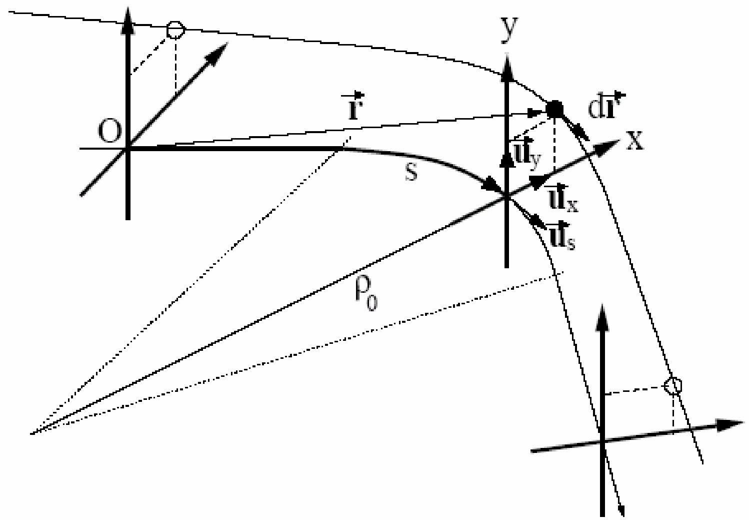
ION OPTICS WITH ELECTROSTATIC LENSES

1. Coordinate System and Matrix Formalism
2. The Paraxial Ray Equation
3. Solution of the Paraxial Ray Equation
4. The Elements of Electrostatic Ion Optics
5. The Transformation of the Longitudinal Coordinates
6. Geometric Optics
7. Electrostatic Quadrupoles and Deflectors
8. Phase Ellipses
9. Space Charge Forces

Coordinate System and Matrix Formalism



$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} x \\ x' \\ y \\ y' \\ l \\ \delta \end{pmatrix} = \begin{pmatrix} \text{horiz. (radial) displacement} \\ \text{horiz. (radial) angle} \\ \text{vert. (axial) displacement} \\ \text{vert. (axial) angle} \\ \text{path length difference} \\ \text{relative momentum deviation} \end{pmatrix} \tag{1}$$

$$\vec{x}(s) = R(s)\vec{x}(0) \tag{2}$$

$$R = \begin{pmatrix} R_{11} & R_{12} & 0 & 0 & 0 & R_{16} \\ R_{21} & R_{22} & 0 & 0 & 0 & R_{26} \\ 0 & 0 & R_{33} & R_{34} & 0 & 0 \\ 0 & 0 & R_{43} & R_{44} & 0 & 0 \\ R_{51} & R_{52} & 0 & 0 & R_{55} & R_{56} \\ 0 & 0 & 0 & 0 & 0 & R_{66} \end{pmatrix} \tag{3}$$

Horizontal and Vertical Transfer Matrix

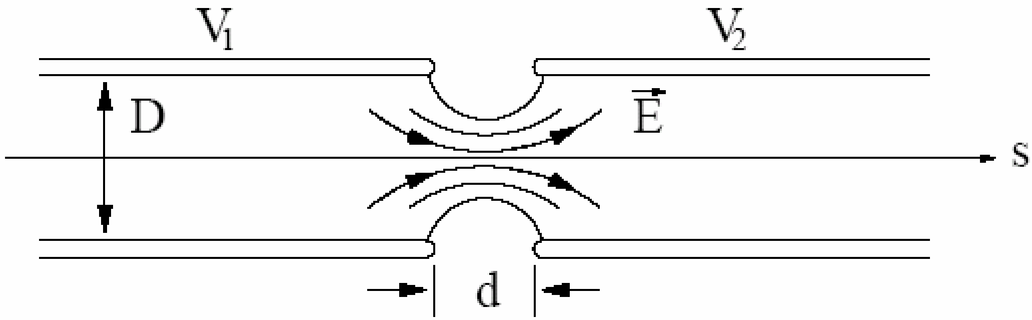
Rotational Symmetric Systems

$$R = \begin{pmatrix} R_{11} & R_{12} & 0 & 0 & 0 & 0 \\ R_{21} & R_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & R_{11} & R_{12} & 0 & 0 \\ 0 & 0 & R_{21} & R_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & R_{55} & R_{56} \\ 0 & 0 & 0 & 0 & 0 & R_{66} \end{pmatrix} \tag{4}$$

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} x(0) \\ x'(0) \end{pmatrix}$$

$$\begin{pmatrix} y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} \tag{5}$$

The Paraxial Ray Equation



$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\dot{x}) &= -q\frac{\partial V}{\partial x}\,,\\ \frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\dot{y}) &= -q\frac{\partial V}{\partial y}\,,\\ \frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\dot{s}) &= -q\frac{\partial V}{\partial s}\,.\end{aligned} \tag{7}$$

