Resonances

- introduction: driven oscillators and resonance condition
- smooth approximation for motion in accelerators
- field imperfections and normalized field errors
- perturbation treatment
- Poincare section
- stabilization via amplitude dependent tune changes
- sextupole perturbation & slow extraction
- chaotic particle motion
Introduction: Damped Harmonic Oscillator

equation of motion for a damped harmonic oscillator:

\[ \frac{d^2}{dt^2} w(t) + \omega_0 \cdot Q^{-1} \cdot \frac{d}{dt} w(t) + \omega_0^2 \cdot w(t) = 0 \]

Q is the damping coefficient

(\text{amplitude decreases with time})

\(\omega_0\) is the Eigenfrequency of the HO

example: weight on a spring \((Q = \infty)\)

\[ \frac{d^2}{dt^2} w(t) + k \cdot w(t) = 0 \quad \rightarrow \quad w(t) = a \cdot \sin(\sqrt{k} \cdot t + \phi_0) \]
Introduction: Driven Oscillators

an external driving force can ‘pump’ energy into the system:

\[ \frac{d^2}{dt^2} w(t) + \omega_0 \cdot Q^{-1} \cdot \frac{d}{dt} w(t) + \omega_0^2 \cdot w(t) = \frac{F}{m} \cdot \cos(\omega \cdot t) \]

general solution:

\[ w(t) = w_{tr}(t) + w_{st}(t) \]

stationary solution:

\[ w_{st}(t) = W(\omega) \cdot \cos[\omega \cdot t - \alpha(\omega)] \]

where ‘\( \omega \)’ is the driving angular frequency!

and \( W(\omega) \) can become large for certain frequencies!
stationary solution

stationary solution follows the frequency of the driving force:

\[ w_{st}(t) = W(\omega) \cdot \cos[\omega \cdot t - \alpha(\omega)] \]

oscillation amplitude can become large for weak damping
Introduction: Pulsed Driven Resonances Example

higher harmonics:

example of a bridge:

2\textsuperscript{nd} harmonic: 3\textsuperscript{rd} harmonic: 4\textsuperscript{th} harmonic:

peak amplitude depends on the excitation frequency and damping
Introduction: Instabilities

- resonance catastrophe without damping:

\[ W(\omega) = W(0) \cdot \frac{1}{\sqrt{[1-(\frac{\omega}{\omega_0})^2]^2 + (\frac{\omega}{Q\omega_0})^2}} \]

- weak damping: resonance condition: \( \omega = \omega_0 \)

Tacoma Narrow bridge
1940

excitation by strong wind on the Eigenfrequencies
Smooth Approximation: Resonances in Accelerators

**revolution frequency:**

\[ \omega_{\text{rev}} = 2\pi f_{\text{rev}} \]

**betatron oscillations:**

**Eigenfrequency:** \( \omega_0 = 2\pi f_\beta \)

\[ Q = \frac{\omega_0}{\omega_{\text{rev}}} \]

**excitation with** \( f_{\text{rev}} \)

**periodic kick**

**driven oscillator**

**weak or no damping!**

(synchrotron radiation damping (single particle) or Landau damping distributions)
Smooth Approximation: Free Parameter

co-moving coordinate system:

choose the longitudinal coordinate as the free parameter for the equations of motion

equations of motion:

\[ \frac{d}{dt} = \frac{ds}{dt} \cdot \frac{d}{ds} \]

with:

\[ \frac{ds}{dt} = v \]

\[ \frac{d^2}{dt^2} = v^2 \cdot \frac{d^2}{ds^2} \]
Smooth Approximation: Equation of Motion I

Smooth approximation for Hills equation:

$$\frac{d^2}{ds^2} w(s) + K(s) \cdot w(s) = 0 \quad \text{K(s) = const} \quad \frac{d^2}{ds^2} w(s) + \omega_0^2 \cdot w(s) = 0$$

(constant $\beta$-function and phase advance along the storage ring)

$$w(s) = A \cdot \cos(\omega_0 \cdot s + \phi_0) \quad \omega_0 = 2\pi \cdot Q_0 / L$$

(Q is the number of oscillations during one revolution)

perturbation of Hills equation:

$$\frac{d^2}{ds^2} w(s) + \omega_0^2 \cdot w(s) = F(w(s), s) / (v \cdot p)$$

in the following the force term will be the Lorenz force of a charged particle in a magnetic field:

$$F = q \cdot \vec{v} \times \vec{B}$$
Field Imperfections: Origins for Perturbations

- **linear magnet imperfections:** derivation from the design dipole and quadrupole fields due to powering and alignment errors

- **time varying fields:** feedback systems (damper) and wake fields due to collective effects (wall currents)

- **non-linear magnets:** sextupole magnets for chromaticity correction and octupole magnets for Landau damping

- **beam-beam interactions:** strongly non-linear field!

