Linear

Imperfections

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Linear Imperfections

equation of motion in an accelerator

→ Hills equation

→ sine and cosine like solutions

→ closed orbit

→ sources for closed orbit perturbations

dipole perturbations

→ closed orbit response

→ dispersion orbit

→ integer resonances

→ BPMs & dipole correctors

quadrupole perturbations

→ one-turn map & tune error

→ beta-beat

→ half-integer resonances

orbit correction

→ local orbit bumps
**Variable Definition**

**Variables in moving coordinate system:**

\[ x' = \frac{d}{ds} x \]

\[ \frac{d}{dt} = \frac{ds}{dt} \cdot \frac{d}{ds} \quad \Rightarrow \quad x' = \frac{p_x}{p_0} \]

**Hill’s Equation:**

\[ \frac{d^2 x}{ds^2} + K(s) \cdot x = 0; \quad K(s) = K(s + L); \]

\[ K(s) = \begin{cases} 
0 \\
1/\rho^2 & \text{drift} \\
0.3 \cdot \frac{B[T/m]}{\rho[GeV]} & \text{dipole} \\
0.3 \cdot \frac{B[T/m]}{\rho[GeV]} & \text{quadrupole} 
\end{cases} \]

**Perturbations:**

\[ \frac{d^2 x}{ds^2} + K(s) \cdot x = G(s); \quad G(s) = \frac{F(s)_{\text{Lorentz}}}{v \cdot p_0} \]
Sinelike and Cosinelike Solutions

system of first order linear differential equations:

\[
\begin{align*}
\mathbf{\dot{y}} &= \begin{pmatrix} x \\ x' \end{pmatrix} \\
\mathbf{y}' + \begin{pmatrix} 0 & 1 \\ K & 0 \end{pmatrix} \cdot \mathbf{y} &= 0
\end{align*}
\]

\(K = \text{const}\)

\[
\begin{align*}
\mathbf{\dot{Y}}_1(s) &= \begin{pmatrix} \sin (\sqrt{K} \cdot s) \\ \sqrt{K} \cdot \cos (\sqrt{K} \cdot s) \end{pmatrix} \\
\mathbf{\dot{Y}}_2(s) &= \begin{pmatrix} \cos (\sqrt{K} \cdot s) \\ -\sqrt{K} \cdot \sin (\sqrt{K} \cdot s) \end{pmatrix}
\end{align*}
\]

initial conditions:

\[
\begin{align*}
\mathbf{\dot{Y}}_1(0) &= \begin{pmatrix} Y_1 \\ Y'_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\mathbf{\dot{Y}}_2(0) &= \begin{pmatrix} Y_2 \\ Y'_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\end{align*}
\]

general solution:

\[
\mathbf{\dot{y}}(s) = a \cdot \mathbf{\dot{Y}}_1(s) + b \cdot \mathbf{\dot{Y}}_2(s)
\]

transport map:

\[
\mathbf{\dot{y}}(s) = \mathbf{M}(s - s_0) \cdot \mathbf{\dot{y}}(s_0)
\]

with:

\[
\begin{pmatrix}
\cos (\sqrt{K} \cdot [s-s_0]) & \sin (\sqrt{K} \cdot [s-s_0]) \\
-\sqrt{K} \cdot \sin (\sqrt{K} \cdot [s-s_0]) & \sqrt{K} \cdot \cos (\sqrt{K} \cdot [s-s_0])
\end{pmatrix}
\]
**Sinelike and Cosinelike Solutions**

Floquet theorem:

\[
\vec{Y}_1 (s) = \begin{pmatrix} \sqrt{\beta(s)} \cdot \sin (\phi(s) + \phi_0) \\ [\cos (\phi(s) + \phi_0) + \alpha(s) \cdot \sin(\phi(s) + \phi_0)] / \sqrt{\beta(s)} \end{pmatrix}
\]

\[
\vec{Y}_2 (s) = \begin{pmatrix} \sqrt{\beta(s)} \cdot \cos (\phi(s) + \phi_0) \\ -[\sin (\phi(s) + \phi_0) + \alpha(s) \cdot \cos(\phi(s) + \phi_0)] / \sqrt{\beta(s)} \end{pmatrix}
\]

\[
\beta (s) = \beta (s + L); \quad \phi (s) = \int \frac{1}{\beta} ds; \quad \alpha (s) = -\frac{1}{2} \beta^1 (s)
\]

