High Voltage Engineering

Enrique Gaxiola

Many thanks to the Electrical Power Systems Group, Eindhoven University of Technology, The Netherlands & CERN AB-BT Group colleagues

Introductory examples

Theoretical foundation and numerical field simulation methods

Generation of high voltages

Insulation and Breakdown

Measurement techniques
Introduction E. Gaxiola:

Studied Power Engineering
Ph.D. on Dielectric Breakdown in Insulating Gases; Non-Uniform Fields and Space Charge Effects
Industry R&D on Plasma Physics / Gas Discharges
CERN Accelerators & Beam, Beam Transfer, Kicker Innovations:
• Electromagnetism
• Beam impedance reduction
• Vacuum high voltage breakdown in traveling wave structures.
• Pulsed power semiconductor applications
CERN Septa and Kicker examples

- Large Hadron Collider
  14 TeV
- Super Proton Synchrotron
  450 GeV
- Proton Synchrotron
  26 GeV

Septum: \( E \leq 12 \text{ MV/m} \quad T = \text{d.c.} \)
\( l = 0.8 - 15 \text{m} \)

Kicker: \( V = 80 \text{kV} \)
\( B = 0.1 - 0.3 \text{ T} \quad T = 10 \text{ ns} - 200 \mu\text{s} \)
\( l = 0.2 - 16 \text{m} \)

RF cavities: High gradients, \( E \leq 150 \text{MV/m} \)

Reference [1]
CAS on Small Accelerators

- SPS injection kicker magnets
- 30 kV spacers
- beam gap magnets
- ferrites
CAS on Small Accelerators

Magnets
- SPS extraction kickers
- 60 kV, 72 kV, 30 kV

Power Semiconductor Diode stack

Thyratron gas discharge switches

Generators
- Pulse Forming Network

Courtesy: E2V Technologies
• Maxwell equations for calculating Electromagnetic fields, voltages, currents
  – Analytical
  – Numerical
Breakdown

Electrical Fields, Geometry

High fields
Field enhancement
Field steering

Insulation and Breakdown

Charges in fields
Ionisation
Breakdown

Medium

Gas
Liquids
Solids
Vacuum
NUMERICAL FIELD SIMULATION METHODS

- **CSM (Charge Simulation Method):** (Coulomb)
  
  Electrode configuration is replaced by a set of discrete charges

- **FDM (Finite Difference Method):**
  
  Laplace equation is discretised on a rectangular grid

- **FEM (Finite Element Method):** Vector Fields (Opera, Tosca), Ansys, Ansoft
  
  Potential distribution corresponds with minimum electric field energy
  \( (w=\frac{1}{2}\varepsilon E^2) \)

- **BEM (Boundary Element Method):** IES (Electro, Oersted)
  
  Potential and its derivative in normal direction on boundary are sufficient
Procedure FEM

1. Generate mesh of triangles:
2. Calculate matrix coefficients: \[ [S]_{ij} = (\nabla \alpha_i \cdot \nabla \alpha_j)A \]
3. Solve matrix equation: \[ [S_{kf} \quad S_{kp}]\begin{bmatrix} U_f \\ U_p \end{bmatrix} = 0 \]
4. Determine equipotential lines and/or field lines

Procedure BEM

1. Generate elements along interfaces
2. Generate matrix coefficients: \[ H_{ij} = \int_{S_j} \frac{\partial \ln r_i}{\partial n} \, ds, \quad G_{ij} = \int_{S_j} \ln r_i \, ds \]
3. Solve matrix equation:
\[ \sum_{j=1}^{n} (H_{ij} - \pi \delta_{ij}) U_j = \sum_{j=1}^{n} G_{ij} Q_j \]
4. Determine potential on arbitrary position:
\[ U(x_0, y_0) = \frac{1}{2\pi} \left( \sum_{j=1}^{n} U_j \int_{S_j} \frac{\partial \ln r}{\partial n} \, ds - \sum_{j=1}^{n} Q_j \int_{S_j} \ln r \, ds \right) \]
Generation of High Voltages

• **AC Sources** (50/60 Hz)
  - High voltage transformer (one coil; divided coils; cascade)
  - Resonance source (series; parallel)

