TRINITY
COLLEGE • OXFORD

Relativity for Accelerator Physicists

Christopher R. Prior

Fellow and Tutor in Mathematics

Trinity College, Oxford

Accelerator Science and Technology Centre

Rutherford Appleton Laboratory, U.K.

Overview

- The principle of special relativity
- Lorentz transformation and its consequences
- 4-vectors: position, velocity, momentum, invariants. Einstein's equation $E=mc^2$
- Examples of the use of 4-vectors
- Inter-relation between β and γ, momentum and energy
- Electromagnetism and Relativity

Reading

- W. Rindler: Introduction to Special Relativity (OUP 1991)
- D. Lawden: An Introduction to Tensor Calculus and Relativity
- N.M.J. Woodhouse: Special Relativity (Springer 2002)
- A.P. French: Special Relativity, MIT Introductory Physics Series (Nelson Thomes)

Historical background

- Groundwork by Lorentz in studies of electrodynamics, with crucial concepts contributed by Einstein to place the theory on a consistent basis.
- Maxwell’s equations (1863) attempted to explain electromagnetism and optics through wave theory
  - light propagates with speed $c = 3\times10^8$ m/s in “ether” but with different speeds in other frames
  - the ether exists solely for the transport of e/m waves
  - Maxwell’s equations not invariant under Galilean transformations
  - To avoid setting e/m apart from classical mechanics, assume light has speed *c* only in frames where source is at rest
  - And the ether has a small interaction with matter and is carried along with astronomical objects

Nonsense! Contradicted by:

- Aberration of star light (small shift in apparent positions of distant stars)
- Fizeau's 1859 experiments on velocity of light in liquids
- Michelson-Morley 1907 experiment to detect motion of the earth through ether
- Suggestion: perhaps material objects contract in the direction of their motion

$$L(v) = L_0\left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

This was the last gasp of ether advocates and the germ of Special Relativity led by Lorentz, Minkowski and Einstein.

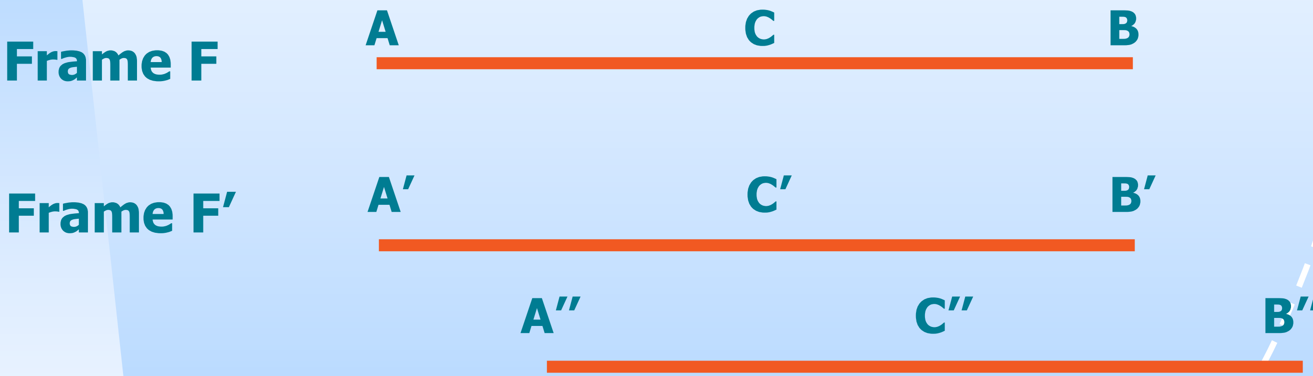
TRINITY
COLLEGE • OXFORD

The Principle of Special Relativity

- A frame in which particles under no forces move with constant velocity is “inertial”
- Consider relations between inertial frames where measuring apparatus (rulers, clocks) can be transferred from one to another.
- Behaviour of apparatus transferred from F to F' is independent of mode of transfer
- Apparatus transferred from F to F', then from F' to F'', agrees with apparatus transferred directly from F to F''.
- *The Principle of Special Relativity states that all physical laws take equivalent forms in related inertial frames, so that we cannot distinguish between the frames.*

Simultaneity

- Two clocks A and B are synchronised if light rays emitted at the same time from A and B meet at the mid-point of AB



- Frame F' moving with respect to F. Events simultaneous in F cannot be simultaneous in F'.
- Simultaneity is not absolute but frame dependent.

The Lorentz Transformation

- Must be linear to agree with standard Galilean transformation in low velocity limit
- Preserves wave fronts of pulses of light,

i.e. $P \equiv x^2 + y^2 + z^2 - c^2t^2 = 0$

whenever $Q \equiv x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$

- Solution is the **Lorentz transformation** from frame F (t,x,y,z) to frame F'(t',x',y',z') moving with velocity v along the x-axis:

$$\left.\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}\right\} \quad \text{where} \quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

TRINITY COLLEGE • OXFORD

Outline of Derivation

Set $t' = \alpha t + \beta x$

$$x' = \gamma x + \delta t$$

$$y' = \varepsilon y$$

$$z' = \varsigma z$$

Then $P = kQ$

$$\Leftrightarrow c^2 t'^2 - x'^2 - y'^2 - z'^2 = k\left(c^2 t^2 - x^2 - y^2 - z^2\right)$$

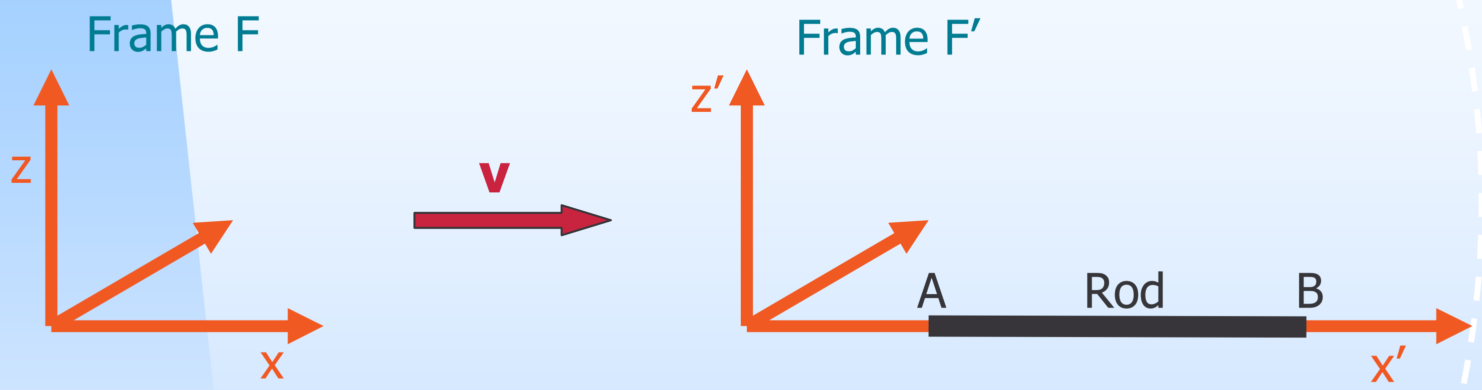
$$\Rightarrow c^2(\alpha t + \beta x)^2 - (\gamma x + \delta t)^2 - \varepsilon^2 y^2 - \varsigma^2 z^2 = k\left(c^2 t^2 - x^2 - y^2 - z^2\right)$$

Equate coefficients of x, y, z, t.