- **non-linear magnetic field imperfections:** particularly difficult to control for super conducting magnets where the field quality is entirely determined by the coil winding accuracy
Field Imperfections: Localized Perturbation

periodic delta function:

$$\delta_L(s - s_0) = \begin{cases} 1 & \text{for } 's' = s_0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\int \delta_L(s - s_0) ds = 1$$

equation of motion for a single perturbation in the storage ring:

$$\frac{d^2}{ds^2} w(s) + \omega_0^2 \cdot w(s) = \delta_L(s - s_0) \cdot l \cdot F(w, s) / (v \cdot p)$$

Fourier expansion of the periodic delta function:

$$\frac{d^2}{ds^2} w(s) + \omega_0^2 \cdot w(s) = \frac{1}{L} \sum_{r=-\infty}^{\infty} \cos(r \cdot 2\pi \cdot s / L) \cdot F(w, s) / (v \cdot p)$$

infinite number of driving frequencies
Field Imperfections: Constant Dipole

normalized field error:
\[
\frac{F}{v \cdot p} = q \cdot \frac{\vec{v} \times \vec{B}}{v \cdot p} \quad \text{\(\rightarrow\) } q \cdot B / p = k_0
\]

equation of motion for single kick:
\[
\frac{d^2}{ds^2} w(s) + \omega_0^2 \cdot w(s) = \frac{lk_0}{L} \sum_{r=-\infty}^{\infty} \cos(r \cdot 2\pi \cdot s / L)
\]

resonance condition:
\[
\omega_0 = r \cdot 2\pi / L \quad \text{\(\rightarrow\) } Q_0 = r
\]

avoid integer tunes!

remember the example of a single dipole imperfection from the ‘Linear Imperfection’ lecture yesterday!
Field Imperfections: Constant Quadrupole

equations of motion:

\[
\frac{d^2}{ds^2} x(s) + \omega_x^2 \cdot x(s) = k_1 \cdot x(s)
\]

\[y(s) \equiv 0\]

\[k_1 = \frac{q}{p} \cdot \frac{\partial B_y}{\partial x}\]

\[
\frac{d^2}{ds^2} x(s) + (\omega_x^2 - k_1) \cdot x(s) = 0
\]

change of tune but no amplitude growth due to resonance excitations!
assume $y = 0$ and $B_x = 0$:  

$$ F(s)/(v \cdot p) = \delta_L (s - s_0) \cdot l \cdot k_1 \cdot x $$

$$ \frac{d^2}{ds^2} x(s) + \omega_{x,0}^2 \cdot x(s) = \frac{lk_1}{L} \sum_{r=-\infty}^{\infty} \cos(2\pi \cdot r \cdot s / L) \cdot x(s) $$

$$ \left[ x(s) = A \cdot \cos(\omega_0 \cdot s) \right] = \frac{lk_1}{2L} \sum_{r=-\infty}^{\infty} \cos(2\pi \cdot r \cdot s / L \pm \omega_0 \cdot s) \cdot x(s) $$

resonance condition:  

$$ \omega_{x,0} = r \cdot 2\pi / L \pm \omega_{x,0} \quad \omega_0 = 2\pi \cdot Q_0 / L \quad Q_0 = r / 2 $$

avoid half integer tunes plus resonance width from tune modulation!

