# *Linear*

# *Imperfections*

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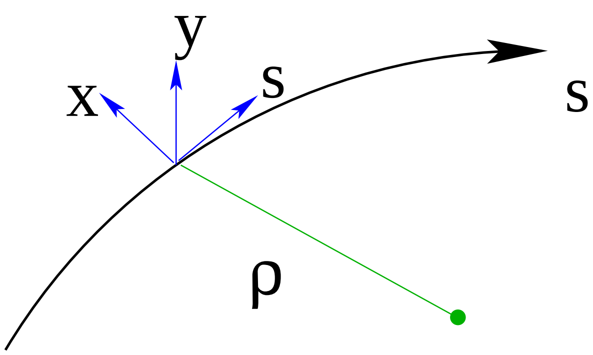
# *Linear Imperfections*

- equation of motion in an accelerator
  - → Hills equation
  - → sine and cosine like solutions + one turn map
- dipole perturbations
  - → closed orbit response
  - → dispersion orbit
  - → integer resonances
- quadrupole perturbations
  - → tune error
  - → beta−beat
  - → half−integer resonances
- orbit correction
  - → local orbit bumps
  - → harmonic filtering
  - → SVD
  - → most effective corrector

# Variable Definition

## Variables in moving coordinate system:




$$x' = \frac{d}{ds} x$$

$$\frac{d}{dt} = \frac{ds}{dt} \cdot \frac{d}{ds} \longrightarrow x' = \frac{p_x}{p_0}$$

($\frac{ds}{dt}$: v)

## Hill's Equation:

$$\frac{d^2x}{d s^2} + K(s) \cdot x = 0; \qquad K(s) = K(s + L);$$

$$K(s) = \begin{cases} 0 & \text{drift} \\ 1/\rho^2 & \text{dipole} \\ 0.3 \cdot \dfrac{B[T/m]}{p[GeV]} & \text{quadrupole} \end{cases}$$

# *Sinelike and Cosinelike Solutions*

- ***system of first order linear differential equations:***

$$\underline{y} = \begin{pmatrix} x \\ x' \end{pmatrix} \longrightarrow \underline{y}' + \begin{pmatrix} 0 & 1 \\ K & 0 \end{pmatrix} \cdot \underline{y} = 0$$

- ***Floquet theorem:***

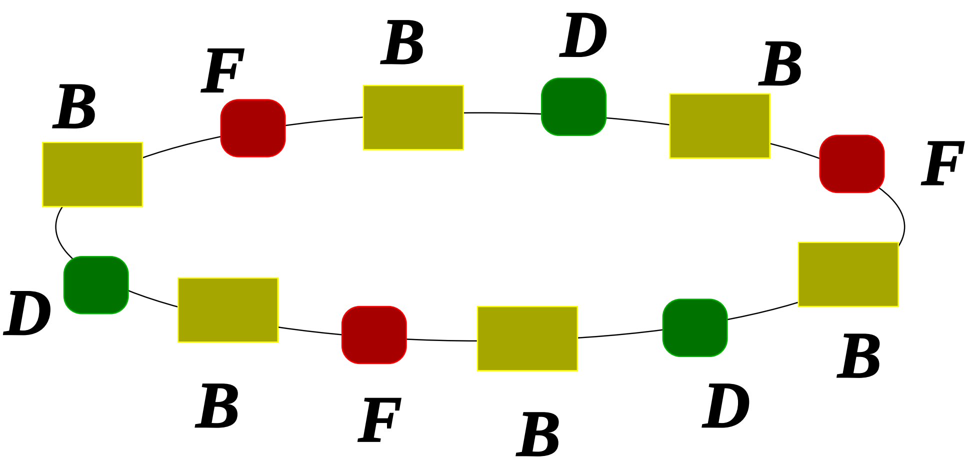
$$\underline{S}(s) = \begin{pmatrix} \sqrt{\beta(s)} \cdot \sin(\phi(s) + \phi_0) \\ [\cos(\phi(s) + \phi_0) + \alpha(s) \cdot \sin(\phi(s) + \phi_0)] / \sqrt{\beta(s)} \end{pmatrix}$$

$$\underline{C}(s) = \begin{pmatrix} \sqrt{\beta(s)} \cdot \cos(\phi(s) + \phi_0) \\ -[\sin(\phi(s) + \phi_0) + \alpha(s) \cdot \cos(\phi(s) + \phi_0)] / \sqrt{\beta(s)} \end{pmatrix}$$

$$\left( \beta(s) = \beta(s + L); \quad \phi(s) = \int \frac{1}{\beta} \, ds; \quad \alpha(s) = -\frac{1}{2} \beta'(s) \right)$$