´sinelike´ and ´cosinelike´ solutions:

\[
\vec{C} (s) = a \cdot \vec{Y}_1 (s) + b \cdot \vec{Y}_2 (s) \quad \vec{S}(s) = c \cdot \vec{Y}_1 (s) + d \cdot \vec{Y}_2 (s)
\]

with: \(\vec{C}(s_0) = \begin{pmatrix} C(s_0) \\ C^\dagger(s_0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\) and \(\vec{S}(s_0) = \begin{pmatrix} S(s_0) \\ S^\dagger(s_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\)

one can generate a transport matrix in analogy to the case with constant K(s)!
\[ \vec{S}(s) = \begin{pmatrix} \sqrt{\beta(s)}{\beta}(s_0) \cdot \sin(\phi(s) + \phi_0) \\ \sqrt{\beta(s)} \cdot [\cos(\phi(s) + \phi_0) + \alpha(s) \cdot \sin(\phi(s) + \phi_0)] / \sqrt{\beta(s)} \end{pmatrix} \]

\[ \vec{C}(s) = \begin{pmatrix} \sqrt{\beta(s)} \cdot [\cos(\phi(s) + \phi_0) + \alpha(s_0) \cdot \sin(\phi(s) + \phi_0)] / \sqrt{\beta(s_0)} \\ -(1 + \alpha \alpha_0) \cdot [\sin(\phi(s) + \phi_0) + (\alpha_0 - \alpha) \cdot \cos(\phi(s) + \phi_0)] / \sqrt{\beta(s_0)} \end{pmatrix} \]

transport map from \( s_0 \) to \( s \): \( \vec{y}(s) = M(s, s_0) \cdot \vec{y}(s_0) \)

with: \( M = \begin{pmatrix} C(s) & S(s) \\ C^t(s) & S^t(s) \end{pmatrix} \)

transport map for \( s = s_0 + L \):

\[ M = I \cdot \cos(2\pi \ Q) + J \cdot \sin(2\pi \ Q) \]

\[ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \]

\( \gamma = [1 + \alpha^2] / \beta \)
Closed Orbit

Particles oscillate around an ideal orbit:

Additional dipole fields perturb the orbit:

- Error in dipole field
- Energy error
- Offset in quadrupole field

\[ B_x = g \cdot y \]
\[ B_y = g \cdot x \]

\[ x = x_0 + \tilde{x} \]
\[ \tilde{B_y} = g \cdot x_0 + g \cdot \tilde{x} \]
**Quadrupole Magnet**

\[ B_x = g \cdot y \]
\[ B_y = g \cdot x \]

\[ F_x = -q \cdot v \cdot B_y \]
\[ F_y = q \cdot v \cdot B_x \]

\[ \frac{d^2 x}{ds^2} + K(s) \cdot x = G(s); \quad G(s) = \frac{F(s)}{v \cdot p_0} \]

**Normalized fields:**

**Dipole:**

\[ k_0(s) = 0.3 \cdot \frac{B_0[T]}{p_0[GeV]} \]

**Quadrupole:**

\[ k_1(s) = 0.3 \cdot \frac{g[T/m]}{p_0[GeV]} \]

Quadrupole misalignment:

\[ \Delta k_0(s) = 0.3 \cdot \frac{g[T/m]}{p[GeV]} \cdot x_0 \]
**Dipole Error and Orbit Stability**

- **Q**: number of $\beta$-oscillations per turn
  
  - $Q = N$

  - The perturbation adds up

  - Amplitude growth and particle loss

  - Watch out for integer tunes!

