RF Cavities

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What is a cavity?

[3. F. cavité, in 13th c. cavité, (= It. cavità, Sp. cavidad), on L. type *cavità-rem (prob. in late L. or Romanic), f. cav-us hollow; see ß8.]

†1. Hollowness. Obs. rare.

2. A hollow place; a void or empty space within a solid body.

3. In naval architecture, the displacement formed in the water by the immersed bottom and sides of the vessel (Smyth Sailor’s Word-bk.).
Lorentz force

A charged particle moving with velocity \( \vec{v} = \frac{\vec{p}}{m\gamma} \) through an electromagnetic field experiences a force

\[
\frac{d\vec{p}}{dt} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)
\]

The total energy of this particle is \( W = \sqrt{(mc^2)^2 + (pc)^2} = \gamma mc^2 \), the kinetic energy is \( W_{kin} = mc^2(\gamma - 1) \)

The role of acceleration is to increase the particle energy!

Change of \( W \) by differentiation:

\[
W \, dW = c^2 \vec{p} \cdot d\vec{p} = qc^2 \vec{p} \cdot \left(\vec{E} + \vec{v} \times \vec{B}\right) \, dt = qc^2 \vec{p} \cdot \vec{E} \, dt
\]

\[
dW = q\vec{v} \cdot \vec{E} \, dt
\]

Note: Only the electric field can change the particle energy!
Maxwell’s equations

The electromagnetic fields inside the “hollow place” obey these equations:

\[
\nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E} = 0 \quad \nabla \cdot \vec{B} = 0
\]

\[
\nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{B} = 0 \quad \nabla \cdot \vec{E} = 0
\]

With the curl of the 3\textsuperscript{rd}, the time derivative of the 1\textsuperscript{st} equation and the vector identity

\[
\nabla \times \nabla \times \vec{E} \equiv \nabla \nabla \cdot \vec{E} - \Delta \vec{E}
\]

this set of equations can be brought in the form

\[
\Delta \vec{E} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{E} = 0
\]

which is the Laplace equation in 4 dimensions.

With the boundaries of the “solid body” around it (the cavity walls), there exist eigensolutions of the cavity at certain frequencies (eigenfrequencies).
Wave vector \( \vec{k} \):
the direction of \( \vec{k} \) is the direction of propagation,
the length of \( \vec{k} \) is the phase shift per unit length.
\( \vec{k} \) behaves like a vector.

\[
\vec{E} \propto \vec{u}_y \cos(\omega t - \vec{k} \cdot \vec{r}) \\
\vec{B} \propto \vec{u}_x \cos(\omega t - \vec{k} \cdot \vec{r}) \\
\vec{k} \cdot \vec{r} = \frac{\omega}{c}(\cos(\varphi)z + \sin(\varphi)x)
\]
Wave length, phase velocity

The components of $\vec{k}$ are related to the wavelength in the direction of that component as $\lambda_z = \frac{2\pi}{k_z}$ etc., to the phase velocity as $v_{\varphi,z} = \frac{\omega}{k_z} = f \lambda_z$.

\[
k \perp = \frac{\omega_c}{c}
\]

\[
k_z = \frac{\omega}{c} \sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}
\]
Superposition of 2 homogeneous plane waves

Metallic walls may be inserted where $E_y = 0$ without perturbing the fields.

Note the standing wave in $x$-direction!

This way one gets a hollow rectangular waveguide
Rectangular waveguide

Fundamental (TE$_{10}$ or H$_{10}$) mode in a standard rectangular waveguide.

**Example:** “S-band” : 2.6 GHz ... 3.95 GHz, Waveguide type WR284 (2.84” wide), dimensions: 72.14 mm x 34.04 mm. Operated at $f = 3$ GHz.

$$\text{power flow: } \frac{1}{2} \text{Re} \left\{ \iint_{\text{cross section}} \vec{E} \times \vec{H}^* \cdot d\vec{A} \right\}$$
Waveguide dispersion

What happens with different waveguide dimensions (different width $a$)?

1: $a = 52$ mm, $f/f_c = 1.04$

2: $a = 72.14$ mm, $f/f_c = 1.44$

3: $a = 144.3$ mm, $f/f_c = 2.88$

$k_z = \frac{\omega}{c} = \frac{2\pi}{\lambda_s} = \frac{\omega}{c} \sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}$

$cutoff \ f_c = \frac{c}{2a}$

$f = 3$ GHz
The phase velocity is the speed with which the crest or a zero-crossing travels in z-direction.

Note on the three animations on the right that, at constant $f$, it is $\propto \lambda_g$. Note that at $f = f_c$, $v_{\varphi, z} = \infty$!