# Closed Orbit

- **particles oscillate around an ideal orbit:**




- **additional dipole fields perturb the orbit:**

- error in dipole field

- energy error

$$\alpha = \frac{l}{\rho} = \frac{q \cdot B \cdot l}{p + \Delta p} \approx \left(1 - \frac{\Delta p}{p}\right) \cdot \frac{q \cdot B \cdot l}{p}$$

- offset in quadrupole field

$$B_x = -g \cdot y \qquad\qquad B_x = -g \cdot \tilde{y}$$

$$B_y = -g \cdot x \qquad x = x_0 + \tilde{x} \rightarrow B_y = -g \cdot x_0 - g \cdot \tilde{x}$$

dipole component

# Sources for Orbit Errors

- **Quadrupole offset:**
  - alignment +/- 0.1 mm
  - ground motion
    - slow drift
    - civilisation
    - moon
    - seasons
    - civil engineering
- **Error in dipole strength**
  - power supplies
  - calibration
- **Energy error of particles**
  - injection energy (RF off)
  - RF frequency
  - momentum distribtion

# Closed Orbit Response

***inhomogeneous equation:***

$$\frac{d^2x}{ds^2} + K(s)\cdot x = G(s); \qquad G(s) = \frac{F(s)_{Lorentz}}{v\cdot p_0}$$

$$\longrightarrow \quad \underline{y}' + \begin{pmatrix} 0 & 1 \\ K & 0 \end{pmatrix}\cdot \underline{y} = \underline{G}; \quad \underline{G} = \begin{pmatrix} 0 \\ G \end{pmatrix}$$

$$\longrightarrow \quad \underline{y}(s) = a\cdot\underline{S}(s) + b\cdot\underline{C}(s) + \underline{\psi}(s)$$

$\longrightarrow$ ***we need to find only one solution!***

***variation of the constant:***

$$\underline{\psi}(s) = \underline{\varphi}(s)\cdot u(s); \qquad \underline{\varphi}(s) = c\cdot\underline{S}(s) + d\cdot\underline{C}(s);$$

# Closed Orbit Response

■ ***variation of the constant in matrix form:***

$$\underline{\psi}(s) = \underline{\varphi}(s) \bullet u(s); \quad \text{with}$$

$$\underline{\varphi}(s) = \begin{pmatrix} \sqrt{\beta(s)} \bullet \sin(\phi(s)+\phi_0) & \sqrt{\beta(s)} \bullet \cos(\phi(s)+\phi_0) \\ \cos(\phi(s)+\phi_0) / \sqrt{\beta(s)} \ldots & -\sin(\phi(s)+\phi_0) / \sqrt{\beta(s)} \ldots \end{pmatrix}$$

■ ***substitute into differential equation:***

$$\longrightarrow \quad \underline{\varphi}(s) \bullet u'(s) = \underline{G}(s)$$

$$\longrightarrow \quad u(s) = \int_{s_0}^{s} \underline{\varphi}(t)^{-1} \bullet \underline{G}(t)\, dt$$

$$\longrightarrow \quad \underline{y}(s) = a \bullet \underline{S}(s) + b \bullet \underline{C}(s) + \underline{\varphi}(s) \bullet \int_{s_0}^{s} \underline{\varphi}(t)^{-1} \bullet \underline{G}(t)\, dt$$

# *Closed Orbit Response*

***periodic boundary conditions:***

$$\underline{y}(s) = a \cdot \underline{S}(s) + b \cdot \underline{C}(s) + \underline{\varphi}(s) \cdot \int_{s0}^{s} \underline{\varphi}(t)^{-1} \cdot \underline{G}(t)\, dt$$

with

$$\underline{y}(s) = \begin{pmatrix} x(s) \\ x'(s) \end{pmatrix}; \quad x(s) = x(s+L); \quad x'(s) = x'(s+L)$$

→

***periodic boundary conditions determine coefficients*** a ***and*** b

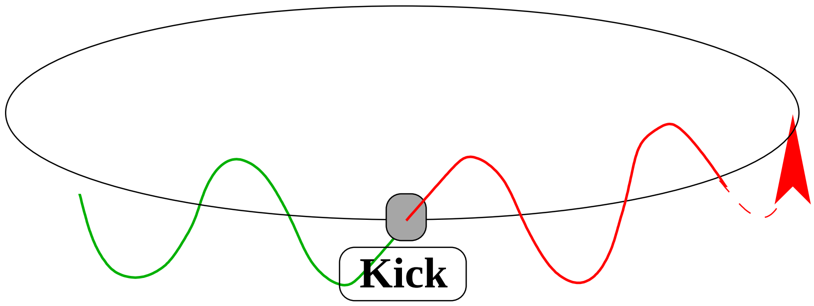
→

$$x(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi \cdot Q)} \cdot \int_{s0}^{s0+circ} \sqrt{\beta(t)} \cdot G(t)\ \cos[\phi(t) - \phi(s) - \pi Q]\, dt$$

