High Voltage Engineering

Enrique Gaxiola

Many thanks to the Electrical Power Systems Group, Eindhoven University of Technology, The Netherlands
& CERN AB-BT Group colleagues

Introductory examples

Theoretical foundation and numerical field simulation methods

Generation of high voltages

Insulation and Breakdown

Measurement techniques

Introduction E.Gaxiola:

Studied Power Engineering

Ph.D. on Dielectric Breakdown in Insulating Gases;

Non-Uniform Fields and Space Charge Effects

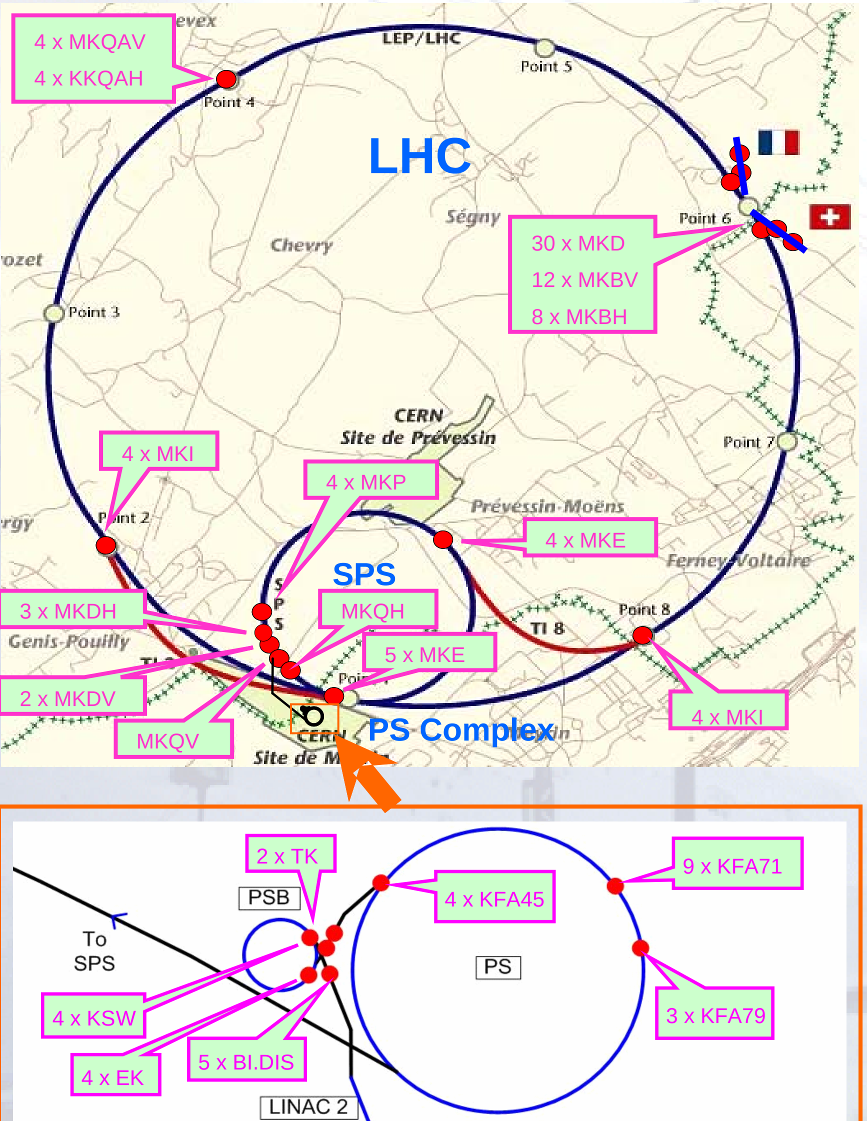
Industry R&D on Plasma Physics / Gas Discharges

CERN Accelerators & Beam, Beam Transfer,

Kicker Innovations:

- Electromagnetism
- Beam impedance reduction
- Vacuum high voltage breakdown in traveling wave structures.
- Pulsed power semiconductor applications

CERN Septa and Kicker examples



Reference [1]

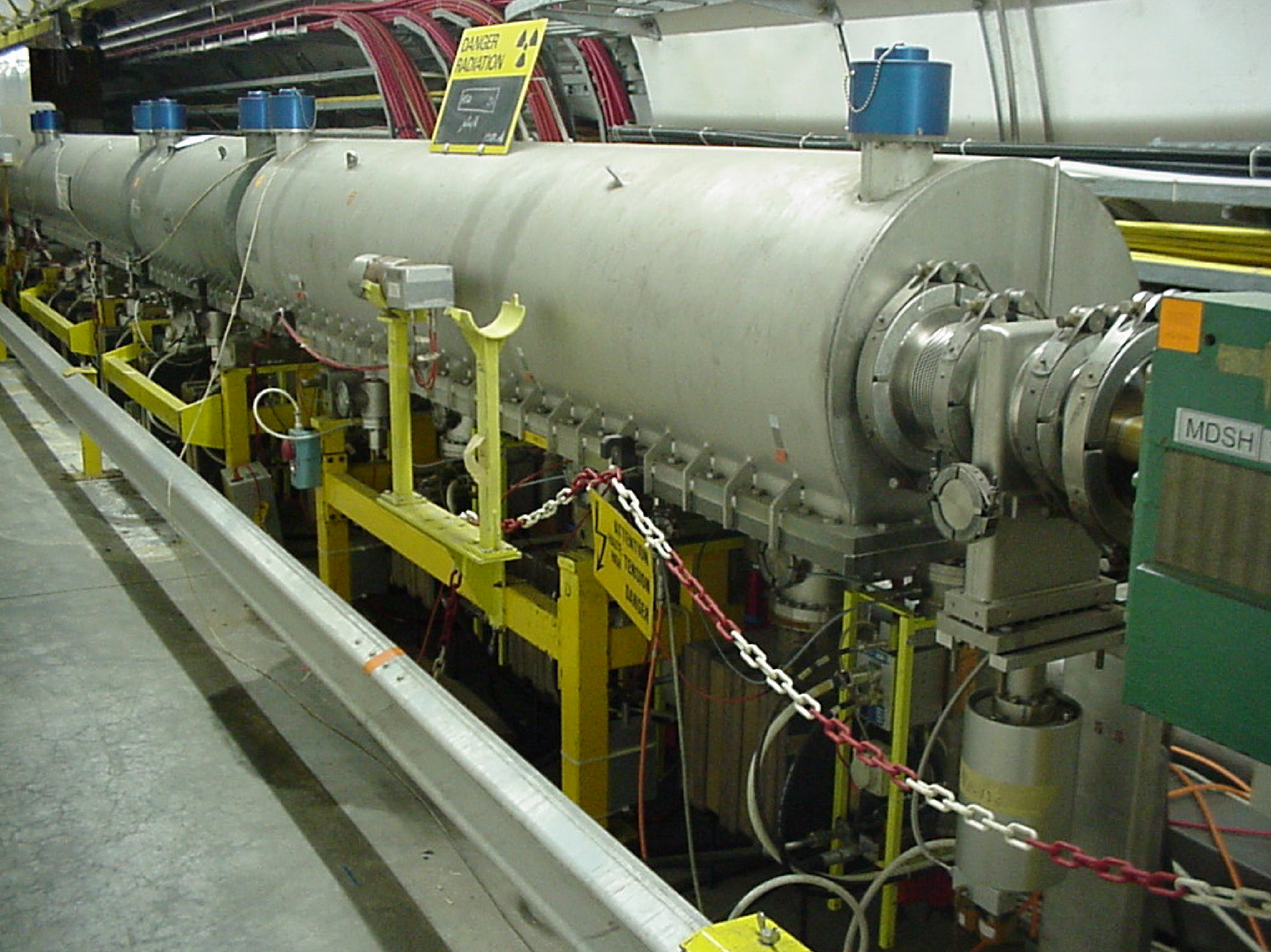
- Large Hadron Collider
  14 TeV
- Super Proton Synchrotron
  450 GeV
- Proton Synchrotron
  26 GeV

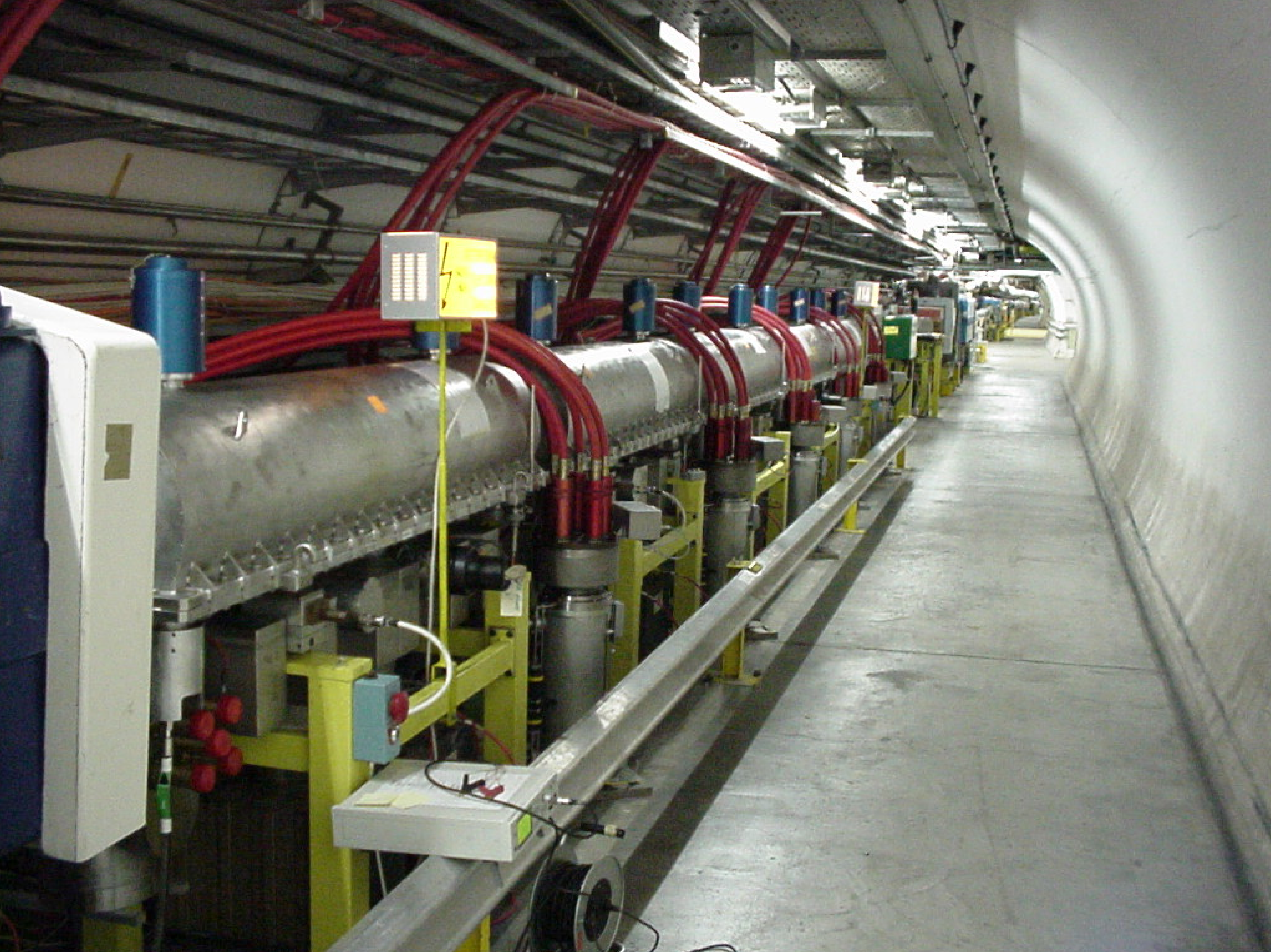
Septum: $E \leq 12$ MV/m T = d.c.
l= 0.8 – 15m

Kicker: V=80kV
B = 0.1-0.3 T T = 10 ns - 200μs
l=0.2 – 16m

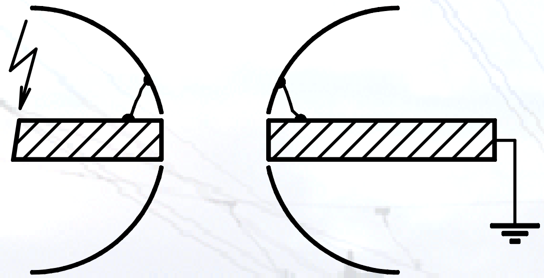
RF cavities: High gradients, $E \leq 150$MV/m