With $f \to \infty$, $v_{\varphi, z} \to c$!

1: $a = 52\, \text{mm}$, $f/f_c = 1.04$

2: $a = 72.14\, \text{mm}$, $f/f_c = 1.44$

3: $a = 144.3\, \text{mm}$, $f/f_c = 2.88$
Rectangular waveguide modes

TE_{10}  TE_{20}  TE_{01}  TE_{11}  
TM_{11}  TE_{21}  TM_{21}  TE_{30}  
TE_{31}  TM_{31}  TE_{40}  TE_{02}  
TE_{12}  TM_{12}  TE_{41}  TM_{41}  
TE_{22}  TM_{22}  TE_{50}  TE_{32}  

plotted: E-field
Radial waves

Also radial waves may be interpreted as superpositions of plane waves. The superposition of an outward and an inward radial wave can result in the field of a round hollow waveguide.

\[ E_z \propto H_n^{(2)}(k_\rho \rho) \cos(n\varphi) \]

\[ E_z \propto H_n^{(1)}(k_\rho \rho) \cos(n\varphi) \]

\[ E_z \propto J_n(k_\rho \rho) \cos(n\varphi) \]
Round waveguide

**TE\textsubscript{11} – fundamental**

\[
\frac{f_c}{\text{GHz}} = \frac{87.9}{a/\text{mm}}
\]

**TM\textsubscript{01} – axial field**

\[
\frac{f_c}{\text{GHz}} = \frac{114.8}{a/\text{mm}}
\]

**TE\textsubscript{01} – low loss**

\[
\frac{f_c}{\text{GHz}} = \frac{182.9}{a/\text{mm}}
\]

\(f/f_c = 1.44\)
Circular waveguide modes

Plotted: $E$-field
General waveguide equations:

Transverse wave equation (membrane equation):
\[
\Delta T + \left( \frac{\omega_c}{c} \right)^2 T = 0
\]

TE (or H) modes

boundary condition:
\[\vec{n} \cdot \nabla T = 0\]

TM (or E) modes

longitudinal wave equations (transmission line equations):
\[
\frac{d U(z)}{dz} + \gamma Z_0 I(z) = 0
\]
\[
\frac{d I(z)}{dz} + \frac{\gamma}{Z_0} U(z) = 0
\]

propagation constant:
\[
\gamma = j \frac{\omega}{c} \sqrt{1 - \left( \frac{\omega_c}{\omega} \right)^2}
\]

characteristic impedance:
\[
Z_0 = \frac{j \omega \mu}{\gamma}
\]
\[
Z_0 = \frac{\gamma}{j \omega \varepsilon}
\]

ortho-normal eigenvectors:
\[\vec{e} = \vec{u}_z \times \nabla T\]
\[\vec{e} = -\nabla T\]

transverse fields:
\[\vec{E} = U(z) \vec{e}\]
\[\vec{H} = I(z) \vec{u}_z \times \vec{e}\]

longitudinal field:
\[H_z = \left( \frac{\omega_c}{c} \right)^2 \frac{T U(z)}{j \omega \mu}\]
\[E_z = \left( \frac{\omega_c}{c} \right)^2 \frac{T I(z)}{j \omega \varepsilon}\]
Rectangular waveguide: transverse eigenfunctions

**TE (H) modes:**

\[
T_{mn}^{(H)} = \frac{1}{\pi} \frac{ab\varepsilon_m\varepsilon_n}{\sqrt{(mb)^2 + (na)^2}} \cos\left(\frac{m\pi}{a} x\right) \cos\left(\frac{n\pi}{b} y\right)
\]

**TM (E) modes:**

\[
T_{mn}^{(E)} = \frac{2}{\pi} \frac{ab}{\sqrt{(mb)^2 + (na)^2}} \sin\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right)
\]

\[
\frac{\omega_c}{c} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}
\]

Round waveguide: transverse eigenfunctions

**TE (H) modes:**

\[
T_{mn}^{(H)} = \frac{\varepsilon_m}{\pi \left(\chi_{mn}^\prime - m^2\right)} \frac{J_m\left(\chi_{mn}^\prime \frac{\rho}{a}\right)}{J_m\left(\chi_{mn}^\prime\right)} \left\{ \cos(m\varphi) \right\}
\]