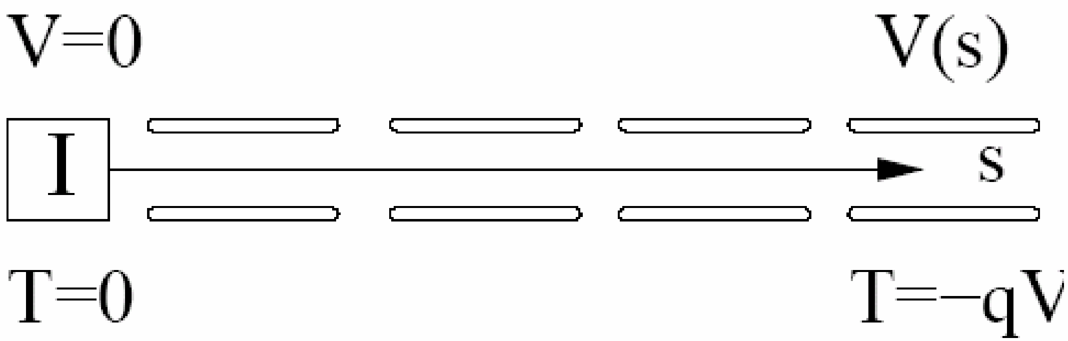
$$v_s = \dot{s} = \frac{\mathrm{d}s}{\mathrm{d}t}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} = v_s\frac{\mathrm{d}}{\mathrm{d}s}. \tag{8}$$

$$\dot{x} = v_s x', \qquad \dot{y} = v_s y', \qquad v^2 = v_s^2(1 + x'^2 + y'^2). \tag{9}$$

$$\begin{aligned}x'' &= \frac{1 + x'^2 + y'^2}{\gamma m v^2} q\left(x'\frac{\partial V}{\partial s} - \frac{\partial V}{\partial x}\right)\,,\\ y'' &= \frac{1 + x'^2 + y'^2}{\gamma m v^2} q\left(y'\frac{\partial V}{\partial s} - \frac{\partial V}{\partial y}\right)\,.\end{aligned} \tag{10}$$

Kinematics

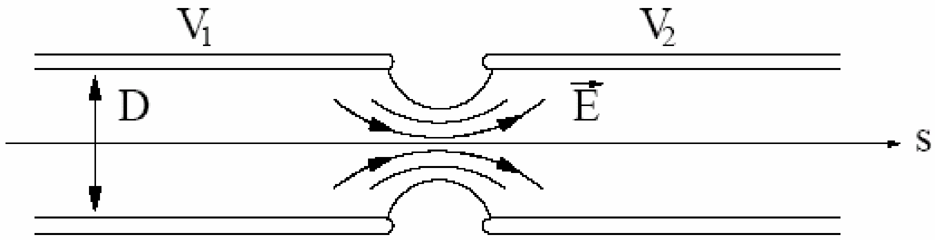
Relation between Kinetic Energy T and Electric Potential V



Electric Rigidity $|\vec{E}|\rho = pv/q$

$$pv = \gamma m v^2 = \frac{(pc)^2}{E} = T\frac{2mc^2 + T}{mc^2 + T} = -qV\frac{2mc^2 - qV}{mc^2 - qV} \tag{11}$$

The Paraxial Ray Equation



Rotational Symmetry and Laplace Equation $\nabla V = 0$

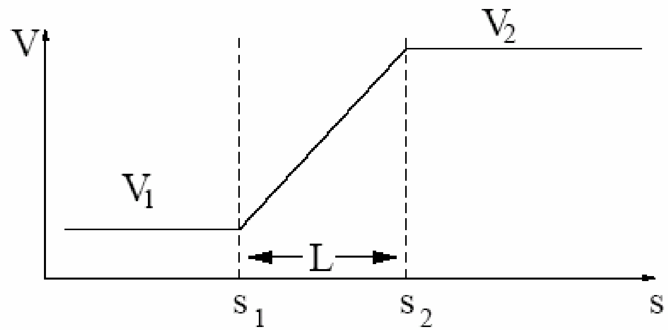
$$V(x, y, s) = V(s) - \frac{x^2 + y^2}{4} V''(s) + \cdots . \tag{17}$$

$$\begin{aligned} \frac{\partial V}{\partial x} &= -\frac{x}{2} V'' , \\ \frac{\partial V}{\partial y} &= -\frac{y}{2} V'' . \end{aligned}$$

Paraxial Ray Equation in Linear Approximation

$$\begin{aligned} x'' &= \frac{q}{pv} \left(x' V' + \frac{x}{2} V'' \right) , \\ y'' &= \frac{q}{pv} \left(y' V' + \frac{y}{2} V'' \right) . \end{aligned} \tag{18}$$

Solution of the Paraxial Ray Equation



$$\Delta x' = \lim_{\Delta s \to 0} \int_{s_i - \Delta s}^{s_i + \Delta s} x'' \mathrm{d}s = \lim_{\Delta s \to 0} \int_{s_i - \Delta s}^{s_i + \Delta s} \frac{xqV''}{2pv} \mathrm{d}s, \quad i = 1, 2. \tag{19}$$

$$s_1 : \quad \Delta x' = +\frac{q}{2p_1 v_1} \frac{V_2 - V_1}{L} x_1 = -\frac{1}{2p_1 v_1} \frac{E_2 - E_1}{L} x_1.$$

$$s_2 : \quad \Delta x' = -\frac{q}{2p_2 v_2} \frac{V_2 - V_1}{L} x_2 = +\frac{1}{2p_2 v_2} \frac{E_2 - E_1}{L} x_2.$$

$$R_x(s_1) = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2p_1 v_1} \frac{E_2 - E_1}{L} & 1 \end{pmatrix} \tag{20}$$

$$R_x(s_2) = \begin{pmatrix} 1 & 0 \\ +\frac{1}{2p_2 v_2} \frac{E_2 - E_1}{L} & 1 \end{pmatrix}$$