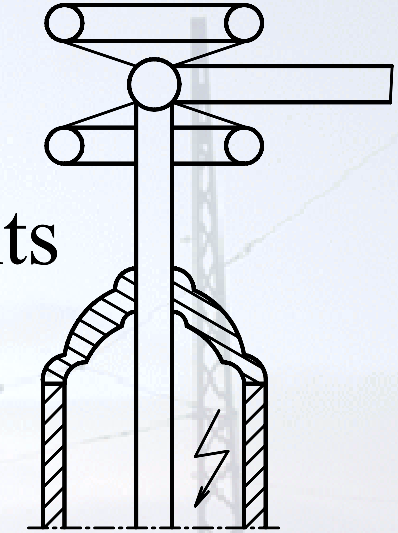

SPS septa ZS

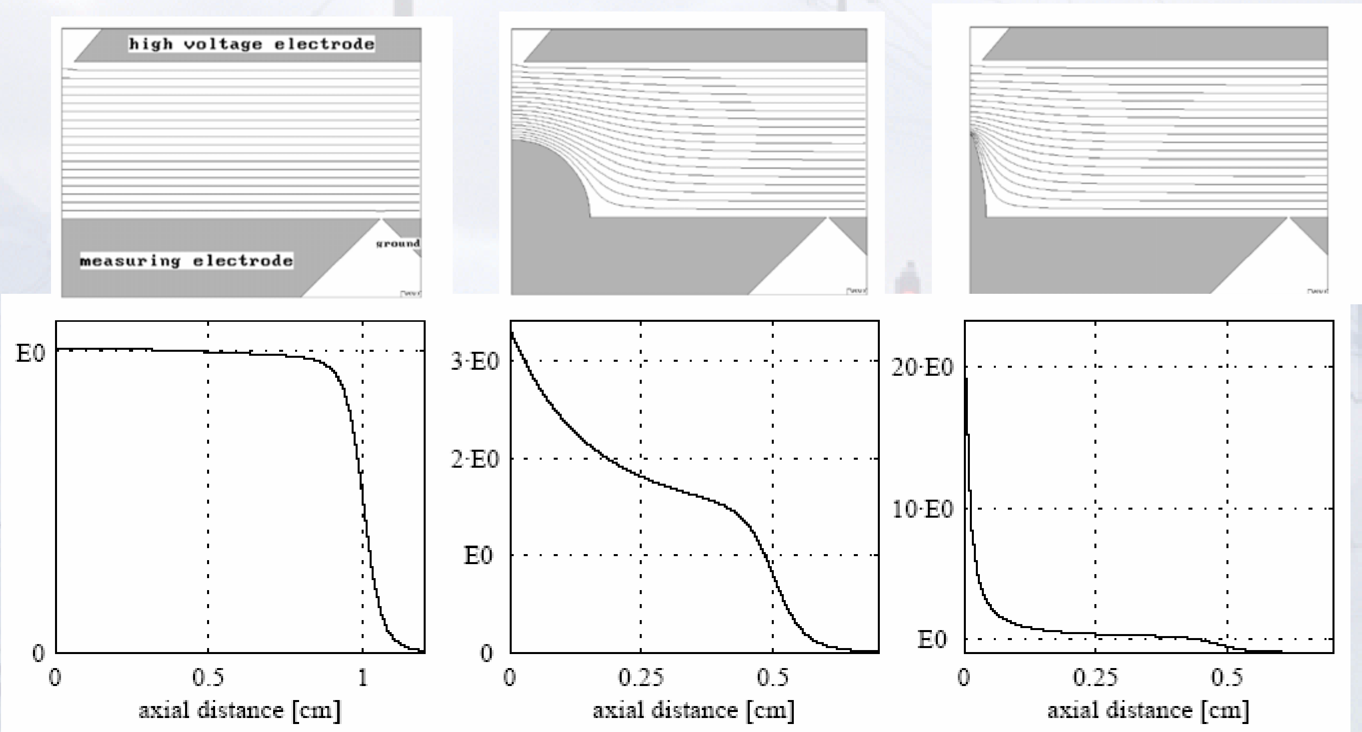

DANGER
RADIATION
MDSH

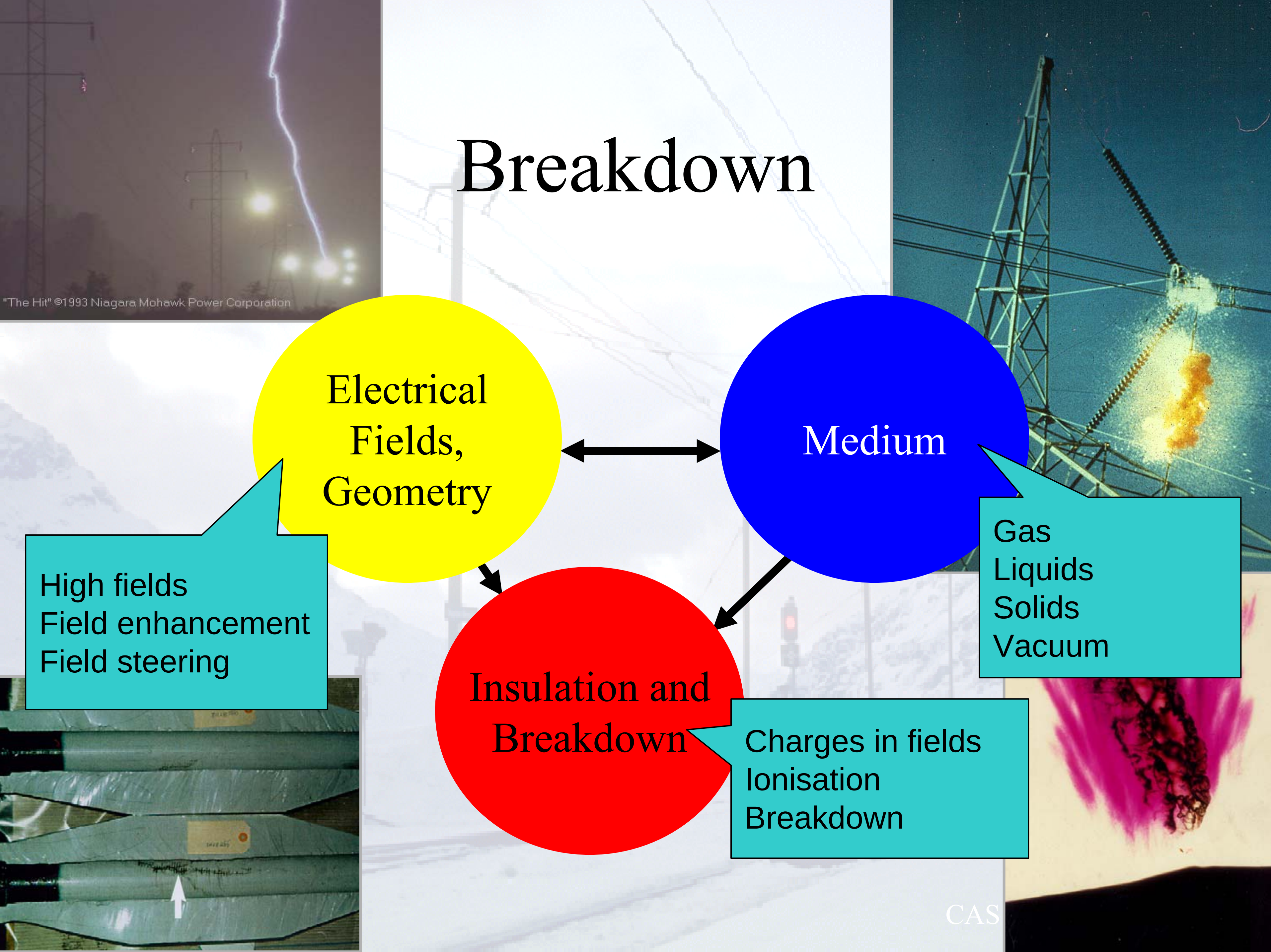


- Maxwell equations for calculating Electromagnetic fields, voltages, currents
  - Analytical
  - Numerical








Breakdown
"The Hit" ©1993 Niagara Mohawk Power Corporation
Electrical Fields, Geometry
Medium
Insulation and Breakdown
High fields
Field enhancement
Field steering
Gas
Liquids
Solids
Vacuum
Charges in fields
Ionisation
Breakdown
CAS

NUMERICAL FIELD SIMULATION METHODS

-**CSM** (Charge Simulation Method): (Coulomb)

Electrode configuration is replaced by a set of discrete charges

- **FDM** (Finite Difference Method):

Laplace equation is discretised on a rectangular grid

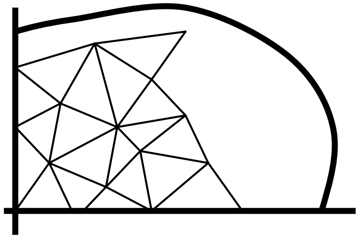
- **FEM (Finite Element Method):** Vector Fields (Opera, Tosca), Ansys, Ansoft

Potential distribution corresponds with minimum electric field energy ($w=\frac{1}{2}\varepsilon E^2$)

- **BEM (Boundary Element Method):** IES (Electro, Oersted)

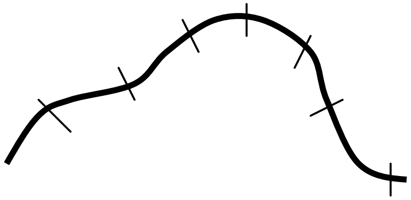
Potential and its derivative in normal direction on boundary are sufficient

Procedure FEM



1. Generate mesh of triangles:
2. Calculate matrix coefficients: $[S]_{ij} = (\nabla\alpha_i \cdot \nabla\alpha_j)A$
3. Solve matrix equation: $\begin{bmatrix} S_{kf} & S_{kp} \end{bmatrix} \begin{bmatrix} U_f \\ U_p \end{bmatrix} = 0$
4. Determine equipotential lines and/or field lines

Procedure BEM



1. Generate elements along interfaces
2. Generate matrix coefficients: $H_{ij} = \int_{S_j} \frac{\partial \ln r_i}{\partial n} ds\,, \quad G_{ij} = \int_{S_j} \ln r_i ds$
3. Solve matrix equation: $\sum_{j=1}^{n} (H_{ij} - \pi\delta_{ij}) U_j = \sum_{j=1}^{n} G_{ij} Q_j$
4. Determine potential on abritary position:

$$U(x_0, y_0) = \frac{1}{2\pi}\left(\sum_{j=1}^{n} U_j \int_{S_j} \frac{\partial \ln r}{\partial n} ds - \sum_{j=1}^{n} Q_j \int_{S_j} \ln r ds \right)$$

Generation of High Voltages

- AC Sources (50/60 Hz)

  High voltage transformer (one coil; divided coils; cascade)
  Resonance source (series; parallel)

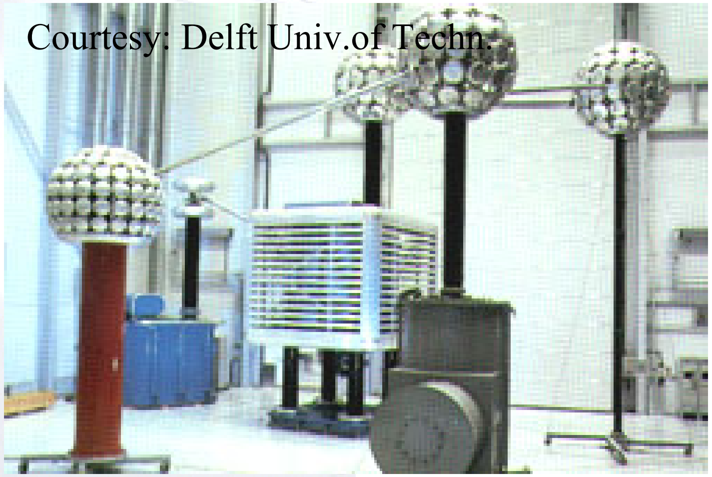
- DC Sources

  Rectifier circuits (single stage; cascade)
  Electrostatic generator (van de Graaff generator)

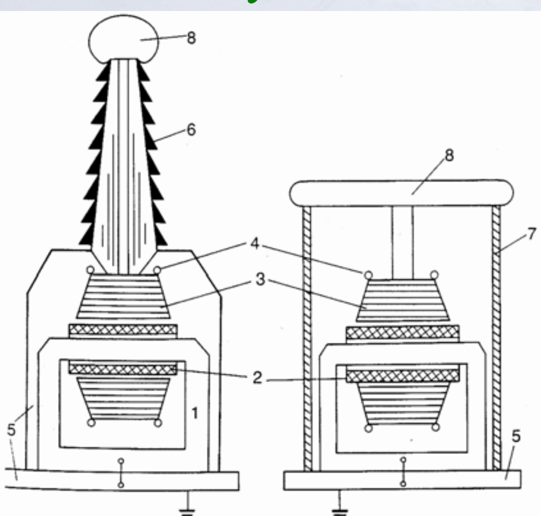
- Pulse sources

  Pulse circuits (single stage; cascade; pulse transformer)
  Traveling wave generators (PFL; PFN; transmission line transformer)

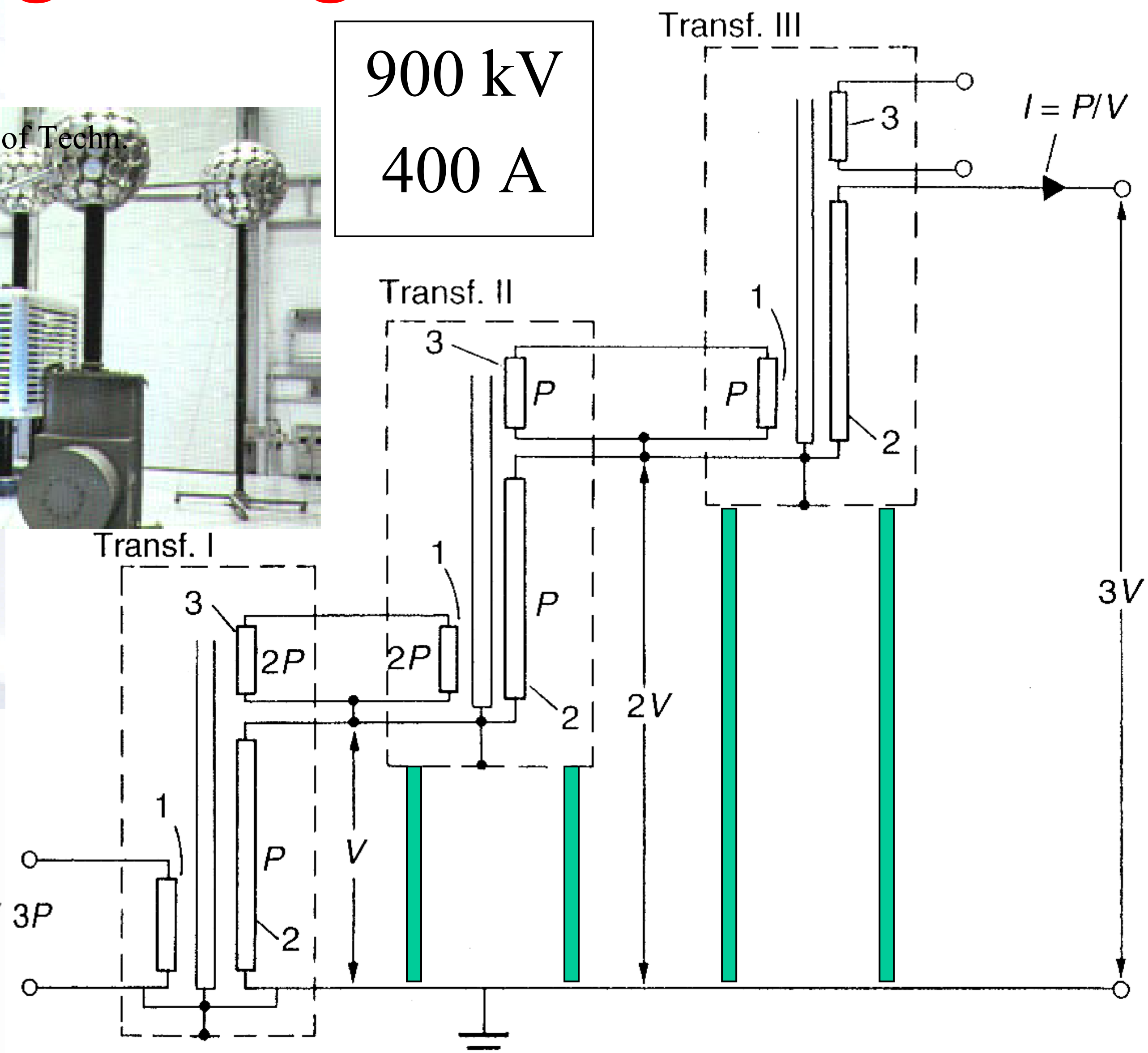
Cascaded High voltage transformer



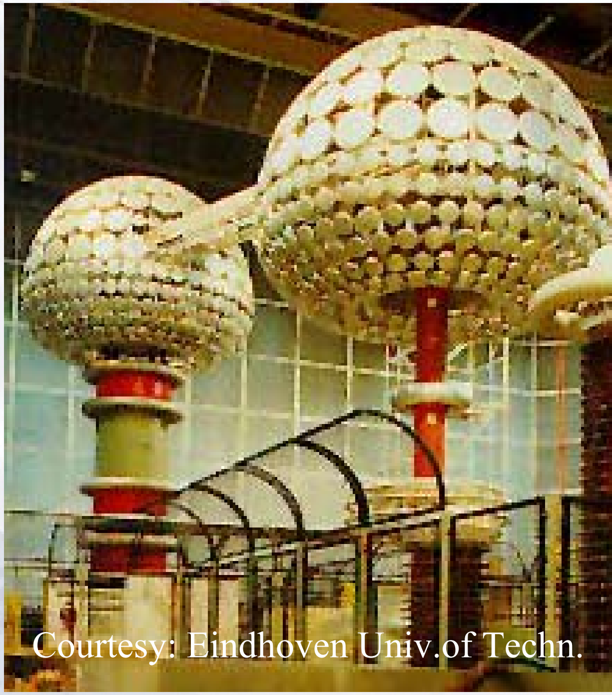
1: primary coil
2: secondary coil
3: tertiary coil

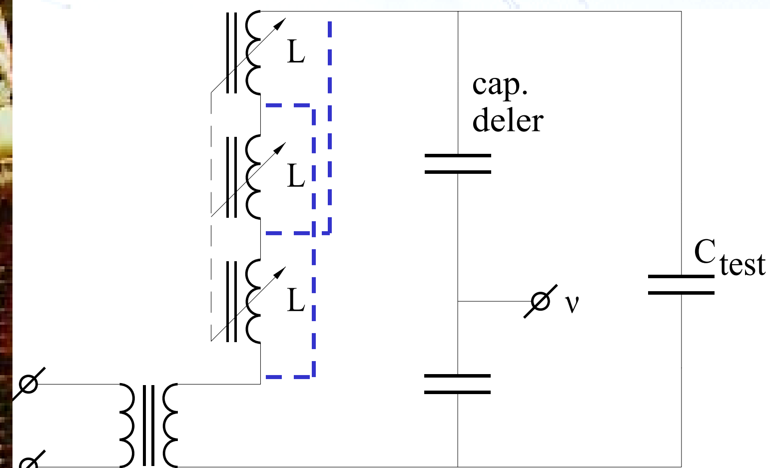


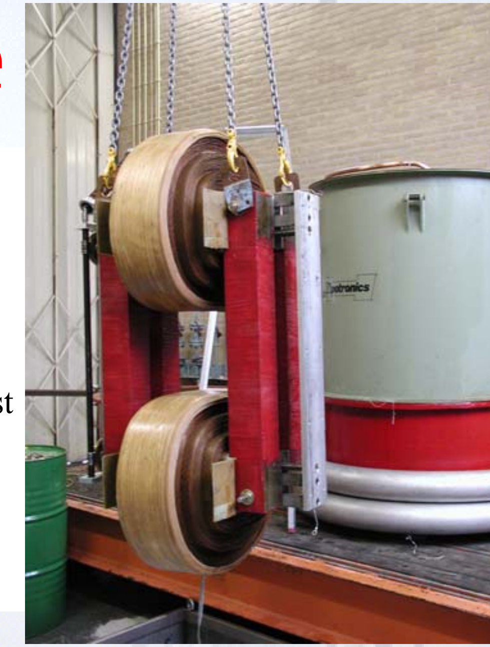
900 kV
400 A

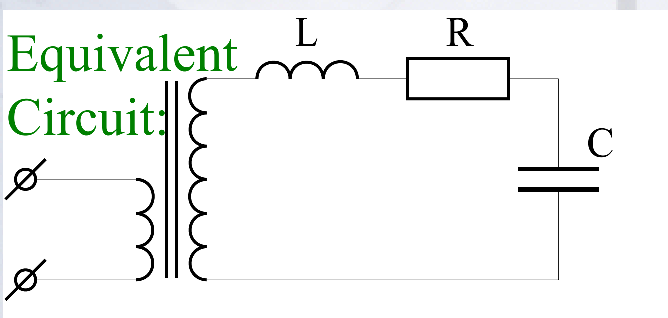


Resonance Source


Courtesy: Eindhoven Univ.of Techn.







