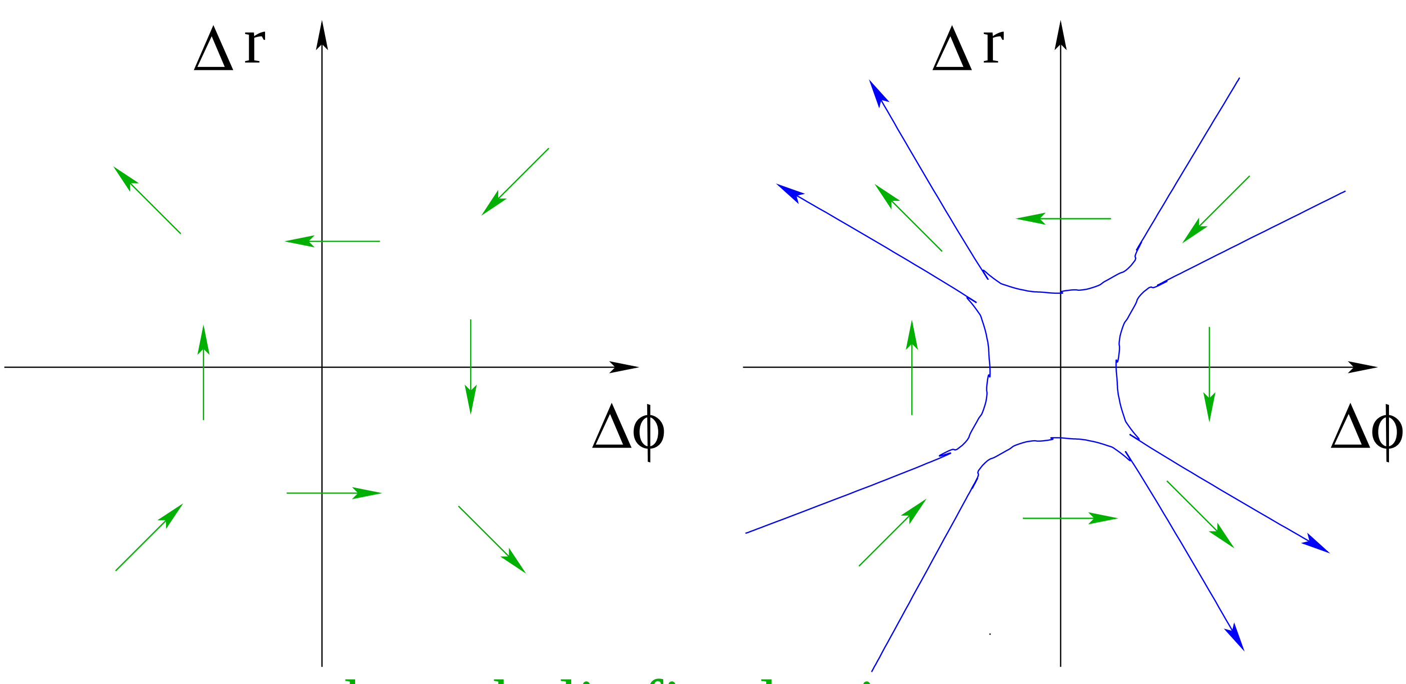
$$\frac{\partial \phi_{i+1}}{\partial r_i} = -\boldsymbol{l} \bullet k_2 \bullet \beta^{3/2} / 8; \qquad \frac{\partial \phi_{i+1}}{\partial \phi_i} = 1$$

$\phi_{\text{fixed point}} = \pi/3; \pi; 5\pi/3;$ and $r_{\text{fixed point}} \neq 0$

$$\Longrightarrow \quad \Delta r_{i+1} = -3\boldsymbol{l} \bullet k_2 \bullet \beta^{3/2} \bullet r^2_{\text{fixed point}} / 8 \bullet \Delta\phi_i$$

$$\Delta\phi_{i+1} = -\boldsymbol{l} \bullet k_2 \bullet \beta^{3/2} / 8 \bullet \Delta r_i$$