Solution of the Paraxial Ray Equation

Between s_1 and s_2 Constant Acceleration ($V'' = 0$):

$$\frac{x''}{x'} = \frac{qV'}{pv} = -\frac{\dot{p}}{pv} = -\frac{p'}{p}, \qquad \frac{x_2'}{x_1'} = \frac{p_1}{p_2}$$

With $x' = x_1' \frac{p_1}{p}$,

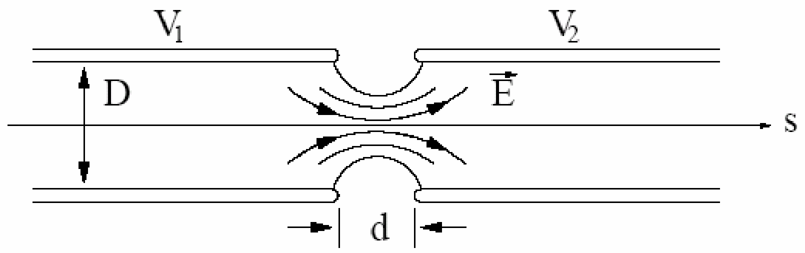
$$x_2 - x_1 = \int_1^2 x' \mathrm{d}s = x_1' p_1 \int_1^2 \frac{\mathrm{d}s}{p}. \tag{21}$$

Relativistic Exact Solution:

$$x_2 - x_1 = x_1' L \frac{p_1 c}{E_2 - E_1} \ln \frac{p_2 c + E_2}{p_1 c + E_1}. \tag{24}$$

$$R_x = \begin{pmatrix} 1 & L_{\mathrm{eff}} \\ 0 & p_1/p_2 \end{pmatrix}, \quad L_{\mathrm{eff}} = L \frac{p_1 c}{E_2 - E_1} \ln \frac{p_2 c + E_2}{p_1 c + E_1}. \tag{25}$$

Transport Matrix of Acceleration Tube



Rough Approximation

$$R_{x,y} \begin{pmatrix} 1 & 0 \\ +\frac{E_2-E_1}{2p_2v_2}\frac{1}{L} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_{\mathrm{eff}} \\ 0 & \frac{p_1}{p_2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{E_2-E_1}{2p_1v_1}\frac{1}{L} & 1 \end{pmatrix}. \tag{26}$$

Accelerating Tube Lens

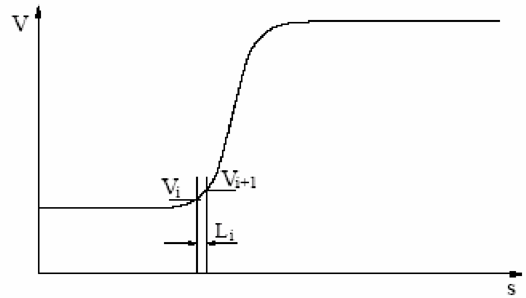
focusing lens – modified drift – defocusing lens

Decelerating Tube Lens

defocusing lens – modified drift – focusing lens

$$\det(R_x) = p_1/p_2. \tag{27}$$

Refined Treatment: Segmentation



Segment i:

$$R_i = \begin{pmatrix} 1 & 0 \\ +\frac{E_{i+1}-E_i}{2p_{i+1}v_{i+1}}\frac{1}{L_i} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_{\mathrm{eff}\,i}\frac{p_i}{p_{i+1}} \\ 0 & \frac{p_i}{p_{i+1}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{E_{i+1}-E_i}{2p_iv_i}\frac{1}{L_i} & 1 \end{pmatrix} \tag{28}$$

Complete Transport Matrix

$$R = R_n\, R_{n-1}\, \cdots\, R_1. \tag{29}$$

Analytical Approximation for $V(s)$

$$V(s) = \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{2}\tanh\left(2,64\frac{s}{D}\right). \tag{30}$$

Driftspace

$$R_{x,y} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} . \tag{31}$$

Constant Energy Gradient ($V' = const$)

Elements of Electrostatic Ion Optics

$$R_{x,y} = \begin{pmatrix} 1 & L_{\text{eff}} \\ 0 & p_1/p_2 \end{pmatrix} \tag{32}$$