$$|H(\omega)| = \frac{Q\,\omega_0/\omega}{\sqrt{1+Q^2(\omega_0/\omega-\omega/\omega_0)^2}}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad \text{and} \quad Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

+ Waveform: almost perfect sinusoidal

+ Power: 1/Q of “normal” transformer

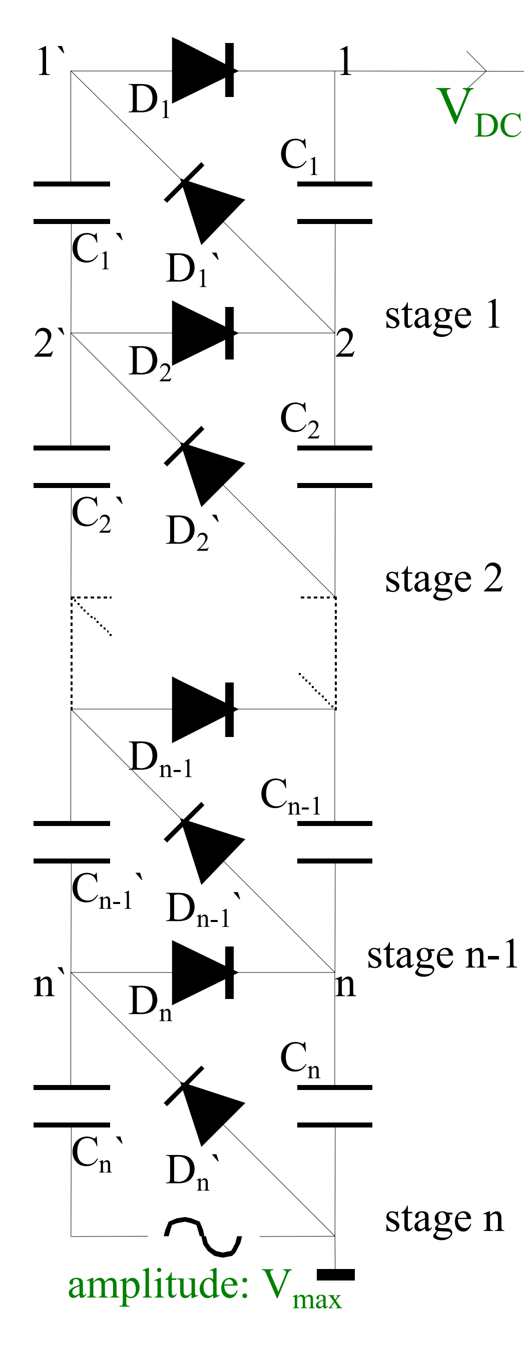
+ Short circuit: Q→0 results in V→0

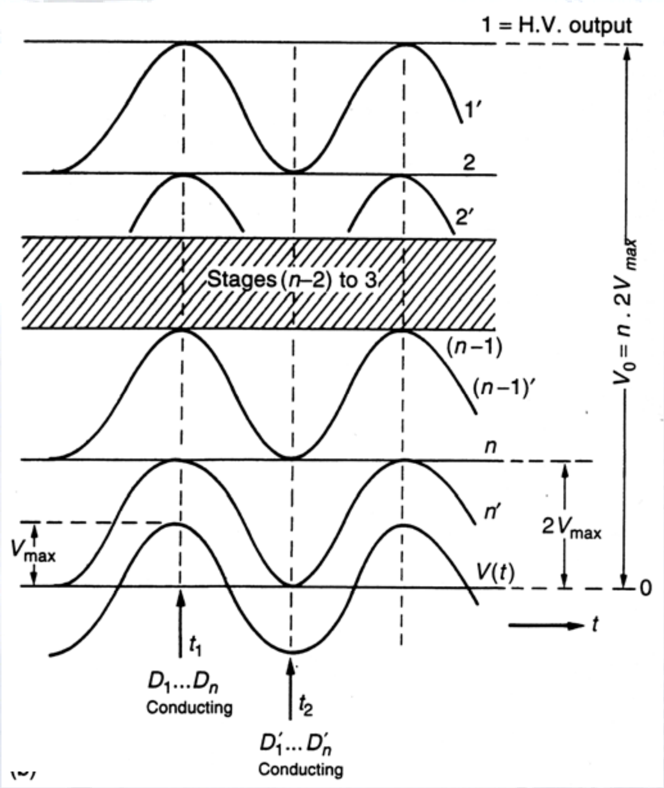
- No resistive load

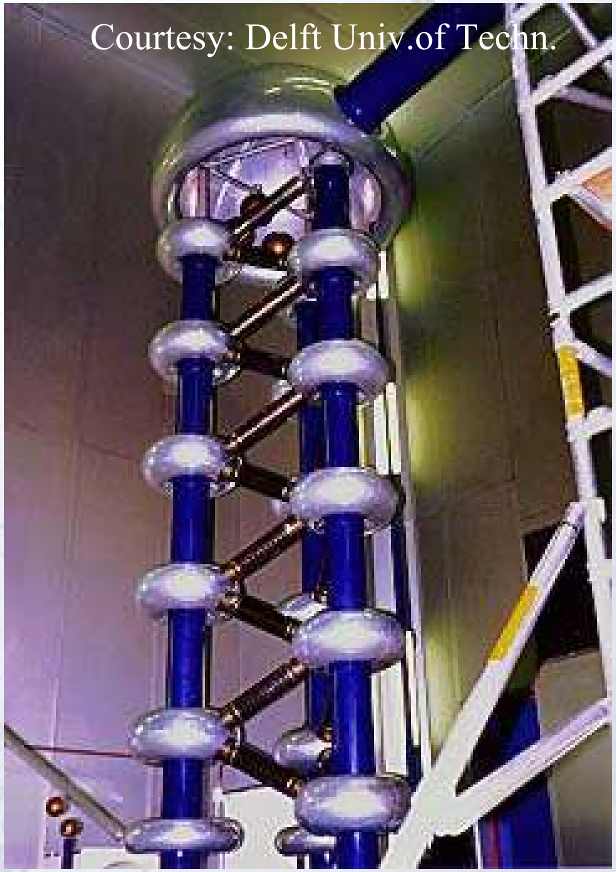
900 kV
100 mA

Cascaded Rectifier

(Greinacher; Cockcroft - Walton)







$$V_{DC} = 2nV_{max}$$

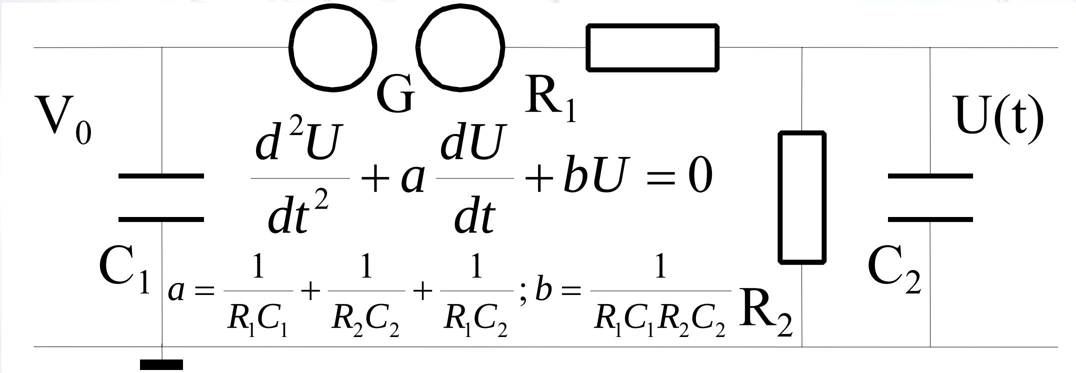
Voltage: 2 MV

Reduce δV (~n^2) and ΔV (~n^3) by:

larger C's (more energy in cascade)

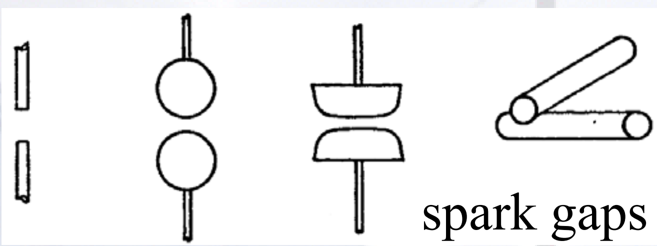
higher f (up to tens of kilohertz)

Single-Stage Pulse Source



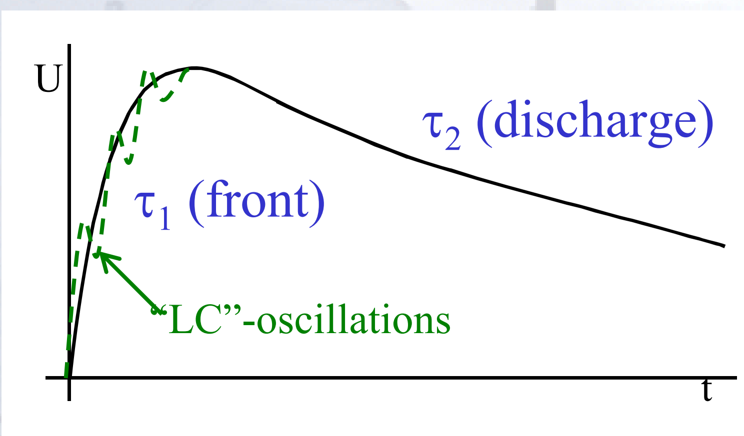
$$\frac{d^2U}{dt^2} + a\frac{dU}{dt} + bU = 0$$

$$a = \frac{1}{R_1C_1} + \frac{1}{R_2C_2} + \frac{1}{R_1C_2}; b = \frac{1}{R_1C_1R_2C_2}$$



60 kV

1 kA



$$U(t) = V_0\left(e^{-t/\tau_2} - e^{-t/\tau_1}\right)$$

if C1 >> C2 and R2 >> R1

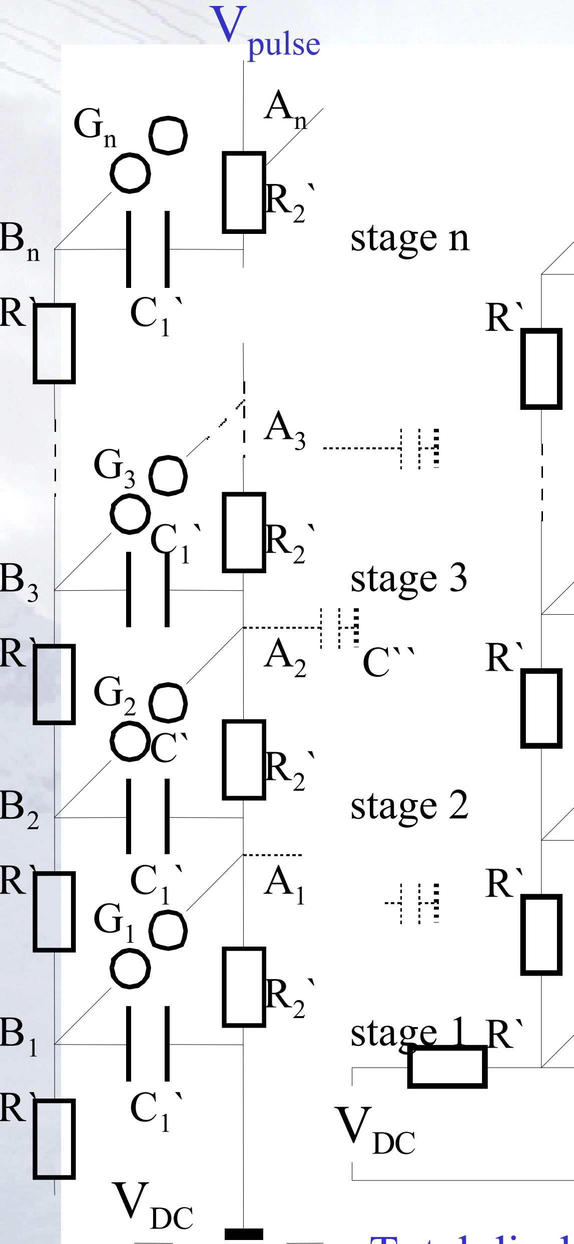
rise time: $\tau 1 = R1C2$

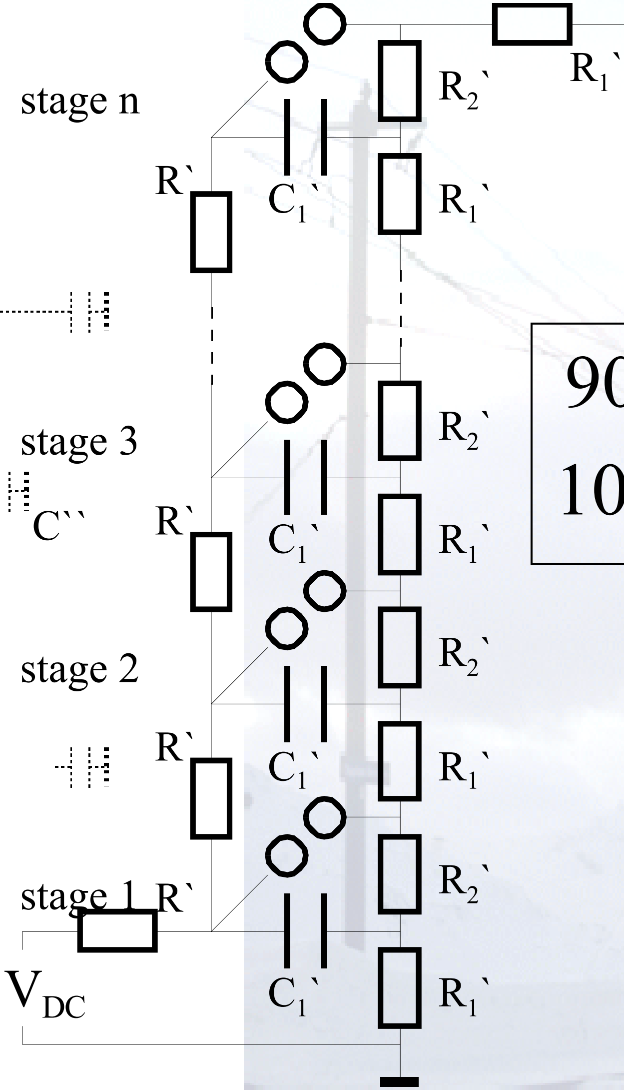
discharge time: $\tau 2 = R2C1$

Standard lightning surge pulse: 1.2 / 50 µs

Cascade Pulse Source

(Marx Generator)

