The CERN Accelerator School

Advanced Accelerator Physics Course

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Low Emittance Machines

Part 1: Beam Dynamics with Synchrotron Radiation

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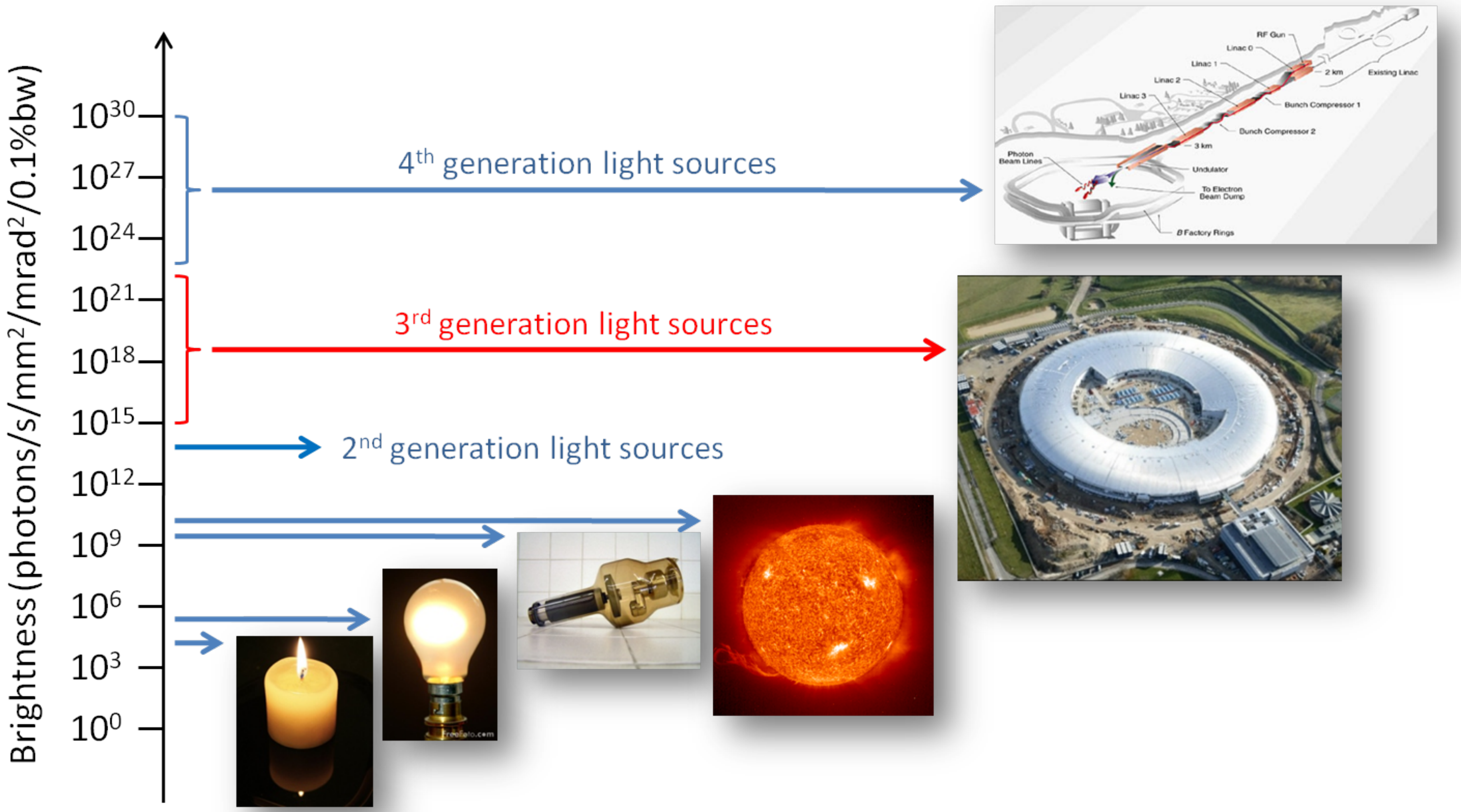
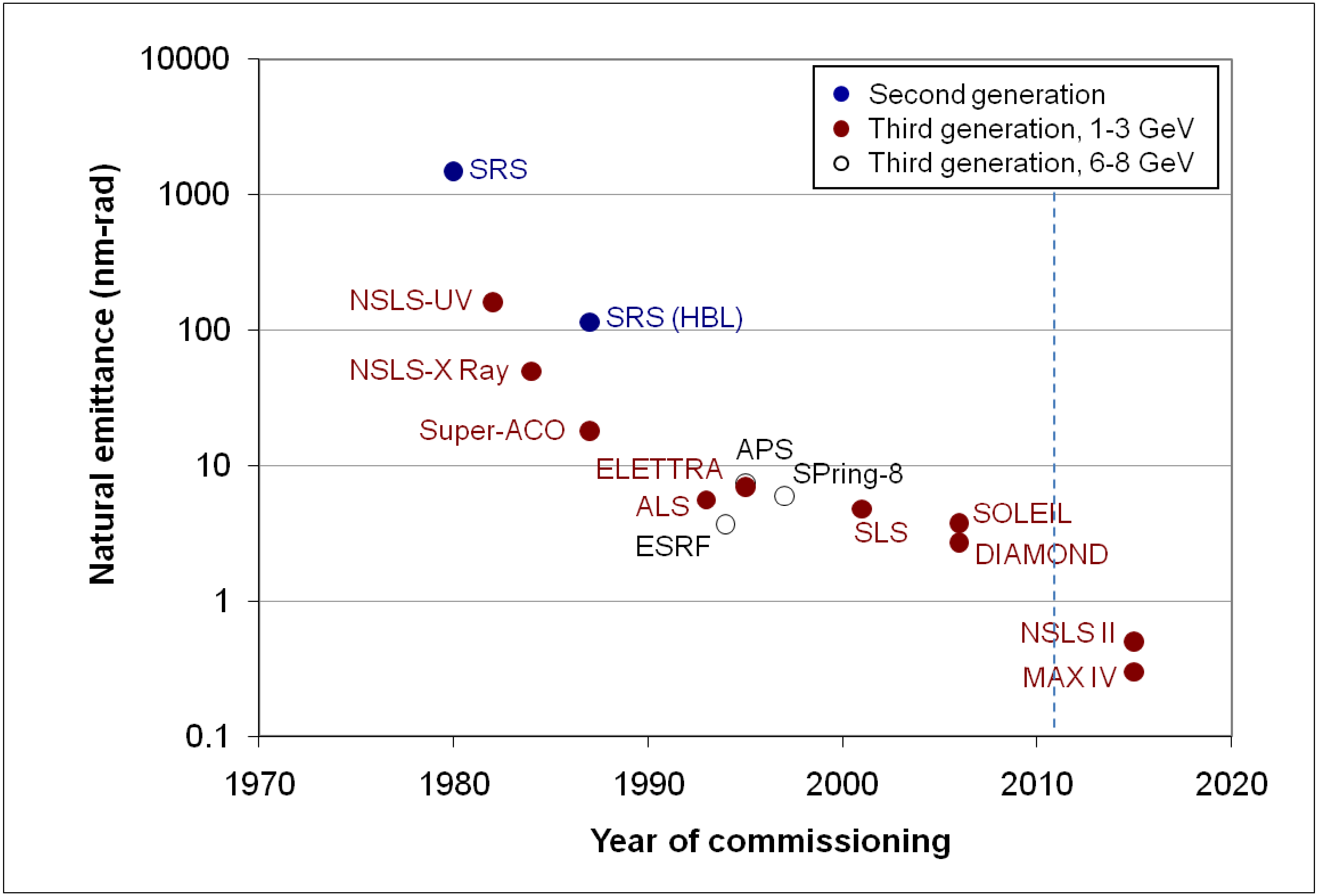
Brightness is a key figure of merit for SR sources



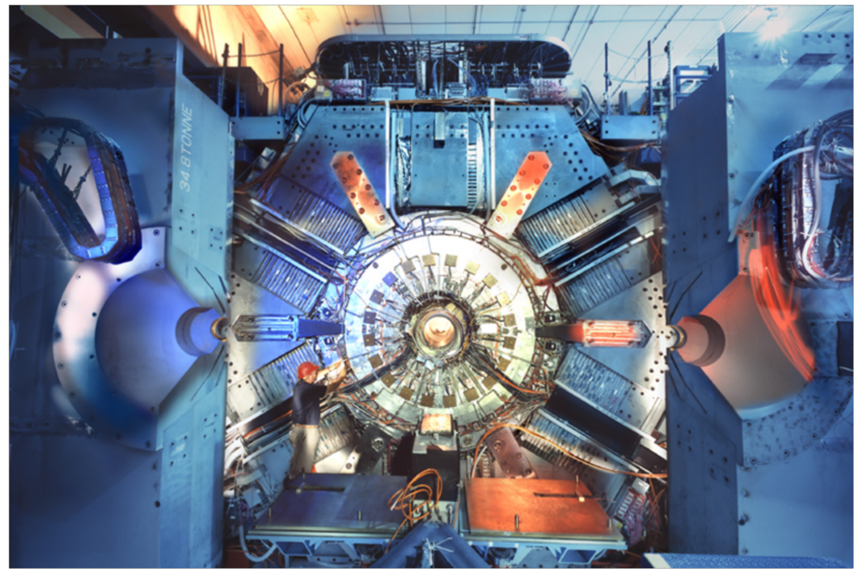
High photon brightness needs low electron beam emittance



