Resonances

- Introduction: driven oscillators and resonance condition
- Smooth approximation for motion in accelerators
- Field imperfections and normalized field errors
- Perturbation treatment
- Poincare section
- Stabilization via amplitude dependent tune changes
- Sextupole perturbation & slow extraction
- Chaotic particle motion
Introduction: Damped Harmonic Oscillator

Equation of motion for a damped harmonic oscillator:

\[
\frac{d^2 w(t)}{dt^2} + \omega_0 \cdot Q^{-1} \cdot \frac{d}{dt} w(t) + \omega_0^2 \cdot w(t) = 0
\]

\(Q\) is the damping coefficient

(\text{amplitude decreases with time})

\(\omega_0\) is the Eigenfrequency of the HO

Example: weight on a spring \((Q = \infty)\)

\[
\frac{d^2 w(t)}{dt^2} + k \cdot w(t) = 0 \quad \rightarrow \quad w(t) = a \cdot \sin(\sqrt{k} \cdot t + \phi_0)
\]
Introduction: Driven Oscillators

an external driving force can ‘pump’ energy into the system:

\[ \frac{d^2}{dt^2} w(t) + \omega_0 \cdot Q^{-1} \cdot \frac{d}{dt} w(t) + \omega_0^2 \cdot w(t) = \frac{F}{m} \cdot \cos(\omega \cdot t) \]

general solution:

\[ w(t) = w_{tr}(t) + w_{st}(t) \]

stationary solution:

\[ w_{st}(t) = W(\omega) \cdot \cos[\omega \cdot t - \alpha(\omega)] \]

where ‘\( \omega \)’ is the driving angular frequency!
and \( W(\omega) \) can become large for certain frequencies!
stationary solution

stationary solution follows the frequency of the driving force:

$$w_{st}(t) = W(\omega) \cdot \cos[\omega \cdot t - \alpha(\omega)]$$

oscillation amplitude can become large for weak damping
Introduction: Pulsed Driven Resonances Example

higher harmonics:

example of a bridge:

2\textsuperscript{nd} harmonic: 3\textsuperscript{rd} harmonic: 5\textsuperscript{th} harmonic:

peak amplitude depends on the excitation frequency and damping

[Bob Barrett; Messiah College]
Introduction: Instabilities

- resonance catastrophe without damping:

\[ W(\omega) = W(0) \cdot \frac{1}{\sqrt{[1-(\omega/\omega_0)^2]^2+(\omega/Q\omega_0)^2}} \]

- weak damping: resonance condition: \( \omega = \omega_0 \)

Tacoma Narrow bridge
1940

excitation by strong wind on the Eigenfrequencies
Smooth Approximation: Resonances in Accelerators

Revolution frequency:

Betatron oscillations:

Eigenfrequency: $\omega_0 = 2\pi f_{\beta}$

$Q = \omega_0 / \omega_{rev}$

Driven oscillator

Weak or no damping!

(synchrotron radiation damping (single particle) or Landau damping distributions)
Smooth Approximation: Free Parameter

co-moving coordinate system:

choose the longitudinal coordinate as the free parameter for the equations of motion

equations of motion:

\[
\frac{d}{dt} = \frac{ds}{dt} \cdot \frac{d}{ds}
\]

with:

\[
\frac{ds}{dt} = v
\]

\[
\frac{d^2}{dt^2} = v^2 \cdot \frac{d^2}{ds^2}
\]
Smooth Approximation: Equation of Motion I

Smooth approximation for Hills equation:

\[
\frac{d^2}{ds^2} w(s) + K(s) \cdot w(s) = 0 \quad \xrightarrow{\text{K(s) = const}} \quad \frac{d^2}{ds^2} w(s) + \omega_0^2 \cdot w(s) = 0
\]

(constant β-function and phase advance along the storage ring)

\[w(s) = A \cdot \cos(\omega_0 \cdot s + \phi_0)\]
\[\omega_0 = 2\pi \cdot Q_0 / L\]

(Q is the number of oscillations during one revolution)

perturbation of Hills equation:

\[
\frac{d^2}{ds^2} w(s) + \omega_0^2 \cdot w(s) = F(w(s), s) / (v \cdot p)
\]

in the following the force term will be the Lorenz force of a charged particle in a magnetic field:

\[F = q \cdot \vec{v} \times \vec{B}\]
Field Imperfections: Origins for Perturbations

- Linear magnet imperfections: derivation from the design dipole and quadrupole fields due to powering and alignment errors.