  - $Q = N + 0.5$

  - The perturbation cancels after each turn
Quadrupole Error and Orbit Stability

Quadrupole Error:

-orbit kick proportional to beam offset in quadrupole

Q = N + 0.5

1. Turn: x > 0

-amplitude increase

2. Turn: x < 0

-amplitude increase

-watch out for half integer tunes!
Sources for Orbit Errors

- Quadrupole offset:
  - alignment $\pm 0.1$ mm
  - ground motion
    - slow drift
    - civilisation
    - moon
    - seasons
    - civil engineering

- Error in dipole strength
  - power supplies
  - calibration

- Energy error of particles
  - injection energy (RF off)
  - RF frequency
  - momentum distribution
Example Quadrupole Alignment in LEP

Transversal tilt dispersion of the 3278 dipoles
\[ \sigma = \pm 0.34 \text{ mrad} \]

Vertical dispersion of the 784 quadrupoles
(with respect to the smoothing polynomial)
\[ \sigma = \pm 0.65 \text{ mm} \]

Figure 1: observed status, end 1992

Figure 2: progressive corrections in 1993 and 1994
Problems Generated by Orbit Errors

**injection errors:**
- aperture → beam losses
- filamentation → beam size

**closed orbit errors:**
- x–y coupling
- aperture
- energy error
- field imperfections
- dispersion → beam size at IP
- beam separation

**Aim:**
\[ \Delta x, \Delta y < 4 \text{ mm} \]
\[ \text{rms} < 0.5 \text{ mm} \]
beam monitors and orbit correctors
**Synchrotron:**

The orbit determines the particle energy!

Assume: \( L > \text{design orbit} \)

Equilibrium:

\[
\begin{align*}
\text{f}_{RF} &= h \cdot \text{f}_{rev} \\
\text{f}_{rev} &= \frac{1}{2\pi} \cdot \frac{q}{m \cdot \gamma} \cdot B
\end{align*}
\]

\( E \) depends on orbit and magnetic field!
**Tidal motion of the Earth:**

Moon

Earth

**Orbit and beam energy modulation:**

\[ f_{\text{mod}} = 24 \text{ h}; 12 \text{ h} \]

\[ \Delta E \approx 10 \text{ MeV} \]

\[ \approx 0.02\% \]

**Aim:** \[ \Delta E \lesssim 0.003\% \]

---

**Requires correction!**
energy modulation due to tidal motion of earth

ΔE \approx 10\ MeV
energy modulation due to lake level changes
changes in the water level of lake Geneva change the position of the LEP tunnel and thus the quadrupole positions

orbit and energy perturbations

ΔE  \approx 20 \text{ MeV}
energy modulation due current perturbations in the main dipole magnets

TGV line between Geneva and Bellegarde
correlation of NMR dipole field measurements with the voltage on the TGC train tracks

ΔE ∼ 5 MeV for LEP operation at 45 GeV
ground motion due to human activity
quadrupole motion in HERA–p (DESY Hamburg)

RMS

peak to peak
Closed Orbit Response

inhomogeneous equation:

\[
\frac{d^2 x}{d s^2} + K(s) \cdot x = G(s); \quad G(s) = \Delta k_0 (s)
\]

\[
\begin{pmatrix}
0 & 1 \\
K & 0
\end{pmatrix} \cdot \vec{y} = \vec{G}; \quad \vec{G} = \begin{pmatrix} 0 \\ G \end{pmatrix}
\]

\[
\vec{y}(s) = a \cdot \vec{S}(s) + b \cdot \vec{C}(s) + \vec{\psi}(s)
\]

we need to find only one solution!

variation of the constant:

\[
\vec{\psi}(s) = c(s) \cdot \vec{S}(s) + d(s) \cdot \vec{C}(s)
\]
Closed Orbit Response

variation of the constant in matrix form:

\[ \vec{\psi}(s) = \phi(s) \cdot \vec{u}(s); \text{ with} \]

\[ \phi(s) = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \]

substitute into differential equation:

\[ \phi(s) \cdot \vec{u}'(s) = \vec{G}(s) \]

\[ \vec{u}(s) = \int_{s_0}^{s} \phi(t)^{-1} \cdot \vec{G}(t) \, dt \]

\[ \vec{y}(s) = a \cdot \vec{S}(s) + b \cdot \vec{C}(s) + \phi(s) \cdot \int_{s_0}^{s} \phi(t)^{-1} \cdot \vec{G}(t) \, dt \]
Closed Orbit Response

**periodic boundary conditions:**

\[
\vec{y}(s) = a \cdot \vec{S}(s) + b \cdot \vec{C}(s) + \phi(s) \cdot \int_{s_0}^{s} \phi(t)^{-1} \cdot \vec{G}(t) \, dt
\]

with

\[
\vec{y}(s) = \begin{pmatrix} x(s) \\ x'(s) \end{pmatrix}; \quad x(s) = x(s + L); \quad x'(s) = x'(s + L)
\]