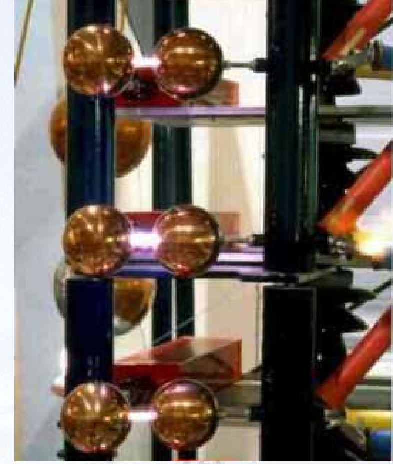




$$V_{pulse} = n \cdot V_{DC}$$

900 kV

100 mA



Courtesy: Kema, The Netherlands

R_1: front resistor

R_2: discharge resistor

Total discharge capacity: $1/C1=\sum 1/C1'$

Front resistance: $R1=R1''+\sum R1'$

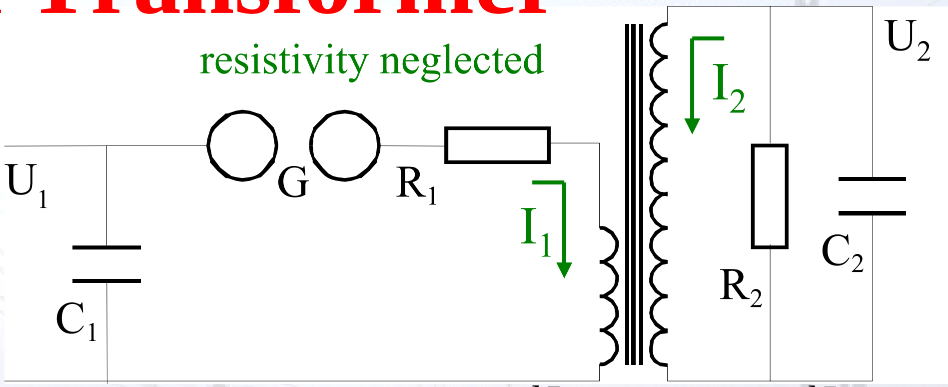
Discharge resistance: $R2=\sum R2'$

Voltage: 300 kV

Pulse Source with Transformer

primary: $L_1C_1\frac{d^2I_1}{dt^2}-MC_1\frac{d^2I_2}{dt^2}+I_1=0$

secondary: $L_2C_2\frac{d^2I_2}{dt^2}-MC_2\frac{d^2I_1}{dt^2}+I_2=0$



$U_1(t)=\frac{d\Phi_1}{dt}$ $U_2(t)=\frac{d\Phi_2}{dt}$

Eigen frequencies from characteristic equation:

$$\begin{pmatrix} L_1C_1 & -MC_1 \\ -MC_2 & L_2C_2 \end{pmatrix}\begin{pmatrix} i_1 \\ i_2 \end{pmatrix}=\frac{1}{\omega^2}\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} \Rightarrow \left(L_1C_1-\frac{1}{\omega^2}\right)\left(L_2C_2-\frac{1}{\omega^2}\right)-M^2C_1C_2=0$$

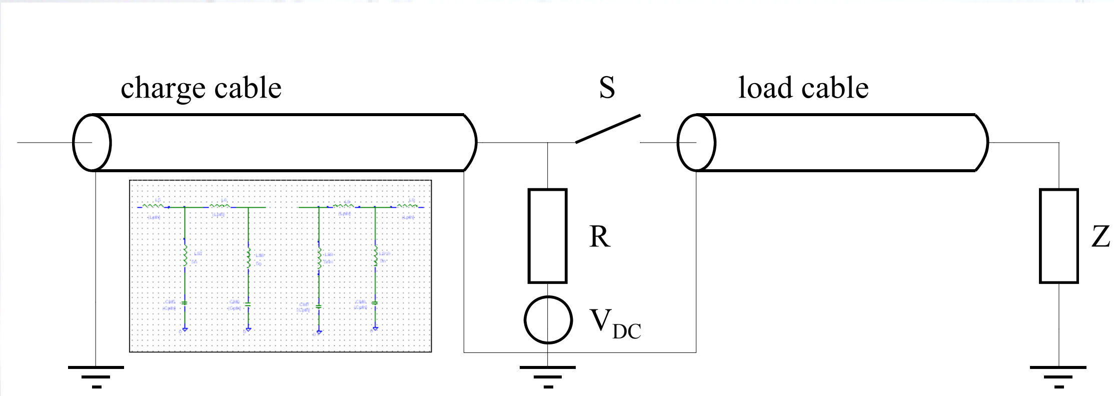
Approximation: transformer almost ideal: $k=M/\sqrt{(L_1L_2)}\rightarrow 1$

$$\omega_1\approx\frac{1}{\sqrt{L_1C_1+L_2C_2}}=\frac{1}{\sqrt{L_1(C_1+C_2')}} \quad , \quad \omega_2\approx\frac{1}{\sqrt{1-k^2}}\sqrt{\frac{1}{L_1C_1}+\frac{1}{L_2C_2}}=\frac{1}{\sqrt{L_{eq}(C_1 // C_2')}}$$

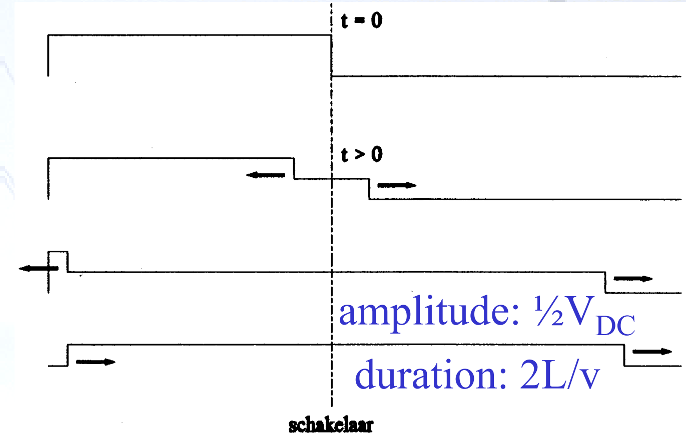
slow oscillation

fast oscillation

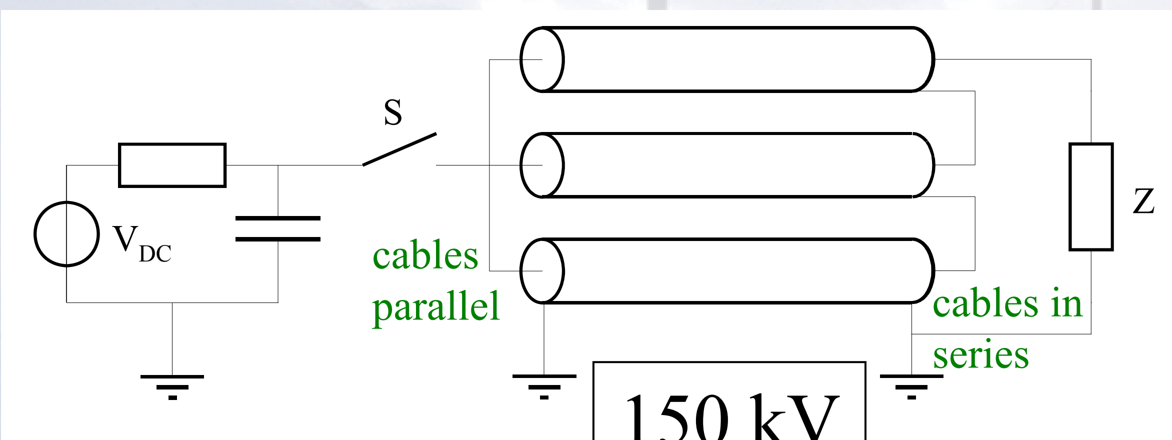
Pulse Forming Line / Network



80 kV, 10 kA, T=20ns - 10μs

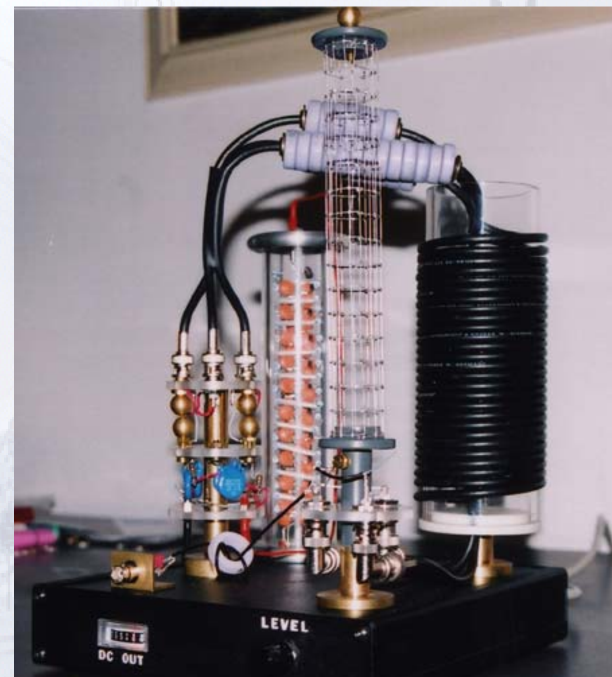


Transmission Line Transformer



150 kV

1 kA



Courtesy: Eindhoven Univ.of Techn.

Insulation and Breakdown

- In Gases

  Ionisation and Avalanche Formation
  Townsend and Streamer Breakdown
  Paschen Law: Gas Type
  Breakdown Along Insulator
  Inhomogeneous Fields, Pulsed Voltages, Corona

- Insulating Liquids

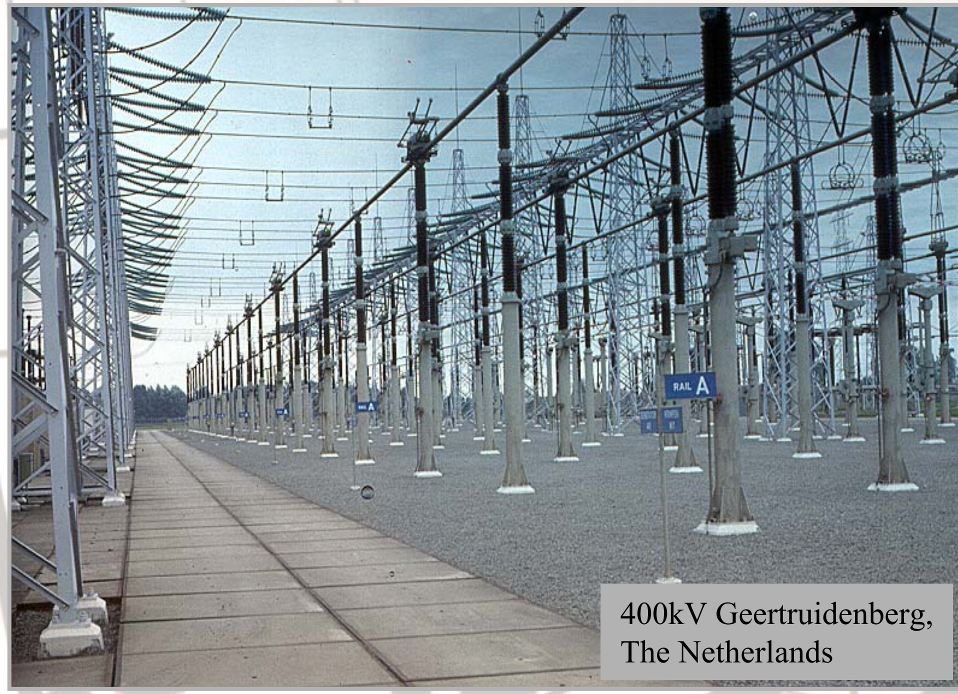
- Solid Insulation

  Breakdown types, Surface tracking, Partial discharges, Polarisation, tan δ

- Vacuum Insulation

  Applications, Breakdown, Cathode Triple-Point, Insulator Surface Charging, Conditioning

400kV



400kV Geertruidenberg, The Netherlands

800kV South Africa

1^{st} free electron

- Cosmic radiation
- Shortwave UV
- Radio active isotopes

In air:
≈ 2.5×10^{19} molecules/cm^3
≈ 1000 ions/cm^3
≈ 10 electrons/cm^3

Free path, effective cross-section

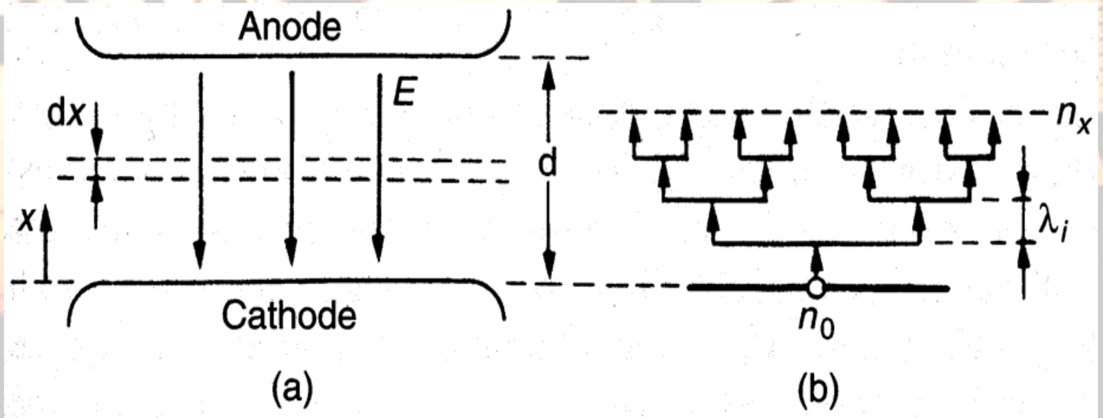
Townsend's 1^{st} ionisation coefficient α

One electron creates α new electrons per unit length

$A + e \rightarrow A^+ + 2e$

$n_e(x=d) = n_0 e^{\alpha d}$

$\alpha/p = f\,(E/P)$



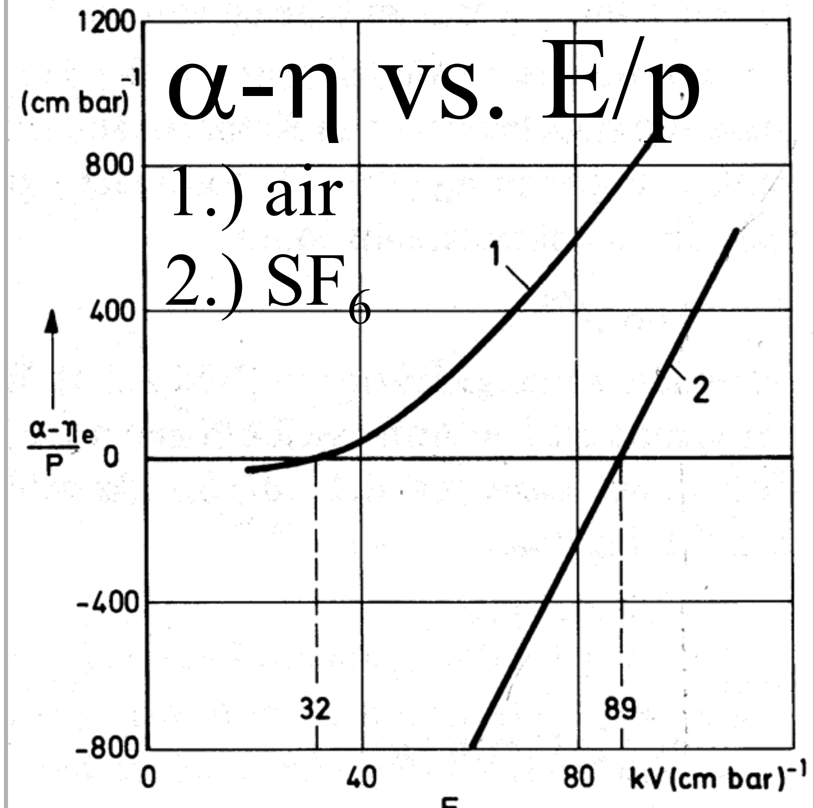
- Electro-negative gasses

Attachment η of electrons to ions

electrons: $n_e(x=d) = n_0 e^{(\alpha-\eta)d}$

negative ions:

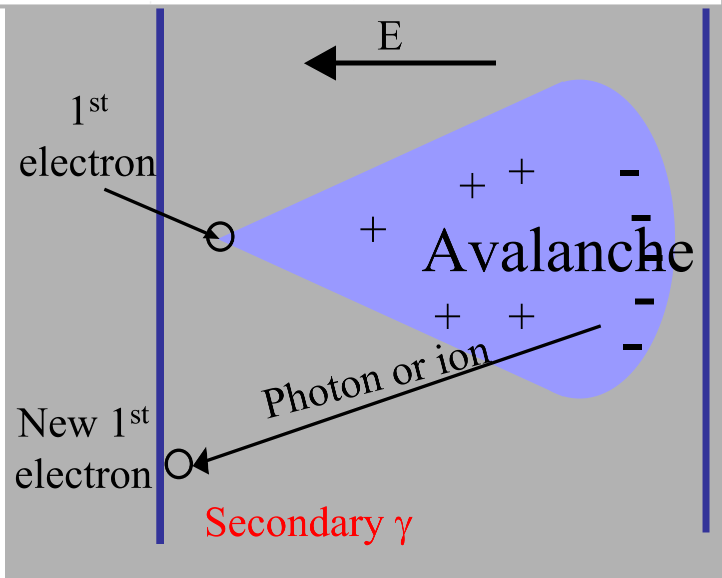
$$n_-(x=d) = \frac{n_0\eta}{\alpha-\eta}\left[e^{(\alpha-\eta)d} - 1\right]$$



Reference [3]

Avalanche ≠ Breakdown; creation of secondaries

Townsend's 2nd ionisation coefficient γ
one ion or photon creates γ new electrons
at cathode $n_e = \gamma n_0(e^{\alpha d}-1)$



Breakdown if: # secondary electrons $\geq n_0$
$\alpha d \geq \ln(1/\gamma + 1)$

steep function of E/p → $e^{\alpha d}$ very steep → $(E/p)_{critical}$ and V_d well defined → γ of weak influence