Isotropy of space $\Rightarrow k = k(\vec{v}) = k(|\vec{v}|) = \pm 1$

Apply some common sense (e.g. $\varepsilon, \varsigma, k = +1$ and not -1)

Consequences: length contraction



Rod AB of length L' fixed in F' at x'_A, x'_B. What is its length measured in F?

Must measure positions of ends in F at the same time, so events in F are (t,x_A) and (t,x_B). From Lorentz:

$$x'_A = \gamma\left(x_A - vt\right) \qquad x'_B = \gamma\left(x_B - vt\right)$$

$$L' = x'_B - x'_A = \gamma\left(x_B - x_A\right) = \gamma L > L$$

Moving objects appear contracted in the direction of the motion

Consequences: time dilatation

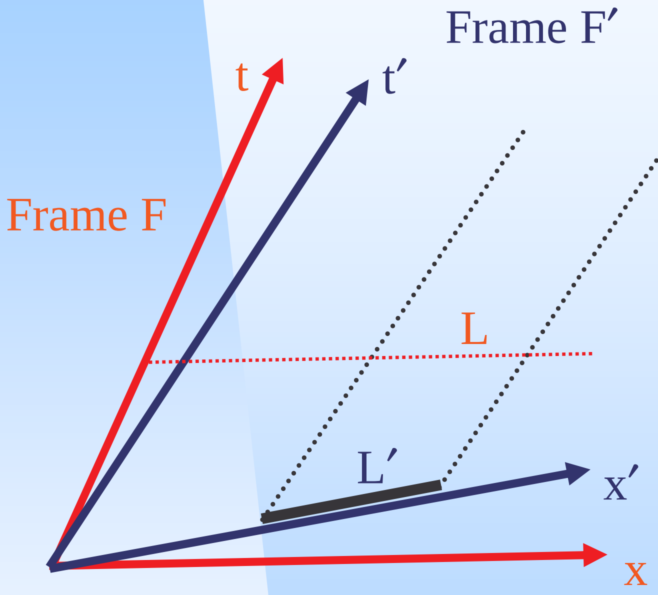
- Clock in frame F at point with coordinates (x,y,z) at different times t_A and t_B
- In frame F' moving with speed v, Lorentz transformation gives

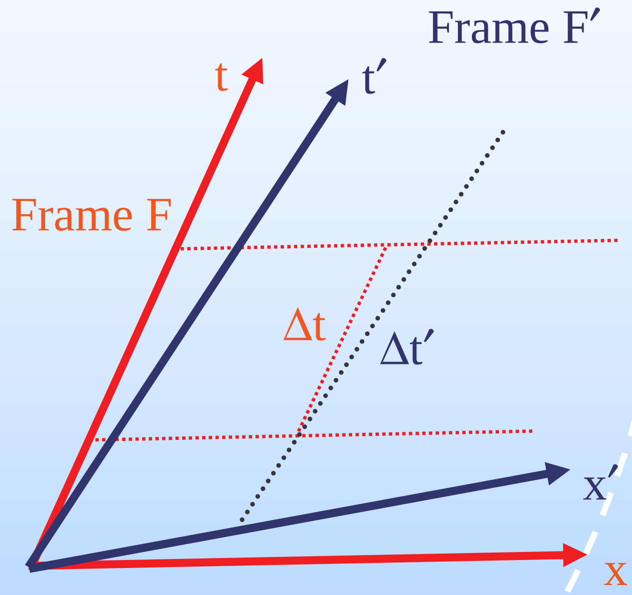
$$t'_A = \gamma\left(t_A - \frac{vx}{c^2}\right) \qquad t'_B = \gamma\left(t_B - \frac{vx}{c^2}\right)$$

- So: $$\Delta t' = t'_B - t'_A = \gamma\left(t_B - t_A\right) = \gamma \Delta t > \Delta t$$

Moving clocks appear to run slow

Schematic Representation of the Lorentz Transformation



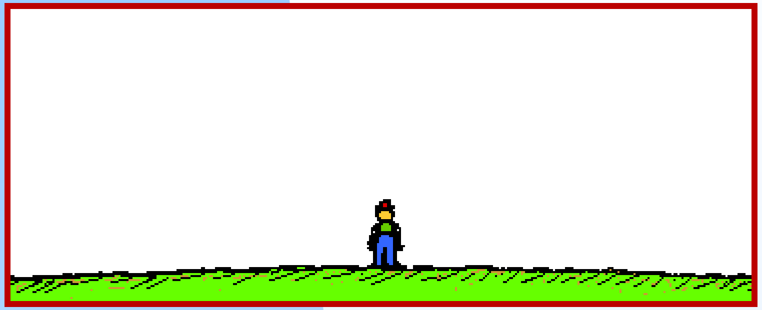


Length contraction L<L′

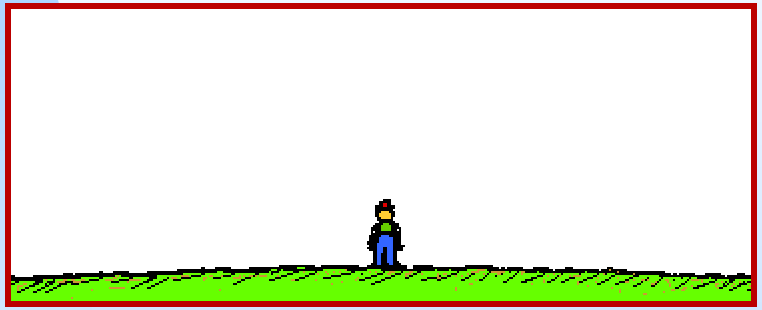
Rod at rest in F′. Measurement in F at fixed time t, along a line parallel to x-axis