# Dipole Error and Orbit Stability

**Q:** ***number of β−oscillations per turn***

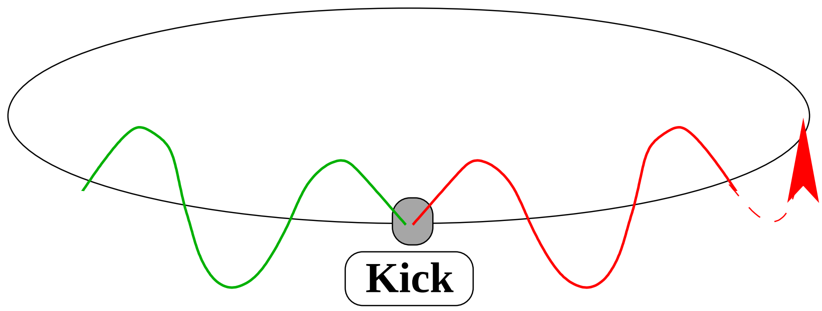
**Q = N**




***the perturbation adds up***

***watch out for integer tunes!***

**Q = N + 0.5**




***the perturbation cancels after each turn***

$$x(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi \cdot Q)} \cdot \oint \sqrt{\beta(t)} \cdot G(t)\ \cos[|\phi(t) - \phi(s)| - \pi Q]\, dt$$

# Closed Orbit Response

***Example:***

$$x(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi \cdot Q)} \cdot \oint \sqrt{\beta(t)} \cdot G(t) \; \cos[|\phi(t) - \phi(s)| - \pi Q] \, dt$$

with

$$G(t) = \frac{-1}{\rho(t)} \cdot \frac{\Delta p}{p}$$

→

$$x(s) = D(s) \cdot \frac{\Delta p}{p}$$

with

$$D(s) = \frac{-\sqrt{\beta(s)}}{2\sin(\pi \cdot Q)} \cdot \oint \frac{\sqrt{\beta(t)}}{\rho(t)} \cdot \cos[|\phi(t) - \phi(s)| - \pi Q] \, dt$$

→ ***Dispersion Orbit***

# $\beta$ − *Beat*

***quadrupole error:***

$$\Delta k(s) = 0.3 \cdot \frac{\Delta B[T/m]}{p_0[GeV]}$$

$$\frac{d^2x}{ds^2} + K(s) \cdot x = x \cdot \Delta k(s);$$

→ ***variation of the constant:***

$$\Delta\beta(s) = \frac{\beta(s)}{2\sin(2\pi \cdot Q)} \cdot \int_{s0}^{s0+circ} \beta(t) \cdot \Delta k(t) \cos[2[\phi(t)-\phi(s)]-2\pi Q]\, dt$$

→

$\beta$ − beat oscillates with twice the betatron frequency

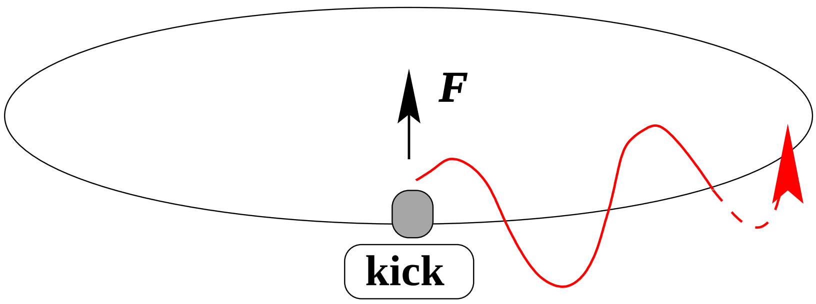
# Quadrupole Error and Orbit Stability

## Quadrupole Error:

→ *orbit kick proportional to beam offset in quadrupole*

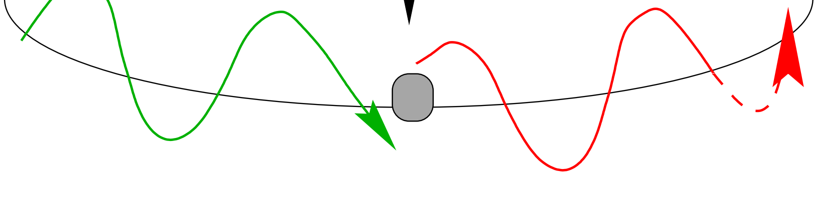
### $Q = N + 0.5$

**1. Turn:** $x > 0$




→ *amplitude increase*

**2. Turn:** $x < 0$




→ *amplitude increase*

→ ***watch out for half integer tunes!***

# *Tune Error*

## ***one turn map:***

cosine− and sine−like solutions to Hill's equation

→ $\underline{z}_{n+1} = M \cdot \underline{z}_n$ $\qquad \underline{z} = \begin{pmatrix} x \\ x' \end{pmatrix}$