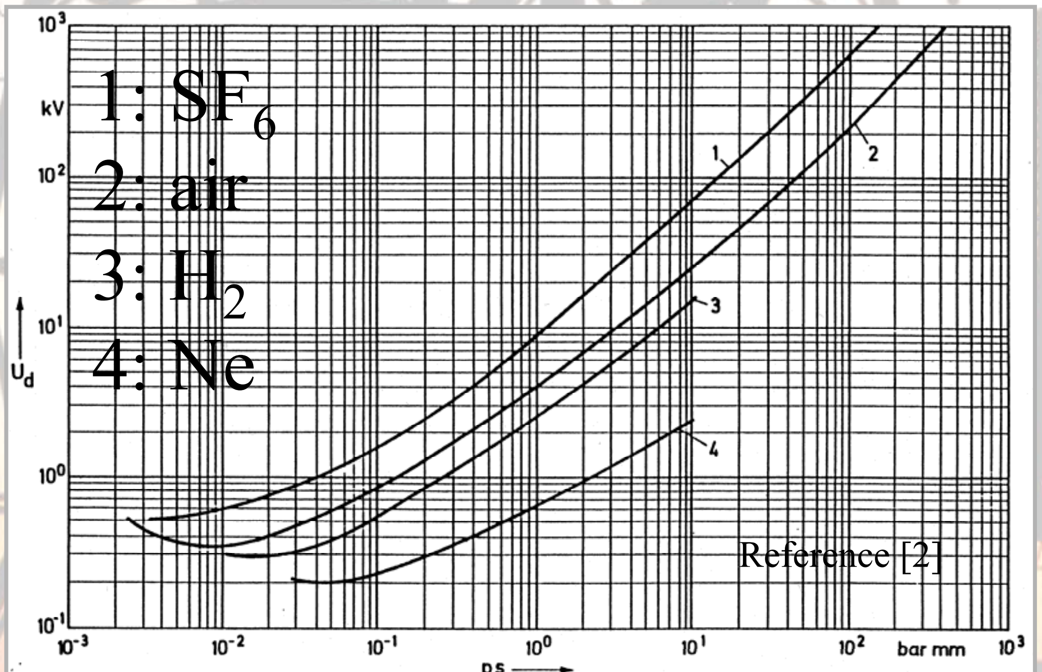
Paschen law / breakdown field

- Townsend breakdown criterion $\alpha d = K$:

$$\frac{E_d}{p} = \frac{B}{\ln(Apd/K)} \qquad V_d = \frac{Bpd}{\ln(Apd/K)}$$

with $A = \sigma_I / kT$
$B = V_i \sigma_I / kT$

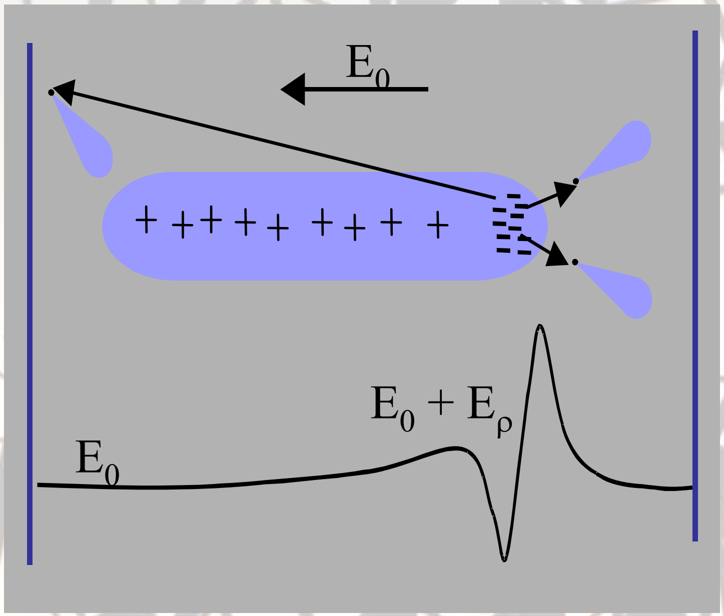
→ E_d and V_d depend only on p*d   p: pressure   d: gap length



Typically practically
E_{bd} = 10 kV/cm
at 1 bar in air

$V_{bd,\text{Paschen min, air}} \approx 300$ V

- Small p*d, d<< λ: few collisions, high field required for ionisation
- Large p*d, d>> λ: collision dominated, small energy build-up, high V_d



Streamer breakdown

Space charge field $E_\rho \approx E_0$

- Field enhancement
  extra ionising collisions ($\alpha\uparrow$)
- High excitation $\Rightarrow$ UV photons

<u>when</u> 1 electron grows into ca. 10^8

<u>then</u> E_ρ large enough for streamer breakdown ($n_e \approx 2 \cdot 10^8$ in avalanche head)

Result:

- Secondary avalanches, directional effect (channel formation)
- Grows out into a breakdown within 1 gap crossing (anode and/or cathode directed)

Characteristic:

- Very fast
- Independent of electrodes (no need for electrode surface secondaries)
- Important at large distances (lightning)

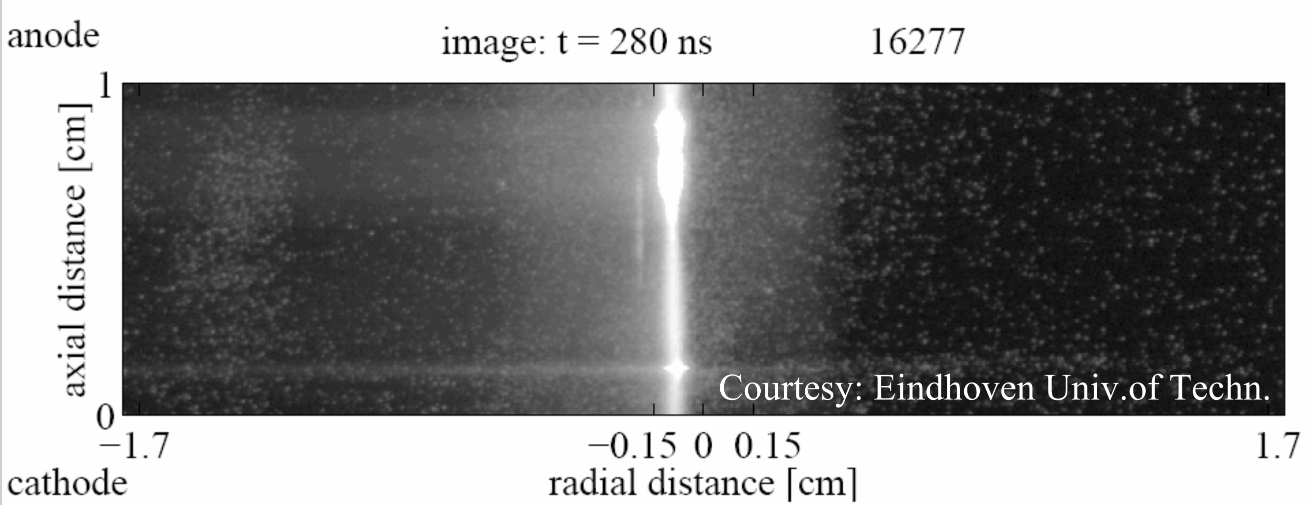
→Townsend, unless:

- Townsend: $\alpha d \geq \ln(1/\gamma + 1) \approx 7...9$
  ($\gamma \approx 10^{-4}...10^{-3}$)
- Streamer: $\alpha d \geq 18...20$

- Strong non-uniform field
  (small electrodes, few secondary electrons)

- Pulsed voltages
  - Townsend slow, ion drift, subsequent gap transitions
  - Streamer fast, photons, 1 gap transition

- High pressure
  - Less diffusion
    E_ρ high
  - photons absorbed in front of cathode
  - positive ions slower



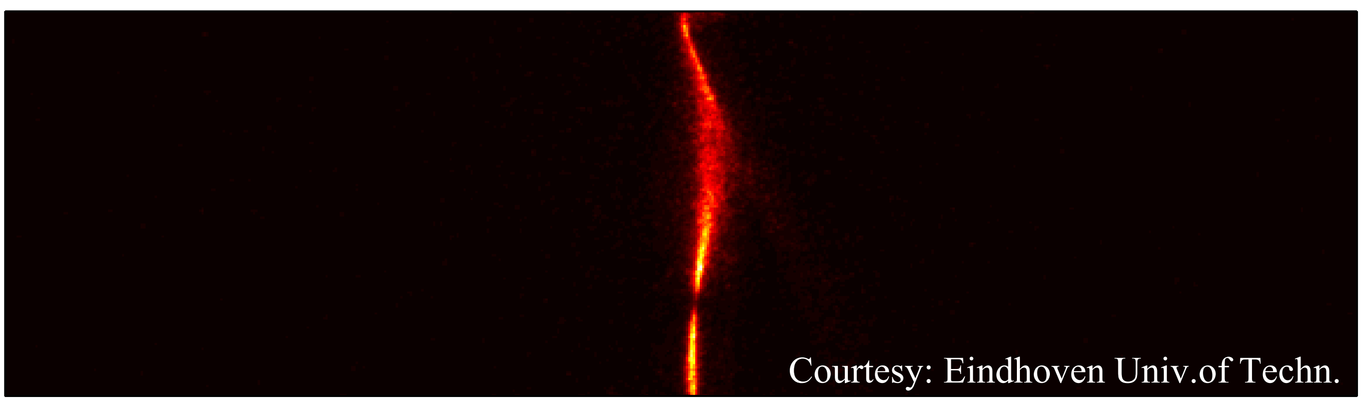
Laser-induced streamer breakdown in air.

Breakdown along insulator

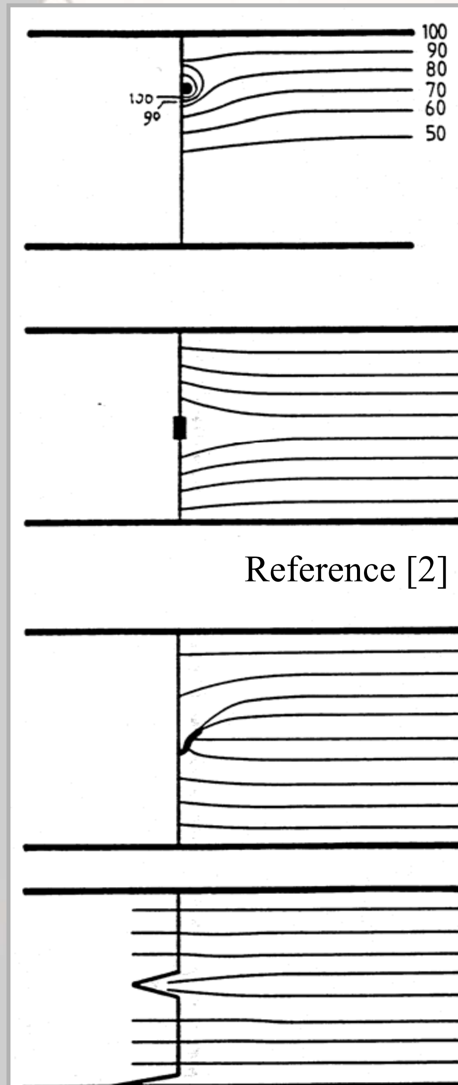
- Surface charge
- (Non-regular) surface conduction
- Particles / contaminations on surface
- Non-regularities (scratches, ridges)

⇒ Field enhancement

⇒ Increased breakdown probability

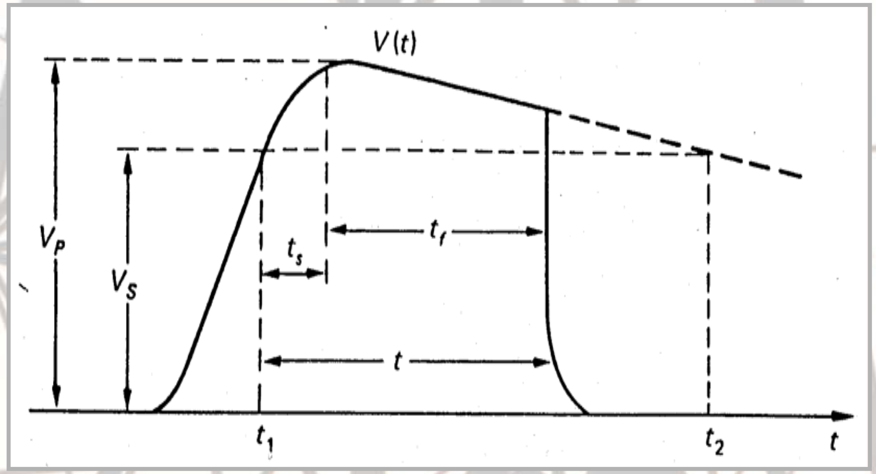


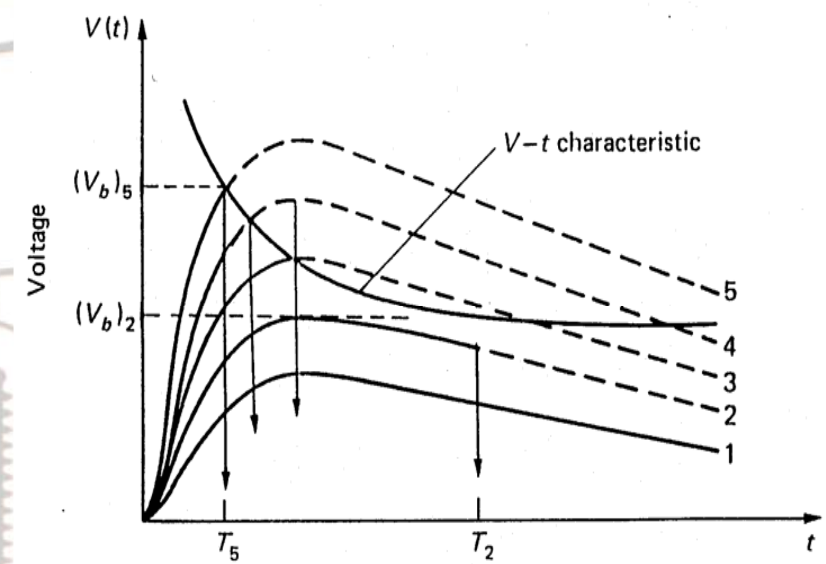
Prebreakdown along insulatior in air.