Luminosity is a key figure of merit for colliders. The luminosity depends directly on the horizontal and vertical emittances.

$$\mathcal{L} = \frac{N_+ N_- f}{2\pi \Sigma_x \Sigma_y}$$

$$\Sigma_{x,y} = \sqrt{\sigma^{*2}_{x,y+} + \sigma^{*2}_{x,y-}}$$



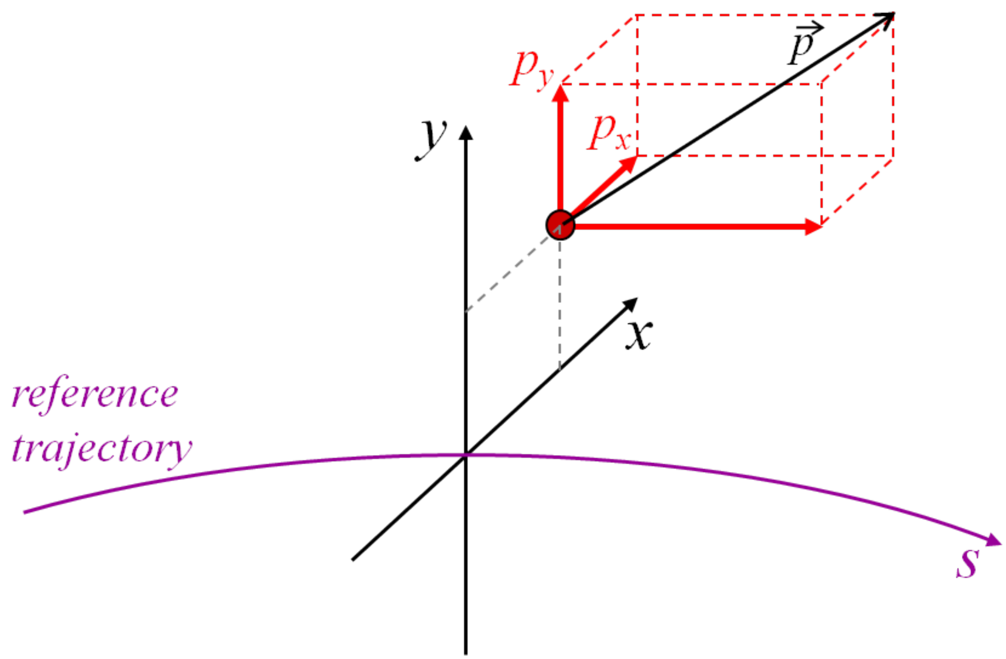
Dynamical effects associated with the collisions mean that it is sometimes helpful to *increase* the horizontal emittance; but generally, reducing the vertical emittance as far as possible helps to increase the luminosity.

1. Beam dynamics with synchrotron radiation
    - Effects of synchrotron radiation on particle motion.
    - The synchrotron radiation integrals.
    - Damping times of the beam emittances.
    - Quantum excitation and equilibrium emittances.
2. Equilibrium emittance and storage ring lattice design
    - Natural emittance in different lattice styles.
    - Achromats and "quasi-achromats".
3. Vertical emittance and coupling
    - Sources of vertical emittance.
    - Emittance computation in coupled storage rings.
    - Low emittance tuning.

In this lecture, we shall:

- define action-angle variables for describing symplectic motion of a particle along a beam line;
- discuss the effect of synchrotron radiation on the (linear) motion of particles in storage rings;
- derive expressions for the damping times of the vertical, horizontal, and longitudinal emittances;
- discuss the effects of quantum excitation, and derive expressions for the equilibrium horizontal and longitudinal beam emittances in an electron storage ring.

We work in a Cartesian coordinate system, with a *reference trajectory* that we define for our own convenience:



In general, the reference trajectory can be curved. At any point along the reference trajectory, the x and y coordinates are perpendicular to the reference trajectory.

The transverse momenta are the *canonical momenta*, normalised by a *reference momentum*, P_0:

$$p_x = \frac{1}{P_0}\left(\gamma m \frac{dx}{dt} + qA_x\right), \qquad (1)$$

and similarly for p_y.

m and q are the mass and charge of the particle, γ is the relativistic factor, and A_x is the x component of the vector potential.

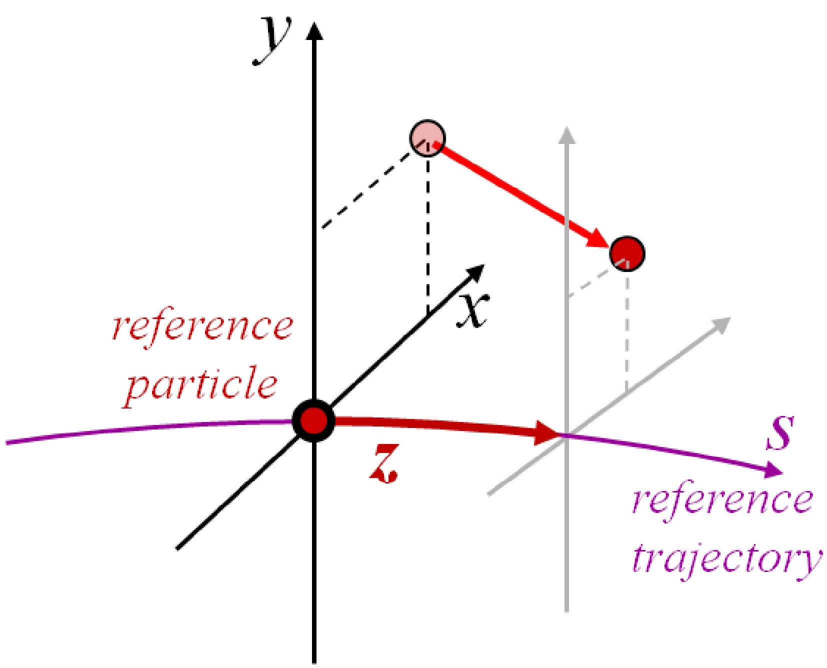
We can choose the reference momentum for our own convenience; usually, we choose P_0 to be equal to the nominal momentum for a particle moving along the beam line.

The transverse dynamics are described by giving the transverse coordinates and momenta as functions of s (the distance along the reference trajectory).

The longitudinal coordinate of a particle is defined by:

$$z = \beta_0 c(t_0 - t), \qquad (2)$$

where β_0 is the normalized velocity of a particle with the reference momentum P_0, t_0 is the time at which the reference particle is at a location s, and t is the time at which the particle of interest arrives at this location.



z is *approximately* the distance along the reference trajectory that a particle is ahead of a reference particle travelling along the reference trajectory with momentum P_0.

The final dynamical variable needed to describe the motion of a particle is the energy of the particle.

Rather than use the absolute energy or momentum, we use the *energy deviation* δ.

The energy deviation is a measure of the difference between the energy of a particle and the energy of a particle with the reference momentum P_0:

$$\delta = \frac{E}{P_0 c} - \frac{1}{\beta_0} = \frac{1}{\beta_0}\left(\frac{\gamma}{\gamma_0} - 1\right), \tag{3}$$

where E is the energy of the particle, and β_0 is the normalized velocity of a particle with the reference momentum P_0.

Note that for a particle whose momentum is equal to the reference momentum, $\gamma = \gamma_0$, and hence $\delta = 0$.

Using the definitions on the previous slides, the coordinates and momenta form *canonical conjugate pairs*:

$$(x, p_x), \qquad (y, p_y), \qquad (z, \delta). \tag{4}$$

The change in the values of the variables when a particle moves along a beam line can be represented by a transfer matrix, M:

$$\begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix}_{s=s_1} = M(s_1; s_0) \cdot \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix}_{s=s_0} \tag{5}$$

Using canonical variables, neglecting radiation effects, *the matrix M is symplectic*.

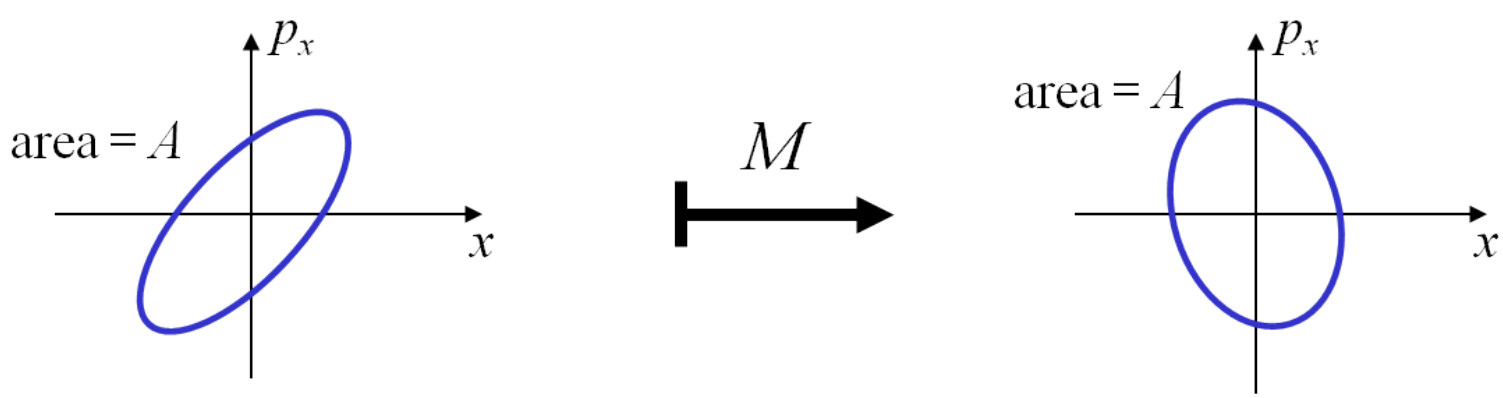
Mathematically, a matrix M is symplectic if it satisfies the relation:

$$M^{\mathsf{T}} \cdot S \cdot M = S, \tag{6}$$

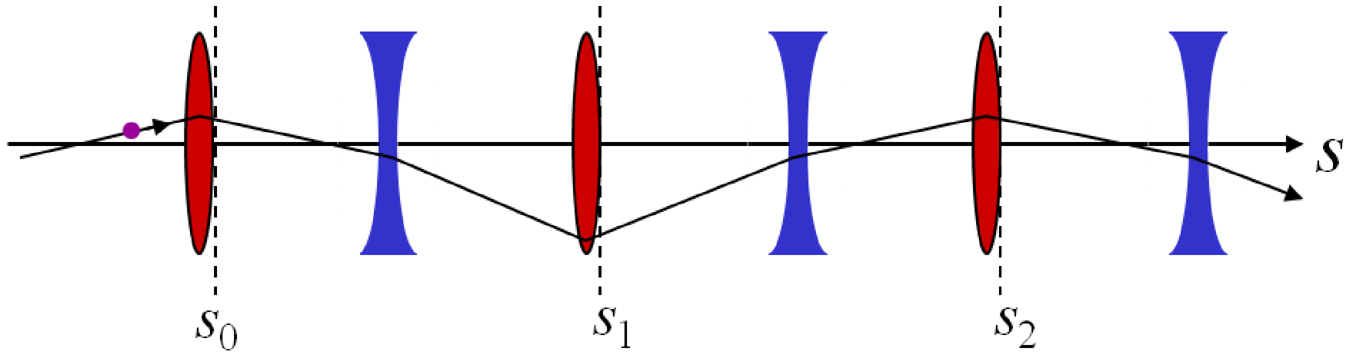
where S is the antisymmetric matrix:

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{7}$$

Physically, symplectic matrices preserve areas in phase space. For example, in one degree of freedom:



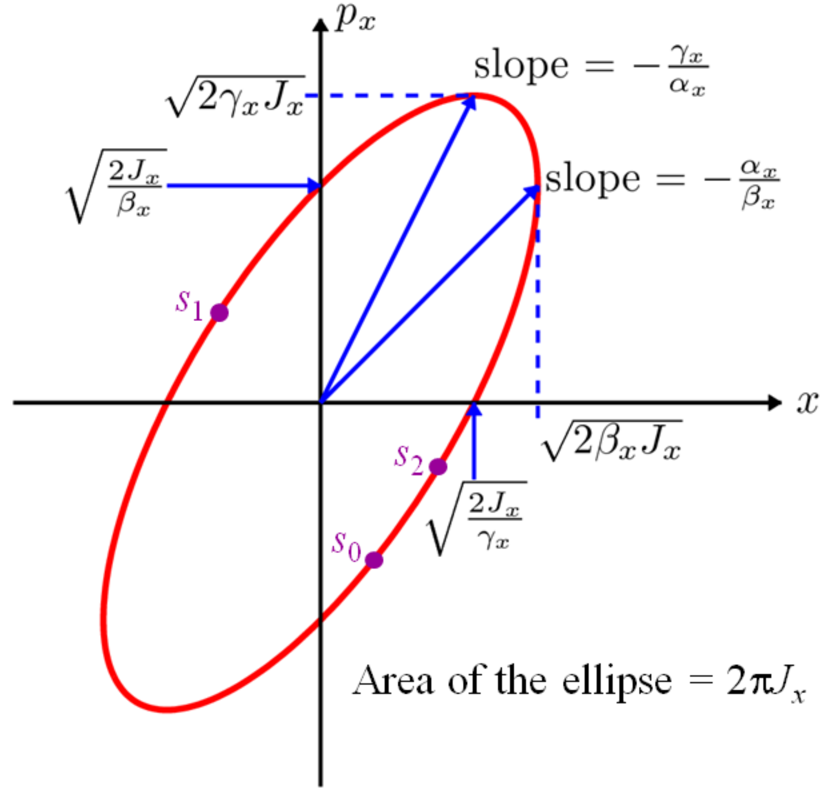
Courant–Snyder parameters and the particle action



In an uncoupled periodic beam line, particles trace out ellipses in phase space with each pass through a periodic cell.

The shape of the ellipse defines the *Courant–Snyder parameters* at the observation point.

The area of the ellipse defines the *action* J_x of the particle.



Applying simple geometry to the phase space ellipse, we find that the action (for uncoupled motion) is related to the Cartesian variables for the particle by:

$$2J_x = \gamma_x x^2 + 2\alpha_x x p_x + \beta_x p_x^2. \tag{8}$$

We also define the *angle* variable ϕ_x as follows:

$$\tan\phi_x = -\beta_x \frac{p_x}{x} - \alpha_x. \tag{9}$$

The action-angle variables form a canonical conjugate pair, and provide an alternative to the Cartesian variables for describing the dynamics.

The advantage of action-angle variables is that, under symplectic transport, the action of a particle is constant.

The action J_x is a variable that is used to describe the amplitude of the motion of a single particle.

In terms of the action-angle variables, the Cartesian coordinate and momentum can be written:

$$x = \sqrt{2\beta_x J_x} \cos\phi_x, \qquad p_x = -\sqrt{\frac{2J_x}{\beta_x}} \left(\sin\phi_x + \alpha_x \cos\phi_x \right). \quad (10)$$

The emittance ε_x is the average action of all particles in a bunch:

$$\varepsilon_x = \langle J_x \rangle. \quad (11)$$

For *uncoupled* motion, it follows that the second-order moments of the particle distribution are related to the Courant–Snyder parameters and the emittance:

$$\langle x^2 \rangle = \beta_x \varepsilon_x, \qquad \langle x p_x \rangle = -\alpha_x \varepsilon_x, \qquad \langle p_x^2 \rangle = \gamma_x \varepsilon_x. \quad (12)$$

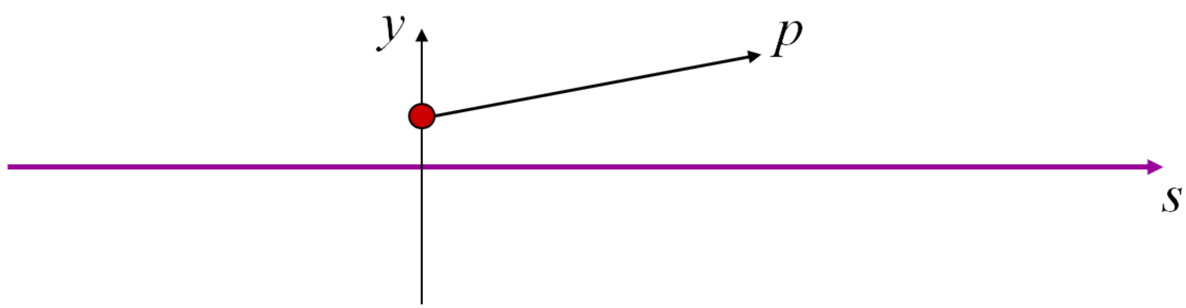
So far, we have considered only symplectic transport, i.e. motion of a particle in the electromagnetic fields of drifts, dipoles, quadrupoles, etc. without any radiation.

However, we know that a charged particle moving through an electromagnetic field will (in general) undergo acceleration, and a charged particle undergoing acceleration will radiate electromagnetic waves.

What impact will the radiation have on the motion of a particle?

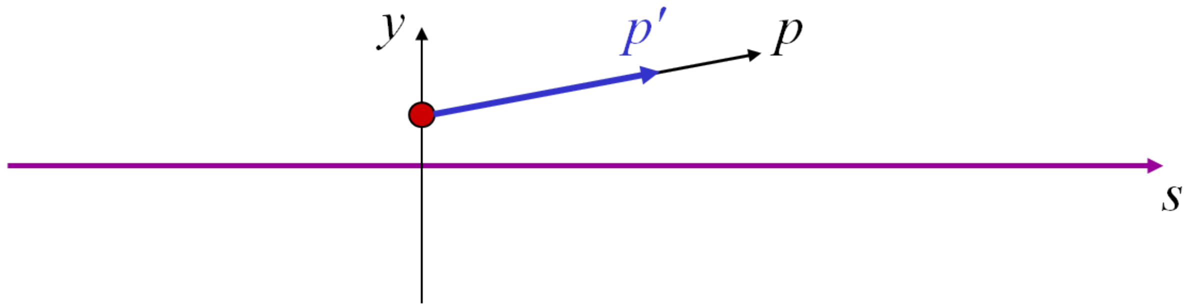
In answering this question, we wil consider first the case of uncoupled vertical motion: for a particle in a storage ring, this turns out to be the simplest case.

Radiation damping of vertical emittance



The radiation emitted by a relativistic particle has an opening angle of $1/\gamma$, where γ is the relativistic factor for the particle.

For an ultra-relativistic particle, $\gamma \gg 1$, and we can assume that the radiation is emitted directly along the instantaneous direction of motion of the particle.