with

$$M = I \cdot \cos(2\pi Q) + J \cdot \sin(2\pi Q)$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}; \quad \gamma = [1 + \alpha^2] / \beta$$

remember: $\cos(2\pi Q) = \frac{1}{2}\,\text{trace}\, M$

→ the coefficients of: $\dfrac{M - I \cdot \cos(2\pi Q)}{\sin(2\pi Q)}$

provide the optic functions at $s_0$

# Tune Error

- ***transfer matrix for single quadrupole:***

$$m_0 = \begin{pmatrix} 1 & 0 \\ -k \cdot l & 1 \end{pmatrix}$$

- ***matrix for single quadrupole with error:***

$$m = \begin{pmatrix} 1 & 0 \\ [-k + \Delta k] \cdot l & 1 \end{pmatrix}$$

- ***one turn matrix with quadrupole error:***

$$M = m \cdot m_0^{-1} \cdot M_0$$

trace M

→

$$\cos(2\pi Q) = \cos(2\pi Q_0) - \frac{1}{2}\beta \cdot \Delta k \cdot l \cdot \sin(2\pi Q_0)$$

# *Tune Error*

## ***distributed perturbation:***

$$\cos(2\pi Q) = \cos(2\pi Q_0) - \frac{\sin(2\pi Q_0)}{2} \cdot \oint \beta \cdot \Delta k \, ds$$

$$\longrightarrow \quad \Delta Q = \frac{1}{4\pi} \cdot \oint \beta \cdot \Delta k \, ds$$

## ***chromaticity:***

$$k_0 = \frac{e \cdot g}{p}$$

momentum error $\longrightarrow$ $\Delta k = -k_0 \cdot \frac{\Delta p}{p}$

$$\Delta Q = -\frac{1}{4\pi} \cdot \oint \beta \cdot k_0 \cdot ds \cdot \frac{\Delta p}{p}$$

$$= \xi \cdot \frac{\Delta p}{p}$$

# *Problems Generated by Orbit Errrors*

- ***injection errors:***
  - *aperture* → *beam losses*
  - *filamentation* → *beam size*

- ***closed orbit errors:***
  - *x−y coupling*
  - *aperture*
  - *energy error*
  - *field imperfections*
  - *dispersion* → *beam size at IP*
  - *beam separation*