**periodic boundary conditions determine coefficients** \( a \) and \( b \)

\[
x(s) = \frac{-\sqrt{\beta(s)}}{2 \sin(\pi \cdot Q)} \cdot \int_{s_0}^{s_0 + \text{circ}} \sqrt{\beta(t)} \cdot G(t) \cos[\phi(t) - \phi(s) - \pi \cdot Q] \, dt
\]
Closed Orbit Response

**Example:** particle momentum error

normalized dipole strength: \[ k_0(s) = 0.3 \cdot \frac{B[T]}{p[GeV]} \]

\[
k_0(s) = \frac{1}{\rho(t)} - \frac{1}{\rho(t)} \cdot \frac{\Delta p}{p_0} \quad \rightarrow \quad G(t) = \frac{1}{\rho(t)} \cdot \frac{\Delta p}{p_0}
\]

\[
x(s) = \frac{\sqrt{\beta(s)}}{2 \sin(\pi \cdot Q)} \cdot \int \sqrt{\beta(t)} \cdot G(t) \cos[\phi(t) - \phi(s)] - \pi \ Q \ dt
\]

\[
\rightarrow \quad x(s) = D(s) \cdot \frac{\Delta p}{p}
\]

with

\[
D(s) = \frac{\sqrt{\beta(s)}}{2 \sin(\pi \cdot Q)} \cdot \int \frac{\sqrt{\beta(t)}}{\rho(t)} \cdot \cos[\phi(t) - \phi(s)] - \pi \ Q \ dt
\]

Dispersion Orbit
the orbit error in a storage ring with conventional magnets is dominated by the contributions from the quadrupole alignment errors.

orbit perturbation is proportional to the local $\beta$-functions at the location of the dipole error.

alignment errors at QF cause mainly horizontal orbit errors.

alignment errors at QD causes mainly vertical orbit errors.
aim at a local correction of the dipole error due to the quadrupole alignment errors

place orbit corrector and BPM next to the main quadrupoles

horizontal BPM and corrector next to QF
vertical BPM and corrector next to QD

orbit in the opposite plane?

relative alignment of BPM and quadrupole?
**LEP Orbit**

**Horizontal Orbit:**
- beam offset in quadrupoles:
  - Lake Geneva
  - moon
- energy error

**Vertical Orbit:**
- beam offset in quadrupoles
- beam separation
- orbit deflection depends on particle energy
- vertical dispersion \([D(s)]\)

\[
\sigma_y = \sqrt{\varepsilon \beta_y + \delta_y^2 D^2}
\]

- small vertical beam size relies on good orbit

- 1994: 13000 vertical orbit corrections in physics
Quadrupole Gradient Error

**one turn map:**

can be generated by matrix multiplication:

\[
\vec{z}_{n+1} = M \cdot \vec{z}_n
\]

\[
\vec{z} = \begin{pmatrix} x \\ x' \end{pmatrix}
\]

and can be expressed in terms of the C and S solutions

\[
M = I \cdot \cos(2\pi Q) + J \cdot \sin(2\pi Q)
\]

\[
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}; \quad \gamma = [1 + \alpha^2] / \beta
\]

remember:

\[
\cos(2\pi Q) = \frac{1}{2} \text{ trace } M
\]

the coefficients of:

\[
\frac{M - I \cdot \cos(2\pi Q)}{\sin(2\pi Q)}
\]

provide the optic functions at \(s_0\)
Quadrupole Gradient Error

Transfer matrix for single quadrupole:

\[ m_0 = \begin{pmatrix} 1 & 0 \\ -k_1 & 1 \end{pmatrix} \]

Matrix for single quadrupole with error:

\[ m = \begin{pmatrix} 1 & 0 \\ -[k_1 + \Delta k_1] & 1 \end{pmatrix} \]

One turn matrix with quadrupole error:

\[ M = m \cdot m_0^{-1} \cdot M_0 \]

Trace \( M \)

\[ \cos(2\pi Q) = \cos(2\pi Q_0) \cdot \frac{1}{2} \beta \Delta k_1 \cdot \sin(2\pi Q_0) \]
**Quadrupole Gradient Error**