- Time varying fields: feedback systems (damper) and wake fields due to collective effects (wall currents).

- Non-linear magnets: sextupole magnets for chromaticity correction and octupole magnets for Landau damping.

- Beam-beam interactions: strongly non-linear field!

- Non-linear magnetic field imperfections: particularly difficult to control for superconducting magnets where the field quality is entirely determined by the coil winding accuracy.
Field Imperfections: Localized Perturbation

periodic delta function:

\[ \delta_L(s - s_0) = \begin{cases} 1 & \text{for } s' = s_0, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \int \delta_L(s - s_0)ds = 1 \]

equation of motion for a single perturbation in the storage ring:

\[ \frac{d^2}{ds^2} w(s) + \omega_0^2 \cdot w(s) = \delta_L(s - s_0) \cdot l \cdot F(w, s)/(v \cdot p) \]

Fourier expansion of the periodic delta function:

\[ \frac{d^2}{ds^2} w(s) + \omega_0^2 \cdot w(s) = \frac{l}{L} \sum_{r=-\infty}^{\infty} \cos(r \cdot 2\pi \cdot s / L) \cdot F(w, s)/(v \cdot p) \]

\[ \rightarrow \text{infinite number of driving frequencies} \]
Field Imperfections: Constant Dipole

normalized field error: \[
\frac{F}{v \cdot p} = q \cdot \frac{\vec{v} \times \vec{B}}{v \cdot p} \stackrel{v \perp B}{\longrightarrow} k_0 = \frac{q \cdot B}{p}
\]

equation of motion for single kick:
\[
\frac{d^2}{ds^2} w(s) + \omega_0^2 \cdot w(s) = \frac{lk_0}{L} \sum_{r=-\infty}^{\infty} \cos(r \cdot 2\pi \cdot s / L)
\]

resonance condition:
\[
\omega_0 = \frac{r \cdot 2\pi}{L} \quad \omega_0 = \frac{2\pi \cdot Q_0}{L} \rightarrow Q_0 = r
\]

avoid integer tunes!

remember the example of a single dipole imperfection from the ‘Linear Imperfection’ lecture yesterday!
Field Imperfections: Constant Quadrupole

Equations of motion:

\[
\frac{d^2}{ds^2} x(s) + \omega_x^2 \cdot x(s) = k_1 \cdot x(s)
\]

\[
y(s) \equiv 0
\]

With:

\[
k_1 = \frac{q \cdot \partial B_y}{p \cdot \partial x}
\]

\[
\frac{d^2}{ds^2} x(s) + (\omega_x^2 - k_1) \cdot x(s) = 0
\]

Change of tune but no amplitude growth due to resonance excitations!
Field Imperfections: Single Quadrupole Perturbation

Assume $y = 0$ and $B_x = 0$:

$$F(s)/(v \cdot p) = \delta_L (s - s_0) \cdot l \cdot k_1 \cdot x$$

$$\frac{d^2}{ds^2} x(s) + \omega_{x,0}^2 \cdot x(s) = \frac{lk_1}{L} \sum_{r=-\infty}^{\infty} \cos(2\pi r \cdot s / L) \cdot x(s)$$

Resonance condition:

$$\omega_{x,0} = n \cdot 2\pi / L \pm \omega_{x,0} \quad \omega_0 = 2\pi Q_0 / L \quad Q_0 = n / 2$$

Avoid half integer tunes plus resonance width from tune modulation!

Exact solution: variation of constants $\Rightarrow$ see the lecture yesterday
Field Imperfections: Time Varying Dipole Perturbation

**time varying perturbation:**

\[
F(t) = F_0 \cdot \cos(\omega_{kick} \cdot t) \xrightarrow{t \to \infty} F_0 \cdot \cos(2\pi \cdot \frac{\omega_{kick}}{\omega_{rev}} \cdot s / L) / (v \cdot p)
\]

\[
\frac{d^2}{ds^2} w(s) + \omega_0^2 \cdot w(s) = \frac{IF_0}{2L} \sum_{r=-\infty}^{\infty} \cos(2\pi \cdot [r \pm \omega_{kick} / \omega_{rev}] \cdot s / L) / (v \cdot p)
\]