The momentum of the particle after emitting radiation is:

$$p' = p - dp \approx p \left(1 - \frac{dp}{P_0} \right), \tag{13}$$

where dp is the momentum carried by the radiation, and we assume that:

$$p \approx P_0. \tag{14}$$

Since there is no change in direction of the particle, we must have:

$$p'_y \approx p_y \left(1 - \frac{dp}{P_0} \right). \tag{15}$$

After emission of radiation, the vertical momentum of the particle is:

$$p'_y \approx p_y \left(1 - \frac{dp}{P_0}\right). \tag{16}$$

Now we substitute this into the expression for the vertical betatron action (valid for *uncoupled* motion):

$$2J_y = \gamma_y y^2 + 2\alpha_y y p_y + \beta_y p_y^2, \tag{17}$$

to find the change in the action resulting from the emission of radiation:

$$dJ_y = -\left(\alpha_y y p_y + \beta_y p_y^2\right) \frac{dp}{P_0}. \tag{18}$$

Then, we average over all particles in the beam, to find:

$$\langle dJ_y \rangle = d\varepsilon_y = -\varepsilon_y \frac{dp}{P_0}, \tag{19}$$

where we have used:

$$\langle y p_y \rangle = -\alpha_y \varepsilon_y, \qquad \langle p_y^2 \rangle = \gamma_y \varepsilon_y, \quad \text{and} \quad \beta_y \gamma_y - \alpha_y^2 = 1. \tag{20}$$

The emittance is conserved under symplectic transport, so if the effects of radiation are "slow", for a particle in a storage ring we can average the momentum loss around the ring.

Then, from (19):

$$\frac{d\varepsilon_y}{dt} = -\frac{\varepsilon_y}{T_0}\oint \frac{dp}{P_0} \approx -\frac{U_0}{E_0 T_0}\varepsilon_y = -\frac{2}{\tau_y}\varepsilon_y, \tag{21}$$

where T_0 is the revolution period, and U_0 is the energy loss in one turn. The approximation is valid for an ultra-relativistic particle, which has $E \approx pc$.

We define the damping time τ_y:

$$\tau_y = 2\frac{E_0}{U_0}T_0. \tag{22}$$

The evolution of the emittance is given by:

$$\varepsilon_y(t) = \varepsilon_y(0)\exp\left(-2\frac{t}{\tau_y}\right). \tag{23}$$

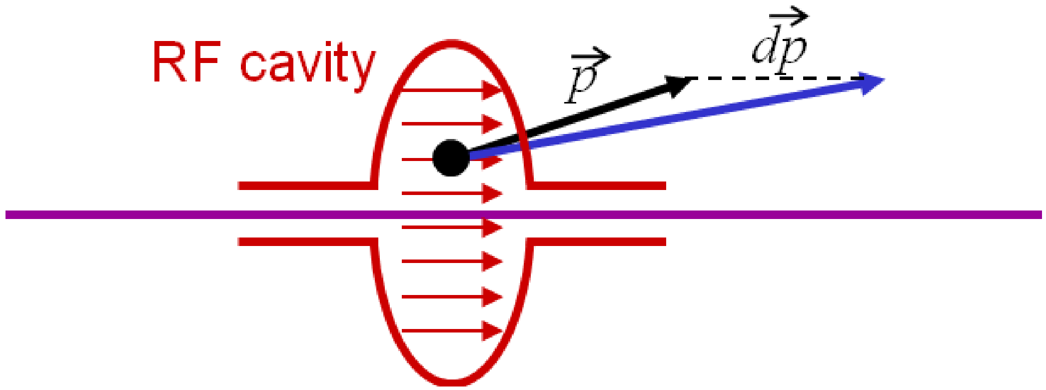
Typically, in an electron storage ring, the damping time is of order several tens of milliseconds, while the revolution period is of order of a microsecond.

Therefore, radiation effects are indeed "slow" compared to the revolution frequency.

But note that we made the assumption that the momentum of the particle was close to the reference momentum, i.e. $p \approx P_0$.

If the particle continues to radiate without any restoration of energy, we will reach a point where this assumption is no longer valid.

However, electron storage rings contain RF cavities to restore the energy lost through synchrotron radiation. But then, we should consider the change in momentum of a particle as it moves through an RF cavity.



Fortunately, RF cavities are usually designed to provide a longitudinal electric field.

This means that particles experience a change in longitudinal momentum as they pass through a cavity, without any change in transverse momentum.

Therefore, we do not have to consider explicitly the effects of RF cavities on the emittance of the beam.

To complete our calculation of the the vertical damping time, we need to find the energy lost by a particle through synchrotron radiation on each turn through the storage ring.

The radiation power from a relativistic particle following a circular trajectory of radius ρ is given by Liénard's formula:

$$P_\gamma = \frac{e^2 c}{6\pi\varepsilon_0}\frac{\beta^4\gamma^4}{\rho^2} \approx \frac{C_\gamma c}{2\pi}\frac{E^4}{\rho^2}, \tag{24}$$

where the particle has charge e, velocity $\beta c \approx c$, and energy $E = \gamma mc^2$.

C_γ is a physical constant given by:

$$C_\gamma = \frac{e^2}{3\varepsilon_0 (mc^2)^4} \approx 8.846 \times 10^{-5}\,\mathrm{m/GeV^3}. \tag{25}$$

For a particle with the reference energy, travelling at (close to) the speed of light along the reference trajectory, we can find the energy loss by integrating the radiation power around the ring:

$$U_0 = \oint P_\gamma \, dt = \oint P_\gamma \frac{ds}{c}. \tag{26}$$

Using the expression (24) for P_γ, we find:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 \oint \frac{1}{\rho^2} \, ds, \tag{27}$$

where ρ is the radius of curvature of the particle trajectory, and we assume that the particle energy is equal to the reference energy E_0.

For convenience, we define the reference trajectory to be the closed orbit for a particle with the reference momentum.

Following convention, we define the *second synchrotron radiation integral*, I_2:

$$I_2 = \oint \frac{1}{\rho^2}\, ds. \tag{28}$$

In terms of I_2, the energy loss per turn U_0 is written:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2. \tag{29}$$

Note that I_2 is a property of the lattice (actually, a property of the reference trajectory), and does not depend on the properties of the beam.

Conventionally, there are five synchrotron radiation integrals used to express the effects of synchrotron radiation on the dynamics of ultra-relativistic particles in an accelerator.

The first synchrotron radiation integral is not, however, directly related to the radiation effects.

It is defined as:

$$I_1 = \oint \frac{\eta_x}{\rho} \, ds, \tag{30}$$

where η_x is the horizontal dispersion.

I_1 is related to the *momentum compaction factor* α_p, which plays an important role in the longitudinal dynamics.

The momentum compaction factor describes the change in the length of the closed orbit with respect to particle energy:

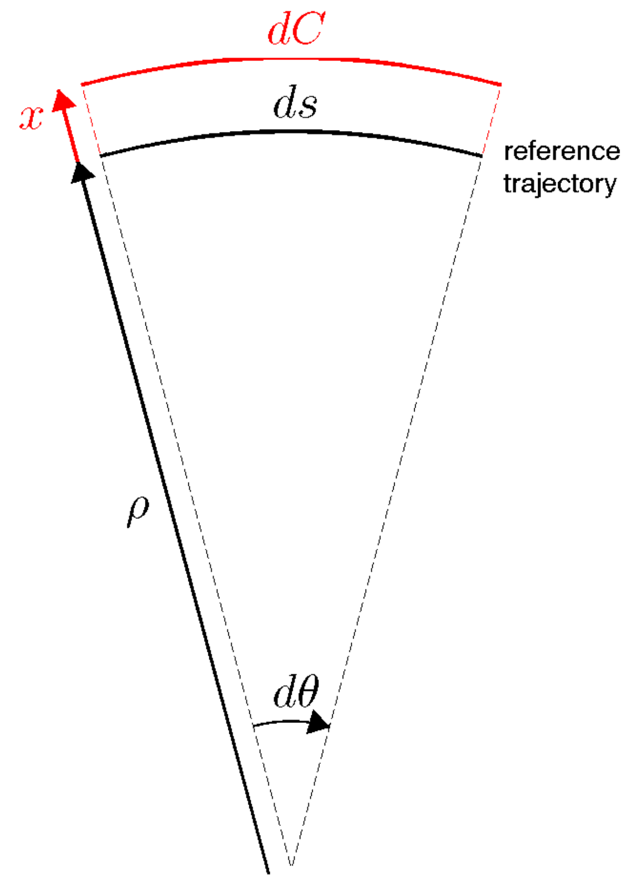
$$\frac{\Delta C}{C_0} = \alpha_p \delta + O(\delta^2). \tag{31}$$

The length of the closed orbit changes with energy because of dispersion in regions where the reference trajectory has some curvature.

$$dC = (\rho + x)\, d\theta = \left(1 + \frac{x}{\rho}\right) ds. \tag{32}$$

If $x = \eta_x \delta$, then:

$$dC = \left(1 + \frac{\eta_x \delta}{\rho}\right) ds. \tag{33}$$



The momentum compaction factor can be written:

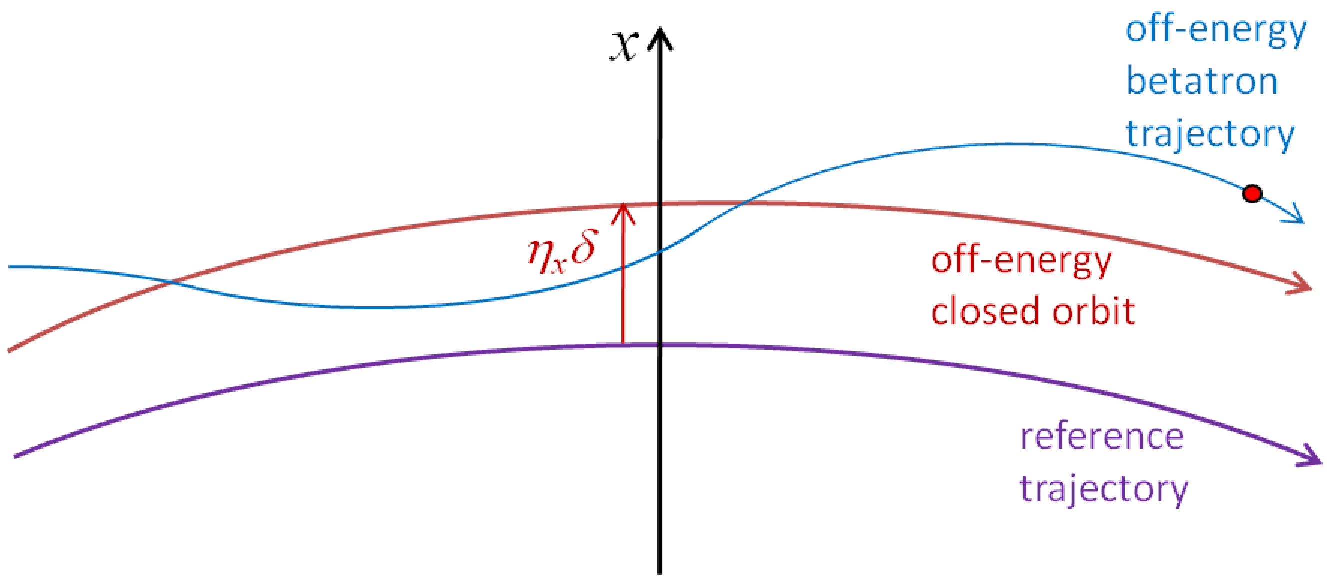
$$\alpha_p = \frac{1}{C_0} \frac{dC}{d\delta}\bigg|_{\delta=0} = \frac{1}{C_0} \oint \frac{\eta_x}{\rho}\, ds = \frac{I_1}{C_0}. \tag{34}$$

Analysis of radiation effects on the vertical emittance was relatively straightforward. When we consider the horizontal emittance, there are three complications that we need to address:

- The horizontal motion of a particle is often strongly coupled to the longitudinal motion (through the dispersion).

- Where the reference trajectory is curved (usually, in dipoles), the path length taken by a particle depends on the horizontal coordinate with respect to the reference trajectory.

- Dipole magnets are sometimes built with a gradient, so that the vertical field seen by a particle in a dipole depends on the horizontal coordinate of the particle.



Coupling between transverse and longitudinal planes in a beam line is usually represented by the dispersion, η_x. So, in terms of the horizontal dispersion and betatron action, the horizontal coordinate and momentum of a particle are given by:

$$x = \sqrt{2\beta_x J_x}\cos\phi_x + \eta_x\delta \tag{35}$$

$$p_x = -\sqrt{\frac{2J_x}{\beta_x}}(\sin\phi_x + \alpha_x\cos\phi_x) + \eta_{px}\delta. \tag{36}$$

When a particle emits radiation, we have to take into account:

- the change in momentum of the particle;
- the change in coordinate x and momentum p_x, resulting from the change in the energy deviation δ.

When we analysed the vertical motion, we ignored the second effect, because we assumed that the vertical dispersion was zero.

Taking all the above effects into account, we can proceed along the same lines as for the analysis of the vertical emittance. That is:

- Write down the changes in coordinate x and momentum p_x resulting from an emission of radiation with momentum dp (taking into account the additional effects of dispersion).

- Substitute expressions for the new coordinate and momentum into the expression for the horizontal betatron action, to find the change in action resulting from the radiation emission.

- Average over all particles in the beam, to find the change in the emittance resulting from radiation emission from each particle.

- Integrate around the ring (taking account of changes in path length and field strength with x in the bends) to find the change in emittance over one turn.

The algebra gets somewhat cumbersome, and is not especially enlightening. See Appendix A for more details. Here, we just quote the result...

The horizontal emittance decays exponentially:

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x, \tag{37}$$

where the horizontal damping time is given by:

$$\tau_x = \frac{2}{j_x}\frac{E_0}{U_0}T_0. \tag{38}$$

The horizontal damping partition number j_x is:

$$j_x = 1 - \frac{I_4}{I_2}, \tag{39}$$

where the fourth synchrotron radiation integral is given by:

$$I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right) ds, \qquad k_1 = \frac{e}{P_0}\frac{\partial B_y}{\partial x}. \tag{40}$$

So far we have considered only the effects of synchrotron radiation on the transverse motion. There are also effects on the longitudinal motion.

Generally, synchrotron oscillations are treated differently from betatron oscillations, because the synchrotron tune in a storage ring is typically much less than 1, while the betatron tunes are typically much greater than 1.

To find the effects of radiation on synchrotron motion, we proceed as follows:

- We write down the equations of motion (for the variables z and δ) for a particle performing synchrotron motion, including the radiation energy loss.

- We express the energy loss per turn as a function of the energy deviation of the particle. This introduces a "damping term" into the equations of motion.

- Solving the equations of motion gives synchrotron oscillations (as expected) with amplitude that decays exponentially.

The change in energy deviation δ and longitudinal coordinate z for a particle in one turn around a storage ring are given by:

$$\Delta\delta = \frac{eV_{RF}}{E_0}\sin\left(\phi_s - \frac{\omega_{RF} z}{c}\right) - \frac{U}{E_0}, \tag{41}$$

$$\Delta z = -\alpha_p C_0 \delta, \tag{42}$$

where V_{RF} is the RF voltage, and ω_{RF} the RF frequency, E_0 is the reference energy of the beam, ϕ_s is the nominal RF phase, and U (which may be different from U_0) is the energy lost by the particle through synchrotron radiation.

If the revolution period is T_0, then we can write the longitudinal equations of motion for the particle:

$$\frac{d\delta}{dt} = \frac{eV_{RF}}{E_0 T_0}\sin\left(\phi_s - \frac{\omega_{RF} z}{c}\right) - \frac{U}{E_0 T_0}, \tag{43}$$

$$\frac{dz}{dt} = -\alpha_p c \delta. \tag{44}$$

To solve the longitudinal equations of motion, we have to make some assumptions.

First, we assume that z is small compared to the RF wavelength:

$$\frac{\omega_{RF}|z|}{c} \ll 1. \tag{45}$$

Particles with higher energy radiate higher synchrotron radiation power. We assume $|\delta| \ll 1$, so we can work to first order in δ:

$$U = U_0 + \Delta E \left.\frac{dU}{dE}\right|_{E=E_0} = U_0 + E_0\delta \left.\frac{dU}{dE}\right|_{E=E_0}. \tag{46}$$

Finally, we assume that the RF phase ϕ_s is set so that for $z = \delta = 0$, the RF cavity restores exactly the amount of energy lost by synchrotron radiation.

With the above assumptions, the equations of motion become:

$$\frac{d\delta}{dt} = -\frac{eV_{RF}}{E_0 T_0}\cos(\phi_s)\frac{\omega_{RF}}{c}z - \frac{1}{T_0}\delta \left.\frac{dU}{dE}\right|_{E=E_0}, \tag{47}$$

$$\frac{dz}{dt} = -\alpha_p c\delta. \tag{48}$$

Combining these equations gives:

$$\frac{d^2\delta}{dt^2} + 2\alpha_E\frac{d\delta}{dt} + \omega_s^2\delta = 0. \tag{49}$$

This is the equation for a damped harmonic oscillator, with frequency ω_s and damping constant α_E given by:

$$\omega_s^2 = -\frac{eV_{RF}}{E_0}\cos(\phi_s)\frac{\omega_{RF}}{T_0}\alpha_p, \tag{50}$$

$$\alpha_E = \frac{1}{2T_0}\left.\frac{dU}{dE}\right|_{E=E_0}. \tag{51}$$

If $\alpha_E \ll \omega_s$, the energy deviation and longitudinal coordinate damp as:

$$\delta(t) = \hat{\delta} \exp(-\alpha_E t) \sin(\omega_s t - \theta_0), \tag{52}$$

$$z(t) = \frac{\alpha_p c}{\omega_s} \hat{\delta} \exp(-\alpha_E t) \cos(\omega_s t - \theta_0). \tag{53}$$

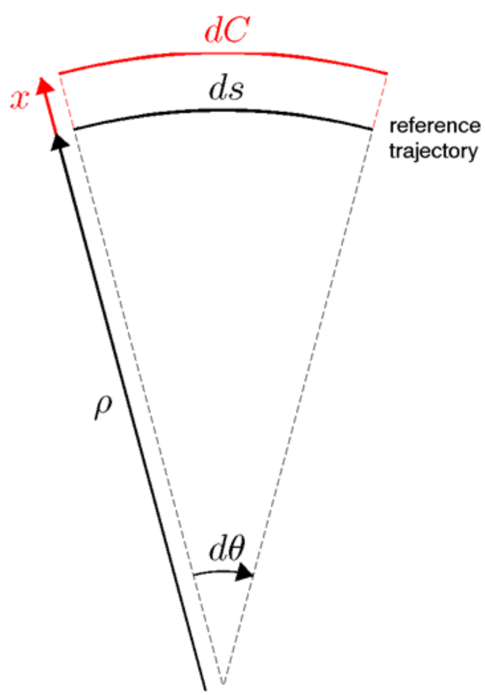
where $\hat{\delta}$ is a constant (the amplitude of the oscillation at $t = 0$).