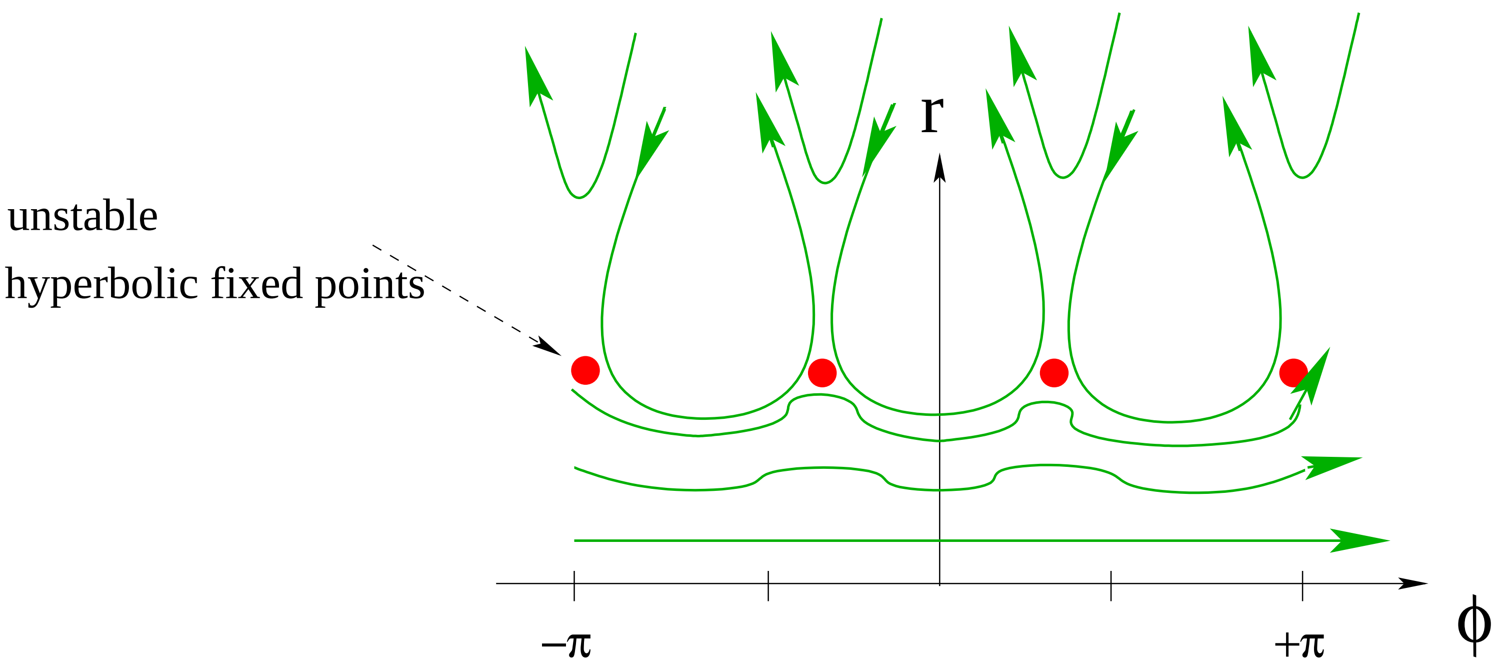
stability?



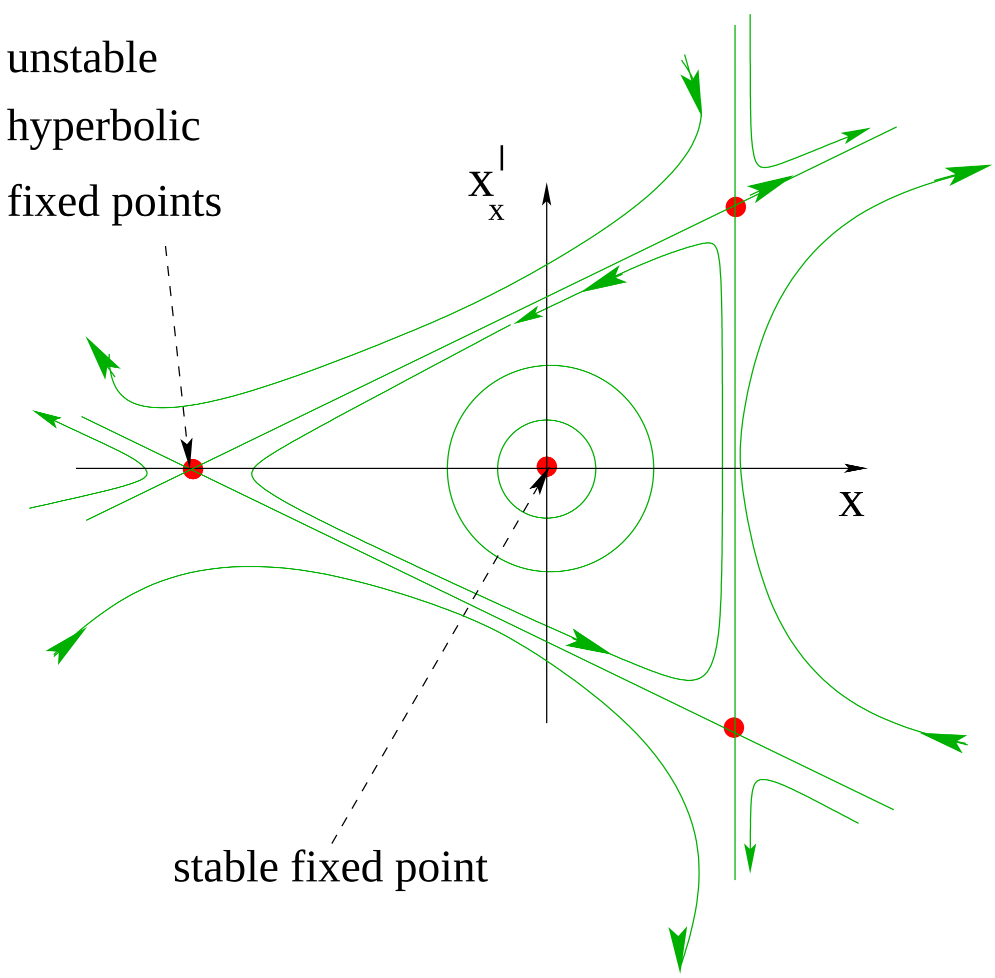
hyperbolic fixed point

Perturbation XIII

- Poincare Section for 'r' and ϕ ':