Time dilatation: Δt<Δt′

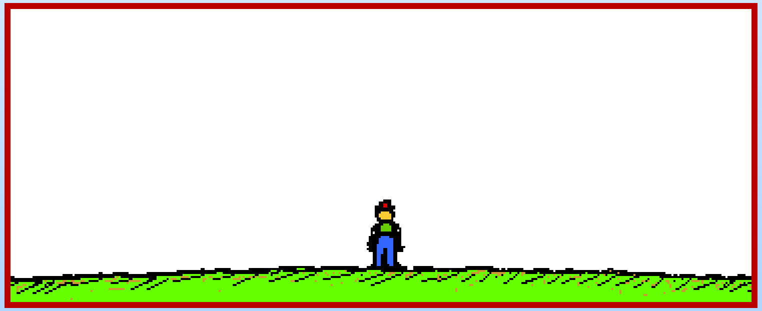
Clock at rest in F. Time difference in F′ from line parallel to x′-axis



v = 0.8c

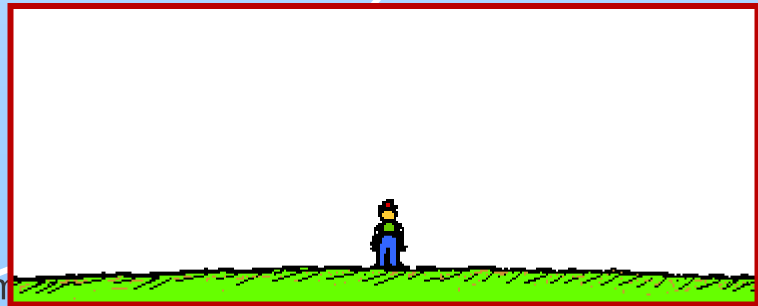


v = 0.9c



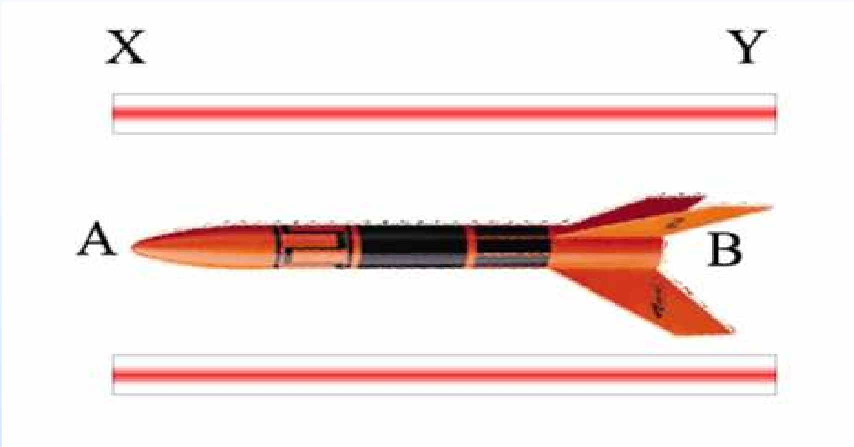
v = 0.99c

v = 0.9999c



Example: Rocket in Tunnel

$$v = \frac{\sqrt{3}}{2}c$$



Tunnel 100m long

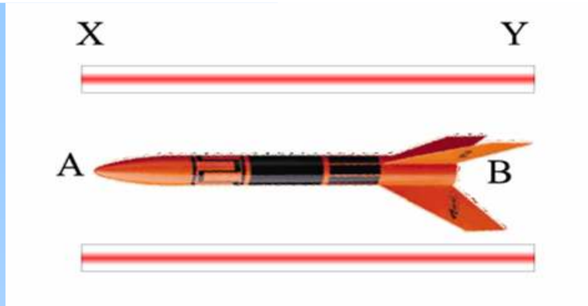
Rocket length 100m

- All clocks synchronised.
- Observers X and Y at exit and entrance of tunnel say the rocket is moving, has contracted and has length

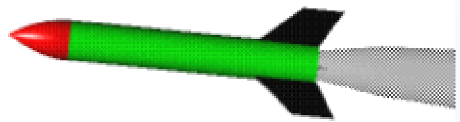
$$\frac{100}{\gamma} = 100\times\left(1-\frac{v^2}{c^2}\right)^{1/2} = 100\times\left(1-\frac{3}{4}\right)^{1/2} = 50\text{m}$$

- But the tunnel is moving relative to the ends A and B of the rocket and obesrvers here say the rocket is 100 m in length but the tunnel has contracted to 50 m

TRINITY
COLLEGE • OXFORD



Questions



- If X's clock reads zero as the A exits tunnel, what does Y's clock read when the B goes in?

Moving rocket length 50m, so B has still 50m to travel before his clock reads 0. Hence clock reading is

$$-\frac{50}{v} = -\frac{100}{\sqrt{3}c} \approx -200\,\text{ns}$$

- What does the B's clock read as he goes in?

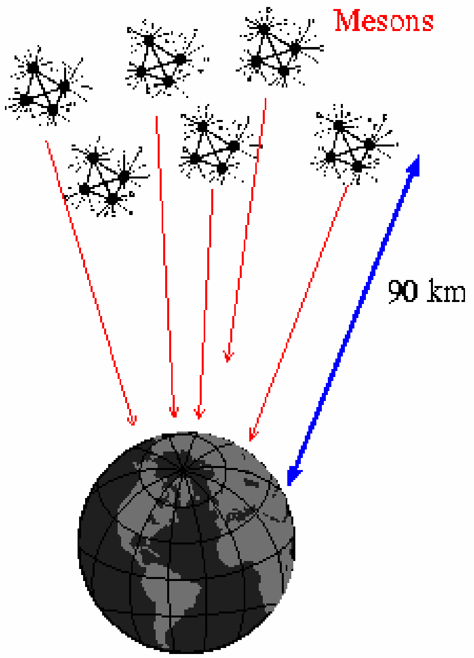
To the B, tunnel is only 50m long, so A is 50m past the exit as B goes in. Hence clock reading is

$$+\frac{50}{v} = +\frac{100}{\sqrt{3}c} \approx +200\,\text{ns}$$

- Where is the B when his clock reads 0?

B's clock reads 0 when A's clock reads 0, which is as A exits the tunnel. To A and B, tunnel is 50m, so B is 50m from the entrance in the rocket's frame, or 100m in tunnel frame.