***Aim:***

$\Delta x,\ \Delta y < 4$ mm

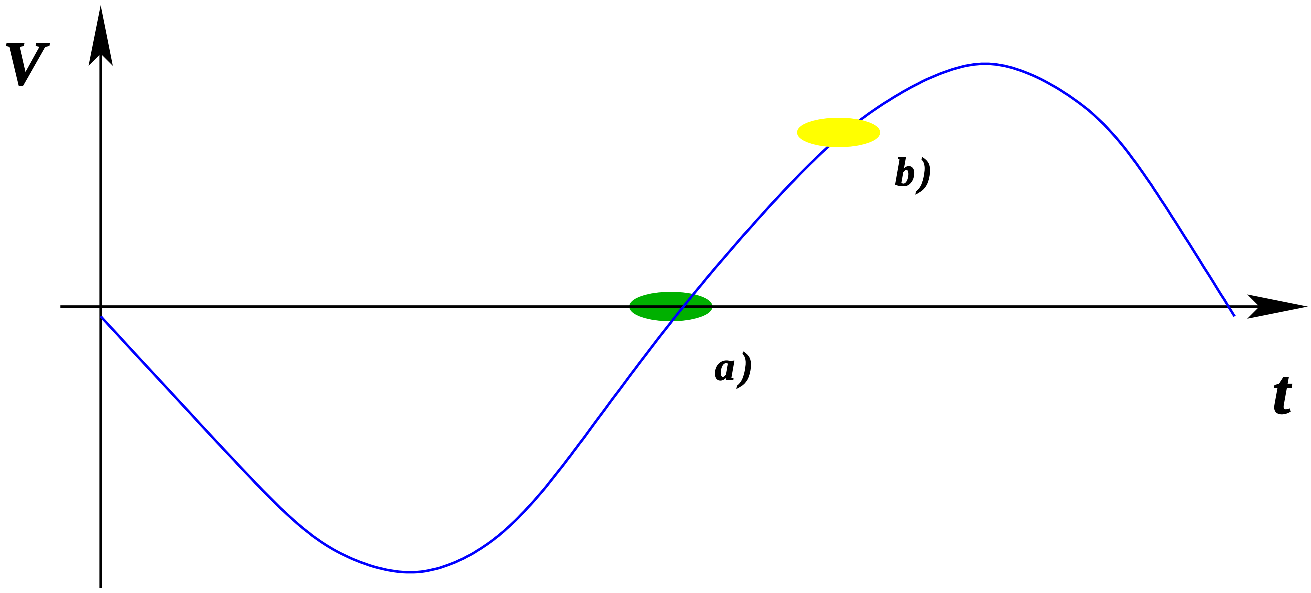
rms $< 0.5$ mm

→ *beam monitors and orbit correctors*

## Synchrotron:

→ *the orbit determines the particle energy!*

*assume: L > design orbit*




→ *energy increase*

## Equilibrium:

$$f_{RF} = h \cdot f_{rev}$$

$$f_{rev} = \frac{1}{2 \cdot \pi} \cdot \frac{q}{m \cdot \gamma} \cdot B$$

→ *E depends on orbit and magnetic field!*

# *Orbit Correction*

- the orbit error in a storage ring with conventional magnets is dominated by the contributions from the quadrupole alignment errors

- orbit perturbation is proportional to the local $\beta$-functions at the location of the dipole error

  → alignment errors at QF cause mainly horizontal orbit errors

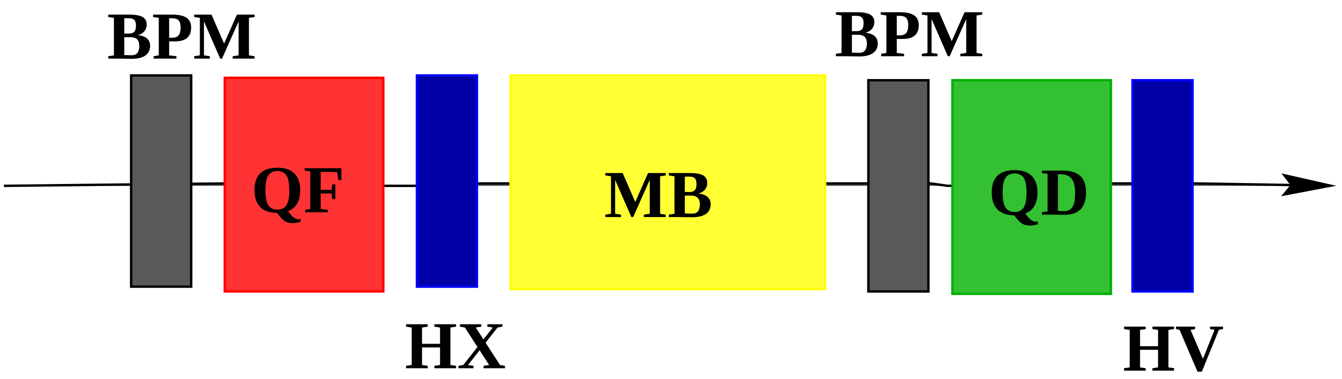
  → alignment errors at QD causes mainly vertical orbit errors

# *Orbit Correction*

- aim at a local correction of the dipole error due to the quadrupole alignment errors

→ place orbit corrector and BPM next to the main quadrupoles

→ horizontal BPM and corrector next to QF
vertical BPM and corrector next to QD




→ **orbit in the opposite plane?**

**relative alignment of BPM and quadrupole?**

# *Local Orbit Bumps I*

## *deflection angle:*

$$\theta_i = \int_{\text{dipole}} G_i(t)\,dt = \frac{0.3 \cdot B_i[T] \cdot l}{p[GeV]}$$

## *trajectory response:*

[no periodic boundary conditions]

$$\longrightarrow \quad x(s) = \sqrt{\beta_i\,\beta(s)} \cdot \theta_i \cdot \sin[\phi(s) - \phi_i]$$

$$\longrightarrow \quad x'(s) = \sqrt{\beta_i / \beta(s)} \cdot \theta_i \cdot \cos[\phi(s) - \phi_i]$$