**resonance condition:**

\[
\omega_0 = 2\pi \cdot (r \pm \omega_{kick} / \omega_{rev}) / L \xrightarrow{\omega_0 = 2\pi \cdot Q_0 / L} f_{kick} = f_{rev} \cdot (Q_0 \pm r)
\]

**avoid excitation on the betatron frequency!**

(the integer multiple of the revolution frequency corresponds to the modes of the bridge in the introduction example)
Field Imperfections: Several Bunches

\[ F(t) = B \cdot \cos(\omega_{\text{kick}} \cdot t); \quad \omega_{\text{kick}} \approx \omega_{\text{rev}} : \]

machine circumference

\[ F(t) = B \cdot \cos(\omega_{\text{kick}} \cdot t); \quad \omega_{\text{kick}} \approx 2 \cdot \omega_{\text{rev}} : \]

higher modes analogous to bridge illustration
Field Imperfections: Multipole Expansion

Taylor expansion of the magnetic field:

\[ B_y + iB_x = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot f_n \cdot (x + iy)^n \]

with:

\[ f_n = \frac{\partial^{n+1} B_y}{\partial x^{n+1}} \]

<table>
<thead>
<tr>
<th>multipole</th>
<th>order</th>
<th>B_x</th>
<th>B_y</th>
</tr>
</thead>
<tbody>
<tr>
<td>dipole</td>
<td>0</td>
<td>0</td>
<td>( B_0 )</td>
</tr>
<tr>
<td>quadrupole</td>
<td>1</td>
<td>( f_1 \cdot y )</td>
<td>( f_1 \cdot x )</td>
</tr>
<tr>
<td>sextupole</td>
<td>2</td>
<td>( f_2 \cdot x \cdot y )</td>
<td>( \frac{1}{2} \cdot f_2 \cdot (x^2 \cdot y^2) )</td>
</tr>
<tr>
<td>octupole</td>
<td>3</td>
<td>( \frac{1}{6} \cdot f_3 \cdot (3yx^2 - y^3) )</td>
<td>( \frac{1}{6} \cdot f_3 \cdot (x^3 - 3xy^2) )</td>
</tr>
</tbody>
</table>

Normalized multipole gradients:

\[ F(s)/(v \cdot p) = \frac{q(\vec{v} \times \vec{B})}{(v \cdot p)} \]

\[ k_n = \frac{q}{p} \cdot f_n \]

\[ k_n = 0.3 \cdot \frac{f_n[T/m^n]}{p[GeV/c]} \]

\[ [k_n] = \frac{1}{m^{n+1}} \]
Field Imperfections: Dipole Magnets

dipole magnet designs:

LEP dipole magnet:
conventional magnet design relying on pole face accuracy of a Ferromagnetic Yoke

LHC dipole magnet:
air coil magnet design relying on precise current distribution
Field Imperfections: Multipole Illustration

upright and skew field errors

upright:

\( n=0 \)

\[ \begin{array}{c}
\text{1} \\
\end{array} \]

\[ \begin{array}{c}
\text{2} \\
\end{array} \]

\( n=1 \)

\[ \begin{array}{c}
\text{4} \quad \text{1} \\
\text{3} \quad \text{2} \\
\end{array} \]

\( n=2 \)

\[ \begin{array}{c}
\text{6} \quad \text{1} \quad \text{2} \\
\text{5} \quad \text{4} \quad \text{3} \\
\end{array} \]

skew:
Field Imperfections: Multipole Illustrations

quadrupole and sextupole magnets

ISR quadrupole

LEP Sextupole
Field Imperfections: Super Conducting Magnets

time varying field errors in super conducting magnets

Luca Bottura CERN, AT-MAS

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Perturbation Treatment: Resonance Condition

**equations of motion:** (nth order Polynomial in x and y for nth order multipole)

\[ \frac{d^2}{ds^2} w(s) + \omega_0^2 \cdot w(s) = \varepsilon \cdot \sum_{l+m<n, r} a_{n,m,r} \cdot x^l \cdot y^m \cdot \cos(2\pi \cdot r \cdot s / L) \]

**perturbation treatment:**

\[ w(s) = x_0 + \varepsilon \cdot x_1 + \varepsilon^2 \cdot x_2 + ... + O(\varepsilon^n) \]

\[ \omega_0 = \frac{2\pi}{L} Q_{x,y} \]

with: \( w = x, y \)

\[ x_0(s) = x_0 \cdot \cos(2\pi \cdot Q_{x,0} \cdot s / L + \phi_{x,0}) \]