• **DC Sources**
  - Rectifier circuits (single stage; cascade)
  - Electrostatic generator (van de Graaff generator)

• **Pulse sources**
  - Pulse circuits (single stage; cascade; pulse transformer)
  - Traveling wave generators (PFL; PFN; transmission line transformer)
Cascaded High voltage transformer

900 kV
400 A

1: primary coil
2: secondary coil
3: tertiary coil

Courtesy: Delft Univ. of Techn.
Resonance Source

Equivalent Circuit:

\[ |H(\omega)| = \frac{Q \omega_0 / \omega}{\sqrt{1 + Q^2 (\omega_0 / \omega - \omega / \omega_0)^2}} \]

\[ \omega_0 = \frac{1}{\sqrt{LC}} \quad \text{and} \quad Q = \frac{1}{R \sqrt{C}} \]

- Waveform: almost perfect sinusoidal
- Power: 1/Q of “normal” transformer
- Short circuit: Q→0 results in V→0
- No resistive load

900 kV
100 mA

Courtesy: Eindhoven Univ. of Techn.
Cascaded Rectifier
(Greinacher; Cockcroft - Walton)

Reduce $\delta V (\sim n^2)$ and $\Delta V (\sim n^3)$ by:
- larger C’s (more energy in cascade)
- higher f (up to tens of kilohertz)

$$V_{DC} = 2nV_{\text{max}}$$

Voltage: 2 MV

amplitude: $V_{\text{max}}$
Single-Stage Pulse Source

\[ V_0 \]

\[ \frac{d^2U}{dt^2} + a \frac{dU}{dt} + bU = 0 \]

\[ a = \frac{1}{R_1C_1} + \frac{1}{R_2C_2} + \frac{1}{R_1C_2}; b = \frac{1}{R_1C_1R_2C_2} R_2 \]

\[ U(t) = V_0 \left( e^{-t/\tau_2} - e^{-t/\tau_1} \right) \]

if \( C_1 >> C_2 \) and \( R_2 >> R_1 \)

rise time: \( \tau_1 = R_1C_2 \)

discharge time: \( \tau_2 = R_2C_1 \)

Standard lightning surge pulse: 1.2 / 50 \( \mu s \)

60 kV

1 kA

Courtesy: Eindhoven Univ.of Techn.
Cascade Pulse Source
(Marx Generator)

\[ V_{\text{pulse}} = n \cdot V_{\text{DC}} \]

Total discharge capacity: \( 1/C1 = \sum 1/C1' \)
Front resistance: \( R1 = R1'' + \sum R1' \)
Discharge resistance: \( R2 = \sum R2' \)

R1: front resistor
R2: discharge resistor

900 kV
100 mA

Courtesy: Kema, The Netherlands
Pulse Source with Transformer

primary: \( L_1 C_1 \frac{d^2 I_1}{dt^2} - MC_1 \frac{d^2 I_2}{dt^2} + I_1 = 0 \)

secondary: \( L_2 C_2 \frac{d^2 I_2}{dt^2} - MC_2 \frac{d^2 I_1}{dt^2} + I_2 = 0 \)

Eigen frequencies from characteristic equation:

\[
\begin{pmatrix}
L_1 C_1 & -MC_1 \\
-MC_2 & L_2 C_2
\end{pmatrix}
\begin{pmatrix}
i_1 \\
i_2
\end{pmatrix}
= \frac{1}{\omega^2}
\begin{pmatrix}
i_1 \\
i_2
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
L_1 C_1 - \frac{1}{\omega^2} \\
L_2 C_2 - \frac{1}{\omega^2}
\end{pmatrix}
- M^2 C_1 C_2 = 0
\]

Approximation: transformer almost ideal: \( k = M / \sqrt{L_1 L_2} \rightarrow 1 \)

\[
\omega_1 \approx \frac{1}{\sqrt{L_1 C_1 + L_2 C_2}} = \frac{1}{\sqrt{L_1 (C_1 + C')}} \quad \text{slow oscillation}
\]

\[
\omega_2 \approx \frac{1}{\sqrt{1 - k^2}} \frac{1}{\sqrt{L_1 C_1}} + \frac{1}{\sqrt{L_2 C_2}} = \frac{1}{\sqrt{L_{eq} (C_1 // C')}} \quad \text{fast oscillation}
\]