exact solution: variation of constants \(\rightarrow\) see the lecture yesterday
Field Imperfections: Time Varying Dipole Perturbation

time varying perturbation:

$$F(t) = F_0 \cdot \cos(\omega_{kick} \cdot t) \xrightarrow{\text{t} \rightarrow \text{s}} F_0 \cdot \cos(2\pi \cdot \frac{\omega_{kick}}{\omega_{rev}} \cdot s / L) / (v \cdot p)$$

$$\frac{d^2}{ds^2} w(s) + \omega_0^2 \cdot w(s) = \frac{lF_0}{2L} \sum_{r=-\infty}^{\infty} \cos(2\pi \cdot [r \pm \omega_{kick} / \omega_{rev}] \cdot s / L) / (v \cdot p)$$

resonance condition:

$$\omega_0 = 2\pi \cdot (r \pm \omega_{kick} / \omega_{rev}) / L \xrightarrow{\omega_0 = 2\pi Q_0 / L} f_{kick} = f_{rev} \cdot (Q_0 \pm r)$$

avoid excitation on the betatron frequency!

(the integer multiple of the revolution frequency corresponds to the modes of the bridge in the introduction example)
Field Imperfections: Several Bunches

\[ F(t) = B \cdot \cos(\omega_{\text{kick}} \cdot t); \omega_{\text{kick}} \approx \omega_{\text{rev}} : \]

higher modes analogous to bridge illustration

\[ F(t) = B \cdot \cos(\omega_{\text{kick}} \cdot t); \omega_{\text{kick}} \approx 2 \cdot \omega_{\text{rev}} : \]

higher modes analogous to bridge illustration
Field Imperfections: Multipole Expansion

Taylor expansion of the magnetic field:

\[ B_y + iB_x = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot f_n \cdot (x + iy)^n \]

with:

\[ f_n = \frac{\partial^{n+1} B_y}{\partial x^{n+1}} \]

<table>
<thead>
<tr>
<th>multipole</th>
<th>order</th>
<th>( B_x )</th>
<th>( B_y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>dipole</td>
<td>0</td>
<td>0</td>
<td>( B_0 )</td>
</tr>
<tr>
<td>quadrupole</td>
<td>1</td>
<td>( f_1 \cdot y )</td>
<td>( f_1 \cdot x )</td>
</tr>
<tr>
<td>sextupole</td>
<td>2</td>
<td>( f_2 \cdot x \cdot y )</td>
<td>( \frac{1}{2} \cdot f_2 \cdot (x^2 - y^2) )</td>
</tr>
<tr>
<td>octupole</td>
<td>3</td>
<td>( \frac{1}{6} \cdot f_3 \cdot (3yx^2 - y^3) )</td>
<td>( \frac{1}{6} \cdot f_3 \cdot (x^3 - 3xy^2) )</td>
</tr>
</tbody>
</table>

normalized multipole gradients:

\[ \frac{F(s)/(v \cdot p)}{(v \cdot p)} = \frac{q \cdot (\vec{v} \times \vec{B})}{(v \cdot p)} \quad k_n = \frac{q}{p} \cdot f_n \quad k_n = 0.3 \cdot \frac{f_n [T/m^n]}{p [GeV/c]} \quad [k_n] = \frac{1}{m^{n+1}} \]
Field Imperfections: DipoleMagnets

dipole magnet designs:

LEP dipole magnet:
conventional magnet design relying on pole face accuracy of a Ferromagnetic Yoke

LHC dipole magnet:
air coil magnet design relying on precise current distribution
Field Imperfections: Multipole Illustration

upright and skew field errors

upright:

n=0

n=1

n=2

skew:
Field Imperfections: Multipole Illustrations

quadrupole and sextupole magnets

ISR quadrupole

LEP Sextupole
Field Imperfections: Super Conducting Magnets

time varying field errors in super conducting magnets

Luca Bottura CERN, AT-MAS
Perturbation Treatment: Resonance Condition

Equations of motion:

\[ \frac{d^2}{ds^2} w(s) + \omega_0^2 \cdot w(s) = \varepsilon \cdot \sum_{l+m<n} \alpha_{n,m,r} \cdot x^l \cdot y^m \cdot \cos(2\pi \cdot r \cdot s / L) \]

with:

\[ w(x, y) \]