To find the damping constant α_E, we need to know how the energy loss per turn U depends on the energy deviation δ...

We can find the total energy lost by integrating over one revolution period:

$$U = \oint P_\gamma \, dt. \tag{54}$$

To convert this to an integral over the circumference, we should recall that the path length depends on the energy deviation; so a particle with a higher energy takes longer to travel around the lattice.



$$dt = \frac{dC}{c} \tag{55}$$

$$dC = \left(1 + \frac{x}{\rho}\right) ds = \left(1 + \frac{\eta_x \delta}{\rho}\right) ds \tag{56}$$

$$U = \frac{1}{c} \oint P_\gamma \left(1 + \frac{\eta_x \delta}{\rho}\right) ds. \tag{57}$$

With the energy loss per turn given by:

$$U = \frac{1}{c} \oint P_\gamma \left(1 + \frac{\eta x}{\rho} \delta \right) ds, \tag{58}$$

and the synchrotron radiation power given by:

$$P_\gamma \approx \frac{C_\gamma}{2\pi} c \frac{E^4}{\rho^2}, \tag{59}$$

we find, after some algebra:

$$\left. \frac{dU}{dE} \right|_{E=E_0} = j_E \frac{U_0}{E_0}, \tag{60}$$

where:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2, \qquad j_E = 2 + \frac{I_4}{I_2}. \tag{61}$$

I_2 and I_4 are the same synchrotron radiation integrals that we saw before, in Eqs. (28) and (40).

Finally, we can write the longitudinal damping time:

$$\tau_z = \frac{1}{\alpha_E} = \frac{2}{j_z}\frac{E_0}{U_0}T_0. \tag{62}$$

U_0 is the energy loss per turn for a particle with the reference energy E_0, following the reference trajectory. It is given by:

$$U_0 = \frac{C_\gamma}{2\pi}E_0^4 I_2. \tag{63}$$

j_z is the longitudinal damping partition number, given by:

$$j_z = 2 + \frac{I_4}{I_2}. \tag{64}$$

The longitudinal emittance can be given by a similar expression to the horizontal and vertical emittance:

$$\varepsilon_z = \sqrt{\langle z^2 \rangle \langle \delta^2 \rangle - \langle z\delta \rangle^2}. \tag{65}$$

Since the amplitudes of the synchrotron oscillations decay with time constant τ_z, the damping of the longitudinal emittance can be written:

$$\varepsilon_z(t) = \varepsilon_z(0) \exp\left(-2\frac{t}{\tau_z}\right). \tag{66}$$

The energy loss per turn is given by:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2, \qquad C_\gamma \approx 8.846 \times 10^{-5} \mathrm{m/GeV}^3. \tag{67}$$

The radiation damping times are given by:

$$\tau_x = \frac{2}{j_x}\frac{E_0}{U_0}T_0, \quad \tau_y = \frac{2}{j_y}\frac{E_0}{U_0}T_0, \quad \tau_z = \frac{2}{j_z}\frac{E_0}{U_0}T_0. \tag{68}$$

The damping partition numbers are:

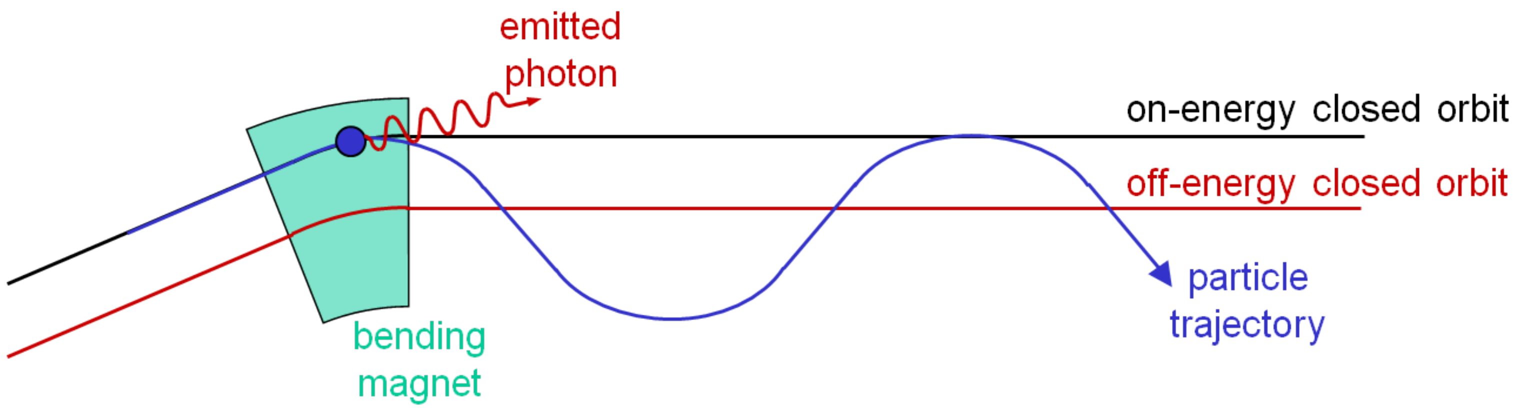
$$j_x = 1 - \frac{I_4}{I_2}, \quad j_y = 1, \quad j_z = 2 + \frac{I_4}{I_2}. \tag{69}$$

The second and fourth synchrotron radiation integrals are:

$$I_2 = \oint \frac{1}{\rho^2}\, ds, \qquad I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right) ds. \tag{70}$$

If radiation were a purely classical process, the emittances would damp to nearly zero.

However radiation is emitted in discrete units (photons), which induces some “noise” on the beam. The effect of the noise is to increase the emittance.



The beam eventually reaches an equilibrium distribution determined by a balance between the radiation damping and the quantum excitation.

By considering the change in the phase-space variables when a particle emits radiation carrying momentum dp, we find that the betatron action changes as:

$$\frac{dJ_x}{dt} = -\frac{w_1}{P_0}\frac{dp}{dt} + \frac{w_2}{P_0^2}\frac{dp^2}{dt}. \tag{71}$$

where w_1 and w_2 are functions of the Courant–Snyder parameters, the dispersion, and the phase-space variables (see Appendix A).

In the classical approximation, we can take $dp \to 0$ in the limit of small time interval, $dt \to 0$.

In this approximation, the second term on the right hand side in the above equation vanishes, and we are left only with damping.

But since radiation is quantized, we cannot take $dp \to 0$.

To take account of the quantization of synchrotron radiation, we write:

$$\frac{dp}{dt} = \frac{1}{c}\int_0^\infty \dot{N}(u)\, u\, du, \tag{72}$$

and:

$$\frac{dp^2}{dt} = \frac{1}{c^2}\int_0^\infty \dot{N}(u)\, u^2\, du. \tag{73}$$

Here, $\dot{N}(u)\, du$ is the number of photons emitted per unit time with energy between u and $u + du$.

In Appendix B, we show that with (71), these relations lead to the equation for the evolution of the emittance:

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x + \frac{2}{j_x \tau_x} C_q \gamma^2 \frac{I_5}{I_2}, \tag{74}$$

The fifth synchrotron radiation integral I_5 is given by:

$$I_5 = \oint \frac{\mathcal{H}}{|\rho^3|}\, ds, \tag{75}$$

where the "curly-H" function $\mathcal{H}$ is defined:

$$\mathcal{H} = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2. \tag{76}$$

The "quantum constant" C_q is given by:

$$C_q = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} \approx 3.832 \times 10^{-13}\mathrm{m}. \tag{77}$$

Using Eq. (74) we see that there is an equilibrium horizontal emittance ε_0, for which the damping and excitation rates are equal:

$$\left.\frac{d\varepsilon_x}{dt}\right|_{\varepsilon_x=\varepsilon_0} = 0, \quad \therefore \quad \frac{2}{\tau_x}\varepsilon_0 = \frac{2}{j_x\tau_x}C_q\gamma^2\frac{I_5}{I_2}. \tag{78}$$

The equilibrium horizontal emittance is given by:

$$\varepsilon_0 = C_q\frac{\gamma^2}{j_x}\frac{I_5}{I_2}. \tag{79}$$

Note that ε_0 is determined by the beam energy, the lattice functions (Courant–Snyder parameters and dispersion) in the dipoles, and the bending radius in the dipoles.

ε_0 is sometimes called the "natural emittance" of the lattice, since it includes only the most fundamental effects that contribute to the emittance: radiation damping and quantum excitation.

Typically, third generation synchrotron light sources have natural emittances of order a few nanometres. With beta functions of a few metres, this implies horizontal beam sizes of tens of microns (in the absence of dispersion).

As the current is increased, interactions between particles in a bunch can increase the emittance above the natural emittance.

Quantum excitation of vertical emittance

In many storage rings, the vertical dispersion in the absence of alignment, steering and coupling errors is zero, so $\mathcal{H}_y = 0$.