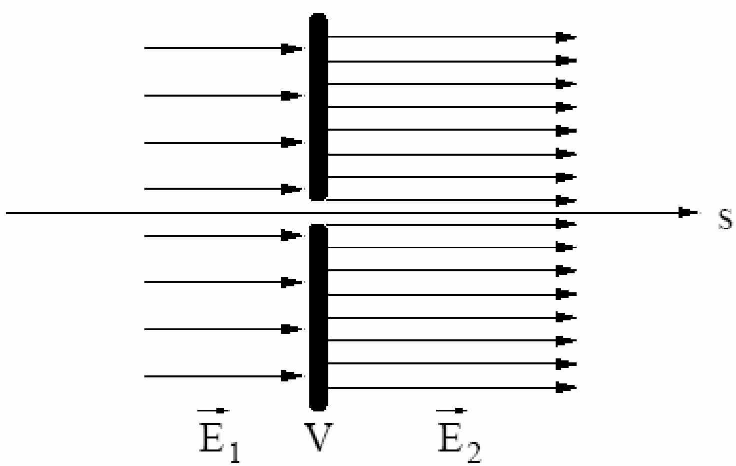
$$L_{\text{eff}} = L \frac{p_1 c}{E_2 - E_1} \ln \frac{p_2 c + E_2}{p_1 c + E_1}.$$

Aperture Lens: Sudden Change of Energy Gradient $E' = \mathrm{d}E/\mathrm{d}s$

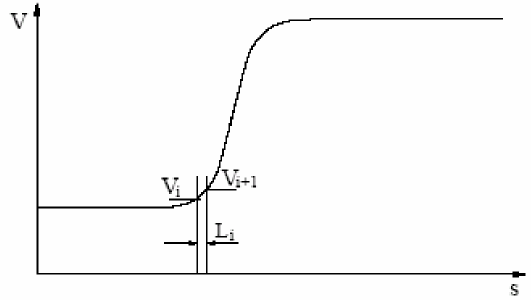
$$R_{x,y} = \begin{pmatrix} 1 & 0 \\ -(E_2' - E_1')/(2pv) & 1 \end{pmatrix} . \tag{33}$$

Focusing Power

$$\frac{1}{f} = \frac{E_2' - E_1'}{2pv}. \tag{34}$$



Elements of Electrostatic Ionoptics



Transition from Segment $i-1$ to Segment i

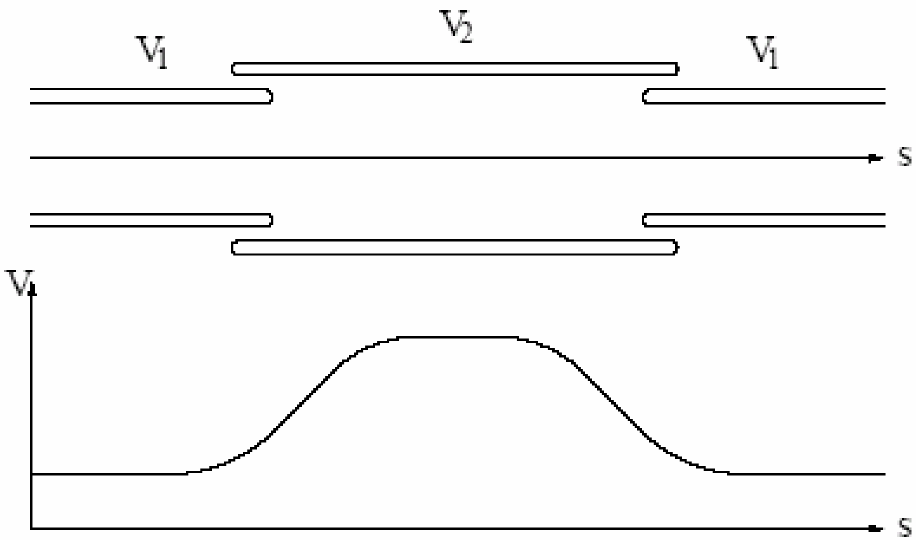
Sudden Change of Energy Gradient $E' = \mathrm{d}E/\mathrm{d}s$ like Aperture Lens

$$E'_i - E'_{i-1} = \frac{E_{i+1} - E_i}{L_i} - \frac{E_i - E_{i-1}}{L_{i-1}} \quad :$$

$$R_x = R_y = \begin{pmatrix} 1 & 0 \\ -\frac{E'_i - E'_{i-1}}{2 p_i v_i} & 1 \end{pmatrix} . \qquad (35)$$

Einzel Lens

Scheme of an Einzel Lens



Einzel Lens: Focusing Lens for $V_2 > V_1$ and $V_2 < V_1$.

Einzel lens does not change the beam energy

Transport Matrix: Product of two Tube Lense Matrices

$$R = R(V_1, V_2) R(V_2, V_1). \tag{36}$$

Longitudinal Transfer Matrix

Drift Space

$$R_l = \begin{pmatrix} R_{55} & R_{56} \\ R_{65} & R_{66} \end{pmatrix} = \begin{pmatrix} 1 & L/\gamma^2 \\ 0 & 1 \end{pmatrix}. \tag{37}$$