Perturbation treatment:

\[ w(s) = w_0 + \varepsilon \cdot w_1 + \varepsilon^2 w_2 + \ldots + O(\varepsilon^n) \]

\[ \omega_0 = \frac{2\pi}{L} Q_0 \]

with:

\[ w_0(s) = w_0 \cdot \cos(2\pi \cdot Q_0 \cdot s / L + \phi_0) \]

with:

\[ w(x) \]

\[ \frac{d^2}{ds^2} x_1 + \omega_0^2 \cdot x_1 = \sum_{\tilde{l} < l, \tilde{m} < m} \alpha_{\tilde{n}, \tilde{m}, r} \cos\left(\frac{2\pi}{L} \cdot [\tilde{l} Q_{x,0} + \tilde{m} Q_{y,0} + r] \cdot s\right) \]
Perturbation Treatment: Tune Diagram I

resonance condition: \[ l \cdot Q_x + m \cdot Q_y = r \]

avoid rational tune values!

there are resonances everywhere!
(the rational numbers lie dens within the real number)

\[
\frac{2\pi}{L} \cdot (\tilde{l} \cdot Q_x + \tilde{m} \cdot Q_y + r) = \frac{2\pi}{L} \cdot Q_{x,y}
\]
Perturbation Treatment: Tune Diagram II

regions with few resonances:

\[ l \cdot Q_x + m \cdot Q_y = r \]

avoid low order resonances!

< 12\text{th} order for a proton beam without damping

< 3\text{rd} \Leftrightarrow 5\text{th} order for electron beams with damping

coupling resonance:

regions without low order resonances are relatively small!
Perturbation Treatment: Single Sextupole Perturbation

perturbed equations of motion: \( F(s)/(\nu \cdot p) = \frac{1}{2} \cdot \delta_L (s-s_0) \cdot lk_2 \cdot x^2 \)

\[ \frac{d^2}{ds^2} x_1(s) + \omega_0^2 \cdot x_1(s) = \frac{1}{2} \cdot lk_2 \cdot x_0^2 \cdot \frac{1}{L} \sum_{r=-\infty}^{\infty} \cos(2\pi \cdot r \cdot s / L) \]

with: \( x_0(s) = A \cdot \cos(\omega_{0,x} \cdot s + \phi_0) \) and \( \omega_{0,x} = 2\pi \cdot Q_{x,0} / L \)

\[ \frac{d^2}{ds^2} x_1(s) + (2\pi Q_{x,0} / L)^2 \cdot x_1(s) = \frac{lk_1}{2L} \cdot A^2 \cdot \sum_{r=-\infty}^{\infty} \cos(2\pi \cdot r \cdot s / L) \]

\[ + \frac{lk_1}{8L} \cdot A^2 \cdot \sum_{r=-\infty}^{\infty} \cos(2\pi \cdot [r \pm 2Q_{x,0}] \cdot s / L) \]
Perturbation Treatment: Sextupole Perturbation

**Resonance Conditions:**

\[
2\pi Q_{x,o} = 2\pi \cdot (r) \rightarrow Q_{x,0} = r
\]

\[
2\pi Q_{x,o} = 2\pi \cdot (r \pm 2Q_{x,0}) \rightarrow Q_{x,0} = \frac{r}{3}
\]

\[
Q_{x,0} = r
\]

Avoid integer and r/3 tunes!

**Perturbation Treatment:**

Contrary to the previous examples, no exact solution exists!

This is a consequence of the non-linear perturbation

(remember the 3 body problem?)

⇒ graphic tools for analyzing the particle motion
Poincare Section: Definition

Poincare Section:

- record the particle coordinates at one location in the storage ring

resonance in the Poincare section:

\[ \Delta \phi_{\text{turn}} = 2\pi \cdot Q \]

fixed point condition: \( Q = n/r \)

points are mapped onto themselves after ‘r’ turns
Poincare Section: Linear Motion

unperturbed solution:

\[ x(s) = \sqrt{R} \cdot \cos(\phi) \]

with \( \frac{d}{ds} \phi = \omega_0 \)

\[ x' = \frac{d}{ds} x = -\sqrt{R} \cdot \omega_0 \cdot \sin(\phi) \]

phase space portrait:

- the motion lies on an ellipse
- linear motion is described by a simple rotation
- consecutive intersections lie on closed curves
Poincare Section: Non-Linear Motion

momentum change due to perturbation:
\[ \Delta x' = \int \frac{F(s)}{v \cdot p} \cdot ds \]

single n-pole kick:
\[ \Delta x' = \frac{1}{n!} \cdot l k_n \cdot x^n \]

phase space portrait with single sextupole:
\[ \Delta x' = \frac{1}{2} \cdot l k_2 \cdot x^2 \]

sextupole kick changes the amplitude and the phase advance per turn!
\[ \Delta Q_{\text{turn}} \propto x^2 \]
Poincare Section: Stability?