Voltage: 300 kV
Pulse Forming Line / Network

charge cable \( \rightarrow \) \( S \) \( \rightarrow \) load cable

\[ R \]

\[ V_{DC} \]

Z

80 kV, 10 kA, \( T=20\text{ns} - 10\mu\text{s} \)

Transmission Line Transformer

\[ V_{DC} \]

S

cables parallel

cables in series

\[ 150 \text{ kV} \]

\[ 1 \text{ kA} \]

amplitude: \( \frac{1}{2} V_{DC} \)
duration: \( 2L/v \)

Courtesy: Eindhoven Univ.of Techn.
Insulation and Breakdown

• In Gases
  - Ionisation and Avalanche Formation
  - Townsend and Streamer Breakdown
  - Paschen Law: Gas Type
  - Breakdown Along Insulator
  - Inhomogeneous Fields, Pulsed Voltages, Corona

• Insulating Liquids

• Solid Insulation
  - Breakdown types, Surface tracking, Partial discharges, Polarisation, \( \tan \delta \)

• Vacuum Insulation
  - Applications, Breakdown, Cathode Triple-Point, Insulator Surface Charging, Conditioning
1\textsuperscript{st} free electron
- Cosmic radiation
- Shortwave UV
- Radio active isotopes

Free path, effective cross-section

Townsend’s 1\textsuperscript{st} ionisation coefficient $\alpha$

One electron creates $\alpha$ new electrons per unit length

$A + e \rightarrow A^+ + 2e$

$n_e(x=d) = n_0 e^{\alpha d}$

$\alpha/p = f(E/P)$

In air:
$\approx 2.5 \times 10^{19}$ molecules/cm$^3$
$\approx 1000$ ions/cm$^3$
$\approx 10$ electrons/cm$^3$
• **Electro-negative gasses**
  Attachment $\eta$ of electrons to ions
electrons: $n_e(x=d) = n_0 e^{(\alpha - \eta)d}$
negative ions:
  $$n_-(x = d) = \frac{n_0 \eta}{\alpha - \eta} [e^{(\alpha-\eta)d} - 1]$$

**Avalanche ≠ Breakdown; creation of secondaries**
Townsend’s 2\textsuperscript{nd} ionisation coefficient $\gamma$
one ion or photon creates $\gamma$ new electrons
at cathode $n_e = \gamma n_0 (e^{\alpha d} - 1)$

Breakdown if: # secondary electrons $\geq n_0$
$\alpha d \geq \ln(1/\gamma + 1)$

steep function of $E/p \rightarrow e^{\alpha d}$ very steep $\rightarrow (E/p)_{\text{critical}}$ and $V_d$
well defined $\rightarrow \gamma$ of weak influence
Paschen law / breakdown field

- Townsend breakdown criterion $\alpha d = K$:
  $$\frac{E_d}{p} = \frac{B}{\ln(\frac{Apd}{K})}$$
  $$V_d = \frac{Bpd}{\ln(\frac{Apd}{K})}$$
  with
  $A = \frac{\sigma}{kT}$
  $B = \frac{V_i\sigma_i}{kT}$

$\Rightarrow E_d$ and $V_d$ depend only on $p*d$

- Typically practically $E_{bd} = 10 \text{ kV/cm}$ at 1 bar in air
  - $V_{bd,Paschen\ min,\ air} \approx 300 \text{ V}$

- Small $p*d$, $d)<< \lambda$: few collisions, high field required for ionisation
- Large $p*d$, $d>> \lambda$: collision dominated, small energy build-up, high $V_d$

1: SF$_6$
2: air
3: H$_2$
4: Ne
Streamer breakdown

Space charge field $E_\rho \approx E_0$
- Field enhancement
- Extra ionising collisions ($\alpha \uparrow$)
- High excitation $\Rightarrow$ UV photons

When 1 electron grows into ca. $10^8$
then $E_\rho$ large enough for streamer breakdown ($n_e \approx 2 \cdot 10^8$ in avalanche head)