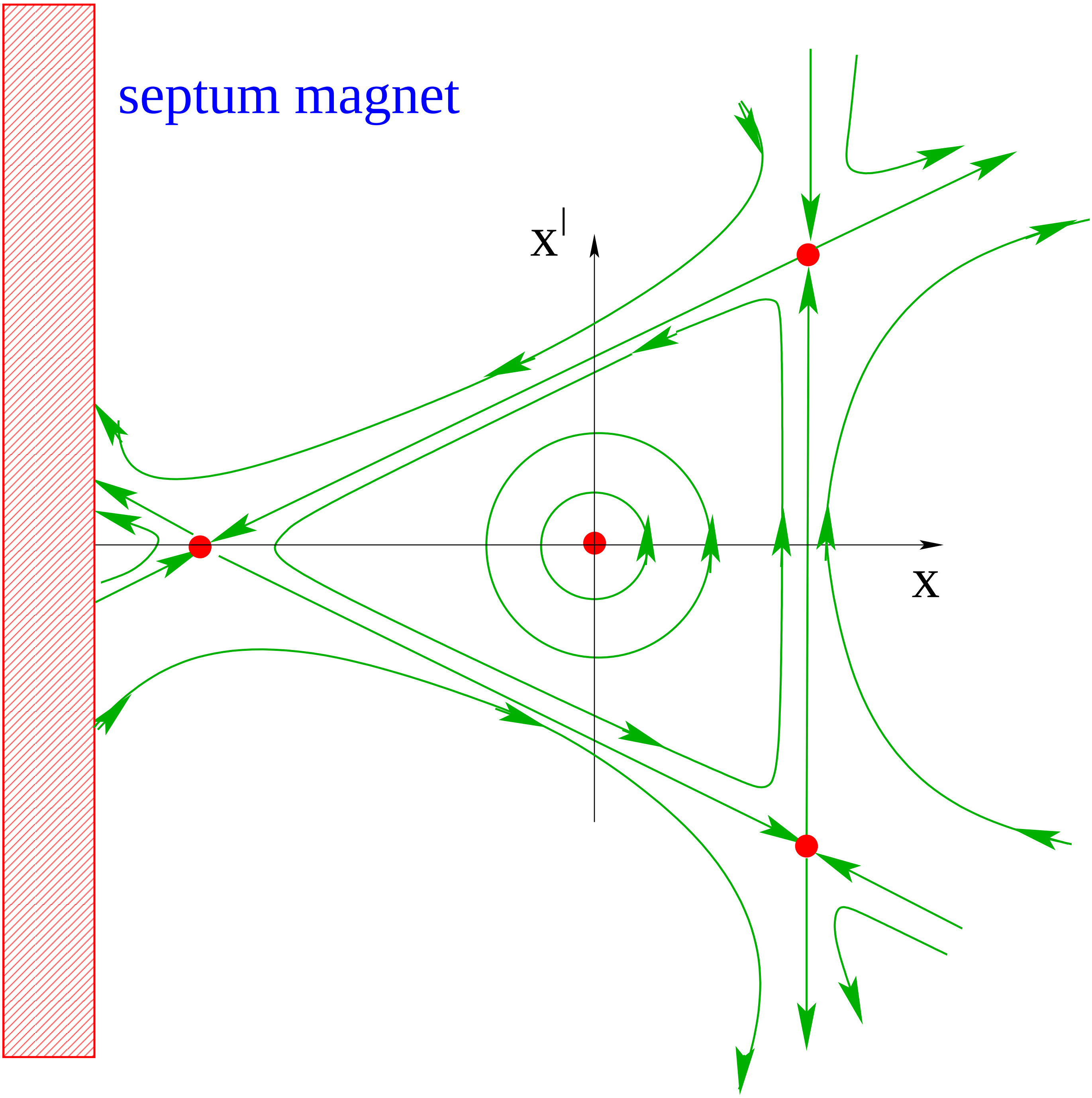


- Poincare section in normalized coordinates:



Perturbation XIV

slow extraction:



fixed point position:

$$r_{\text{fixed point}} = \frac{16\,\pi\,(Q - \frac{p}{3})}{l \cdot k_2 \cdot \beta^{3/2}} \longrightarrow$$

changing the tune during extraction!

Perturbation XV

- octupole perturbation:

$$\Delta r_i = l \cdot k_3 \cdot x_i^3 \cdot \sqrt{\beta} \cdot \sin(\phi_i)/6$$

with: $x_i = \sqrt{\beta} \cdot r \cdot \cos(\phi_i)$

$$\Delta r_i = l \cdot k_3 \cdot r_i^3 \cdot \beta^2 \cdot [4 \sin(2\phi_i) + \sin(4\phi_i)] / 48$$

sum over many turns: $\phi_i = 2\pi Q \cdot i + \phi_0$

$$\boxed{r = 0 \quad \text{unless:} \quad Q = p, p/2, p/4}$$

- tune change (first order in the perturbation):

$$2\pi \Delta Q_i = l \cdot k_3 \cdot r_i^2 \cdot \beta^2 \cdot [4 \cos(4\pi Q i + 2\phi_0) + 3 + \cos(8\pi Q i + 4\phi_0)]/48$$

sum over many turns (unless: $Q = p$ or $Q = p/4$):

$$\boxed{\langle \Delta Q \rangle = l \cdot k_3 \cdot r^2 \cdot \beta^2 / 16 / 2\pi}$$