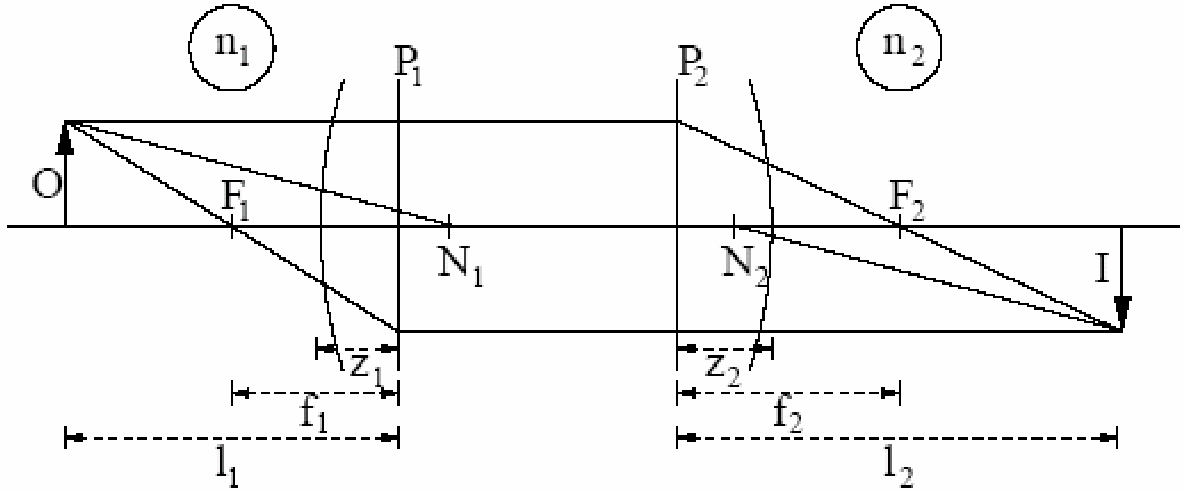
Constant Energy Gradient ($V' = const$)

$$R_l = \begin{pmatrix} R_{55} & R_{56} \\ R_{65} & R_{66} \end{pmatrix} = \begin{pmatrix} \frac{v_2}{v_1} & p_1 \frac{v_2 - v_1}{E_2 - E_1} L \\ 0 & \frac{p_1 v_1}{p_2 v_2} \end{pmatrix}. \tag{38}$$

Aperture Lens

$$R_l = \begin{pmatrix} R_{55} & R_{56} \\ R_{65} & R_{66} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{39}$$

Geometric Optics



$$n = \frac{n_2}{n_1} = \frac{f_2}{f_1} = \frac{p_2}{p_1} \tag{40}$$

$$\begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = \begin{pmatrix} 1 & z_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f_2 & 1/n \end{pmatrix} \begin{pmatrix} 1 & z_1 \\ 0 & 1 \end{pmatrix}, \tag{42}$$

$$\begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = \begin{pmatrix} 1 - \frac{z_2}{f_2} & z_1 + \frac{z_2}{n} - \frac{z_1 z_2}{f_2} \\ -\frac{1}{f_2} & \frac{1}{n} - \frac{z_1}{f_2} \end{pmatrix}. \tag{43}$$

$$z_1 = \frac{R_{22} - \det(R)}{R_{21}}, \quad z_2 = \frac{R_{11} - 1}{R_{21}}, \quad f_1 = -\frac{\det(R)}{R_{21}}, \quad f_2 = -\frac{1}{R_{21}} \tag{44}$$

$$\overline{\mathrm{F}_1\mathrm{N}_1} = f_2, \qquad \overline{\mathrm{N}_2\mathrm{F}_2} = f_1. \tag{45}$$

Telescopic Systems

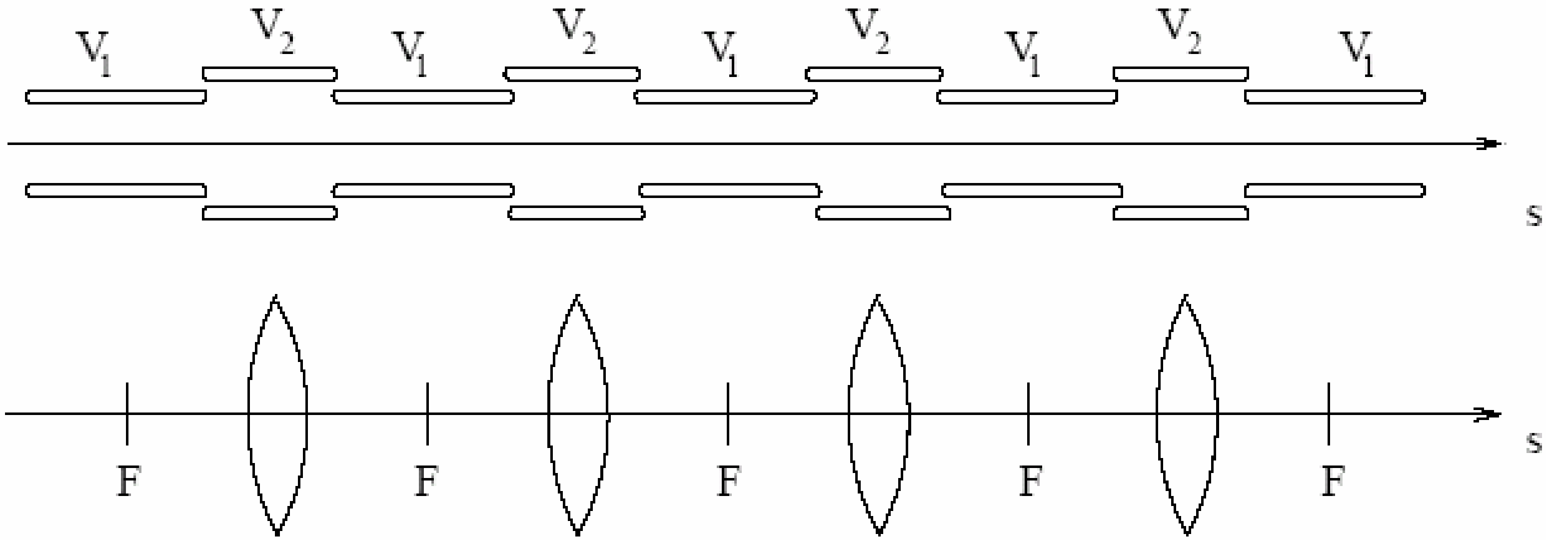
Transport Matrix between Focal Planes of a Thick Focusing Lens