Result:
- Secondary avalanches, directional effect (channel formation)
- Grows out into a breakdown within 1 gap crossing (anode and/or cathode directed)

Characteristic:
- Very fast
- Independent of electrodes (no need for electrode surface secondaries)
- Important at large distances (lightning)
 Townsend, unless:

- Strong non-uniform field  
  (small electrodes, few secondary electrons)
- Pulsed voltages
  - Townsend: slow, ion drift, subsequent gap transitions
  - Streamer: fast, photons, 1 gap transition
- High pressure
  - Less diffusion
    - $E_p$ high
  - Photons absorbed
    - in front of cathode
  - Positive ions
    - Slower

Townsend: $\alpha d \geq \ln(1/\gamma + 1) \approx 7...9$  
($\gamma \approx 10^{-4}...10^{-3}$)

Streamer: $\alpha d \geq 18...20$

Laser-induced streamer breakdown in air.  

Courtesy: Eindhoven Univ.of Techn.
Breakdown along insulator

- Surface charge
- (Non-regular) surface conduction
- Particles / contaminations on surface
- Non-regularities (scratches, ridges)

⇒ Field enhancement
⇒ Increased breakdown probability

Prebreakdown along insulator in air.

Reference [2]
Breakdown at pulse voltages; time-lag

\[ t_s, \text{ wait for first elektron} \]
\[ t_f, \text{ breakdown formation} \]

- Townsend or Streamer

Short pulses, high breakdown voltage
Non-uniform fields; Corona

Breakdown conditions:
- Global → Full breakdown
- Local → Streamer breakdown
  → Partial discharge

Non-uniform field:
- Discharge starts in high field region
- ....and “extinguishes” in low field region

Reference [2]
Corona
- Power loss; EM noise
- Chemical corrosion
+ Useful applications

Pulsed corona discharges:
- Fast, short duration HV pulses
- Many streamers, high density
- Generation of electrons, radicals, excited molecules, UV
- E.g. Flue gas cleaning

Transmission line transformer

Courtesy: Eindhoven Univ.of Techn.
### Solid insulation

**Breakdown field strength:**
- Very clean (lab): high
- Practical: lower due to imperfections
  - Voids
  - Absorbed water
  - Contaminations
  - Structural deformations

**Requirements:**
- Mechanical strength
- Contact with electrodes and semiconducting layers
- Resistant to high T, UV, dirt, contamination, rain, ice, desert sand

**Problems:**
- Surface tracking
- Partial discharges
  - In voids (in material or at electrodes, often created at production).

### Types of solid insulation materials

<table>
<thead>
<tr>
<th>Anorganic</th>
<th>Quartz, mica, glass Porcelain</th>
<th>Disc insulator Feedthrough Spacer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Natural</td>
<td>Al₂O₃</td>
<td>Cable Capacitor</td>
</tr>
<tr>
<td>Synthetic</td>
<td>Polyethylene HD, LD, XL – PE Teflon Polystyrene, PVC, polypropene, etc</td>
<td>Spec. properties: Moisture content high T losses bonding</td>
</tr>
<tr>
<td>Paper + Oil</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Synthetic</td>
<td>Polyethylene HD, LD, XL – PE Teflon Polystyrene, PVC, polypropene, etc</td>
<td>Spec. properties: Moisture content high T losses bonding</td>
</tr>
<tr>
<td>Epoxy</td>
<td>Hardener Filler</td>
<td>Moulding in mold</td>
</tr>
</tbody>
</table>

---

- **Types of solid insulation materials**
Surface tracking

Remedies:
- Geometry
- Creeping distance (IEC-norm)
- Surface treatment (emaillate, coating)
- Washing
Vacuum insulation

Applications:
- Vacuum circuit breaker
- Cathode Ray Tubes / accelerators
- Elektron microscope
- X-ray tube
- Transceiver tube

What is vacuum?
- “Pressure at which no collisions for Brownian “temperature” movements of electrons”
- $\lambda \gg$ characteristic distances
- E.g. $p = 10^{-6}$ bar, $\lambda = 400$ m

Advantages:
- “Self healing”
- No dielectric losses
- High breakdown fieldstrength
- Non flammable
- Non toxic, non contaminating