Perturbation XVI

- detuning with amplitude:

particle tune depends on particle amplitude

→ tune spread for particle distribution

→ stabilization of collective instabilities

→ install octupoles in the storage ring

→ distribution covers more resonances in the tune diagram

→ avoid octupoles in the storage ring

→ requires a delicate compromise

- Poincare section topology:

Q = p/4 and apply method of averaging

→ $$\Delta r_i = \boldsymbol{l} \cdot k_3 \cdot r_i^3 \cdot \beta^2 \cdot \sin(4\phi_{\boldsymbol{i}}) / 48$$

$$\Delta\phi_i = \boldsymbol{l} \cdot k_3 \cdot r_i^2 \cdot \beta^2 \cdot [3 + \cos(4\phi_{\boldsymbol{i}})] / 48 + 2\pi Q$$

Perturbation XVII

- fixed point conditions: $Q_0 \lesssim$ p/4; $k_3 > 0$

Δ r / turn = 0 and $\Delta\phi$ / turn = 2πp / 4

with: $\Delta r_i = \boldsymbol{l} \cdot k_3 \cdot r_i^3 \cdot \beta^2 \cdot \sin(4\,\phi_{\boldsymbol{i}}\,)\,/\,48$

$$\Delta\phi_i = 2\pi\,Q_0 + \boldsymbol{l} \cdot k_3 \cdot r_i^2 \cdot \beta^2 \cdot [3 + \cos(4\,\phi_{\boldsymbol{i}}\,)]\,/\,48$$

$\longrightarrow$ $\phi_{\text{fixed point}} = \pi/2;\ \pi;\ 3\pi/2;\ 2\pi$

$$r_{\text{fixed point}} = \sqrt{\frac{96\,\pi\,(\text{p}/4 - Q_0\,)}{\boldsymbol{l}\ k_3\ \beta^2\ (3{+}1)}}$$

$\longrightarrow$ $\phi_{\text{fixed point}} = \pi/4;\ 3\pi/4;\ 5\pi/4;\ 7\pi/4$

$$r_{\text{fixed point}} = \sqrt{\frac{96\,\pi\,(\text{p}/4 - Q_0\,)}{\boldsymbol{l}\ k_3\ \beta^2\ (3{-}1)}}$$

Perturbation XVIII

- fixed point stability for single octupole kick:

Jacobian matrix

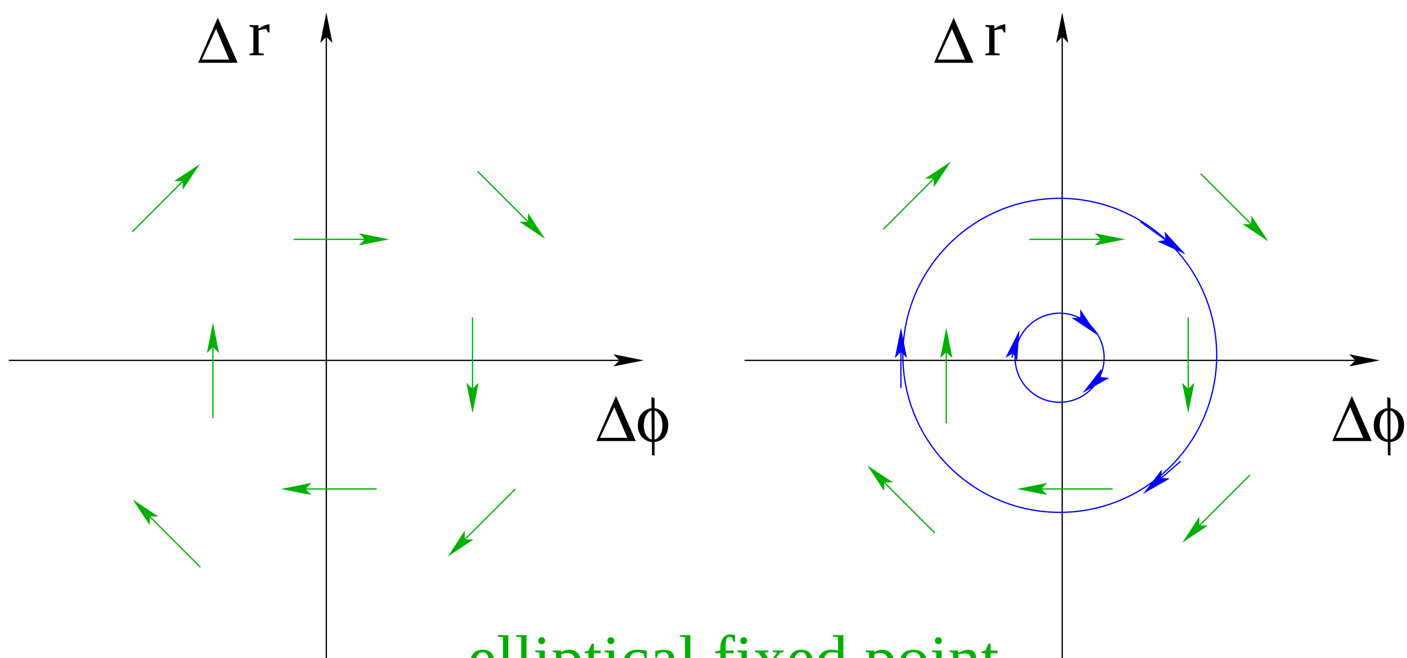
$$\frac{\partial r_{i+1}}{\partial r_i} = 1; \qquad \frac{\partial r_{i+1}}{\partial \phi_i} = \pm\, 4\, \boldsymbol{l} \cdot k_3 \cdot \beta^2 \cdot r^3_{\text{fixed point}} / 48$$