Reference [2]

Breakdown at pulse voltages; time-lag



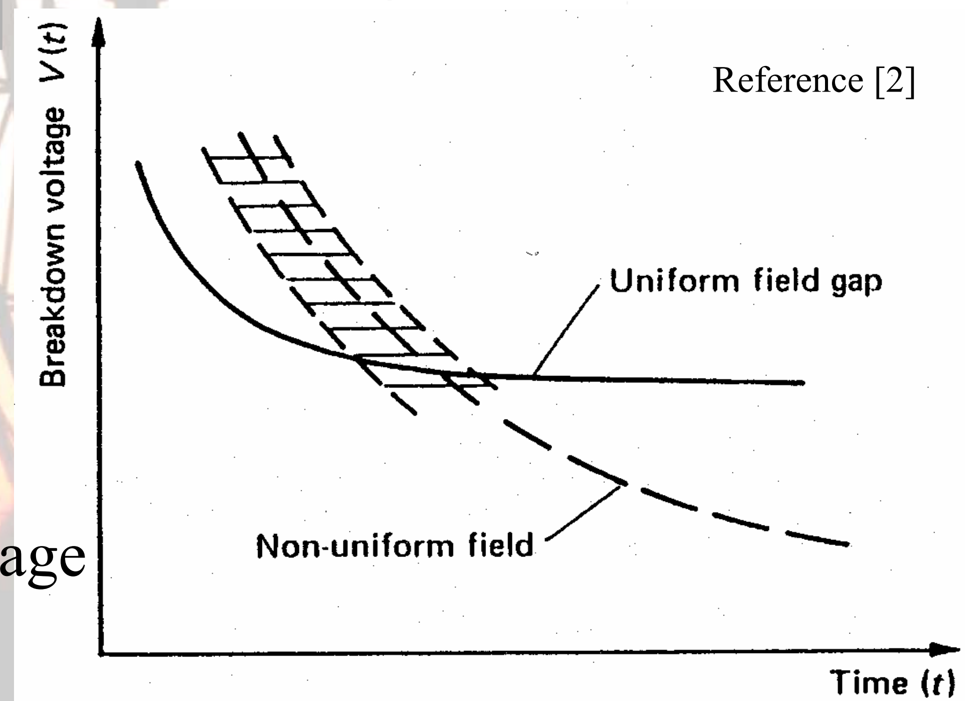


t_s, wait for first elektron

t_f, breakdown formation

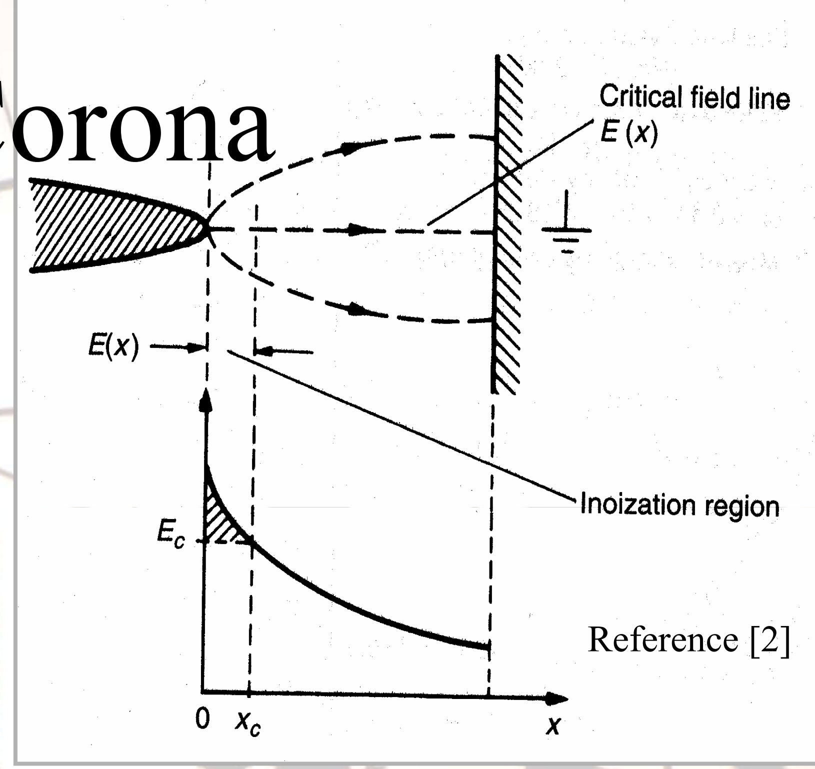
- Townsend or Streamer

Short pulses, high breakdown voltage



Reference [2]

Non-uniform fields; Corona



Reference [2]

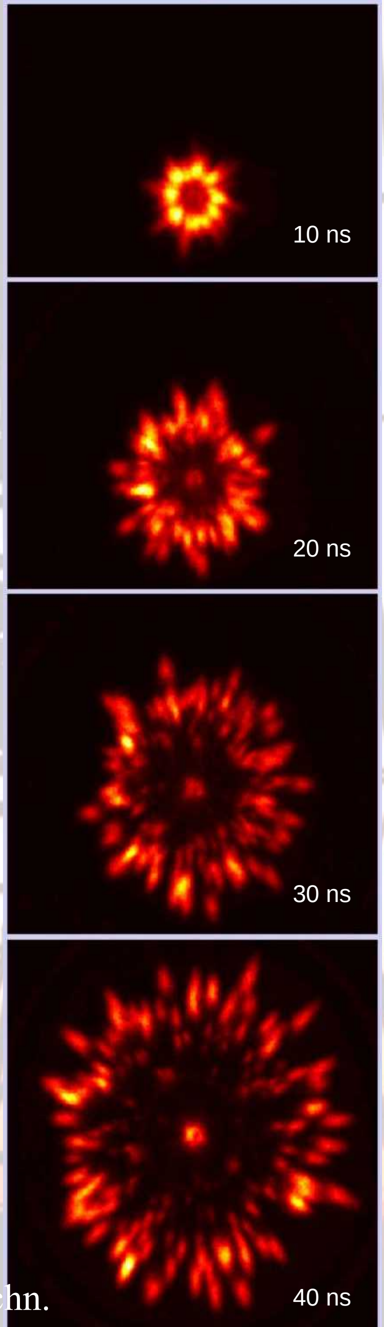
Breakdown conditions:

- Global → Full breakdown
- Local → Streamer breakdown
  → Partial discharge

Non-uniform field:

- Discharge starts in high field region
- ....and “extinguishes” in low field region

Courtesy: Eindhoven Univ.of Techn.



Corona

- Power loss; EM noise
- Chemical corrosion
+ Useful applications

Pulsed corona discharges:

- Fast, short duration HV pulses
- Many streamers, high density
- Generation of electrons, radicals, excited molecules, UV
- E.g. Flue gas cleaning

Transmission line transformer

Solid insulation

Breakdown field strength:

- Very clean (lab): high
- Practical: lower due to imperfections
  - Voids
  - Absorbed water
  - Contaminations
  - Structural deformations

Anorganic Natural Synthetic	Quartz,mica,glas Porcelain Al_2O_3	Disc insulator Feedthrough Spacer
Paper + Oil		Cable Capacitor
Synthetic Organic Polymerisation	Polyethelene HD,LD,XL – PE Teflon Polystyrene, PVC, polypropene,etc	Spec. properties: Moisture content high T losses bonding
Epoxy	Hardener Filler	Moulding in mold

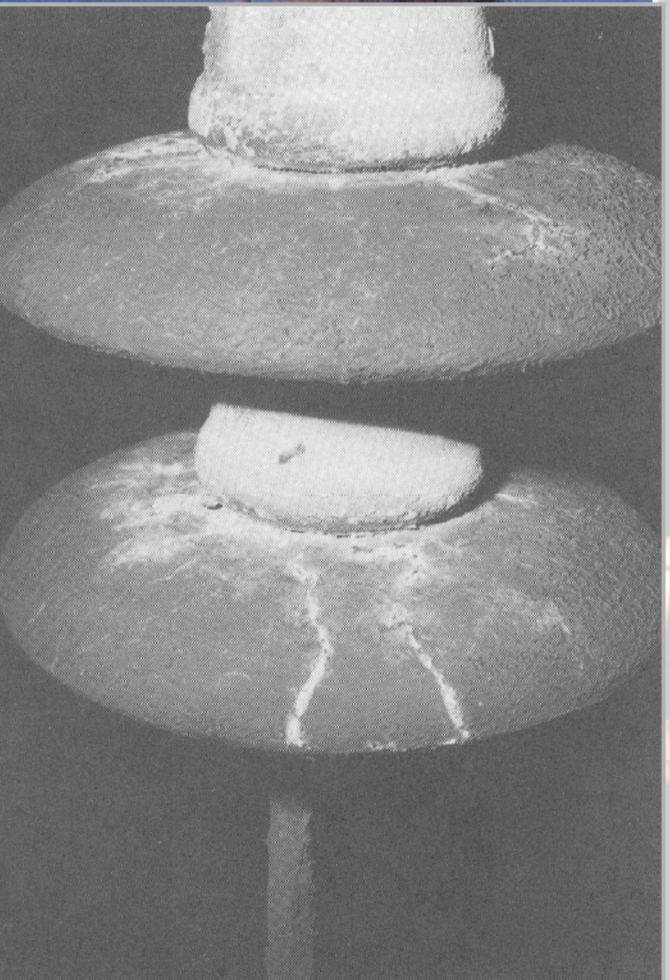
Requirements:

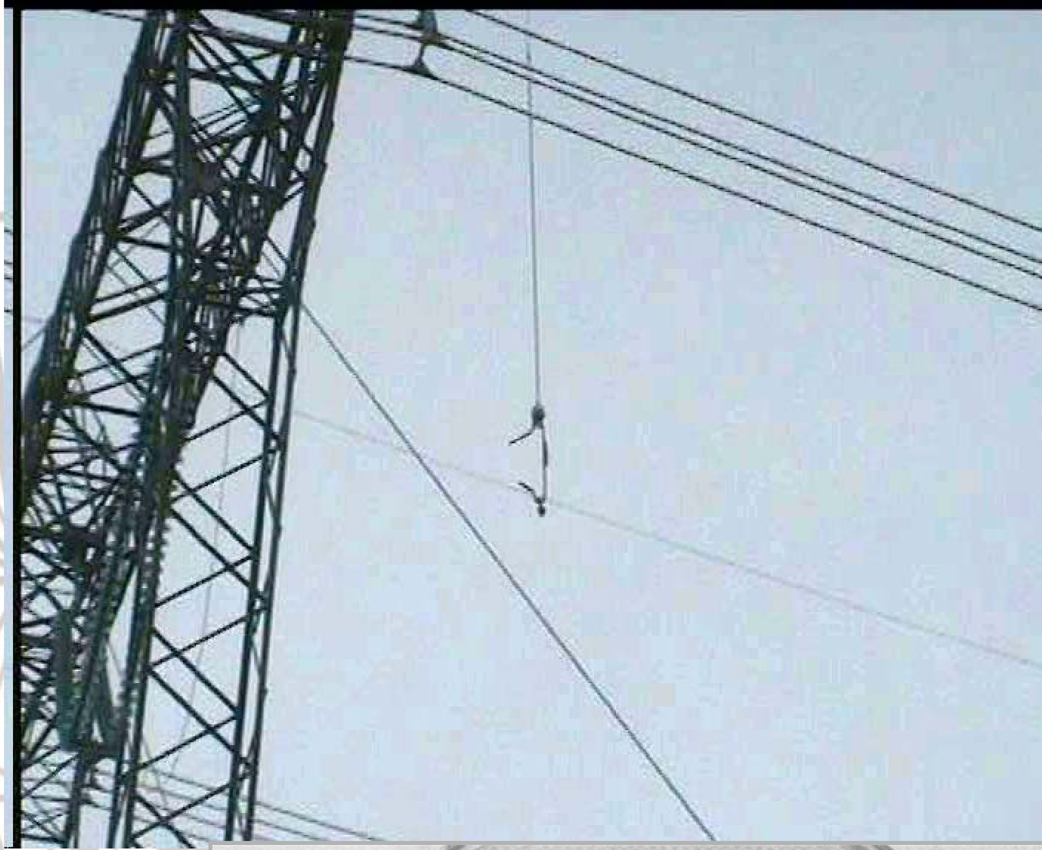
- Mechanical strength
- Contact with electrodes and semiconducting layers
- Resistant to high T, UV, dirt, contamination, rain, ice, desert sand

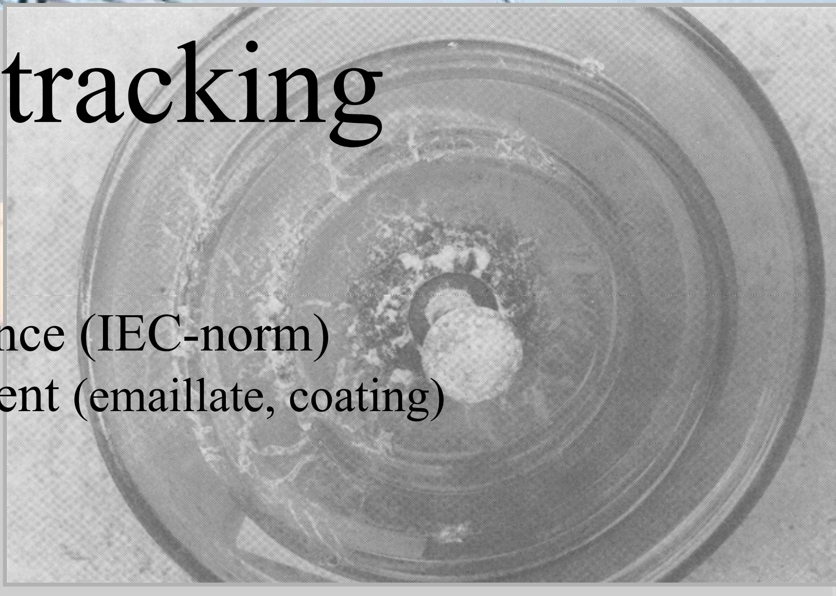
Problems:

- Surface tracking
- Partial discharges
  - In voids
    (in material or at electrodes, often created at production).

Types of solid insulation materials





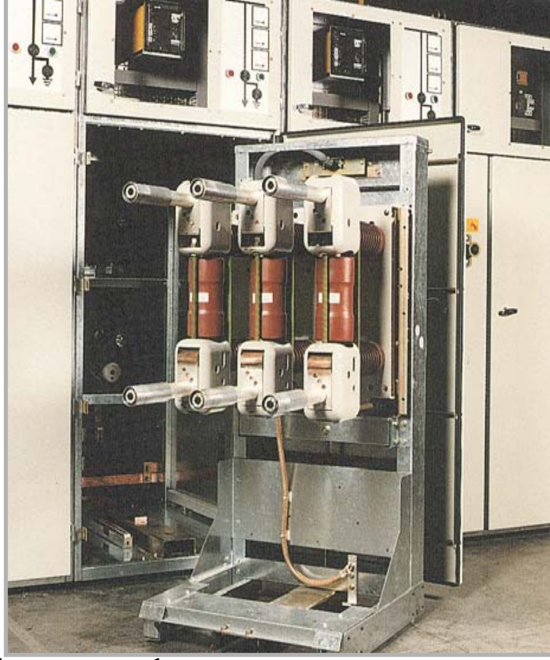


Surface tracking

Remedies:

- Geometry
- Creeping distance (IEC-norm)
- Surface treatment (emaillate, coating)
- Washing

Vacuum insulation



Applications:

- Vacuum circuit breaker
- Cathode Ray Tubes / accelerators
- Elektron microscope
- X-ray tube
- Transceiver tube

What is vacuum?

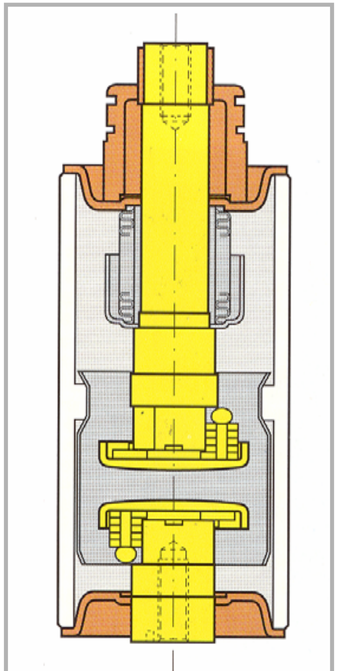
- “Pressure at which no collisions for Brownian “temperature” movements of electrons”
- $\lambda >>$ characteristic distances
- E.g. $p = 10^{-6}$ bar, $\lambda = 400$ m

Advantages:

- “Self healing”
- No dielectric losses
- High breakdown fieldstrength
- Non flammable
- Non toxic, non contaminating

Disadvantages:

- Requires hermetic containment and mechanical support
- Quality determind by:
  - electrodes and insulators
  - Material choice, machining
  - Contaminations, conditioning



Characteristics of vacuum breakdown

No 1st electron from “gas”

- Cathode emission
  - primary: photoemission, thermic emission, field emission, Schottky-emission
  - secondary: e.g. e^- bombarded anode → $^+$ion collides at cathode → e^-