$$R_x = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & f \\ -\frac{1}{f} & 0 \end{pmatrix}. \tag{48}$$

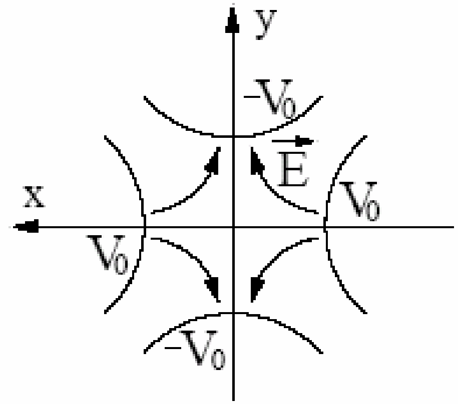
Telescope of Two Identical Focusing Lenses: -1 Transformation

$$R_x = \begin{pmatrix} 0 & f \\ -1/f & 0 \end{pmatrix} \begin{pmatrix} 0 & f \\ -1/f & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{49}$$

Telescope of 4 Einzel Lenses and Optical Analogon: $+1$ Transformation



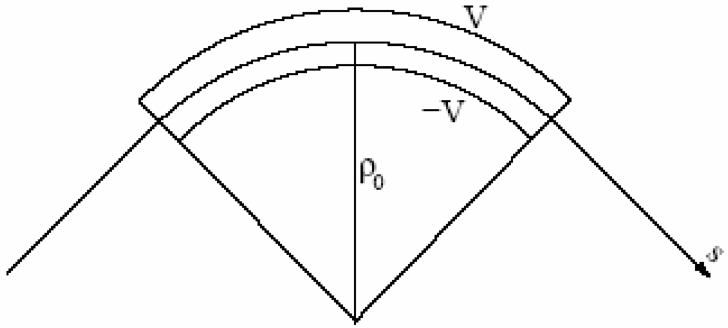
Electrostatic Quadrupole



$$R = \begin{pmatrix} \cos\sqrt{k}L & \frac{\sin\sqrt{k}L}{\sqrt{k}} & 0 & 0 & 0 & 0 \\ -\sqrt{k}\sin\sqrt{k}L & \cos\sqrt{k}L & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh\sqrt{k}L & \frac{\sinh\sqrt{k}L}{\sqrt{k}} & 0 & 0 \\ 0 & 0 & \sqrt{k}\sinh\sqrt{k}L & \cosh\sqrt{k}L & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L/\gamma^2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{50}$$

$$k = \frac{|g|}{(E\rho)_0} = \frac{2|V_0|}{a^2}\frac{1}{(E\rho)_0}. \tag{51}$$

Electrostatic Deflector

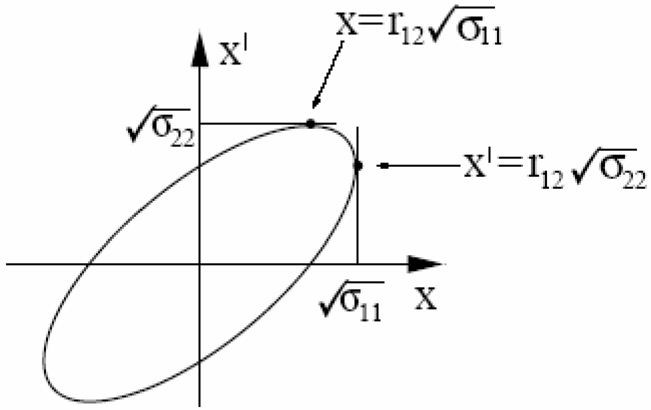


$$R = \begin{pmatrix} \cos\sqrt{k_x}L & \frac{\sin\sqrt{k_x}L}{\sqrt{k_x}} & 0 & 0 & 0 & \frac{2-\beta_0^2}{\rho_0 k_x}(1-\cos\sqrt{k_x}L) \\ -\sqrt{k_x}\sin\sqrt{k_x}L & \cos\sqrt{k_x}L & 0 & 0 & 0 & \frac{2-\beta_0^2}{\rho_0\sqrt{k_x}}\sin\sqrt{k_x}L \\ 0 & 0 & \cos\sqrt{k_y}L & \frac{\sin\sqrt{k_y}L}{\sqrt{k_y}} & 0 & 0 \\ 0 & 0 & -\sqrt{k_y}\sin\sqrt{k_y}L & \cos\sqrt{k_y}L & 0 & 0 \\ -\frac{\sin\sqrt{k_x}L}{\rho_0\sqrt{k_x}} & \frac{\cos(\sqrt{k_x}L)-1}{\rho_0 k_x} & 0 & 0 & 1 & \frac{L}{\gamma^2} - \frac{2-\beta_0^2}{\rho_0^2 k_x}\left(L - \frac{\sin\sqrt{k_x}L}{\sqrt{k_x}}\right) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{52}$$