# *Local Orbit Bumps II*

## *closed orbit bump:*

compensate the trajectory perturbation with

additional corrector kicks further down stream

→ closure of the perturbation within one turn

→ local orbit excursion

→ possibility to correct orbit errors locally

→ closure with one additional corrector magnet

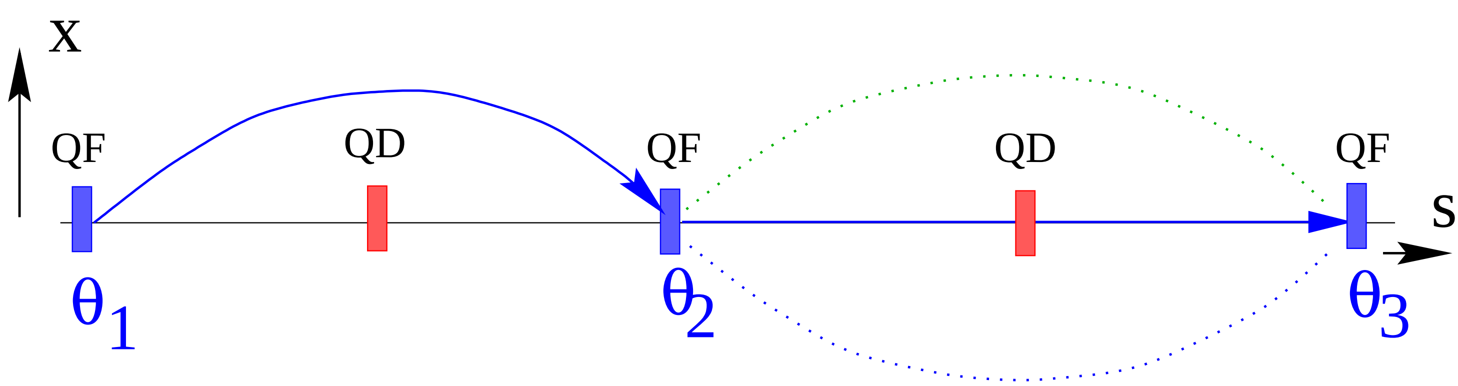
→ $\pi$ - bump

→ closure with two additional corrector magnets

→ three corrector bump

# *Local Orbit Bumps III*

■ $\pi$ - bump: (quasi local correction of error)




→ $$\theta_2 = \frac{\sqrt{\beta_1}}{\sqrt{\beta_2}} \cdot \theta_1$$

■ limits / problems:

→ closure depends on lattice phase advance

→ requires 90$^o$ lattice

→ sensitive to lattice errors

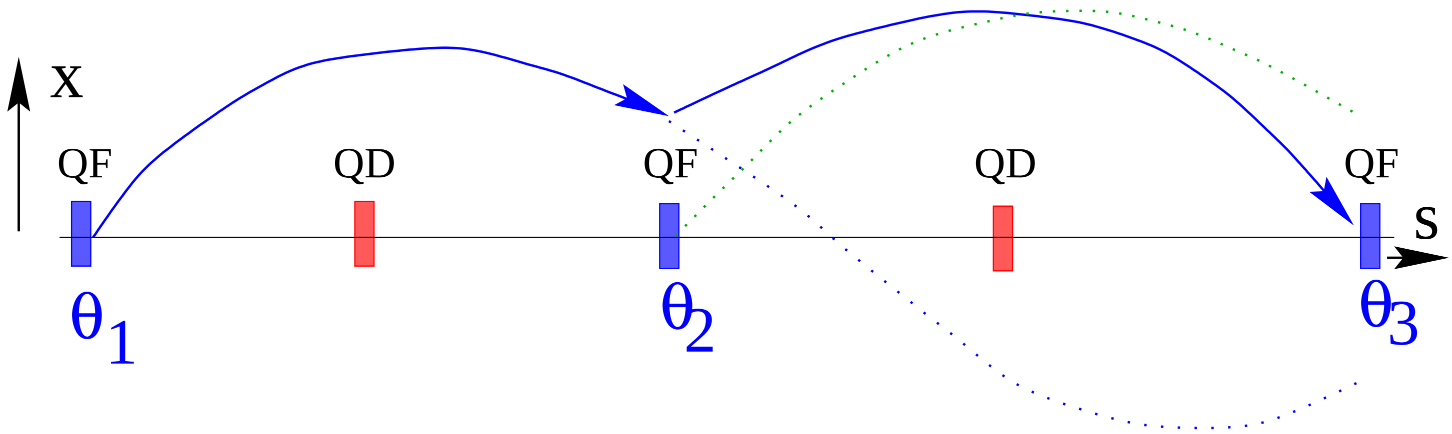
→ requires horizontal BPMs at QF and QD

→ sensitive to BPM errors

→ requires large number of correctors

# Local Orbit Bumps IV

- **3 corrector bump:** (quasi local correction of error)




$$\theta_2 = -\frac{\sqrt{\beta_1}}{\sqrt{\beta_2}} \cdot \frac{\sin(\Delta\phi_{3-1})}{\sin(\Delta\phi_{3-2})} \cdot \theta_1$$

$$\theta_3 = \left( \frac{\sin(\Delta\phi_{3-1})}{\tan(\Delta\phi_{3-2})} - \cos(\Delta\phi_{3-1}) \right) \cdot \frac{\sqrt{\beta_1}}{\sqrt{\beta_3}} \cdot \theta_1$$

- works for any lattice phase advance
- requires only horizontal BPMs at QF

- **limits / problems:**
  - sensitive to BPM errors
  - large number of correctors
  - can not control $x'$

# *Harmonic Filtering I*

- unperturbed solution (smooth approximation):

$$x'' + \left[\frac{2\pi}{C} \cdot Q\right]^2 \cdot x = 0 \longrightarrow x(s) = A \cdot e^{i \frac{2\pi}{C} \cdot Q \cdot s}$$