Example: π-mesons



Half-life = 2 x 10^{-6} sec

so required velocity = 90/2 x 10^{-6} = 4.5 x 10^{7} km/sec

= 150 c

- Mesons are created in the upper atmosphere, 90km from earth. Their half life is τ=2 μs, so they can travel at most 2 ×10^{-6}c=600m before decaying. So how do more than 50% reach the earth's surface undecayed?
- Mesons see distance contracted by γ, so

$$v\tau \approx \left(\frac{90}{\gamma}\right)\text{km}$$

- Earthlings say mesons' clocks run slow so their half-life is γτ and

$$v(\gamma\tau) \approx 90\,\text{km}$$

- Both give

$$\frac{\gamma v}{c} = \frac{90\text{ km}}{c\tau} = 150, \quad v \approx c, \quad \gamma \approx 150$$

Invariants

- An invariant is a quantity that has the same value in all inertial frames.
  - Examples: phase of a wave, rate of radiation of moving charged particle
- Lorentz transformation is based on invariance of

$$c^2t^2 - (x^2 + y^2 + z^2) = (ct)^2 - \vec{x}\cdot\vec{x}$$

- Write this in terms of the 4-position vector $\mathrm{X} = \left(ct, \vec{x}\right)$ as $\mathrm{X}\cdot\mathrm{X}$

If $\mathrm{X} = (x_0, \vec{x}),\ \mathrm{Y} = (y_0, \vec{y}),$ define the invariant product $\mathrm{X}\cdot\mathrm{Y} = x_0 y_0 - \vec{x}\cdot\vec{y}$

- Fundamental invariant (preservation of speed of light):

$$c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = c^2\Delta t^2\left(1 - \frac{\Delta x^2 + \Delta y^2 + \Delta z^2}{c^2\Delta t^2}\right)$$

$$= c^2\Delta t^2\left(1 - \frac{v^2}{c^2}\right) \quad = c^2\left(\frac{\Delta t}{\gamma}\right)^2$$

- Write $\Delta\tau = \Delta t/\gamma$, τ is the proper time
- When $v=0$, $\tau = t$, so τ is the time in the rest-frame.

4-Vectors

The Lorentz transformation can be written in matrix form

$$\boxed{\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}} \quad \Rightarrow \quad \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{\gamma v}{c} & 0 & 0 \\ -\frac{\gamma v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

An object made up of 4 elements which transforms like X is called a 4-vector

(analogous to the 3-vector of classical mechanics)

Position 4-vector $\mathrm{X} = (ct, \vec{x})$

4-Vectors in Special Relativity Mechanics

- Velocity: $$V = \frac{dX}{d\tau} = \gamma\frac{dX}{dt} = \gamma\frac{d}{dt}(ct,\vec{x}) = \gamma(c,\vec{v})$$

- Note invariant $$V \cdot V = \gamma^2(c^2 - \vec{v}^{\,2}) = c^2$$

- Momentum $$P = m_0 V = m_0\gamma(c,\vec{v}) = (mc,\vec{p})$$

$$m = m_0\gamma \quad \text{is relativistic mass}$$
$$\vec{p} = m_0\gamma\vec{v} = m\vec{v} \quad \text{is the 3-momentum}$$

Example of Transformation: Addition of Velocities

- A particle moves with velocity $\vec{u} = (u_x, u_y, u_z)$ in frame F, so has 4-velocity $V = \gamma_u (c, \vec{u})$
- Add velocity $\vec{v} = (v,0,0)$ by transforming to frame F′ to get new velocity $\vec{w}$.
- Lorentz transformation gives $\left(t \leftrightarrow \gamma, \quad \vec{x} \leftrightarrow \gamma\vec{u}\right)$

$$\gamma_w = \gamma_v \left(\gamma_u + \frac{v\gamma_u u_x}{c^2} \right)$$

$$\gamma_w w_x = \gamma_v (\gamma_u u_x + v\gamma_u)$$

$$\gamma_w w_y = \gamma_u u_y$$

$$\gamma_w w_z = \gamma_u u_z$$

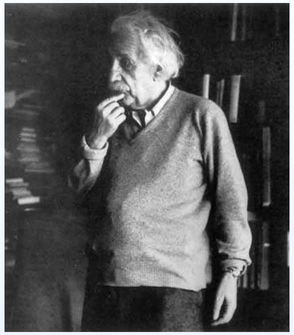
$$\Rightarrow$$

$$w_x = \frac{u_x + v}{1 + \frac{vu_x}{c^2}}$$

$$w_y = \frac{u_y}{\gamma_v \left(1 + \frac{vu_x}{c^2}\right)} \qquad w_z = \frac{u_z}{\gamma_v \left(1 + \frac{vu_x}{c^2}\right)}$$

TRINITY COLLEGE • OXFORD

Einstein's relation



- Momentum invariant $\qquad P \cdot P = m_0^2 \,(V \cdot V) = m_0^2 c^2$
- Differentiate $\qquad P \cdot \frac{dP}{d\tau} = 0 \quad \Rightarrow \quad V \cdot \frac{dP}{d\tau} = 0$
- From Newton's 2nd Law expect 4-Force given by

$$F = \frac{dP}{d\tau} = \gamma \frac{dP}{dt} = \gamma \frac{d}{dt}(mc, \vec{p}) = \gamma\left(c\frac{dm}{dt}, \frac{d\vec{p}}{dt}\right) = \gamma\left(c\frac{dm}{dt}, \vec{f}\right)$$

- But $\qquad V \cdot \frac{dP}{d\tau} = 0 \quad \Rightarrow \quad V \cdot F = 0$
- So $\qquad \frac{d}{dt}\left(mc^2\right) - \vec{v} \cdot \vec{f} = 0$