$$n_E = 1 + \frac{\rho_0}{r_0}, \quad k_x = \frac{3 - n_E - \beta_0^2}{\rho_0^2}, \quad k_y = \frac{n_E - 1}{\rho_0^2}. \tag{53}$$

Double Focusing: $1 < n_E < 3 - \beta_0^2$

Phase Ellipses



Transverse and Longitudinal Phase Ellipses

$$
\begin{aligned}
\sigma_x &= \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}, \\
\sigma_y &= \begin{pmatrix} \sigma_{33} & \sigma_{34} \\ \sigma_{34} & \sigma_{44} \end{pmatrix}, \\
\sigma_l &= \begin{pmatrix} \sigma_{55} & \sigma_{56} \\ \sigma_{56} & \sigma_{66} \end{pmatrix}.
\end{aligned}
\tag{54}
$$

$$
(x, x')\sigma^{-1} \begin{pmatrix} x \\ x' \end{pmatrix} = 1. \tag{55}
$$

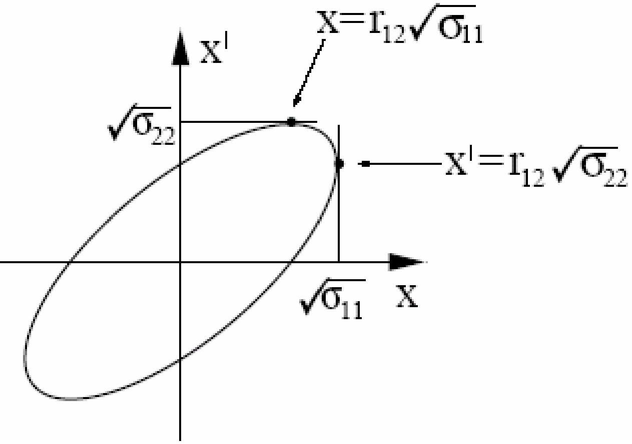
$$
\sigma_{22}x^2 - 2\sigma_{12}xx' + \sigma_{11}x'^2 = \det(\sigma_x) = \epsilon_x^2. \tag{56}
$$

Phase Ellipse

Emittance: Area of the Ellipse

$$E_x = \pi \epsilon_x = \pi \sqrt{\sigma_{11}\sigma_{22} - \sigma_{12}^2}. \tag{57}$$

Maximum Displacement in x and x':



$$x_{max} = \sqrt{\sigma_{11}}, \quad x'_{max} = \sqrt{\sigma_{22}}. \tag{58}$$

Correlation Parameter r_{12}:

$$r_{12} = \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}. \tag{59}$$

Often Definition of σ_x, σ_y and σ_l as Covariance Matrix of Beam Density Distribution $\Longrightarrow$ RMS-Values $\sqrt{\sigma_{11}}$, $\sqrt{\sigma_{22}}$, ...

Gaussian Beam Density Distribution $\rho(\vec{x})$ with $\vec{x} = (x, x')$:

$$\rho(\vec{x}) = \frac{1}{2\pi\epsilon_x} \exp\left(-\frac{1}{2}\vec{x}^T \sigma_x^{-1} \vec{x}\right). \tag{60}$$

Phase Ellipse

Transformation of the Phase Ellipse

$$
\begin{aligned}
\sigma_x(s) &= R_x(s)\sigma_x(0)R_x^{\mathrm{T}}(s)\,, \\
\sigma_y(s) &= R_y(s)\sigma_y(0)R_y^{\mathrm{T}}(s)\,, \\
\sigma_l(s) &= R_l(s)\sigma_l(0)R_l^{\mathrm{T}}(s)\,.
\end{aligned}
$$

Beam Envelopes

$$
\begin{aligned}
x\mathrm{max}(s) &= \sqrt{\sigma_{11}(s)}\,, \\
y\mathrm{max}(s) &= \sqrt{\sigma_{33}(s)}\,, \\
l\mathrm{max}(s) &= \sqrt{\sigma_{55}(s)}\,.
\end{aligned}
$$

Electrostatic Acceleration: Transverse and Longitudinal Beam Emittances

$$
\begin{aligned}
\epsilon_x(s) &= \frac{p(0)}{p(s)}\epsilon_x(0)\,, \\
\epsilon_y(s) &= \frac{p(0)}{p(s)}\epsilon_y(0)\,, \qquad (63) \\
\epsilon_l(s) &= \frac{p(0)}{p(s)}\epsilon_l(0)\,.
\end{aligned}
$$