- orbit perturbation due to random kicks:

$$x'' + K(s) \cdot x = \sum_{i=1}^{m} \theta_i \cdot \delta(s - s_i)$$

$$= F(s)$$

- periodic boundary conditions:

$$\longrightarrow x(s) = \sum_n d_n \cdot e^{i \frac{2\pi}{C} \cdot n \cdot s}$$

$$\longrightarrow F(s) = \sum_n f_n \cdot e^{i \frac{2\pi}{C} \cdot n \cdot s}$$

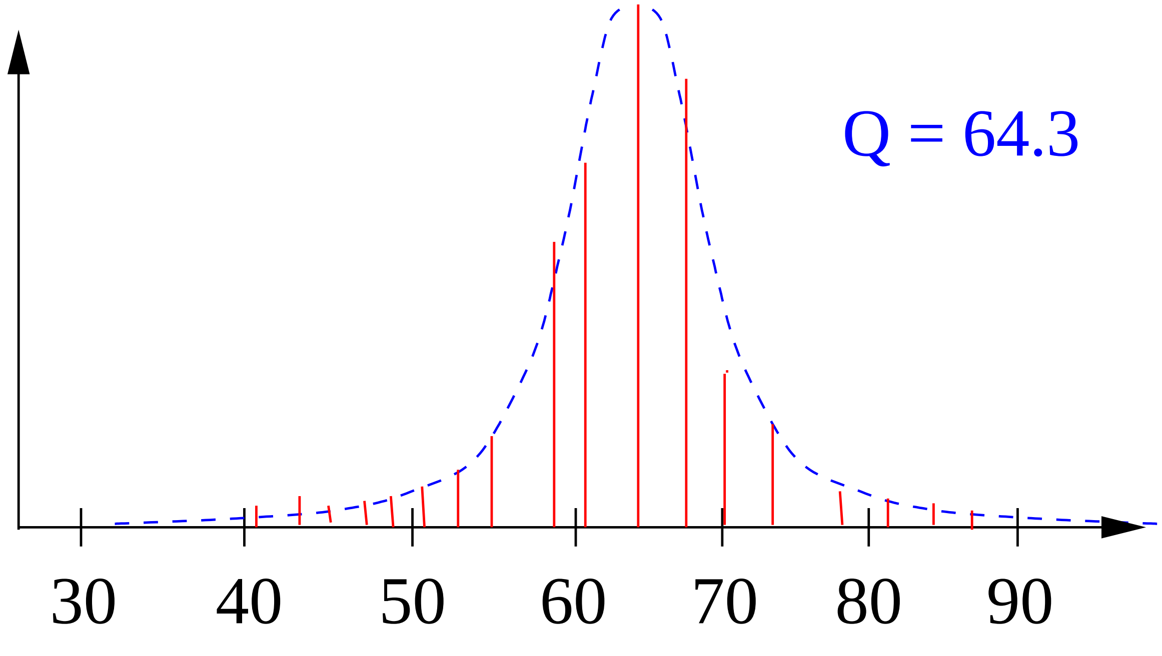
# *Harmonic Filtering II*

- inserting Ansatz into Hill's equation:

→ $$x(s) = \sum_n d_n \cdot e^{i \frac{2\pi}{C} \cdot n \cdot s}$$

with: $$d_n = \frac{f_n}{\left(\frac{2\pi}{C} \cdot Q\right)^2 - \left(\frac{2\pi}{C} \cdot n\right)^2}$$

→ the spectrum peaks around the tune




→ small number of correctors are efficient

# ***SVD I***

- linear relation between BPM and corrector data:

COR: vector of corrector amplitudes

→ COR = $[a_1, a_2, \ldots, a_m]$

BPM: vector of all BPM data

→ BPM = $[b_1, b_2, \ldots, b_n]$

→ BPM = A · COR; A = n x m matrix

- global orbit correction:

ORB: vector of all measured orbit data

→ ORB = $[c_1, c_2, \ldots, c_n]$

→ find a set of corrector settings that satisfies:

ORB - A ·COR = 0

# *SVD II*

- mathematical solution:

$$\underline{COR} = A^{-1} \cdot \underline{ORB}$$

- problem:

A is normally not invertible

(A is normally not even a square matrix)