Disadvantages:
- Requires hermetic containment and mechanical support
- Quality determind by:
  - electrodes and insulators
  - Material choice, machining
  - Contaminations, conditioning
Characteristics of vacuum breakdown

No 1st electron from “gas”
• Cathode emission
  – primary: photoemission, thermic emission, field emission, Schottky-emission
  – secondary: e.g. e⁻ bombarded anode → +ion collides at cathode → e⁻

No breakdown medium
• No multiplication through collision ionisation
• Medium in which the breakdown occurs has to be created (“evaporated” from electrodes, insulators)

Important: prevent field emission
• Keep field at cathode and “cathode triple point” as low as possible
• Insulator surface charging, conditioning
Breakdown vs. field cathode-triple-point

Insulator surface charging

Conditioning

# Conditioning breakdowns needed to reach 50kV hold voltage

Ref [12]
Insulating liquids

Requirements:
- Pure, dry and free of gases
- \( \varepsilon_r \) (high for C’s, low for trafo) (demi water \( \varepsilon_{r,d.c.} = 80 \))
- Stable (T), non-flammable, non toxic (pcb’s), ageing, viscosity
- No interface problems
- Combined cooling/insulation
- “Cheap” (no mould)
- Liquid tight housing

Applications:
- Transformers
- Cables
- Capacitors
- Bushings

Courtesy: Sandia labs, U.S.A.
Breakdown fieldstrength:

- Very clean (lab): high 1 - 4 MV/cm (In practice much lower)
- Important at production: outgassing, filtering, drying
- Mineral oil (“old” time application, cheap, flammable)
  - Synthetic oil (purer, specifically made, more expensive)
    - Silicon oil (very stable up to high T, non-toxic, expensive)
- Liquid H2, N2, Ar, He (supra-conductors)
- Demi-water (incidental applications, pulsed power)
- Limitation $V_{bd}$:
  - Inclusions: Partial discharges $\rightarrow$ Oil decomposition $\rightarrow$ Breakdown
  - Growth (pressure increase)
  - “extension” in field direction
- Particles drift to region with highest E $\rightarrow$ bridge formation $\rightarrow$ breakdown
Transformer:

- **Mineral oil**: Insulation and cooling

- **Paper**: Barrier for charge carriers and chain formation
  - Mechanical strength

- **Ageing**
  - Thermical and electrical (partial discharges)
  - Lifetime: 30 years, strongly dependent on temperature, short-circuits, over-loading, over-voltages
  - Breakage of oil molecules, Creation of gasses, Concentration of various gas components indication for exceeded temperature (as specified in IEC599)

- **Lifetime**
  - Time in which paper loses 50% of its mechanical strength
  - Strongly dependent on:
    - Moisture (from 0.2% to 2% accelerated ageing factor 20)
    - Oxygen (presence accelerates ageing by a factor 2)
Partial discharges

- UV, fast electrons, ions, heat
- Deterioration void:
  - Oxidation, degradation through ion-impact
  - “Pitting”, followed by treeing
- Eventually breakdown


- High sensitivity measurements on often large objects
- $Q_{\text{app}} \neq Q_{\text{real}}$, still useful, because measure for dissipated energy, thereby for induced damage
- Relative measurement

AC voltage phase resolved discharge pattern detection $\rightarrow$ Type of defect
Partial discharges

- \( Q_{\text{app}} \) gives \( i_t \) (\( Q_{\text{app}} = \int i_t \, dt \))
- \( C_c \) gives \( i_t \) if \( C_c \gg C_{\text{object}} \)
- Calibration through injecting known charge

<table>
<thead>
<tr>
<th>Before</th>
<th>After</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q = aV + b(V - \Delta V) + c \Delta V )</td>
<td>( Q - c \Delta V = (a + b)(V - \delta V) )</td>
</tr>
<tr>
<td>( \Delta V = \Delta V b/(a+b) \approx \Delta V b/a )</td>
<td></td>
</tr>
</tbody>
</table>