Rate of doing work, $\vec{v} \cdot \vec{f}$ = rate of change of kinetic energy

Therefore kinetic energy

$$T = mc^2 + \text{constant} = m_0 c^2 (\gamma - 1)$$

$E=mc^2$ is total energy

Basic quantities used in Accelerator calculations

Relative velocity $\beta = {}^{\mathrm{v}}\!/\!{}_{\mathrm{c}}$

Velocity $v = \beta \mathrm{c}$

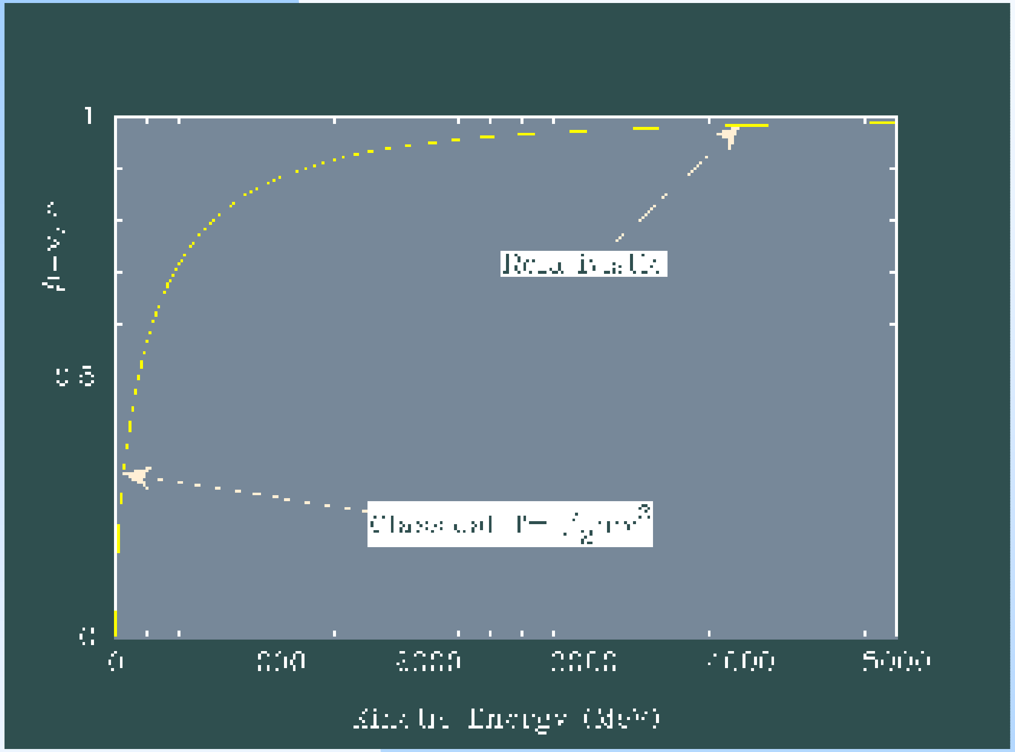
Momentum $p = mv = m_0 \gamma \beta c$

Kinetic energy $T = (m\text{-}m_0)c^2 = m_0 c^2 (\gamma - 1)$

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = \left(1 - \beta^2\right)^{-\frac{1}{2}}$$

$$\Rightarrow \quad (\beta\gamma)^2 = \frac{\gamma^2 v^2}{c^2} = \gamma^2 - 1 \quad \Rightarrow \quad \beta^2 = \frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2}$$

Velocity as a Function of Energy



$$T = m_0(\gamma - 1)c^2$$

$$\gamma = 1 + T/m_0c^2$$

$$\beta = \sqrt{1 - 1/\gamma^2}$$

$$p = m_o c\beta\gamma$$

For small v/c, $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \approx 1 + \frac{1}{2}\frac{v^2}{c^2}$

so $T = m_0c^2(\gamma - 1) \approx \frac{1}{2}m_0v^2$

TRINITY COLLEGE • OXFORD

Relationships between small variations in parameters ΔE, ΔT, Δp, Δβ, Δγ

$$(\beta\gamma)^2 = \gamma^2 - 1$$

$$\Rightarrow \quad \beta\gamma\Delta(\beta\gamma) = \gamma\Delta\gamma$$

$$\Rightarrow \quad \beta\Delta(\beta\gamma) = \Delta\gamma \qquad (1)$$

$$\frac{1}{\gamma^2} = 1 - \beta^2$$

$$\Rightarrow \quad \frac{1}{\gamma^3}\Delta\gamma = \beta\Delta\beta \qquad (2)$$

$$\frac{\Delta p}{p} = \frac{\Delta(m_0\gamma\beta c)}{m_0\gamma\beta c} = \frac{\Delta(\beta\gamma)}{\beta\gamma}$$

$$= \frac{1}{\beta^2}\frac{\Delta\gamma}{\gamma} = \frac{1}{\beta^2}\frac{\Delta E}{E}$$

$$= \gamma^2\frac{\Delta\beta}{\beta}$$

$$= \frac{\gamma}{\gamma+1}\frac{\Delta T}{T} \qquad \text{(exercise)}$$

	$\frac{\Delta\beta}{\beta}$	$\frac{\Delta p}{p}$	$\frac{\Delta T}{T}$	$\frac{\Delta E}{E}=\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta\beta}{\beta} =$	$\frac{\Delta\beta}{\beta}$	$\frac{1}{\gamma^2}\frac{\Delta p}{p}$ $\frac{\Delta p}{p}-\frac{\Delta\gamma}{\gamma}$	$\frac{1}{\gamma(\gamma+1)}\frac{\Delta T}{T}$	$\frac{1}{\beta^2\gamma^2}\frac{\Delta\gamma}{\gamma}$ $\frac{1}{\gamma^2-1}\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta p}{p} =$	$\gamma^2\frac{\Delta\beta}{\beta}$	$\frac{\Delta p}{p}$	$\frac{\gamma}{\gamma+1}\frac{\Delta T}{T}$	$\frac{1}{\beta^2}\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta T}{T} =$	$\gamma(\gamma+1)\frac{\Delta\beta}{\beta}$	$\left(1+\frac{1}{\gamma}\right)\frac{\Delta p}{p}$	$\frac{\Delta T}{T}$	$\frac{\gamma}{\gamma-1}\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta E}{E} =$	$(\beta\gamma)^2\frac{\Delta\beta}{\beta}$	$\beta^2\frac{\Delta p}{p}$	$\left(1-\frac{1}{\gamma}\right)\frac{\Delta T}{T}$	$\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta\gamma}{\gamma} =$	$(\gamma^2-1)\frac{\Delta\beta}{\beta}$	$\frac{\Delta p}{p}-\frac{\Delta\beta}{\beta}$		

4-Momentum Conservation

- Equivalent expression for 4-momentum $$P = m_0\gamma(c,\vec{v}) = (mc,\vec{p}) = \left(E/c,\vec{p}\right)$$

- Invariant $$m_0^2c^2 = P\cdot P = \frac{E^2}{c^2} - \vec{p}^{\,2}$$

$$\boxed{E^2/c^2 = m_0^2c^2 + \vec{p}^{\,2}}$$

- Classical momentum conservation laws $\rightarrow$ conservation of 4-momentum. Total 3-momentum and total energy are conserved.

$$\sum_{\text{particles, i}} P_i = \text{constant}$$

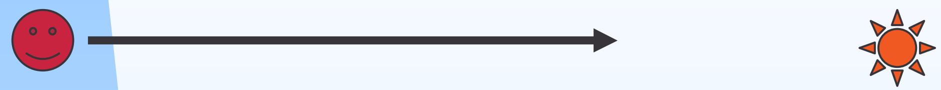
$$\Rightarrow \sum_{\text{particles, i}} E_i \text{ and } \sum_{\text{particles,i}} \vec{p}_i \text{ constant}$$

TRINITY
COLLEGE • OXFORD

Example of use of invariants

- Two particles have equal rest mass m_0.
  - Frame 1: one particle at rest, total energy is E_1.
  - Frame 2: centre of mass frame where velocities are equal and opposite, total energy is E_2.