**TM (E) modes:**

\[
T_{mn}^{(E)} = \frac{\varepsilon_m}{\pi \chi_{mn}} \frac{J_m\left(\chi_{mn} \frac{\rho}{a}\right)}{J_{m-1}\left(\chi_{mn}\right)} \left\{ \sin(m\varphi) \right\}
\]

\[
\frac{\omega_c}{c} = \frac{\chi_{mn}}{a}
\]

where

\[
\varepsilon_i = \begin{cases} 
1 & \text{for } i = 0 \\
2 & \text{for } i \neq 0 
\end{cases}
\]

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Waveguide perturbed by notches

Reflections from notches lead to a superimposed standing wave pattern. “Trapped mode”
Short-circuited waveguide

$\text{TM}_{010}$ (no axial dependence)  $\text{TM}_{011}$  $\text{TM}_{012}$
Single WG mode between two shorts

Eigenvalue equation for field amplitude $a$:

$$a = e^{-jk_z \ell} a$$

Non-vanishing solutions exist for $2k_z \ell = 2\pi m$:

With $k_z = \frac{\omega}{c} \sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}$, this becomes $f_0^2 = f_c^2 + \left(c \frac{m}{2\ell}\right)^2$
Simple pillbox

TM_{010}-mode

electric field (purely axial)
magnetic field (purely azimuthal)
Pillbox cavity field (w/o beam tube)

\[ T(\rho, \varphi) = \sqrt{\frac{1}{\pi}} \frac{J_0\left(\frac{\chi_{01}\rho}{a}\right)}{\chi_{01} J_1\left(\frac{\chi_{01}}{a}\right)} \quad \chi_{01} = 2.40483... \]

The only non-vanishing field components:

\[ E_z = \frac{1}{j \omega \varepsilon_0} \frac{\chi_{01}}{a} \sqrt{\frac{1}{\pi}} \frac{J_0\left(\frac{\chi_{01}\rho}{a}\right)}{a J_1\left(\frac{\chi_{01}}{a}\right)} \]

\[ B_\varphi = \mu_0 \sqrt{\frac{1}{\pi}} \frac{J_1\left(\frac{\chi_{01}\rho}{a}\right)}{a J_1\left(\frac{\chi_{01}}{a}\right)} \]

\[ \omega_0 \bigg|_{\text{pillbox}} = \frac{\chi_{01} c}{a} \quad \eta = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377 \, \Omega \]

\[ Q \bigg|_{\text{pillbox}} = \frac{\sqrt{2a \eta \sigma \chi_{01}}}{2 \left(1 + \frac{a}{h}\right)} \]

\[ \frac{R}{Q \bigg|_{\text{pillbox}}} = \frac{4\eta}{\chi_{01}^3 \pi J_1^2\left(\frac{\chi_{01}}{a}\right)} \sin^2\left(\frac{\chi_{01} h}{2a}\right) \]
Pillbox with beam pipe

One needs a hole for the beam pipe – circular waveguide below cutoff

TM\textsubscript{010}-mode

(only 1/4 shown)
A more practical pillbox cavity

Round of sharp edges (field enhancement!)

TM_{010}-mode (only 1/4 shown)

electric field

magnetic field
The energy stored in the electric field is
\[ \int\int\int_{\text{cavity}} \frac{\varepsilon}{2} |\vec{E}|^2 \, dV \]

The energy stored in the magnetic field is
\[ \int\int\int_{\text{cavity}} \frac{\mu}{2} |\vec{H}|^2 \, dV \]

Since \( \vec{E} \) and \( \vec{H} \) are 90° out of phase, the stored energy continuously swaps from electric energy to magnetic energy. On average, electric and magnetic energy must be equal.

The (imaginary part of the) Poynting vector describes this energy flux.

In steady state, the total stored energy \( W = \int\int\int_{\text{cavity}} \left( \frac{\varepsilon}{2} |\vec{E}|^2 + \frac{\mu}{2} |\vec{H}|^2 \right) dV \) is constant in time.
Stored energy & Poynting vector

electric field energy

Poynting vector

magnetic field energy
Losses & Q factor

The losses $P_{\text{loss}}$ are proportional to the stored energy $W$.

The cavity quality factor $Q$ is defined as the ratio $Q = \frac{\omega_0 W}{P_{\text{loss}}}$.

In a vacuum cavity, losses are dominated by the ohmic losses due to the finite conductivity of the cavity walls.

If the losses are small, one can calculate them with a perturbation method:

- The tangential magnetic field at the surface leads to a surface current.
- This current will see a wall resistance $R_A = \sqrt{\frac{\omega \mu}{2\sigma}}$.
- $\{R_A$ is related to the skin depth $\delta$ by $\delta \sigma R_A = 1$ .}$
- The cavity losses are given by $P_{\text{loss}} = \iint_{\text{wall}} R_A |H_t|^2 \, dA$.
- If other loss mechanisms are present, losses must be added. Consequently, the inverses of the $Q$‘s must be added!
Acceleration voltage & $R$-upon-$Q$

I define $V_{acc} = \int E_z e^{j \frac{\omega}{c} z} \, dz$. The exponential factor accounts for the variation of the field while particles with velocity $\beta c$ are traversing the gap (see next page).