However, the equilibrium vertical emittance is larger than zero, because the vertical opening angle of the radiation excites some vertical betatron oscillations.

The fundamental lower limit on the vertical emittance, from the opening angle of the synchrotron radiation, is given by*:

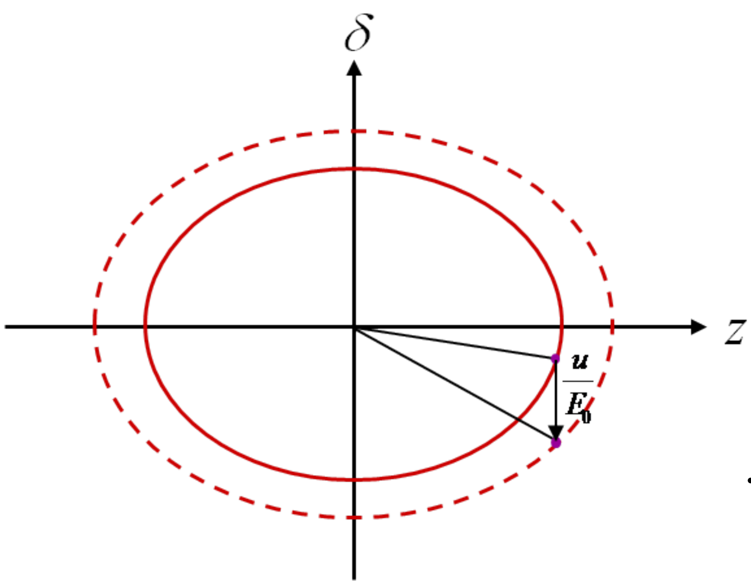
$$\varepsilon_y = \frac{13}{55} \frac{C_q}{j_y I_2} \oint \frac{\beta_y}{|\rho^3|}\, ds. \tag{80}$$

In most storage rings, this is an extremely small value, typically four orders of magnitude smaller than the natural (horizontal) emittance.

In practice, the vertical emittance is dominated by magnet alignment errors. Storage rings typically operate with a vertical emittance that is of order 1% of the horizontal emittance, but many can achieve emittance ratios somewhat smaller than this.

*T. Raubenheimer, SLAC Report 387, p.19 (1991).

Quantum effects excite longitudinal emittance as well as transverse emittance. Consider a particle with longitudinal coordinate z and energy deviation δ, which emits a photon of energy u.



$$\delta' = \hat{\delta}' \sin\theta' = \hat{\delta}\sin\theta - \frac{u}{E_0}. \quad (81)$$

$$z' = \frac{\alpha_p c}{\omega_s}\hat{\delta}'\cos\theta' = \frac{\alpha_p c}{\omega_s}\hat{\delta}\cos\theta. \quad (82)$$

$$\therefore\ \hat{\delta}'^2 = \hat{\delta}^2 - 2\hat{\delta}\frac{u}{E_0}\sin\theta + \frac{u^2}{E_0^2}. \quad (83)$$

Averaging over the bunch gives:

$$\Delta\sigma_\delta^2 = \frac{\langle u^2\rangle}{2E_0^2} \quad \text{where} \quad \sigma_\delta^2 = \langle\delta^2\rangle = \frac{1}{2}\langle\hat{\delta}^2\rangle. \quad (84)$$

Including radiation damping, the energy spread evolves as:

$$\frac{d\sigma_\delta^2}{dt} = \frac{1}{2E_0^2}\left\langle \int_0^\infty \dot{N}(u)u^2\,du \right\rangle_C - \frac{2}{\tau_z}\sigma_\delta^2, \tag{85}$$

where the brackets $\langle\,\rangle_C$ represent an average around the ring.

Using Eq. (130) from Appendix B for $\int \dot{N}(u)u^2\,du$, we find:

$$\frac{d\sigma_\delta^2}{dt} = C_q\gamma^2\frac{2}{j_z\tau_z}\frac{I_3}{I_2} - \frac{2}{\tau_z}\sigma_\delta^2. \tag{86}$$

The equilibrium energy spread is found from $d\sigma_\delta^2/dt = 0$:

$$\sigma_{\delta 0}^2 = C_q\gamma^2\frac{I_3}{j_z I_2}. \tag{87}$$

The third synchrotron radiation integral I_3 is defined:

$$I_3 = \oint \frac{1}{|\rho^3|}\,ds. \tag{88}$$

The equilibrium energy spread determined by radiation effects is:

$$\sigma_{\delta 0}^2 = C_q \gamma^2 \frac{I_3}{j_z I_2}. \tag{89}$$

This is often referred to as the "natural" energy spread, since collective effects can often lead to an increase in the energy spread with increasing bunch charge.

The natural energy spread is determined essentially by the beam energy and by the bending radii of the dipoles.

Note that the natural energy spread *does not depend on the RF parameters (either voltage or frequency)*.

The bunch length σ_z in a *matched* distribution with energy spread σ_δ is:

$$\sigma_z = \frac{\alpha_p c}{\omega_s} \sigma_\delta. \tag{90}$$

We can increase the synchrotron frequency ω_s, and hence reduce the bunch length, by increasing the RF voltage, or by increasing the RF frequency.

Note: in a matched distribution, the shape of the distribution in phase space is the same as the path mapped out by a particle in phase space when observed on successive turns. Neglecting radiation effects, a matched distribution stays the same on successive turns of the bunch around the ring.

Including the effects of radiation damping and quantum excitation, the emittances vary as:

$$\varepsilon(t) = \varepsilon(0) \exp\left(-2\frac{t}{\tau}\right) + \varepsilon(\infty)\left[1 - \exp\left(-2\frac{t}{\tau}\right)\right]. \tag{91}$$

The damping times are given by:

$$j_x \tau_x = j_y \tau_y = j_z \tau_z = 2\frac{E_0}{U_0} T_0. \tag{92}$$

The damping partition numbers are given by:

$$j_x = 1 - \frac{I_4}{I_2}, \qquad j_y = 1, \qquad j_z = 2 + \frac{I_4}{I_2}. \tag{93}$$

The energy loss per turn is given by:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2, \qquad C_\gamma = 9.846 \times 10^{-5}\ \mathrm{m/GeV^3}. \tag{94}$$

The natural emittance is:

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{j_x I_2}, \qquad C_q = 3.832 \times 10^{-13}\ \text{m}. \tag{95}$$

The natural energy spread and bunch length are given by:

$$\sigma_\delta^2 = C_q \gamma^2 \frac{I_3}{j_z I_2}, \qquad \sigma_z = \frac{\alpha_p c}{\omega_s} \sigma_\delta. \tag{96}$$

The momentum compaction factor is:

$$\alpha_p = \frac{I_1}{C_0}. \tag{97}$$

The synchrotron frequency and synchronous phase are given by:

$$\omega_s^2 = -\frac{eV_{RF}}{E_0} \frac{\omega_{RF}}{T_0} \alpha_p \cos\phi_s, \qquad \sin\phi_s = \frac{U_0}{eV_{RF}}. \tag{98}$$

The synchrotron radiation integrals are:

$$I_1 = \oint \frac{\eta_x}{\rho}\, ds, \tag{99}$$

$$I_2 = \oint \frac{1}{\rho^2}\, ds, \tag{100}$$

$$I_3 = \oint \frac{1}{|\rho|^3}\, ds, \tag{101}$$

$$I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right) ds, \qquad k_1 = \frac{e}{P_0}\frac{\partial B_y}{\partial x}, \tag{102}$$

$$I_5 = \oint \frac{\mathcal{H}_x}{|\rho|^3}\, ds, \qquad \mathcal{H}_x = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2. \tag{103}$$

Appendix A: Damping of horizontal emittance

In this Appendix, we derive the expression for radiation damping of the horizontal emittance:

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x, \tag{104}$$

where:

$$\tau_x = \frac{2}{j_x}\frac{E_0}{U_0}T_0, \qquad j_x = 1 - \frac{I_4}{I_2}. \tag{105}$$

To derive these formulae, we proceed as follows:

1. We find an expression for the change of horizontal action of a single particle when emitting radiation with momentum dp.

2. We integrate around the ring to find the change in action per revolution period.

3. We average the action over all the particles in the bunch, to find the change in emittance per revolution period.

Appendix A: Damping of horizontal emittance

To begin, we note that, in the presence of dispersion, the action J_x is written:

$$2J_x = \gamma_x \tilde{x}^2 + 2\alpha_x \tilde{x}\tilde{p}_x + \beta_x \tilde{p}_x^2, \tag{106}$$

where:

$$\tilde{x} = x - \eta_x \delta, \qquad \text{and} \qquad \tilde{p}_x = p_x - \eta_{px}\delta. \tag{107}$$

After emission of radiation carrying momentum dp, the variables change by:

$$\delta \mapsto \delta - \frac{dp}{P_0}, \quad \tilde{x} \mapsto \tilde{x} + \eta_x \frac{dp}{P_0}, \quad \tilde{p}_x \mapsto \tilde{p}_x \left(1 - \frac{dp}{P_0}\right) + \eta_{px}(1-\delta)\frac{dp}{P_0}. \tag{108}$$

We write the resulting change in the action as:

$$J_x \mapsto J_x + dJ_x. \tag{109}$$

The change in the horizontal action is:

$$dJ_x = -\frac{w_1}{P_0}dp + \frac{w_2}{P_0^2}dp^2 \qquad \therefore \qquad \frac{dJ_x}{dt} = -\frac{w_1}{P_0}\frac{dp}{dt} + \frac{w_2}{P_0^2}\frac{dp^2}{dt}, \tag{110}$$

where, in the limit $\delta \to 0$:

$$w_1 = \alpha_x x p_x + \beta_x p_x^2 - \eta_x(\gamma_x x + \alpha_x p_x) - \eta_{px}(\alpha_x x + \beta_x p_x), \tag{111}$$

and:

$$w_2 = \frac{1}{2}\left(\gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2\right) - \left(\alpha_x \eta_x + \beta_x \eta_{px}\right) p_x + \frac{1}{2}\beta_x p_x^2. \tag{112}$$

Treating radiation as a classical phenomenon, we can take the limit $dp \to 0$ in the limit of small time interval, $dt \to 0$.