Apparent charge:
\[
Q_{\text{tot}} \Delta V = (a + bc/(b+c)) \Delta V
\approx (a+b) \delta V \approx a\delta V
\]

\[
\frac{Q_{\text{app}}}{Q_{\text{real}}} = \frac{a\delta V}{c\Delta V} = \frac{b}{c} \approx \frac{\varepsilon_r d_{\text{void}}}{d_{\text{insul}}}
\]

High sensitivity measurement because \( b/c \ll 1 \)

- Measure with resonant RLC circuit:
  - Excitation by short pulse \( i_t \)
  - No 50 Hz problem
  - \( V = q/C \exp(-\alpha t) \) \{ \( \cos\beta t - \alpha/\beta \sin\beta t \) \}

\[
\alpha = \frac{1}{2RC} \quad \beta = \left[\frac{1}{LC} - \alpha^2\right]^{-1/2}
\]
Loss angle, tan(δ)

Sources:
- Conduction σ (for DC or LF)
- Partial discharges
- Polarisation

Schering bridge:
- i=0, RC=R₄C₄
- Gives: tan(δ)
  - parallel: 1/ωRC
  - série: ωRC

Tan δ:
- “Bulk” parameter
- No difference between phases

PD:
- Detection of weakest spot
- Largest activity and asymmetry in “blue” phase (ridge discharges)
Summary

Seen many basic high voltage engineering technology aspects here:
- High voltage generation
- Field calculations
- Discharge phenomena

The above to be applied in your practical accelerator environments as needed:
- Vacuum feed through: Triple points
- Breakdown field strength in air 10kV/cm
- Challenging calculations for real practical geometries.
References

Appendix I

Maxwell equations in integral form

**Electrostatic**
\[ \oint D \cdot dA = \int \int \int \rho \, dV = Q_{\text{omsl.}} \]  
\[ \oint E \cdot dl = -\frac{d}{dt} \int \int B \cdot dA = -\frac{d\Phi_{\text{omsl.}}}{dt} \]  
\[ \oint B \cdot dA = 0 \]  
\[ \oint H \cdot dl = \int \int (J + \frac{\partial D}{\partial t}) \cdot dA \]  

**Magnetostatic**
\[ \oint B \cdot dA = 0 \]  
\[ \oint H \cdot dl = I_{\text{omsl.}} \]  

Maxwell equations in differential form

**Electrostatic**
\[ \nabla \cdot D = \rho \]  
\[ \nabla \times E = -\frac{\partial B}{\partial t} \]  
\[ \nabla \cdot B = 0 \]  
\[ \nabla \times H = J + \frac{\partial D}{\partial t} \]  

**Magnetostatic**
\[ \nabla \cdot B = 0 \]  
\[ \nabla \times H = J \]  

**No space charge**
\[ \nabla \cdot D = 0 \]  
\[ \nabla \times E = 0 \]  
\[ \nabla \cdot B = 0 \]  
\[ \nabla \times H = 0 \]  

**In area without source**
\[ \nabla \cdot B = 0 \]  
\[ \nabla \times H = 0 \]
Field energy minimal inside each closed region $G$:

$$W = \int \frac{1}{2} \varepsilon |\vec{E}|^2 dV = \int \frac{1}{2} \varepsilon |\nabla U|^2 dV$$

Assume $U$ satisfies Laplace equation, but $U'$ does not, then

$$W_{U'} - W_U \geq 0:$$

$$W_{U'} - W_U = \frac{1}{2} \varepsilon \int \int \int (|\nabla U'|^2 - |\nabla U|^2) dV = \cdots = \frac{1}{2} \varepsilon \int \int \int |\nabla U' - \nabla U|^2 dV \geq 0$$

### Field energy for one element (2-dim)

Potential is linear inside element: $U = a + bx + cy = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

on corners:

$$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Potential can be written as:

$$U = (1 \ x \ y) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^{-1} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \sum_{i=1}^{3} U_i \alpha_i(x, y)$$

Field energy in element (e):

$$W^{(e)} = \frac{1}{2} \varepsilon U^T \left[ S^{(e)} \right] U$$

where $S^{(e)}_{ij} = \int \int (\nabla \alpha_i \cdot \nabla \alpha_j) dxdy$

$\alpha$’s are linear in $x$ and $y \Rightarrow \nabla \alpha$ is constant: $S_{ij} = (\nabla \alpha_i \cdot \nabla \alpha_j) A^{(e)}$
Total field energy of n elements