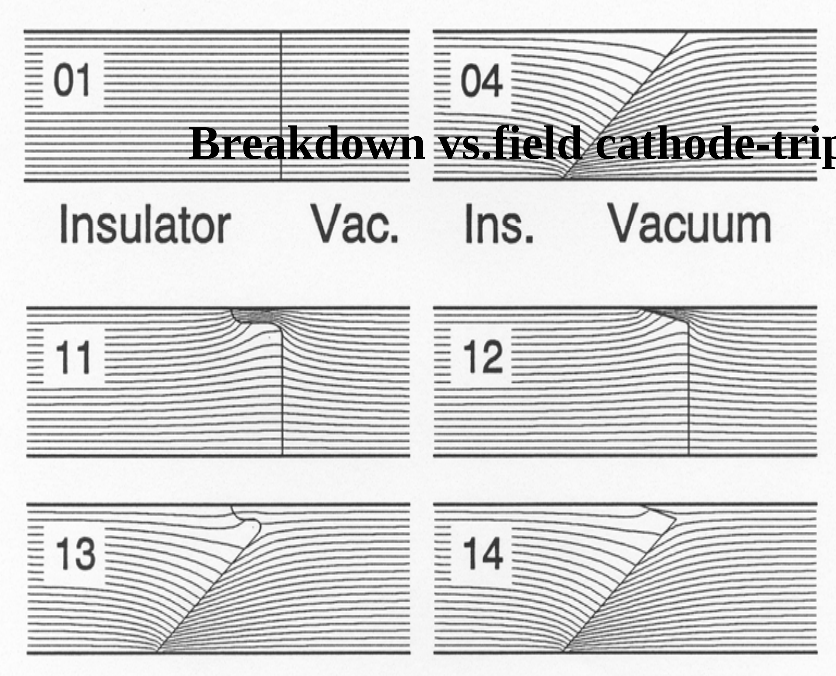
No breakdown medium

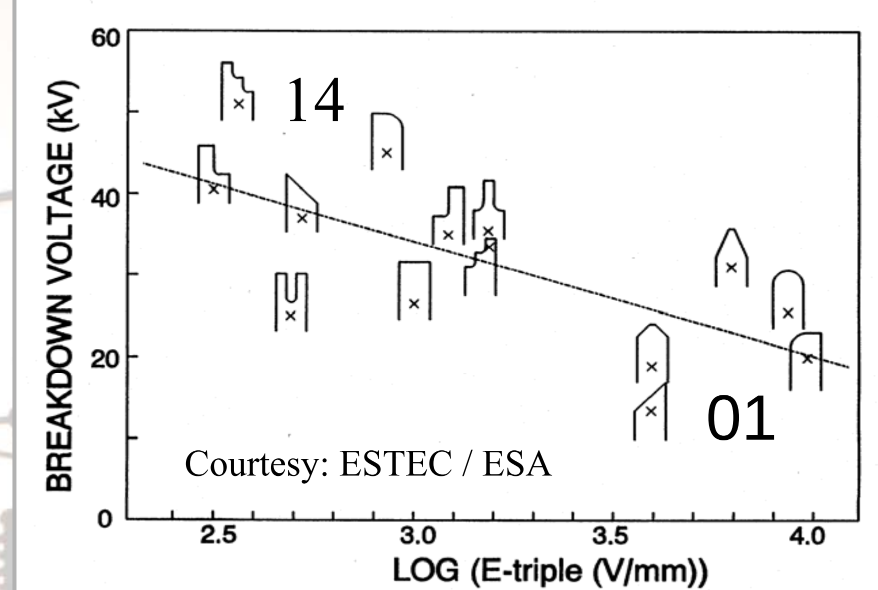
- No multiplication through collision ionisation
- Medium in which the breakdown occurs has to be created (“evaporated” from electrodes, insulators)

Important: prevent field emission

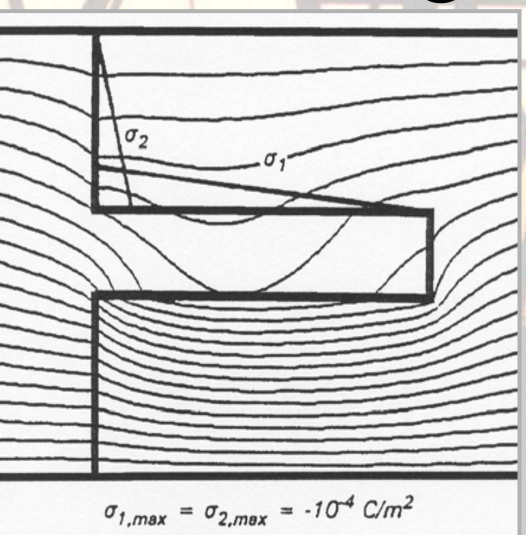
- Keep field at cathode and “cathode triple point” as low as possible
- Insulator surface charging, conditioning

Breakdown vs.field cathode-triple-point

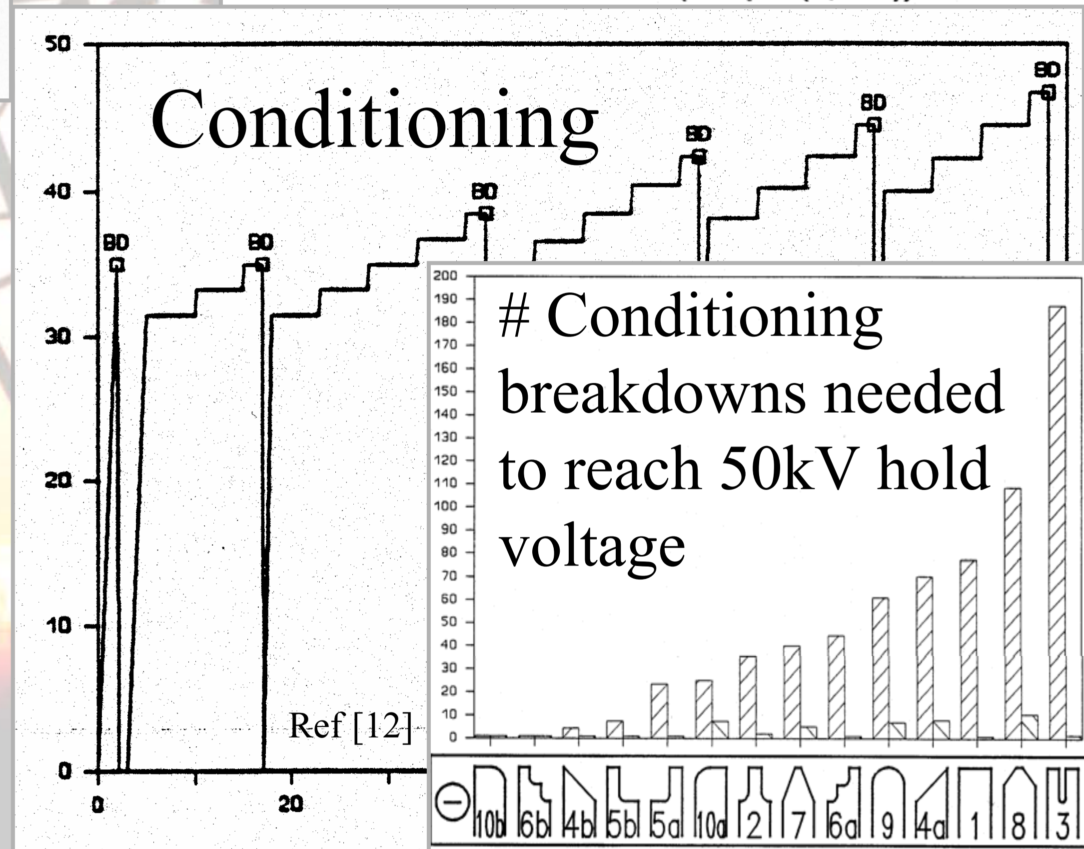


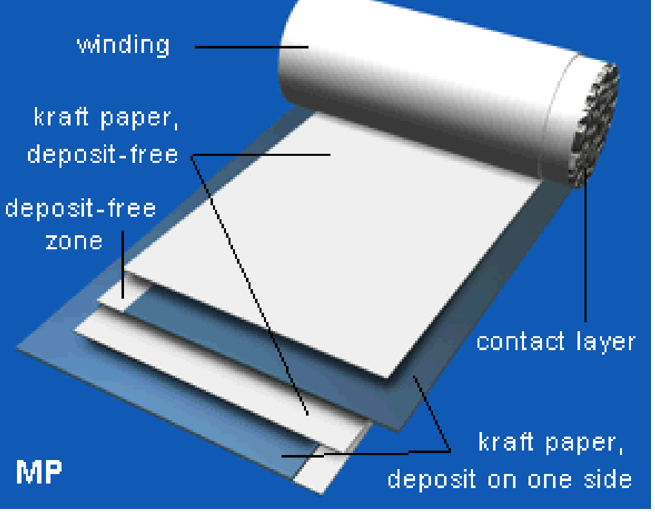


Insulator surface charging



Conditioning effect lost when charge compensated





Insulating liquids

Requirements:

- Pure, dry and free of gases
- ε_r (high for C's, low for trafo) (demi water $\varepsilon_{r,d.c.} = 80$)
- Stable (T), non-flammable, non toxic (pcb's), ageing, viscosity

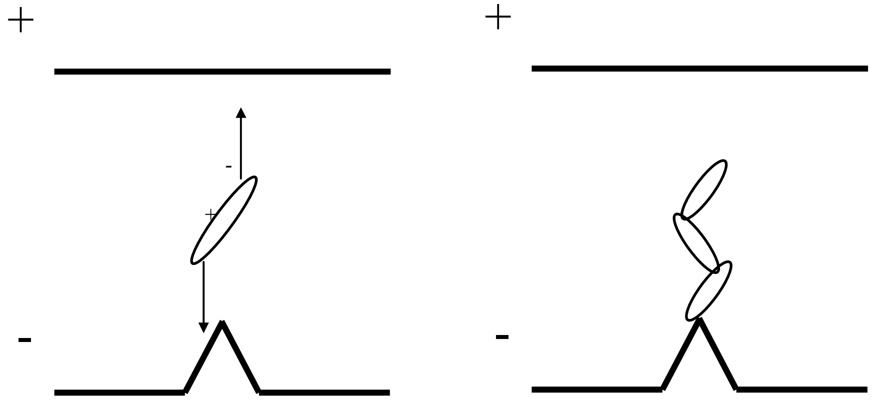
Applications:

- Transformers
- Cables
- Capacitors
- Bushings

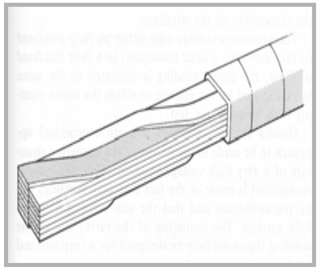
- No interface problems
- Combined cooling/insulation
- "Cheap" (no mould)
- Liquid tight housing

Breakdown fieldstrength:

- **Very clean (lab): high 1 - 4 MV/cm (In practice much lower)**
- **Important at production:outgassing, filtering, drying**
- **Mineral oil** ("old" time application, cheap, flammable)
- **Synthetic oil** (purer, specifically made, more expensive)
  - **Silicon oil** (very stable up to high T, non-toxic, expensive)
- **Liquid H2, N2, Ar, He** (supra-conductors)
- **Demi-water** (incidental applications, pulsed power)
- **Limitation V_{bd}:**
  - Inclusions: Partial discharges → Oil decomposition → Breakdown
  - Growth (pressure increase)
  - "extension" in field direction"
- **Particles drift** to region with highest E → bridge formation → breakdown



Transformer:

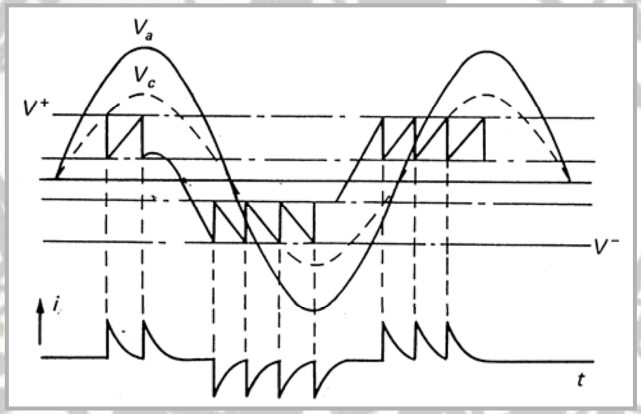


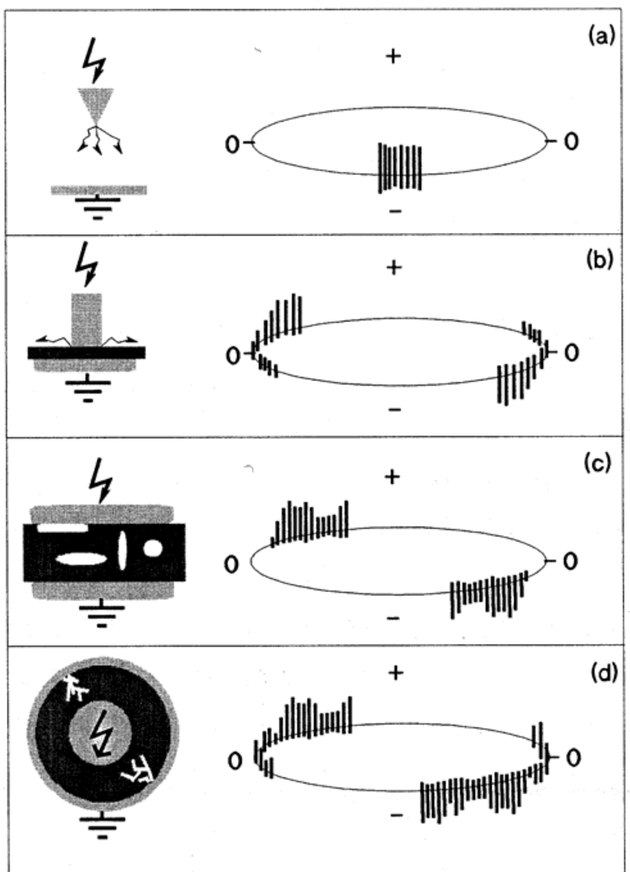
- **Mineral oil**: Insulation and cooling
- **Paper**: Barrier for charge carriers and chain formation
  - Mechanical strength
- **Ageing**
  - Thermical and electrical (partial discharges)
  - Lifetime: 30 years, strongly dependent on temperature, short-circuits, over-loading , over-voltages
  - Breakage of oil moleculs, Creation of gasses, Concentration of various gas components indication for exceeded temperature

    (as specified in IEC599)
- **Lifetime**
  - Time in which paper looses 50 % of its mechanical strenght
  - Strongly dependent on:
    - Moisture (from 0.2 % to 2 % accelerated ageing factor 20)
    - Oxygen (presence accelerates ageing by a factor 2)

Measurement techniques

Partial discharges





- UV, fast electrons, ions, heat
- Deterioration void:
  - Oxidation, degradation through ion-impact
  - “Pitting”, followed by treeing
- Eventually breakdown

Acceptable lifetime? Preferably no partial discharges.

- High sensitivity measurements on often large objects
- $Q_{app.} \neq Q_{real}$, still useful, because measure for dissipated energy, thereby for induced damage
- relative measurement

AC voltage phase resolved discharge pattern detection → Type of defect

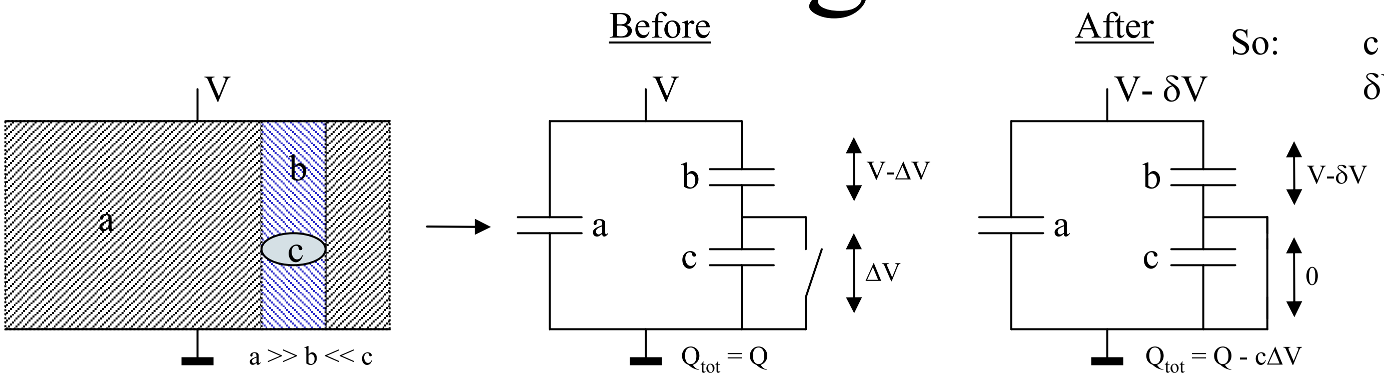
Partial discharges

Before: $Q = aV + b(V - \Delta V) + c\,\Delta V$

After: $Q - c\,\Delta V = (a + b)(V - \delta V)$

So: $c\,\Delta V = a\,\delta V + b(\delta V - \Delta V) + c\,\Delta V$

$\delta V = \Delta V\, b/(a+b) \approx \Delta V\, b/a$

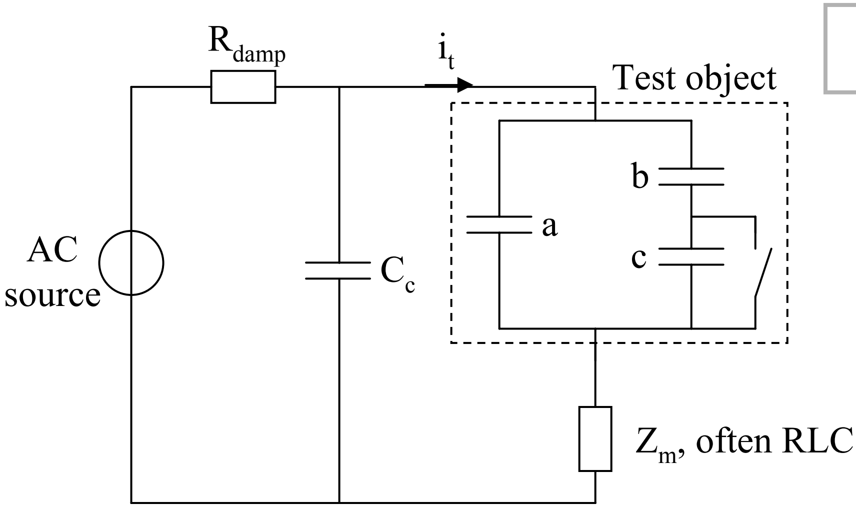


Apparent charge:

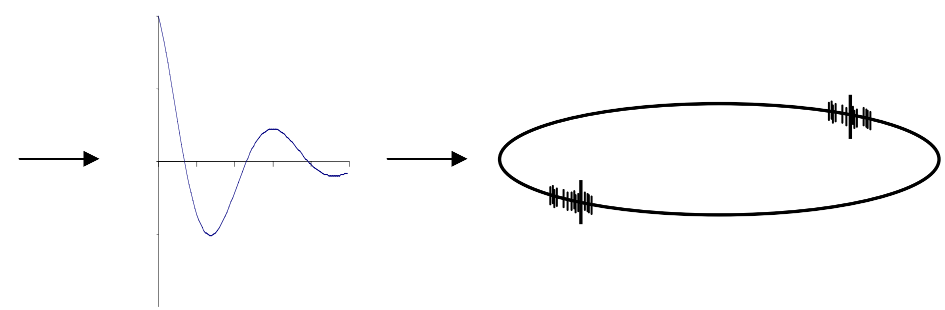
$C_{tot}\,\delta V = (a + bc/(b+c))\,\delta V$

$\approx (a+b)\,\delta V \approx a\delta V$

$$\frac{Q_{app}}{Q_{real}} = \frac{a\delta V}{c\Delta V} = \frac{b\Delta V}{c\Delta V} = \frac{b}{c} \approx \frac{\varepsilon_r . d_{void}}{d_{insul}}$$



High sensitivity measurement because b/c << 1



- Measure with resonant RLC circuit:
  - Excitation by short pulse i_t
  - No 50 Hz problem
  - $V = q/C \exp(-\alpha t)\ \{ \cos\beta t - \alpha/\beta \sin\beta t \}$

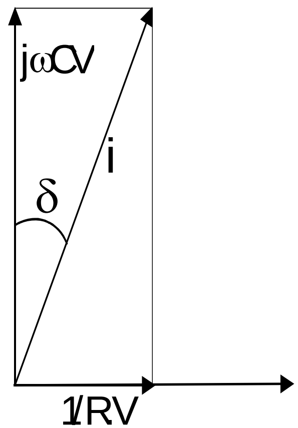
    $\alpha = 1/(2RC) \quad \beta = [1/(LC) - \alpha^2]^{-1/2}$

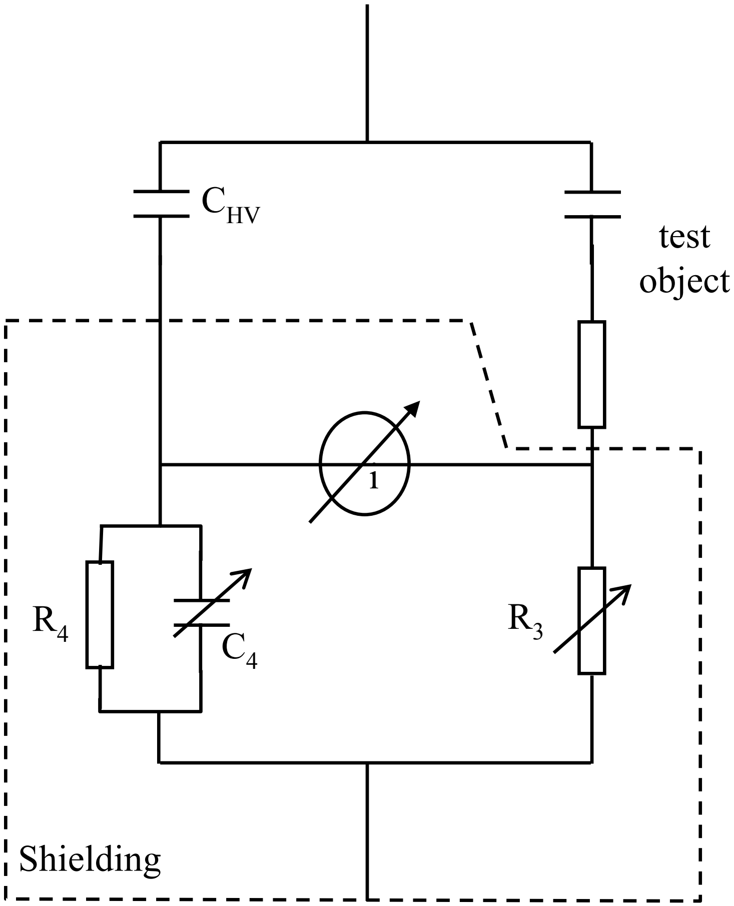
- Q_{app} gives i_t ($Q_{app} = \int i_t dt$)
- C_c gives i_t if $C_c >> C_{object}$
- Calibration through injecting known charge

Loss angle, $\tan(\delta)$

Sources:

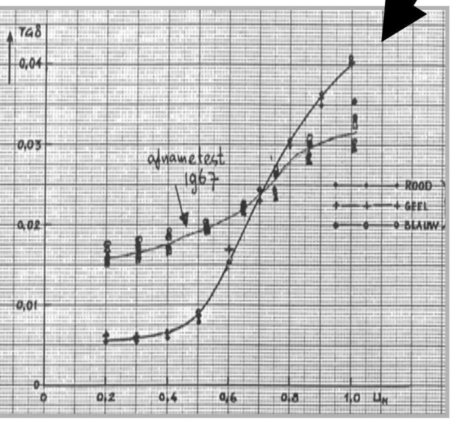
- Conduction σ (for DC or LF)
- Partial discharges
- Polarisation





Schering bridge:

- i=0, $RC=R_4C_4$
- Gives: $\tan(\delta)$
  - parallel: $1/\omega RC$
  - serie: ωRC

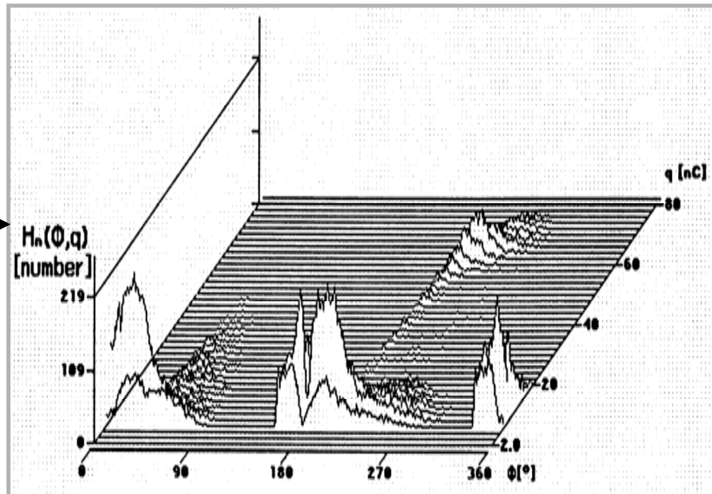


Tan δ:

- "Bulk" parameter
- No difference between phases

PD:

- Detection of weakest spot
- Largest activity and asymmetry in "blue" phase (ridge discharges)



Summary

Seen many basic high voltage engineering technology aspects here:

- High voltage generation
- Field calculations
- Discharge phenomena

The above to be applied in your practical accelerator environments as needed:

- Vacuum feed through: Triple points
- Breakdown field strength in air 10kV/cm
- Challenging calculations for real practical geometries.

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Appendix I

Maxwell equations in integral form

$$\oiint \vec{D}\cdot\vec{dA} = \iiint \rho\, dV = Q_{omsl.}$$

$$\oint \vec{E}\cdot\vec{dl} = -\frac{d}{dt}\iint \vec{B}\cdot\vec{dA} = -\frac{d\phi_{omsl.}}{dt}$$

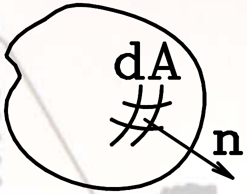
$$\oiint \vec{B}\cdot\vec{dA} = 0$$

$$\oint \vec{H}\cdot\vec{dl} = \iint \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)\cdot\vec{dA}$$

Electrostatic

$$\oiint \vec{D}\cdot\vec{dA} = Q_{omsl.} \qquad (1)$$

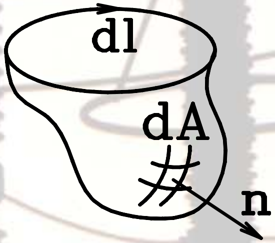
$$\oint \vec{E}\cdot\vec{dl} = 0 \qquad (2)$$



Magnetostatic

$$\oiint \vec{B}\cdot\vec{dA} = 0 \qquad (3)$$

$$\oint \vec{H}\cdot\vec{dl} = I_{omsl.} \qquad (4)$$



Maxwell equations in differential form

$$\vec{\nabla}\cdot\vec{D} = \rho$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Electrostatic

$$\vec{\nabla}\cdot\vec{D} = \rho$$

$$\vec{\nabla}\times\vec{E} = 0$$

No space charge

$$\vec{\nabla}\cdot\vec{D} = 0 \qquad (5)$$

$$\vec{\nabla}\times\vec{E} = 0 \qquad (6)$$

Magnetostatic

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{H} = \vec{J}$$

In area without source

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad (7)$$

$$\vec{\nabla}\times\vec{H} = 0 \qquad (8)$$

Appendix III Finite Element Method (FEM)

Field energy minimal inside each closed region G:

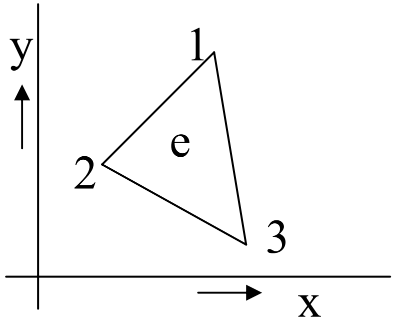
$$W = \int_G \tfrac{1}{2}\varepsilon \left|\vec{E}\right|^2 dV = \int_G \tfrac{1}{2}\varepsilon \left|\nabla U\right|^2 dV$$

Assume U satisfies Laplace equation, but U' does not, then

$$W_{U'} - W_U \geq 0:$$

$$W_{U'} - W_U = \tfrac{1}{2}\varepsilon \iiint_G \left(\left|\nabla U'\right|^2 - \left|\nabla U\right|^2\right) dV = \cdots = \tfrac{1}{2}\varepsilon \iiint_G \left|\nabla U' - \nabla U\right|^2 dV \geq 0$$

Field energy for one element (2-dim)



Potential is linear inside element: $U = a + bx + cy = \begin{pmatrix} 1 & x & y \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

on corners: $\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Potential can be written as:

$$U = \begin{pmatrix} 1 & x & y \end{pmatrix}\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}^{-1}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \sum_{i=1}^{3} U_i \alpha_i(x,y)$$

Field energy in element (e):

$$W^{(e)} = \tfrac{1}{2}\varepsilon U^T \left[\mathrm{S}^{(e)}\right]\mathrm{U} \quad \text{with} \quad \mathrm{S}_{ij}^{(e)} = \iint \left(\nabla\alpha_i \cdot \nabla\alpha_j\right) dxdy$$

α's are linear in x and y ⇒ ∇α is constant: $S_{ij} = (\nabla\alpha_i \cdot \nabla\alpha_j)\, A^{(e)}$

Total field energy of n elements

All elements together: $U^T = \left(U_1 \quad \cdots \quad U_m \quad U_{m+1} \quad \cdots \quad U_n\right) \equiv \left(U_f \quad U_p\right)$

free *prescribed*

free: potential values to be determined

prescribed: potential according to boundary conditions

$$W = \tfrac{1}{2}\varepsilon U^T [S] U = \tfrac{1}{2}\varepsilon \left[U_{f'}^T \quad U_{p'}^T\right] \begin{bmatrix} S_{f'f} & S_{f'p} \\ S_{p'f} & S_{p'p} \end{bmatrix} \begin{bmatrix} U_f \\ U_p \end{bmatrix}$$

Partial derivatives of W to U_k are zero for $1 \le k \le m$ (m equations):

$$\frac{\partial W}{\partial U_k} = 0 \quad \Rightarrow \quad \left[S_{kf} \quad S_{kp}\right] \begin{bmatrix} U_f \\ U_p \end{bmatrix} = 0$$

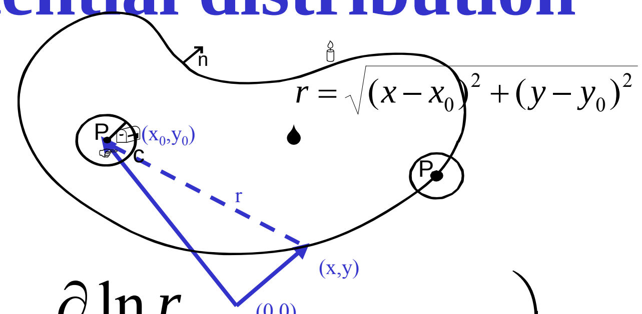
Boundary Element Method (BEM)

Boundaries uniquely prescribe potential distribution

Laplace equation : $\Delta u = 0$

border $\Gamma_1 : u(x, y)$ (Dirichlet)

border $\Gamma_2 : q(x, y) \equiv \dfrac{\partial u}{\partial n}$ (Neumann)



$P_0=(x_0,y_0)$ inside Γ: $u(x_0, y_0) = \dfrac{1}{2\pi} \oint_\Gamma \left(u(x, y) \dfrac{\partial \ln r}{\partial n} - q(x, y) \ln r \right) ds$

Problem: $u(x,y)$ and $q(x,y)$ not both known at the same time

Green II (2-dim):

$$\oint (u\nabla v - v\nabla u)\cdot \hat{n}\,ds = \iint (u\Delta v - v\Delta u)\,dxdy$$

Choose $v(x,y)=\ln(1/r)$ $\Delta v(x,y)=0$ for $P\neq P_0(x_0,y_0)$

Exclude region σ around P_0 by means of circle c

$$\oint_{\Gamma+c} \left(u\frac{\partial \ln r^{-1}}{\partial n} - \ln r^{-1}\frac{\partial u}{\partial n}\right)ds = \iint_{\Omega-\sigma} (u\,\Delta \ln r^{-1} - \ln r^{-1}\Delta u)\,dxdy = 0$$

$$-\oint_{\Gamma}\left(u\frac{\partial \ln r}{\partial n} - \ln r\frac{\partial u}{\partial n}\right)ds + \oint_{c}\left(u\frac{\partial \ln r^{-1}}{\partial n} - \ln r^{-1}\frac{\partial u}{\partial n}\right)ds = 0$$

$$\lim_{\varepsilon \downarrow 0}\int_0^{2\pi}\left(u\frac{1}{\varepsilon} + \frac{\partial u}{\partial n}\ln\varepsilon\right)\varepsilon\, d\vartheta = 2\pi u(x_0,y_0)$$

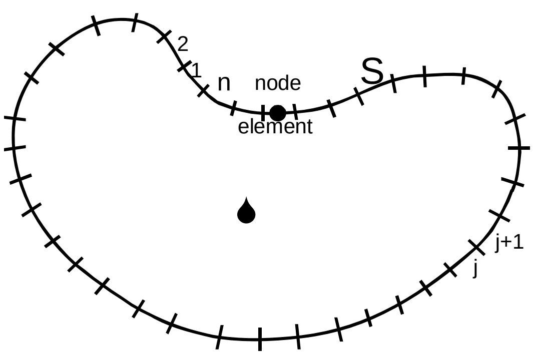
$P_i=(x_i,y_i)$ on border Γ: $$u(x_i,y_i) = \frac{1}{\pi}\oint_{\Gamma}\left(u(x,y)\frac{\partial \ln r}{\partial n} - q(x,y)\ln r\right)ds$$

Discretisation:

$$\pi U(x_i,y_i) = \sum_{j=1}^{n} U_j \int_{S_j}\frac{\partial \ln r_{ij}}{\partial n}ds - \sum_{j=1}^{n} Q_j \int_{S_j}\ln r_{ij}\,ds$$

H G

ij ij



In matrix notation:

$$\sum_{j=1}^{n}\left(H_{ij} - \pi\delta_{ij}\right)U_j = \sum_{j=1}^{n} G_{ij}Q_j$$

Generates missing information