- instability can be fixed by ‘detuning’:
  - overall stability depends on the balance between amplitude increase per turn and tune change per turn:
    - \( \Delta Q_{\text{turn}}(x) \) ➔ motion moves eventually off resonance
    - \( \Delta R_{\text{turn}}(x) \) ➔ motion becomes unstable

- sextupole kick:
  - amplitudes increases faster then the tune can change
  - ➔ overall instability!
Poincare Section: Illustration of Topology

Poincare section:

- Fixed points and separatrix
- \( Q < r/3 \)

Small amplitudes:
- Regular motion

Large amplitudes:
- Instability & particle loss

Fixed points and separatrix
- Border between stable and unstable motion
- Chaotic motion

Equation:
\[
F(s)/((v \cdot p)) = \frac{1}{2} \cdot \delta_L (s - s_0) \cdot l k_2 \cdot x^2
\]
Poincare Section: Simulations for a Sextupole Perturbation

- Poincare Section right after the sextupole kick

  - for small amplitudes the intersections still lie on closed curves ➔ regular motion!

  - separatrix location depends on the tune distance from the exact resonance condition (Q < n/3)

- for large amplitudes and near the separatrix the intersections fill areas in the Poincare Section ➔ chaotic motion;
  ➔ no analytical solution exist!
Stabilization of Resonances

Instability can be fixed by stronger ‘detuning’:

- If the phase advance per turn changes uniformly with increasing $R$, the motion moves off resonance and stabilizes.

Octupole perturbation:

- $F(s)/(v \cdot p) = \frac{1}{6} \cdot l k_3 \cdot x^3$

Perturbation treatment:

- $x(s) = x_0(s) + \varepsilon \cdot x_1(s) + \ldots$

\[
\frac{d^2}{ds^2} x_1(s) + \left(\frac{2 \pi Q_{x,0}}{L}\right)^2 \cdot x_1(s) = \frac{1}{6} \cdot l k_3 \cdot x_0^2 \cdot x_1
\]

\[
x_0 = A \cdot \cos(\omega_0 \cdot s + \phi_0) \Rightarrow x_0^2 = \frac{A^2}{2} [1 + \cos(2\omega_0 \cdot s + 2\phi_0)]
\]

\[
\frac{d^2}{ds^2} x_1(s) + \left[\left(\frac{2 \pi Q_{x,0}}{L}\right)^2 - \frac{A^2 \cdot l k_3}{2 \cdot 6}\right] \cdot x_1(s) = \frac{A^2 \cdot l k_3}{2 \cdot 6} \cdot \cos(2\omega_0 \cdot s) \cdot x_1
\]
Stabilization of Resonances

resonance stability for octupole:

- an octupole perturbation generates phase-independent detuning and amplitude growth of the same order

- amplitude growth and detuning are balanced and the overall motion is stable!

- this is not generally true in case of several resonance driving terms and coupling between the horizontal and vertical motion!
Chaotic Motion

→ Octupole + sextupole perturbation:

→ The interference of the octupole and sextupole perturbations generate additional resonances → Additional island chains in the Poincare Section!

→ Intersections near the resonances lie no longer on closed curves → Local chaotic motion around the separatrix & instabilities → Slow amplitude growth (Arnold diffusion)

→ Neighboring resonance islands start to ‘overlap’ for large amplitudes → Global chaos & fast instabilities
Chaotic Motion

‘Russian Doll’ effect:

$\Rightarrow$ magnifying sections of the Poincare Section reveals always the same pattern on a finer scale $\Rightarrow$ renormalization theory!
Summary

- field imperfections drive resonances
- higher order than quadrupole field imperfections generate non-linear equations of motion (no closed analytical solution)
  (three body problem of Sun, Earth and Jupiter)
  - solutions only via perturbation treatment
- Poincare Section as a graphical tool for analyzing the stability
- slow extraction as example of resonance application in accelerator
- island chains as signature for non-linear resonances
- island overlap as indicator for globally chaotic & unstable motion