With this definition, $V_{acc}$ is generally complex – this becomes important with more than one gap. For the time being we are only interested in $|V_{acc}|$.

Attention, different definitions are used!

The square of the acceleration voltage is proportional to the stored energy $W$.

The proportionality constant defines the quantity called $R$-upon-$Q$:

$$\frac{R}{Q} = \frac{|V_{acc}|^2}{2 \omega_0 W}$$

Attention, also here different definitions are used!
The transit time factor is the ratio of the acceleration voltage to the (non-physical) voltage a particle with infinite velocity would see.

\[ TT = \left| \frac{V_{\text{acc}}}{\int E_z \, dz} \right| = \frac{\left| \int E_z e^{j\frac{\omega z}{\beta c}} \, dz \right|}{\left| \int E_z \, dz \right|} \]

The transit time factor of an ideal pillbox cavity (no axial field dependence) of height (gap length) \( h \) is:

\[ TT = \sin \left( \frac{\chi_0 h}{2a} \right) \bigg/ \left( \frac{\chi_0 h}{2a} \right) \]

Field rotates by 360° during particle passage.
Shunt impedance

The square of the acceleration voltage is proportional to the power loss $P_{\text{loss}}$. The proportionality constant defines the quantity “shunt impedance”

$$R = \frac{|V_{\text{acc}}|^2}{2P_{\text{loss}}}$$

Attention, also here different definitions are used!

Traditionally, the shunt impedance is the quantity to optimize in order to minimize the power required for a given gap voltage.
Equivalent circuit

\[ \frac{R}{\beta} \quad C \quad L \quad R \]

\( I_G \quad V_{acc} \quad I_B \)

\( P \)

\( \beta : \) coupling factor

\( R : \) Shunt impedance

\( \sqrt{\frac{L}{C}} : R\text{-upon-}Q \)

\( L = \frac{R}{Q\omega_0} \)

\( C = \frac{Q}{R\omega_0} \)

Simplification: single mode
Resonance

![Graph showing resonance with different Q values](chart.png)
Reentrant cavity

Nose cones increase transit time factor, round outer shape minimizes losses.

Example: KEK photon factory 500 MHz
- $R$ probably as good as it gets -

<table>
<thead>
<tr>
<th></th>
<th>this cavity</th>
<th>optimized pillbox</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R/Q$:</td>
<td>111 $\Omega$</td>
<td>107.5 $\Omega$</td>
</tr>
<tr>
<td>$Q$:</td>
<td>44270</td>
<td>41630</td>
</tr>
<tr>
<td>$R$:</td>
<td>4.9 $M\Omega$</td>
<td>4.47 $M\Omega$</td>
</tr>
</tbody>
</table>

Nose cone example

Freq = 500.003

nose cone
Loss factor

\[ k_{loss} = \frac{\omega_0 R}{2Q} = \frac{|V_{acc}|^2}{4W} = \frac{1}{2C} \]

Energy deposited by a single charge \( q \): \( k_{loss} q^2 \)

Voltage induced by a single charge \( q \):

\[ \frac{V_{acc}}{2k_{loss} q} \]

Impedance seen by the beam

Cavity

\( L = R/(Q\omega_0) \)

\( C = Q/(R\omega_0) \)
Summary: relations between $V_{acc}$, $W$, $P_{loss}$

- Energy stored inside the cavity: $W$
- Gap voltage: $V_{acc}$
- Power lost in the cavity walls: $P_{loss}$
- Shunt impedance: $R = \frac{|V_{acc}|^2}{2P_{loss}}$
- $Q$ factor: $Q = \frac{\omega_0 W}{P_{loss}}$

R-upon-$Q$

\[
\frac{R}{Q} = \frac{|V_{acc}|^2}{2\omega_0 W}
\]

\[
k_{loss} = \frac{\omega_0}{2} \frac{R}{Q} = \frac{|V_{acc}|^2}{4W}
\]
Beam loading – RF to beam efficiency

The beam current “loads” the generator, in the equivalent circuit this appears as a resistance in parallel to the shunt impedance. If the generator is matched to the unloaded cavity, beam loading will cause the accelerating voltage to decrease.