Problem: Relate E_1 to E_2



Total energy E_1

(Fixed target experiment)

$$P_1 = \left(\frac{E_1 - m_0 c^2}{c}, \vec{p} \right) \qquad P_2 = (m_0 c, 0)$$

Total energy E_2

(Colliding beams expt)

$$P_1 = \left(\frac{E_2}{2c}, \vec{p}' \right) \qquad P_2 = \left(\frac{E_2}{2c}, -\vec{p}' \right)$$

$$\text{Invariant:} \quad P_2 \cdot (P_1 + P_2)$$

$$m_0 c \times \frac{E_1}{c} - 0 \times p = \frac{E_2}{2c} \times \frac{E_2}{c} + p' \times 0$$

$$\Rightarrow \qquad 2 m_0 c^2 E_1 = E_2^2$$

TRINITY COLLEGE • OXFORD

Electromagnetism and Relativity

- Maxwell's equations are relativistically invariant

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad \text{source - free}$$

$$\nabla \cdot \vec{D} = \rho \qquad \nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{j} \qquad \text{charge and current densities}$$

$$\vec{B} = \mu_0 \vec{H} \qquad \vec{D} = \varepsilon_0 \vec{E} \qquad \text{in vacuum } \varepsilon_0 \mu_0 = \frac{1}{c^2}$$

- Lorentz force law:

$$\vec{f} = q\left(\vec{E} + \vec{v} \wedge \vec{B}\right) \quad \text{for single particle charge } q$$

$$\vec{f} = \rho \vec{E} + \vec{j} \wedge \vec{B} \quad \text{for charge distribution}$$

Lorentz force law

- Relativistic equation of motion
  - 4-vector form:

$$F = \frac{dP}{d\tau} \Rightarrow \gamma\left(\frac{\vec{v}\cdot\vec{f}}{c}, \vec{f}\right) = \gamma\left(\frac{1}{c}\frac{dE}{dt}, \frac{d\vec{p}}{dt}\right)$$

  - 3-vector component:

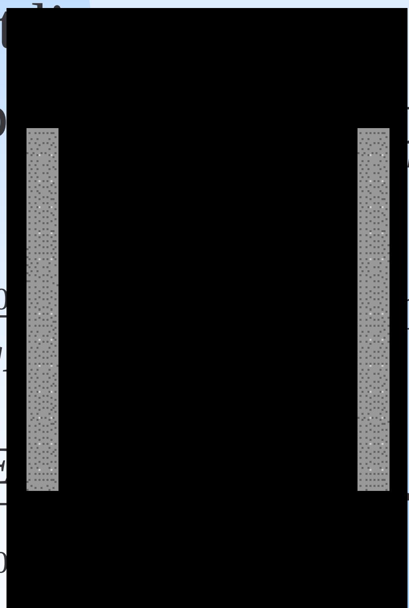
$$\boxed{\frac{d}{dt}\left(m_0\gamma\vec{v}\right) = \vec{f} = q\left(\vec{E} + \vec{v}\wedge\vec{B}\right)}$$

  - Lorentz force derives naturally from relativistic 4-vector (4×4 matrix) formulation of Maxwell’s equations and is not an additional hypothesis.

Motion of charged particles in constant electromagnetic fields

$$\boxed{\frac{d}{dt}(m_0\gamma\vec{v}) = \vec{f} = q\left(\vec{E} + \vec{v}\wedge\vec{B}\right)}$$

- Constant E-field gives uniform acceleration in straight
- Solutio



$\frac{q}{m_0}E$

$x = \frac{m_0}{q}$ … 1

$\approx \frac{1}{2}\frac{qE}{m_0}$

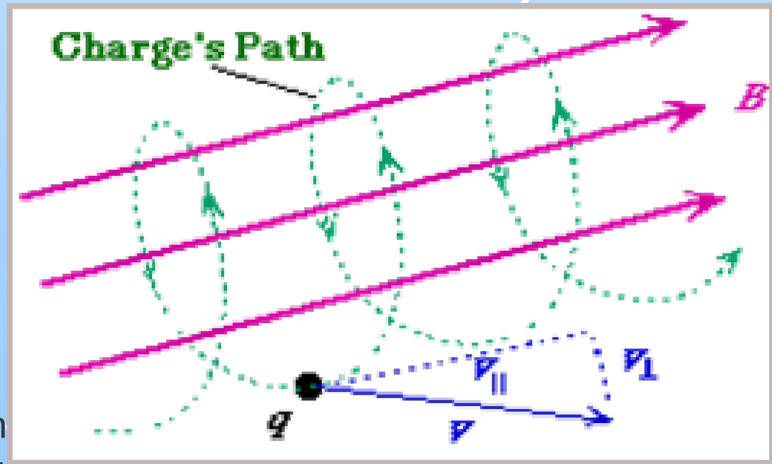
- Energy gain $= qEx$

- Constant magnetic field gives uniform spiral about B with constant energy (and γ).

$$\frac{d\vec{v}}{dt} = \frac{q}{m_0\gamma}\vec{v}\wedge\vec{B}$$

$$\vec{v}_{//} = \text{constant}$$

$$\left|\vec{x}_{\perp}\right| = \text{constant}$$



Relativistic Transformations of E and B

- In a frame F, a particle moves with velocity $\vec{v}$, under fields $\vec{E}$ and $\vec{B}$, and Lorentz force is

$$\vec{f} = q\left(\vec{E} + \vec{v}\wedge\vec{B}\right)$$

  - In particle's rest frame F′, … $\vec{f}' = q'\vec{E}'$
  - Assume measure…

idea

Exact:

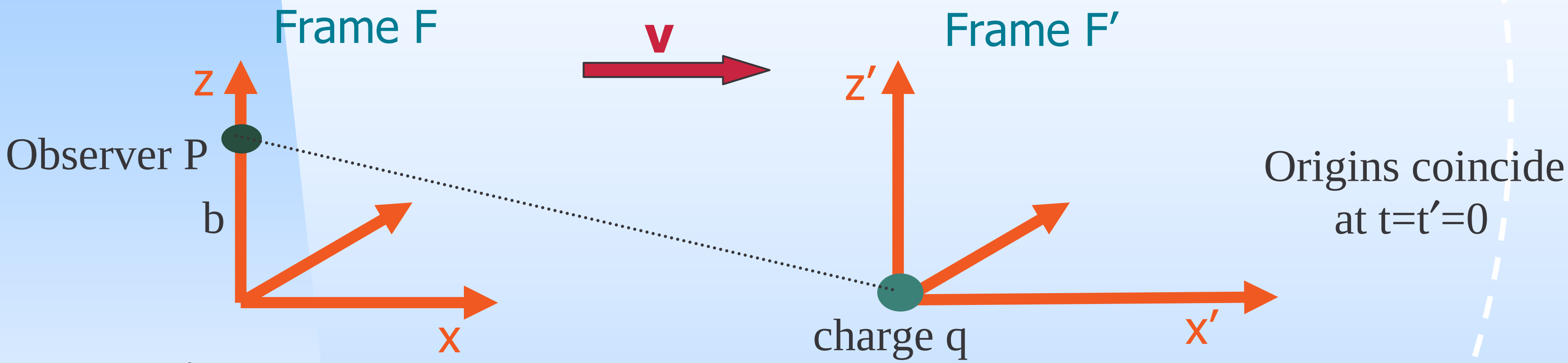
$$\boxed{\begin{aligned} \vec{E}'_\perp &= \gamma\left(E_\perp + \vec{v}\wedge\vec{B}\right), & \vec{E}'_{//} &= \vec{E}_{//} \\ \vec{B}'_\perp &= \gamma\left(B_\perp - \frac{\vec{v}\wedge\vec{E}}{c^2}\right), & \vec{B}'_{//} &= \vec{B}_{//} \end{aligned}}$$

- Point charge q at re… $\frac{\vec{r}}{r^3}$, $\vec{B} = 0$
  - See a current in F′, giving a field (Biot-Savart) $\vec{B} = -\frac{\mu_0 q}{4\pi}\frac{\vec{v}\wedge\vec{r}}{r^3} = -\frac{1}{c^2}\vec{v}\wedge\vec{E}$
  - Suggests $\vec{B}' = \vec{B} - \frac{1}{c^2}\vec{v}\wedge\vec{E}$

TRINITY COLLEGE • OXFORD

Electromagnetic Field of a Single Particle

- Charged particle moving along x-axis of Frame F



- In F′, P is at

$$\vec{x}'_P = (-vt', 0, b), \text{ so } \left|\vec{x}_p\right| = r' = \sqrt{b^2 + v^2 t'^2}, \quad t'_P = \gamma\left(t_p - \frac{vx_p}{c^2}\right) = \gamma t_P$$

- And fields are only electrostatic ($B=0$), given by

$$\vec{E}' = \frac{q}{r'^3}\vec{x}'_P \quad \Rightarrow \quad E'_x = -\frac{qvt'}{r'^3},\ E'_y = 0,\ E'_z = \frac{qb}{r'^3}$$

- Transform to laboratory frame *F*:

$$E_x = E'_x = -\frac{q\gamma v t}{\left(b^2+\gamma^2 v^2 t^2\right)^{3/2}} \qquad B_y = -\frac{\gamma v}{c^2} E'_z = -\frac{\beta}{c} E_z$$

$$E_y = 0 \qquad B_x = B_z = 0$$

$$E_z = \gamma E'_z = \frac{q\gamma b}{\left(b^2+\gamma^2 v^2 t^2\right)^{3/2}}$$

- As $v \rightarrow c$, $\beta \rightarrow 1$, and magnetic induction $cB_y \approx -E_z$
- At non-relativistic energies, $\gamma \approx 1$, and restores the Biot-Savart law:

$$\vec{B} \propto q \frac{\vec{v} \wedge \vec{r}}{r^3}$$

Electromagnetic Field of a Beam of Particles

- Coasting beam, momentum p±Δp
- In effective rest frame, see only an electrostatic field, $E'_\perp$, and $B'_\perp=0$
- Transform to laboratory frame: $E_{//} = E'_{//} = 0, \quad E_\perp = \gamma E'_\perp$

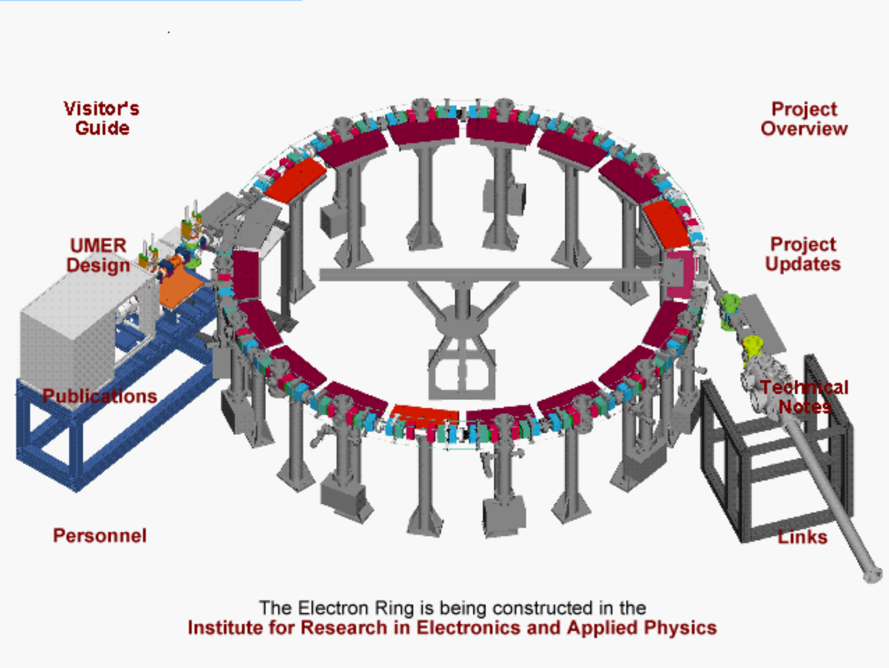
$$B_{//} = B'_{//} = 0, \quad B_\perp = \gamma \frac{\vec{v}\times\vec{E}'_\perp}{c^2}$$