$$\frac{\partial \phi_{i+1}}{\partial r_i} = +\, \boldsymbol{l} \cdot k_3 \cdot \beta^2 \cdot r \;(3 \pm 1) / 24; \qquad \frac{\partial \phi_{i+1}}{\partial \phi_i} = 1$$

$$\longrightarrow \quad \Delta r_{i+1} = \pm\, 4\, \boldsymbol{l} \cdot k_3 \cdot \beta^2 \cdot r^3_{\text{fixed point}} / 48 \cdot \Delta\phi_i$$

$$\Delta\phi_{i+1} = \boldsymbol{l} \cdot k_3 \cdot \beta^2 \;(3 \pm 1) / 24 \cdot \Delta r_i$$

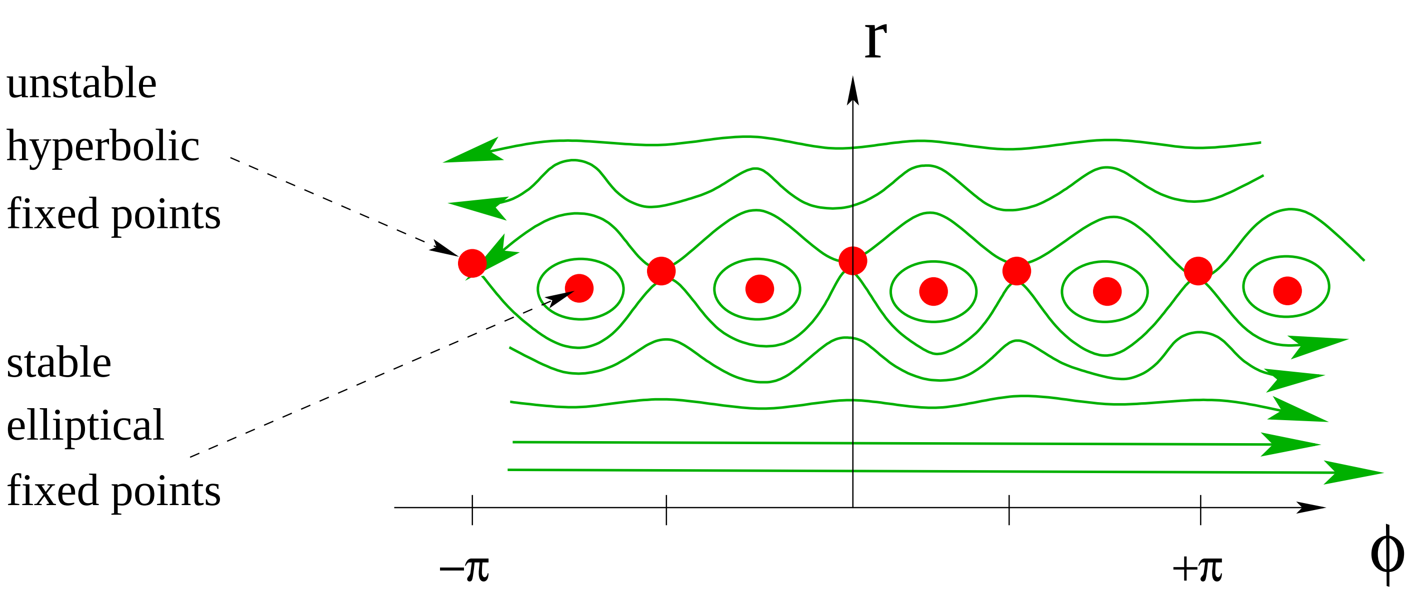
Stability for '−' sign and $k_3 > 0$?