[same for ‘y(s)’]

\[ \frac{d^2}{ds^2} x_1 + \omega_0^2 \cdot x_1 = \sum_{\tilde{l}<l, \tilde{m}<m} a_{\tilde{n},\tilde{m},r} \cos\left(\frac{2\pi}{L} \cdot [\tilde{l} Q_{x,0} + \tilde{m} Q_{y,0} + r] \cdot s \right) \]
Perturbation Treatment: Tune Diagram I

resonance condition: \[ \frac{2\pi}{L} \cdot (\tilde{t} \cdot Q_x + \tilde{m} \cdot Q_y + r) = \frac{2\pi}{L} \cdot Q_{x,y} \]

avoid rational tune values!

\[ l \cdot Q_x + m \cdot Q_y = r \]

up to 11 order (p+1 < 12)

there are resonances everywhere!
(\text{the rational numbers lie dense within the real number})
Perturbation Treatment: Tune Diagram II

regions with few resonances:

\[ l \cdot Q_x + m \cdot Q_y = r \]

avoid low order resonances!

< 12\textsuperscript{th} order for a proton beam without damping

< 3\textsuperscript{rd} ⇔ 5\textsuperscript{th} order for electron beams with damping

coupling resonance:

regions without low order resonances are relatively small!
Perturbation Treatment: Single Sextupole Perturbation

perturbed equations of motion: \[ F(s)/(v \cdot p) = \frac{1}{2} \delta_L (s - s_0) \cdot l k_2 \cdot x^2 \]

\[ \frac{d^2}{ds^2} x_1(s) + \omega_0^2 \cdot x_1(s) = \frac{1}{2} \cdot l k_2 \cdot x_0^2 \cdot \frac{1}{L} \sum_{r=-\infty}^{\infty} \cos(2\pi \cdot r \cdot s / L) \]

with: \[ x_0(s) = A \cdot \cos(\omega_{0,x} \cdot s + \phi_0) \] and \[ \omega_{0,x} = 2\pi \cdot Q_{x,0} / L \]

\[ \frac{d^2}{ds^2} x_1(s) + (2\pi Q_{x,0} / L)^2 \cdot x_1(s) = \frac{l k_1}{2L} \cdot A^2 \cdot \sum_{r=-\infty}^{\infty} \cos(2\pi \cdot r \cdot s / L) \]

\[ + \frac{l k_1}{8L} \cdot A^2 \cdot \sum_{r=-\infty}^{\infty} \cos(2\pi \cdot [r \pm 2Q_{x,0}] \cdot s / L) \]
Perturbation Treatment: Sextupole Perturbation

resonance conditions:

\[ 2\pi Q_{x,0} = 2\pi \cdot (r) \rightarrow Q_{x,0} = r \]
\[ 2\pi Q_{x,0} = 2\pi \cdot (r \pm 2Q_{x,0}) \rightarrow Q_{x,0} = r / 3 \]
\[ Q_{x,0} = r \]

avoid integer and r/3 tunes!

perturbation treatment:

contrary to the previous examples no exact solution exist!
this is a consequence of the non-linear perturbation
(remember the 3 body problem?)

graphic tools for analyzing the particle motion
**Poincare Section: Definition**

**Poincare Section:**

- Record the particle coordinates at one location in the storage ring.

**Resonance in the Poincare section: 
\[ \Delta \phi_{\text{turn}} = 2\pi \cdot Q \]

**Fixed point condition:** 
\[ Q = \frac{n}{r} \]

Points are mapped onto themselves after ‘r’ turns.
Poincare Section: Linear Motion

unperturbed solution:

\[ x(s) = \sqrt{R} \cdot \cos(\phi) \quad \text{with} \quad \frac{d}{ds} \phi = \omega_0 \]

\[ x' = \frac{d}{ds} x = -\sqrt{R} \cdot \omega_0 \cdot \sin(\phi) \]

phase space portrait:

\[ \text{the motion lies on an ellipse} \]

\[ \text{linear motion is described by a simple rotation} \]

\[ \text{consecutive intersections lie on closed curves} \]
Poincare Section: Non-Linear Motion

momentum change due to perturbation:
\[ \Delta x' = \int \frac{F(s)}{v \cdot p} \cdot ds \]

single n-pole kick:
\[ \Delta x' = \frac{1}{n!} \cdot lk_n \cdot x^n \]

phase space portrait with single sextupole:
\[ \Delta x' = \frac{1}{2} \cdot lk_2 \cdot x^2 \]

sextupole kick changes the amplitude and the phase advance per turn!
\[ \Delta Q_{\text{turn}} \propto x^2 \]
Poincare Section: Stability?