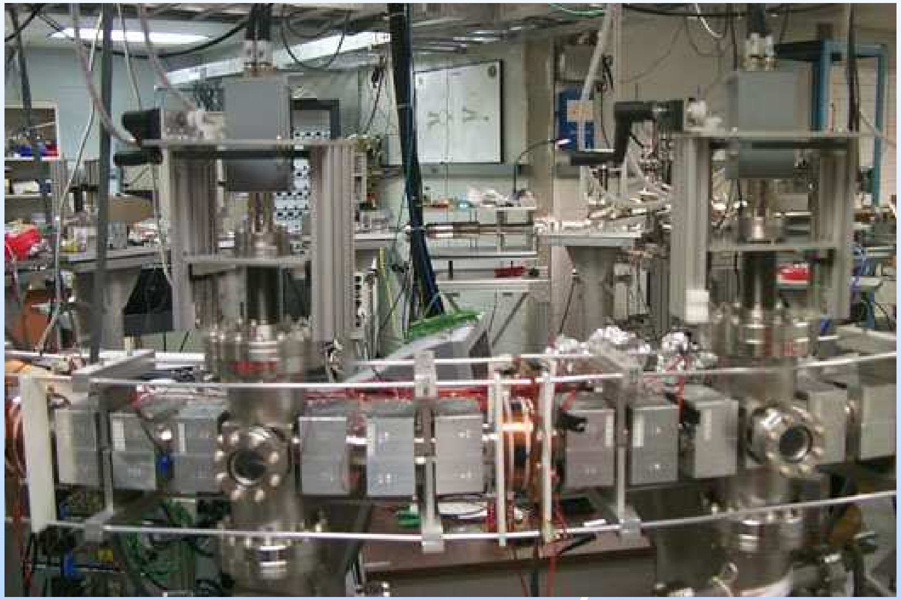
- So particle in beam is affected by a space charge force proportional to

$$\left(\vec{E}+\vec{v}\times\vec{B}\right)_\perp = \gamma\left(E'_\perp + \vec{v}\times\frac{\vec{v}\times\vec{E}'_\perp}{c^2}\right) = \gamma\left(1-\frac{v^2}{c^2}\right)\vec{E}'_\perp$$

Electrostatic repulsion between particles minus Magnetostatic attraction between thin current wires

University of Maryland Electron Ring

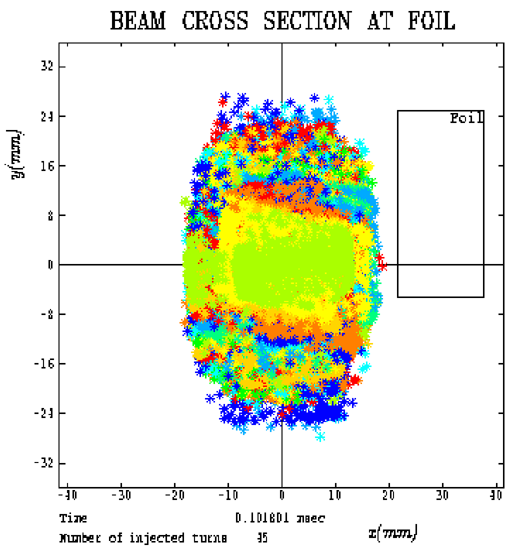




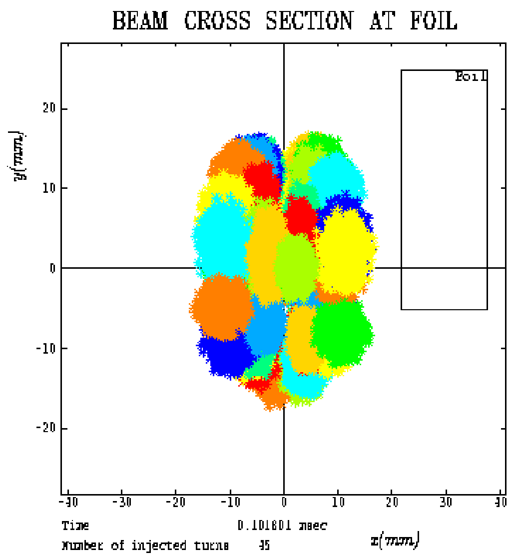
A small ring used to explore aspects of beam dynamics, including effects of strong space-charge forces in the beam.

Effect of Space Charge on an Intense Beam

Injected beam in Proposed Fermilab Booster

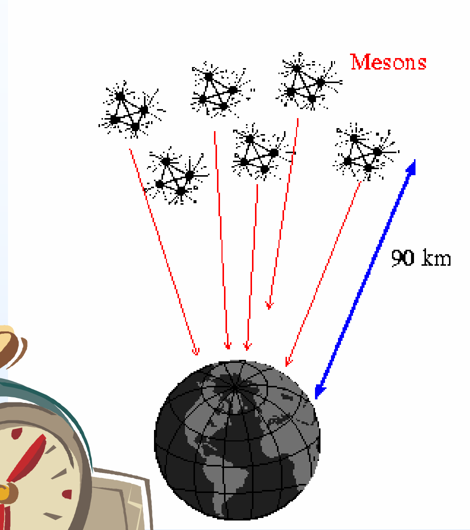


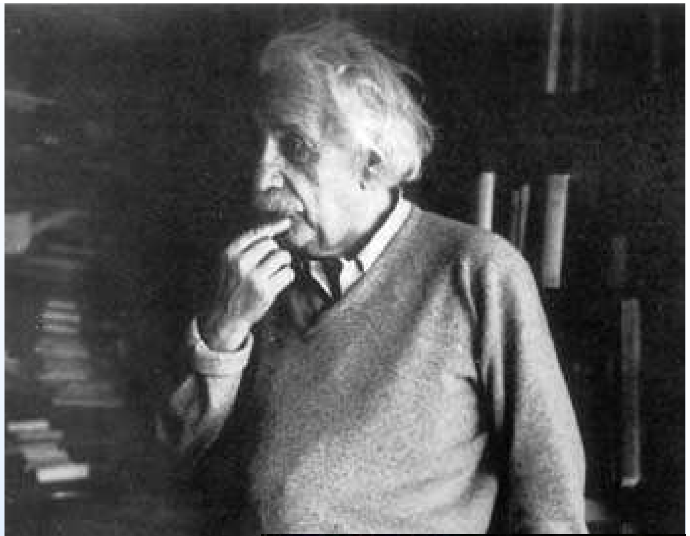
With space charge

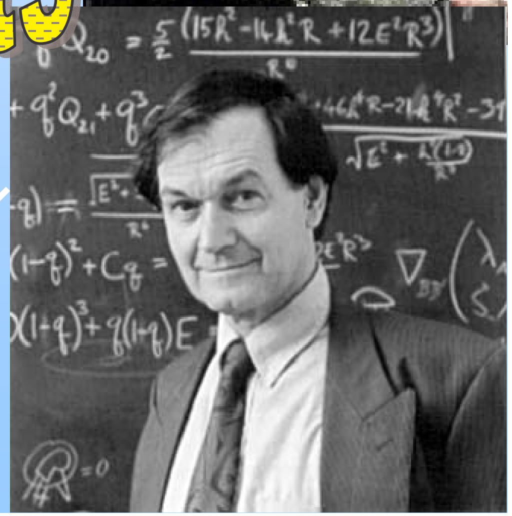


Without space charge

Electromagnetism and Relativity







Invariants

4-vectors

CERN School on Small Accelerators