In this approximation:

$$\frac{dJ_x}{dt} \approx -w_1 \frac{1}{P_0}\frac{dp}{dt} \approx -w_1 \frac{P_\gamma}{P_0 c}, \tag{113}$$

where P_γ is the *rate of energy loss* of the particle through synchrotron radiation.

To find the *average* rate of change of horizontal action, we integrate over one revolution period:

$$\frac{dJ_x}{dt} = -\frac{1}{T_0} \oint w_1 \frac{P_\gamma}{P_0 c} \, dt. \tag{114}$$

We have to be careful changing the variable of integration where the reference trajectory is curved:

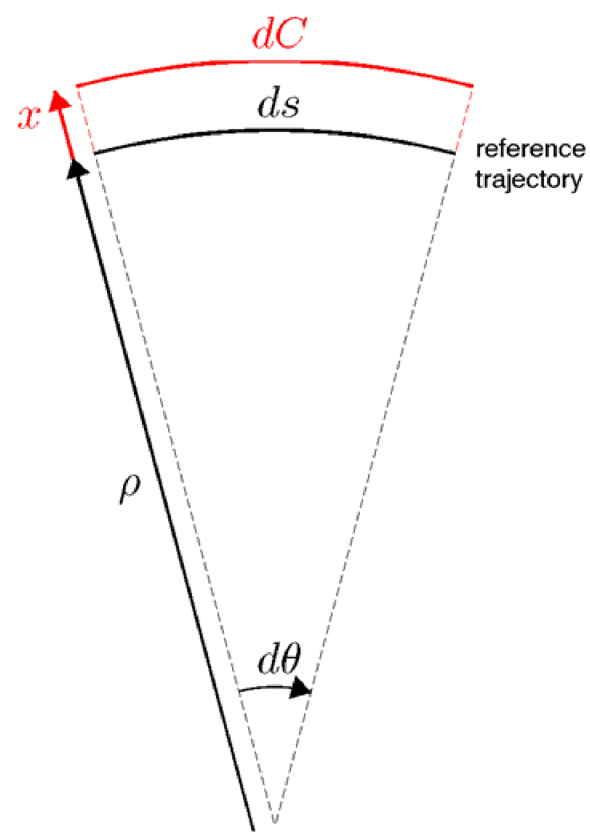
$$dt = \frac{dC}{c} = \left(1 + \frac{x}{\rho}\right) \frac{ds}{c}. \tag{115}$$

So:

$$\frac{dJ_x}{dt} = -\frac{1}{T_0 P_0 c^2} \oint w_1 P_\gamma \left(1 + \frac{x}{\rho}\right) ds, \tag{116}$$

where the rate of energy loss is:

$$P_\gamma = \frac{C_\gamma}{2\pi} c^3 e^2 B^2 E^2. \tag{117}$$



We have to take into account the fact that the field strength in a dipole can vary with position. To first order in x we can write:

$$B = B_0 + x\frac{\partial B_y}{\partial x}. \tag{118}$$

Substituting Eq. (118) into (117), and with the use of (111), we find (after some algebra!) that, averaging over all particles in the beam:

$$\oint \left\langle w_1 P_\gamma \left(1 + \frac{x}{\rho}\right)\right\rangle ds = cU_0 \left(1 - \frac{I_4}{I_2}\right)\varepsilon_x, \tag{119}$$

where:

$$U_0 = \frac{C_\gamma}{2\pi} c E_0^4 I_2, \qquad I_2 = \oint \frac{1}{\rho^2} ds, \qquad I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right) ds, \tag{120}$$

and k_1 is the normalised quadrupole gradient in the dipole field:

$$k_1 = \frac{e}{P_0}\frac{\partial B_y}{\partial x}. \tag{121}$$

Combining Eqs. (116) and (119) we have:

$$\frac{d\varepsilon_x}{dt} = -\frac{1}{T_0}\frac{U_0}{E_0}\left(1 - \frac{I_4}{I_2}\right)\varepsilon_x. \tag{122}$$

Defining the horizontal damping time τ_x:

$$\tau_x = \frac{2}{j_x}\frac{E_0}{U_0}T_0, \qquad j_x = 1 - \frac{I_4}{I_2}, \tag{123}$$

the evolution of the horizontal emittance can be written:

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x. \tag{124}$$

The quantity j_x is called the *horizontal damping partition number*.

For most synchrotron storage ring lattices, if there is no gradient in the dipoles then j_x is very close to 1.

In deriving the equation of motion (116) for the action of a particle emitting synchrotron radiation, we made the classical approximation that in a time interval dt, the momentum dp of the radiation emitted goes to zero as dt goes to zero.

In reality, emission of radiation is quantized, so writing "$dp \to 0$" actually makes no sense.

Taking into account the quantization of radiation, the equation of motion for the action (110) should be written:

$$\frac{dJ_x}{dt} = -\frac{w_1}{P_0 c}\int_0^\infty \dot{N}(u)\,u\,du + \frac{w_2}{P_0^2 c^2}\int_0^\infty \dot{N}(u)\,u^2\,du, \tag{125}$$

where $\dot{N}(u)$ is the number of photons emitted per unit time in the energy range from u to $u + du$.

The first term on the right hand side of Eq. (125) just gives the same radiation damping as in the classical approximation.

The second term on the right hand side of Eq. (125) is an excitation term that we previously neglected.

To proceed, we find expressions for the integrals $\int \dot{N}(u)\, u\, du$ and $\int \dot{N}(u)\, u^2\, du$.

The required expressions can be found from the spectral distribution of synchrotron radiation from a dipole magnet. This is given by:

$$\frac{d\mathcal{P}}{d\vartheta} = \frac{9\sqrt{3}}{8\pi} P_\gamma \vartheta \int_\vartheta^\infty K_{5/3}(x)\, dx, \tag{126}$$

where $d\mathcal{P}/d\vartheta$ is the energy radiated per unit time per unit frequency range, and $\vartheta = \omega/\omega_c$ is the radiation frequency ω divided by the critical frequency ω_c:

$$\omega_c = \frac{3}{2}\frac{\gamma^3 c}{\rho}. \tag{127}$$

P_γ is the total energy radiated per unit time, and $K_{5/3}(x)$ is a modified Bessel function.

Since the energy of a photon of frequency ω is $u = \hbar\omega$, it follows that:

$$\dot{N}(u)\,du = \frac{1}{\hbar\omega}\frac{d\mathcal{P}}{d\vartheta}\,d\vartheta. \qquad (128)$$

Using (126) and (128), we find:

$$\int_0^\infty \dot{N}(u)\,u\,du = P_\gamma, \qquad (129)$$

and:

$$\int_0^\infty \dot{N}(u)\,u^2\,du = 2C_q\gamma^2\frac{E_0}{\rho}P_\gamma. \qquad (130)$$

C_q is a constant given by:

$$C_q = \frac{55}{32\sqrt{3}}\frac{\hbar}{mc} \approx 3.832 \times 10^{-13}\,\mathrm{m}. \qquad (131)$$

Appendix B: Quantum excitation of horizontal emittance

The final step is to substitute for the integrals in (125) from (129) and (130), substitute for w_1 and w_2 from (111) and (112), average over the circumference of the ring, and average also over all particles in the beam.

Then, since $\varepsilon_x = \langle J_x \rangle$, we find (for $x \ll \eta_x$ and $p_x \ll \eta_{px}$):

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x + \frac{2}{j_x \tau_x} C_q \gamma^2 \frac{I_5}{I_2} \tag{132}$$

where the fifth synchrotron radiation integral I_5 is given by:

$$I_5 = \oint \frac{\mathcal{H}_x}{|\rho^3|} \, ds, \tag{133}$$

The "curly-H" function $\mathcal{H}_x$ is given by:

$$\mathcal{H}_x = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2. \tag{134}$$

The damping time and horizontal damping partition number are given by:

$$j_x \tau_x = 2\frac{E_0}{U_0} T_0, \qquad U_0 = \frac{C_\gamma}{2\pi} c E_0^4 I_2, \tag{135}$$

(U_0 is the energy loss per turn) and:

$$j_x = 1 - \frac{I_4}{I_2}. \tag{136}$$

Note that the excitation term is independent of the emittance.

The quantum excitation does not simply modify the damping time, but leads to a non-zero equilibrium emittance.

The equilibrium emittance ε_0 is determined by the condition:

$$\left.\frac{d\varepsilon_x}{dt}\right|_{\varepsilon_x=\varepsilon_0} = 0. \tag{137}$$

From (132), we see that the equilibrium emittance is given by:

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{j_x I_2}. \tag{138}$$