Phase Ellipsoid

Extension of the Phase Ellipse Formalism: Phase Ellipsoid

$$\vec{x}^T \sigma^{-1} \vec{x} = 1 \tag{64}$$

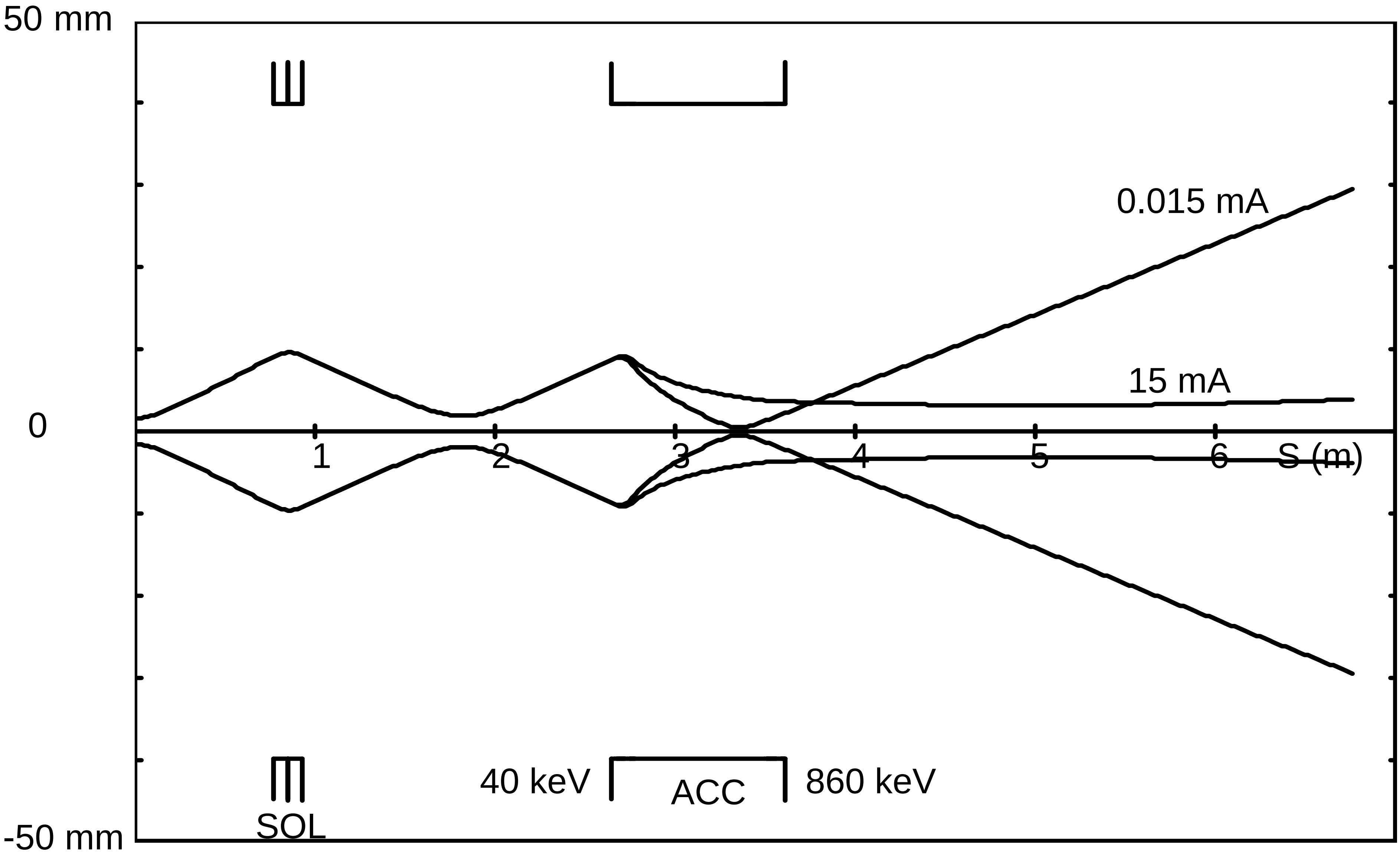
Systems with Midplane Symmetry:

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 & 0 & \sigma_{15} & \sigma_{16} \\ \sigma_{12} & \sigma_{22} & 0 & 0 & \sigma_{25} & \sigma_{26} \\ 0 & 0 & \sigma_{33} & \sigma_{34} & 0 & 0 \\ 0 & 0 & \sigma_{34} & \sigma_{44} & 0 & 0 \\ \sigma_{15} & \sigma_{25} & 0 & 0 & \sigma_{55} & \sigma_{56} \\ \sigma_{16} & \sigma_{26} & 0 & 0 & \sigma_{56} & \sigma_{66} \end{pmatrix} \tag{65}$$

Gaussian Beam Density Distribution $\rho(\vec{x})$:

$$\rho(\vec{x}) = \frac{1}{(2\pi)^3 \sqrt{\det(\sigma)}} \exp\left(-\frac{1}{2}\vec{x}^T \sigma^{-1} \vec{x}\right)$$

Acceleration Tube: Beam Envelopes



PSI Graphic Transport Framework by U. Rohrer

Summary

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