→ minimise the norm: $\| \underline{ORB} - A \cdot \underline{COR} \|$

with $\| \underline{X} \| = \left( \sum_{i=1}^{m} |x_i|^p \right)^{1/p}$

→ find a matrix B such that:

$$\| \underline{ORB} - A \cdot B \cdot \underline{ORB} \|$$

attains a minimum

# *SVD III*

singular value decomposition SVD:

$$\longrightarrow \quad A = O_1 \cdot D \cdot O_2$$

$O_1$ and $O_2$ are orthogonal matrices: $O^{-1} = O^t$

$(a_{ij} \longrightarrow a_{ji})$

D is a diagonal matrix:

$$D = \begin{pmatrix} \sigma_{11} & & 0 & \cdots & 0 \\ & \sigma_{22} & & & \vdots \\ 0 & & \ddots & & \\ & & \sigma_{kk} & \cdots & 0 \end{pmatrix} \qquad k \leqslant \min(n,m)$$

define:

$$\hat{D} = \begin{pmatrix} 1/\sigma_{11} & & 0 & \\ & 1/\sigma_{22} & & \\ 0 & & \ddots & \\ \vdots & & & 1/\sigma_{kk} \\ & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \longrightarrow D \cdot \hat{D} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

# ***SVD IV***

- define the correction matrix:

→ $B = O_2^t \cdot \hat{D} \cdot O_1^t$

→ $A \cdot B = (O_1 \cdot D \cdot O_2) \cdot (O_2^t \cdot \hat{D} \cdot O_1^t)$

- SVD allows you to adjust k corrector magnets

  k = min(m,n)

- if k = m = n one obtains a zero orbit
  (by using a all possible corrector magnets)

- if m $\neq$ n SVD algorithm minimises the norm
  (by using all possible corrector magnets)

- algorithm is not stable if $\underline{D}$ has small eigenvalues

# *Most Effective Corrector*

- orbit is perturbed by a few large perturbations:

→ minimise

$$\| BPM - A \cdot COR \| \quad \text{with} \quad \| \underline{X} \| = \left( \sum_{i=1}^{m} |x_i|^p \right)^{1/p}$$

with a small set of 'k' corrector magnets

- brut force: select all possible combinations

→ time consuming but good result

- selective: keep the already selected correctors

→ much faster!

→ finite chance to miss best choice

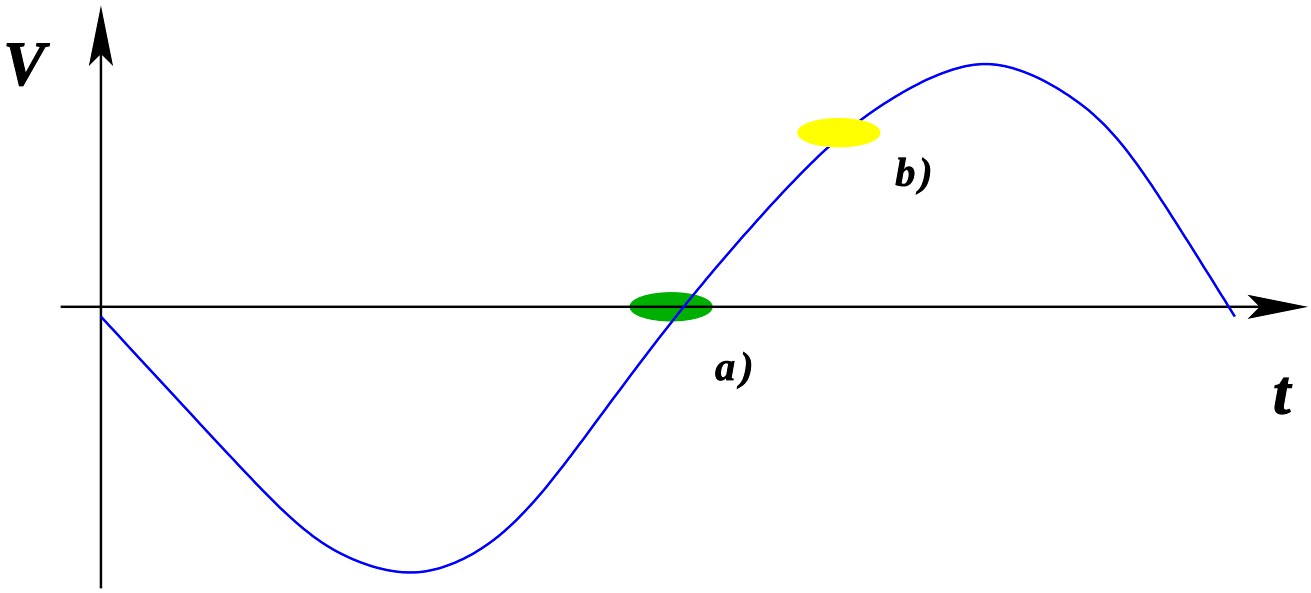
→ can generate orbit bumps

- MICADO: selective + cross-correlation between orbit residues and remaining correctors

## Synchrotron:

→ the orbit determines the particle energy!

assume: L > design orbit




→ energy increase

## Equilibrium:

$$f_{RF} = h \bullet f_{rev}$$

$$f_{rev} = \frac{1}{2 \bullet \pi} \bullet \frac{q}{m \bullet \gamma} \bullet B$$

→ E depends on orbit and magnetic field!