**distributed perturbation:**

\[
\cos(2\pi Q) = \cos(2\pi Q_0) - \frac{\sin(2\pi Q_0)}{2} \cdot \oint \beta \cdot \Delta k_1 ds
\]

\[
\Delta Q = \frac{1}{4\pi} \cdot \oint \beta \cdot \Delta k_1 ds
\]

**chromaticity:**

\[
k_1 = \frac{e \cdot g}{p}
\]

**momentum error**

\[
\Delta k_1 = -k_1 \cdot \frac{\Delta p}{p}
\]

\[
\Delta Q = -\frac{1}{4\pi} \cdot \oint \beta \cdot k_1 ds \cdot \frac{\Delta p}{p}
\]

\[
= \xi \cdot \frac{\Delta p}{p}
\]
**quadrupole error:**

\[ \vec{z}_{n+1} = M \cdot \vec{z}_n \]

with

\[ M = I \cdot \cos(2\pi Q) + J \cdot \sin(2\pi Q) \]

\[ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}; \quad \gamma = [1 + \alpha^2] / \beta \]

\[ \Delta \beta(s) = \frac{\beta(s)}{2 \sin(2\pi Q)} \int_{s_0}^{s_0+\text{circ}} \beta(t) \cdot \Delta k(t) \cos[2\phi(t) - \phi(s)] - 2\pi Q \, dt \]

**\( \beta - \text{Beat} \)**

\( \beta - \text{beat oscillates with twice the betatron frequency} \)
**Local Orbit Bumps I**

**Deflection Angle:**

\[
\theta_i = \int G_i(t) \, dt = \frac{0.3 \cdot B_i [T] \cdot l}{p [\text{GeV}]}
\]

**Trajectory Response:**

[no periodic boundary conditions]

\[
\begin{align*}
  x(s) &= \sqrt{\beta_i / \beta(s)} \cdot \theta_i \cdot \sin[\phi(s) - \phi_i] \\
  x^l(s) &= \sqrt{\beta_i / \beta(s)} \cdot \theta_i \cdot \cos[\phi(s) - \phi_i]
\end{align*}
\]
Local Orbit Bumps II

closed orbit bump:

compensate the trajectory perturbation with additional corrector kicks further down stream

closure of the perturbation within one turn

local orbit excursion

possibility to correct orbit errors locally

closure with one additional corrector magnet

$\pi$ - bump

closure with two additional corrector magnets

three corrector bump
Local Orbit Bumps III

π - bump: (quasi local correction of error)

\[ \theta_2 = \frac{\sqrt{\beta_1}}{\sqrt{\beta_2}} \cdot \theta_1 \]

limits / problems:

- closure depends on lattice phase advance
- requires 90° lattice
- sensitive to lattice errors
- requires horizontal BPMs at QF and QD
- sensitive to BPM errors
- requires large number of correctors
3 corrector bump: (quasi local correction of error)

Local Orbit Bumps IV

\[ \theta_2 = -\frac{\sqrt{\beta_1}}{\sqrt{\beta_2}} \cdot \frac{\sin(\Delta \phi_{3-1})}{\sin(\Delta \phi_{3-2})} \cdot \theta_1 \]

\[ \theta_3 = \left( \frac{\sin(\Delta \phi_{3-1})}{\tan(\Delta \phi_{3-2})} - \cos(\Delta \phi_{3-1}) \right) \cdot \frac{\sqrt{\beta_1}}{\sqrt{\beta_3}} \cdot \theta_1 \]

- works for any lattice phase advance
- requires only horizontal BPMs at QF

limits / problems:

- sensitive to BPM errors
- large number of correctors
- cannot control $x'$
Summary Linear Imperfections

Avoid machine tunes near integer resonances:

- They amplify the response to dipole field errors
- A closed orbit perturbation propagates with the betatron phase around the storage ring
- Discontinuities in the derivative of the closed orbit response at the location of the perturbation

Avoid storage ring tunes near half-integer resonances:

- They amplify the response to quadrupole field errors
- Betafunction perturbations propagate with twice the betatron phase advance around the storage ring

Integral expressions are mainly used for estimates, numerical programs mainly rely on maps

Closed orbit = fixed point of '1-turn' map
Dispersion = eigenvector of extended '1-turn' map
Tune is given by the trace of the '1-turn' map
Twiss functions are given by the matrix elements