elliptical fixed point

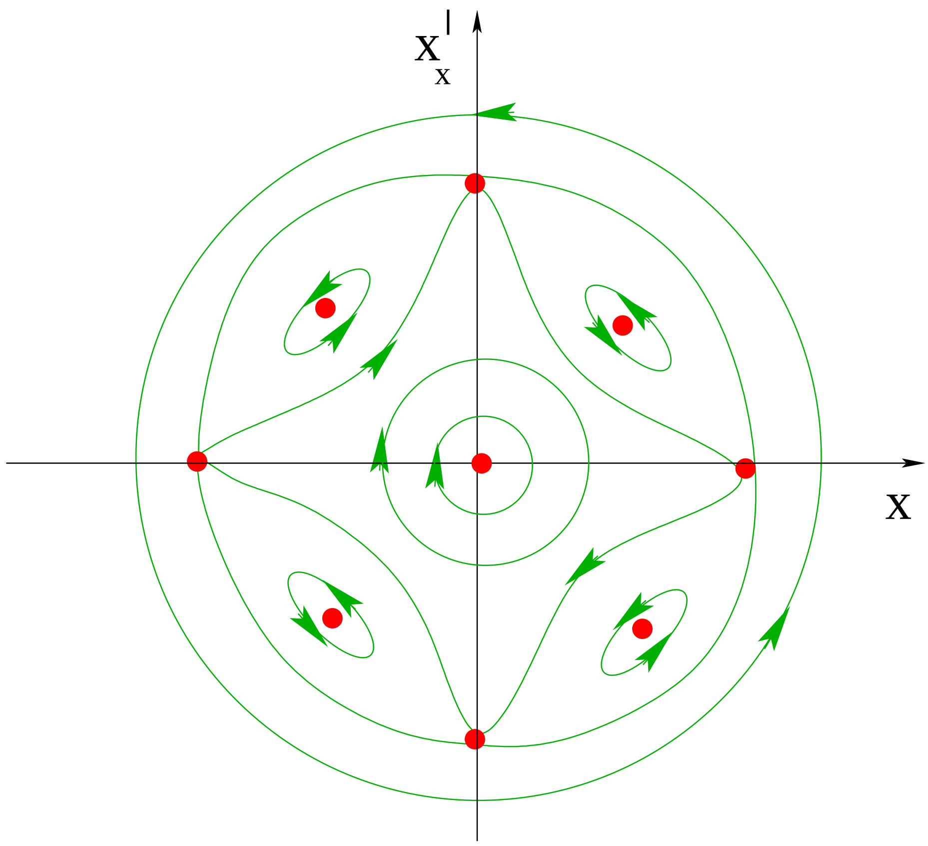
Perturbation XIX

- Poincare Section for 'r' and ϕ ':



island structure

- Poincare section in normalized coordinates:



Perturbation XX

- generic signature of non−linear resonances:

  → chain of resonance islands

- pendulum dynamics:

  expand equation of motion around resonance amplitude

$$\frac{dr}{ds} = -F \cdot \sin(\phi) \qquad \frac{d\phi}{ds} = G \cdot r$$

→ generic equation of motion near resonances

→ resonance width: $\Delta r_{res/max} = 4\sqrt{F / nG}$

island oscillation frequency: $\omega_{island} = \sqrt{F \cdot G/n}$

- pendulum motion:

  libration: oscillation around stable fixed point

  rotation: continous increase of phase variable

  separatrix: separation between the two types

Integrable Systems

- trajectories in phase space do not intersect

  deterministic system

- integrable systems:

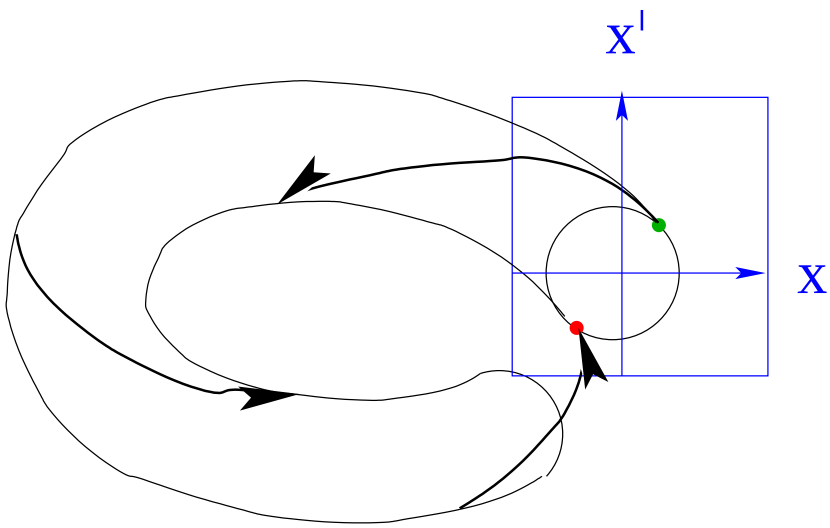
  all trajectories lie on invariant surfaces

  n degrees of freedom

  → n dimensional surfaces

- two degrees of freedom:

  x, s → motion lies on a torus



- Poincare section for two degrees of freedom:

  → motion lies on closed curves

  → indication of integrability

Perturbation XXI

'chaos' and non−integrability:

so far we removed all but one resonance

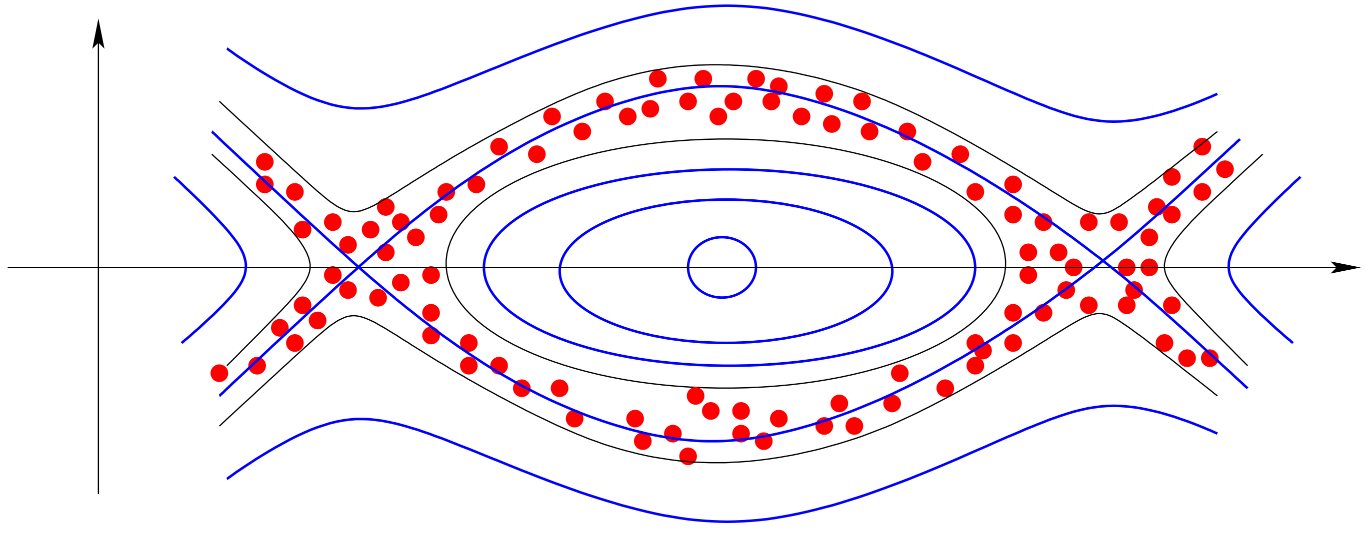
(method of averaging)

→ dynamics is integrable and therefore predictable

re−introduction of the other resonances 'perturbs' the separatrix motion

→ motion can 'change' from libration to rotation

→ generation of a layer of 'chaotic motion'



no hope for exact deterministic solution in this area!

Perturbation XXII

- slow particle loss:

  particles can stream along the ’stochastic layer’

  for 1 degree of freedom (plus ’s’ dependence) the particle amplitude is bound by neighboring integrable lines

  not true for more than one degree of freedom

- global ’chaos’ and fast particle losses:

  if more than one resonance are present their resonance islands can overlap

  $\longrightarrow$ the particle motion can jump from one resonance to the other

  $\longrightarrow$ ’global chaos’

  $\longrightarrow$ fast particle losses and dynamic aperture

Long Term Stability

- **Non-linear Perturbation:**
  - amplitude growth
  - detuning with amplitude
  - coupling

→ **Complex dynamics:**

3 degrees of freedom

\+ 1 invariant of the motion

\+ non-linear dynamics

→ **no global analytical solution!**

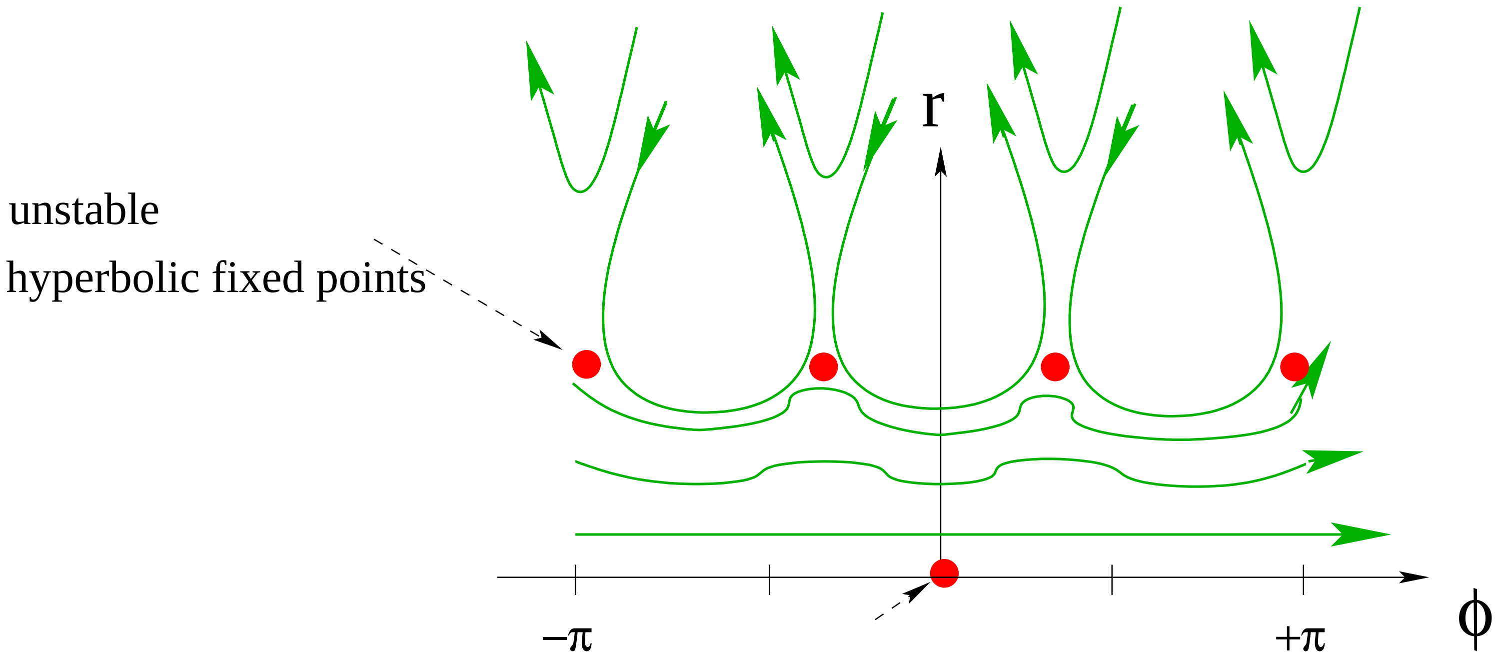
→ analytical analysis relies on perturbation theory