All elements together:

\[ U^T = \begin{pmatrix} U_1 & \cdots & U_m & U_{m+1} & \cdots & U_n \end{pmatrix} \equiv \begin{pmatrix} U_f & U_p \end{pmatrix} \]

free \quad \text{prescribed}

\textbf{free}: potential values to be determined

\textbf{prescribed}: potential according to boundary conditions

\[ W = \frac{1}{2} \varepsilon U^T [S] U = \frac{1}{2} \varepsilon [U_f^T, U_p^T] \begin{bmatrix} S_{f'f} & S_{f'p} \\ S_{p'f} & S_{p'p} \end{bmatrix} \begin{bmatrix} U_f \\ U_p \end{bmatrix} \]

Partial derivatives of \( W \) to \( U_k \) are zero for \( 1 \leq k \leq m \) (m equations):

\[ \frac{\partial W}{\partial U_k} = 0 \quad \Rightarrow \quad \begin{bmatrix} S_{kf} & S_{kp} \end{bmatrix} \begin{bmatrix} U_f \\ U_p \end{bmatrix} = 0 \]

\textbf{Boundary Element Method (BEM)}

\textbf{Boundaries uniquely prescribe potential distribution}

\textbf{Laplace equation} : \quad \Delta u = 0

\textbf{border} \ \Gamma_1 : u(x, y) \quad (\text{Dirichlet})

\textbf{border} \ \Gamma_2 : q(x, y) = \frac{\partial u}{\partial n} \quad (\text{Neumann})

\textbf{inside} \ \Gamma : \quad u(x_0, y_0) = \frac{1}{2\pi} \oint_{\Gamma} \left( u(x, y) \frac{\partial \ln r}{\partial n} - q(x, y) \ln r \right) ds

\textbf{Problem}: \ u(x, y) \ and \ q(x, y) \ \text{not both known at the same time}
**Green II (2-dim):**

\[
\int (u \nabla v - v \nabla u) \cdot \hat{n} ds = \iint (u \Delta v - v \Delta u) dxdy
\]

Choose \( v(x, y) = \ln(1/r) \)

\( \Delta v(x, y) = 0 \) for \( P \neq P_0(x_0, y_0) \)

Exclude region \( \sigma \) around \( P_0 \) by means of circle \( c \)

\[
\int_{\Gamma + c} (u \frac{\partial \ln r^{-1}}{\partial n} - \ln r^{-1} \frac{\partial u}{\partial n}) ds = \iint_{\Omega - \sigma} (u \Delta \ln r^{-1} - \ln r^{-1} \Delta u) dxdy = 0
\]

\[
-\int_{\Gamma} (u \frac{\partial \ln r}{\partial n} - \ln r \frac{\partial u}{\partial n}) ds + \int_{c} (u \frac{\partial \ln r^{-1}}{\partial n} - \ln r^{-1} \frac{\partial u}{\partial n}) ds = 0
\]

\[
\lim_{\epsilon \downarrow 0} 2\pi \int_{0}^{2\pi} \left( u \frac{1}{\epsilon} + \frac{\partial u}{\partial n} \ln \epsilon \right) \epsilon d\theta = 2\pi u(x_0, y_0)
\]

\( P_i = (x_i, y_i) \) on border \( \Gamma \):

\[
u(x_i, y_i) = \frac{1}{\pi} \int_{\Gamma} (u(x, y) \frac{\partial \ln r}{\partial n} - q(x, y) \ln r) ds
\]

Discretisation:

\[
\pi U(x_i, y_i) = \sum_{j=1}^{n} U_j \int_{s_j} \frac{\partial \ln r_{ij}}{\partial n} ds - \sum_{j=1}^{n} Q_j \int_{s_j} \ln r_{ij} ds
\]

In matrix notation:

\[
\sum_{j=1}^{n} (H_{ij} - \pi \delta_{ij}) U_j = \sum_{j=1}^{n} G_{ij} Q_j
\]

Generates missing information