The power absorbed by the beam is

\[ P_{\text{loss}} = \frac{1}{2} \Re \{ V_{\text{acc}} I_B^* \}, \]

the power loss

\[ P_{\text{loss}} = \frac{|V_{\text{acc}}|^2}{2 R}. \]

For high efficiency, beam loading should be high.

The RF to beam efficiency is

\[ \eta = \frac{1}{1 + \frac{V_{\text{acc}}}{R |I_B|}} = \frac{|I_B|}{|I_G|}. \]
Characterizing cavities

- Resonance frequency
  \[ \omega_0 = \frac{1}{\sqrt{L \cdot C}} \]

- Transit time factor
  field varies while particle is traversing the gap

- Shunt impedance
  gap voltage – power relation

- \( Q \) factor

- \( R/Q \)
  independent of losses – only geometry!

- Loss factor

Linac definition

\[ |V_{acc}|^2 = 2 \frac{R}{Q} P_{loss} \]

\[ \frac{R}{Q} = \frac{|V_{acc}|^2}{2 \omega_0 W} = \sqrt{\frac{L}{C}} \]

\[ \frac{R}{Q} = \frac{|V_{acc}|^2}{\omega_0 W} \]

\[ k_{loss} = \frac{\omega_0 R}{2 Q} = \frac{|V_{acc}|^2}{4 W} \]

\[ k_{loss} = \frac{\omega_0 R}{4 Q} = \frac{|V_{acc}|^2}{4 W} \]
Higher order modes

external dampers

\[ R_1, Q_1, \omega_1 \]
\[ R_2, Q_2, \omega_2 \]
\[ R_3, Q_3, \omega_3 \]

\[ n_1 \]
\[ n_2 \]
\[ n_3 \]

\[ I_B \]
Higher order modes (measured spectrum)

without dampers

with dampers
Pillbox: dipole mode

\[ \text{TM}_{110}-\text{mode} \quad \text{(only 1/4 shown)} \]

electric field

magnetic field
CERN/PS 80 MHz cavity (for LHC)

inductive (loop) coupling, low self-inductance
Higher order modes

Example shown: 80 MHz cavity PS for LHC.
Color-coded:
What do you gain with many gaps?

• The $R/Q$ of a single gap cavity is limited to some 100 $\Omega$. Now consider to distribute the available power to $n$ identical cavities: each will receive $P/n$, thus produce an accelerating voltage of $\sqrt{2RP/n}$. The total accelerating voltage thus increased, equivalent to a total equivalent shunt impedance of $nR$.

\[
|V_{\text{acc}}| = n \sqrt{2R \frac{P}{n}} = \sqrt{2(nR)P}
\]
Standing wave multicell cavity

- Instead of distributing the power from the amplifier, one might as well couple the cavities, such that the power automatically distributes, or have a cavity with many gaps (e.g. drift tube linac).

- Coupled cavity accelerating structure (side coupled)

- The phase relation between gaps is important!
Brillouin diagram
Travelling wave structure

speed of light line, \( \omega = \beta / c \)

synchronous

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Examples of cavities

PEP II cavity
476 MHz, single cell,
1 MV gap with 150 kW,
strong HOM damping,

LEP normal-conducting Cu RF cavities,
350 MHz. 5 cell standing wave + spherical cavity
for energy storage, 3 MV
CERN PS 200 MHz cavities
PS 19 MHz cavity (prototype, photo: 1966)
CERN PS 80 MHz Cavity (1997)
Ferrite cavity – CERN PSB, 0.6 ... 1.8 MHz

PS Booster, ‘98
0.6 – 1.8 MHz,
< 10 kV gap
NiZn ferrites
CERN PS 10 MHz cavity (1 of 10)
Drift-tube linac (JPARC JHF, 324 MHz)
CERN SPS 200 MHz TW cavity
Travelling wave cavities

CLIC “T18”, 12 GHz

CLIC “HDS”, 12 GHz

“Shintake” structure, 5.7 GHz
Side-coupled cavity (JHF, 972 MHz)
Single- and multi-cell SC cavities (1.3 GHz)
SC cavities in a cryostat (CERN LHC 400 MHz)
SC deflecting cavity (KEK-B, 508 MHz)

Asymmetric shape to split the two polarizations.
Summary  RF Cavities

• The EM fields inside a hollow cavity are superpositions of homogeneous plane waves.
• When operating near an eigenfrequency, one can profit from a resonance phenomenon (with high $Q$).
• $R$-upon-$Q$, Shunt impedance and $Q$ factor were are useful parameters, which can also be understood in an equivalent circuit.
• The perturbation method allows to estimate losses and sensitivity to tolerances.
• Many gaps can increase the effective impedance.