instability can be fixed by ‘detuning’:

- overall stability depends on the balance between amplitude increase per turn and tune change per turn:

  \[ \Delta Q_{\text{turn}}(x) \Rightarrow \text{motion moves eventually off resonance} \]

  \[ \Delta R_{\text{turn}}(x) \Rightarrow \text{motion becomes unstable} \]

sextupole kick:

amplitudes increase faster than the tune can change

\[ \Rightarrow \text{overall instability!} \]
Poincare Section: Illustration of Topology

- **Poincare section:**
  - $Q < r/3$
  - $F(s)/(v \cdot p) = \frac{1}{2} \cdot \delta_L (s - s_0) \cdot l k_2 \cdot x^2$

- **Small amplitudes:** ➔ regular motion
- **Large amplitudes:** ➔ instability & particle loss
- **Fixed points and separatrix:** border between stable and unstable motion ➔ chaotic motion
Poincare Section: Simulations for a Sextupole Perturbation

- Poincare Section right after the sextupole kick

- For small amplitudes, the intersections still lie on closed curves → regular motion!

- Separatrix location depends on the tune distance from the exact resonance condition \((Q < n/3)\)

- For large amplitudes and near the separatrix, the intersections fill areas in the Poincare Section → chaotic motion; → no analytical solution exists!
Stabilization of Resonances

- instability can be fixed by stronger ‘detuning’:

  \[ F(s)/(v \cdot p) = \frac{1}{6} \cdot lk_3 \cdot x^3 \]

- if the phase advance per turn changes uniformly with increasing R the motion moves off resonance and stabilizes

- octupole perturbation:

  \[ x(2) = x_0(s) + \varepsilon \cdot x_1(2) + \ldots \]

- perturbation treatment:

  \[ \frac{d^2}{ds^2} x_1(s) + \left(2\pi Q_{x,0} / L\right)^2 \cdot x_1(s) = \frac{1}{6} \cdot lk_3 \cdot x_0^2 \cdot x_1 \]

  \[ x_0 = A \cdot \cos(\omega_0 \cdot s + \phi_0) \Rightarrow x_0^2 = \frac{A^2}{2} \cdot [1 + \cos(2\omega_0 \cdot s + 2\phi_0)] \]

  \[ \frac{d^2}{ds^2} x_1(s) + \left[\left(2\pi Q_{x,0} / L\right)^2 - \frac{A^2 \cdot lk_3}{2 \cdot 6}\right] \cdot x_1(s) = \frac{A^2 \cdot lk_3}{2 \cdot 6} \cdot \cos(2\omega_0 \cdot s) \cdot x_1 \]
Stabilization of Resonances

- resonance stability for octupole:

  - an octupole perturbation generates phase independent detuning and amplitude growth of the same order

  - amplitude growth and detuning are balanced and the overall motion is stable!

  - this is not generally true in case of several resonance driving terms and coupling between the horizontal and vertical motion!
Chaotic Motion

- octupole + sextupole perturbation:

- the interference of the octupole and sextupole perturbations generate additional resonances
- additional island chains in the Poincare Section!

- intersections near the resonances lie no longer on closed curves
- local chaotic motion around the separatrix & instabilities
- slow amplitude growth (Arnold diffusion)

- neighboring resonance islands start to ‘overlap’ for large amplitudes
- global chaos & fast instabilities
Chaotic Motion

‘Russian Doll’ effect:

→ magnifying sections of the Poincare Section reveals always the same pattern on a finer scale → renormalization theory!
field imperfections drive resonances

higher order than quadrupole field imperfections generate non-linear equations of motion (no closed analytical solution)

(three body problem of Sun, Earth and Jupiter)

solutions only via perturbation treatment

Poincare Section as a graphical tool for analyzing the stability

slow extraction as example of resonance application in accelerator

island chains as signature for non-linear resonances

island overlap as indicator for globally chaotic & unstable motion