Perturbation XXIII

- why did we not find islands for a sextupole?

  → the pendulum approximation requires an amplitude dependent tune!

  → $$\frac{d\phi}{ds} = G \cdot r$$



- the sextupole detuning term appears only in second order of the kick strength

  → higher order perturbation calculation

Perturbation XXIV

- so far we assumed on the right−hand side:

$$\phi_i = 2\pi\, Q_0 \cdot i + \phi_0$$

this provides only first order solutions

- second order perturbation:

$$r(s) = r_0(s) + \varepsilon\, r_1(s) + \varepsilon^2 r_2(s) + O(\varepsilon^3)$$

$$\phi(s) = \phi_0(s) + \varepsilon\, \phi_1(s) + \varepsilon^2 \phi_2(s) + O(\varepsilon^3)$$

with: $\varepsilon = \beta^{3/2} \cdot \boldsymbol{l} \cdot r_0 \cdot k_2$

- smooth approximation:

$$\frac{dr}{ds} = \frac{\Delta r}{L} \quad \text{and} \quad \frac{d\phi}{ds} = \frac{\Delta\phi}{L}$$

and assume:

β = constant along the machine

Perturbation XXV

- expand equation of motion into a Taylor series around zero order solution

$$\frac{dr}{ds} = f(r, \phi) \qquad \frac{d\phi}{ds} = g(r, \phi)$$

→ single sextupole kick:

$$f = \frac{r^2}{r_0} \bullet [\sin(3\phi) + 3\sin(\phi)] / 8$$

$$g = \frac{r}{r_0} \bullet [\cos(3\phi) + 3\cos(\phi)] / 8$$

→
$$\frac{dr}{ds} = \varepsilon \cdot f + \left[\frac{\partial f}{\partial r} \cdot r_1 + \frac{\partial f}{\partial \phi} \cdot \phi_1\right] \cdot \varepsilon^2 + O(\varepsilon^3)$$

$$\frac{d\phi}{ds} = \frac{2\pi Q}{L} + \varepsilon \cdot g + \left[\frac{\partial g}{\partial r} \cdot r_1 + \frac{\partial g}{\partial \phi} \cdot \phi_1\right] \cdot \varepsilon^2 + O(\varepsilon^3)$$

Perturbation XXVI

- match powers of ε and solve equation of motion in ascending order of ε^n:

- zero order: $$\phi_0(s) = \frac{2\pi p}{3\,L}\cdot s + \frac{2\pi\nu}{3\,L}\cdot s + \phi_0$$

$$r_0(s) = r_0 \qquad (Q = p + \nu)$$

→ substitute into equation of motion and solve for $\phi_1(s)$ and $r_1(s)$

- first order:

$$\phi_1(s) = \frac{1}{2\pi\nu}\cdot\frac{1}{8}\cdot\left[\sin\left(\frac{6\pi\nu}{L}\cdot s + \phi_0\right)/3 + \sin\left(\frac{2\pi\nu}{L}\cdot s + \phi_0\right)\right]$$

$$r_1(s) = \frac{-r_0}{2\pi\nu}\cdot\frac{1}{8}\cdot\left[\cos\left(\frac{6\pi\nu}{L}\cdot s + \phi_0\right)/3 + \cos\left(\frac{3\pi\nu}{L}\cdot s + \phi_0\right)\right]$$

Perturbation XXVII

- second order:

→ substitute $\phi_1(s)$ and $r_1(s)$ into equation

of motion and order powers of ε^2

you get terms of the form: $\frac{dr_2}{ds} = \left[\frac{\partial f}{\partial r} \cdot r_1 + \frac{\partial f}{\partial \phi} \cdot \phi_1 \right]$

$$\frac{d\phi}{ds} = \left[\frac{\partial g}{\partial r} \cdot r_1 + \frac{\partial g}{\partial \phi} \cdot \phi_1 \right]$$

→

$\cos(3\phi) \cdot \cos(3\phi)$; $\cos(3\phi) \cdot \cos(\phi)$; $\cos(\phi) \cdot \cos(\phi)$

→ $\frac{dr}{ds} \propto \cos(6\phi)$; $\cos(4\phi)$; $\cos(2\phi)$; 1

- higher order resonances: ε^n

a single perturbation generates ALL resonances

driving term strength and resonance width

decrease with increasing order!

→ avoid low order resonances!