Just to give a kind of definition ...

A synchrotron is a type of circular accelerator that needs:

- **A bending field**
  - to keep the particles on a closed orbit
- **A mechanism to lock this B-field to the particle energy**
  - → constant orbit
- **Focusing fields** that follow the energy gain to keep the particles together
  - → well defined beam size
- **A RF structure** to accelerate the particles
  - → energy gain per turn
- **A mechanism to synchronise the rf frequency to the particle timing**
  - → phase focusing / synchrotron principle

ADA, Frascati
LHC, 27km, 1232 dipole magnets, 7 TeV
**Introduction and Basic Ideas**

“... in the end and after all it should be a kind of circular machine“

→ need transverse deflecting force

Lorentz force

\[
\vec{F} = q*(\vec{E} + \vec{v} \times \vec{B})
\]

typical velocity in high energy machines:

\[
\nu \approx c \approx 3 \times 10^8 \text{ m/s}
\]

Example:

\[
B = 1 \text{ T} \quad \rightarrow \quad F = q*3 \times 10^8 \frac{m}{s} * 1 \frac{Vs}{m^2}
\]

\[
F = q*300 \frac{MV}{m}
\]

\[
equivalent \text{ el field} \quad E
\]

\[
technical \text{ limit for a el. field} \quad E \leq 1 \frac{MV}{m}
\]

Unlike to a cyclotron, we localise & optimise the magnets at the location of the beam ... to keep the machine “small”.
**The ideal circular orbit**

Condition for circular orbit:

**Lorentz force**

\[ F_L = e \gamma m_0 v^2 B \]

**Centrifugal force**

\[ F_{\text{centr}} = \frac{\gamma m_0 v^2}{\rho} \]

\[ \frac{\gamma m_0 v^2}{\rho} = e \sqrt{B} \]

\[ \frac{p}{e} = B \rho \]

\[ B \rho = \text{"beam rigidity"} \]

---

**The arrangement and strength of the dipole magnets define the maximum particle momentum that can be carried by the synchrotron.**

**LHC:** 7000 GeV Proton storage ring

dipole magnets \( N = 1232 \)

\[ l = 15 \text{ m} \]

\[ q = +1 \text{ e} \]

\[ \int B \, dl \approx N \, l \, B = 2\pi \, \frac{p}{e} \]

\[ B \approx \frac{2\pi \times 7000 \times 10^9 \text{ eV}}{1232 \times 15 \text{ m} \times 3 \times 10^8 \text{ m/s}} = 8.3 \text{ Tesla} \]
**The Magnetic Guide Field**

Normalise magnetic field to momentum:

\[
\frac{p}{e} = B \rho \quad \Rightarrow \quad \frac{1}{\rho} = \frac{eB}{p}
\]

**Example LHC:**

\[
B = 8.3 \, T \\
p = 7000 \, \frac{GeV}{c}
\]  

\[\rho = 2.53 \, km\]

**Dipole Magnets:**

define the ideal orbit via their **homogeneous** field, which is created by two flat pole shoes

convenient units:

\[
B = [T] = \left[ \frac{Vs}{m^2} \right] \quad p = \left[ \frac{GeV}{c} \right]
\]

field map of a storage ring dipole magnet
Focusing Properties - Transverse Beam Optics

Classical Mechanics: pendulum

there is a **restoring force**, proportional to the elongation $x$:

$$ F = m \frac{d^2 x}{dt^2} = -k x $$

**Ansatz**

$$ x(t) = A \cos(\omega t + \varphi) $$

general solution: free harmonic oscillation

Storage Ring: we need a Lorentz force that rises as a function of the distance to the design orbit

$$ F(x) = q v B(x) $$

Quadrupole Magnets:

required: focusing forces to keep trajectories in vicinity of the ideal orbit

linear increasing Lorentz force

linear increasing magnetic field

$$ B_y = g x \quad B_x = g y $$
Normalised quadrupole field:

Field in a quadrupole \( B_y = g \ x \) \( B_x = g \ y \)

Normalised gradient of a quadrupole \( k = \frac{g}{p / e} \)

what about the vertical plane: \( \nabla \times \vec{B} = \vec{j} + \frac{\partial \vec{E}}{\partial t} \)

\[ \Rightarrow \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y} \]

The Equation of Motion:

\[ \frac{B(x)}{p / e} = \frac{1}{\rho} + k \ x + \frac{1}{2!} m x^2 + \frac{1}{3!} n x^3 + \ldots \]

only terms linear in \( x, y \) taken into account
dipole fields
quadrupole fields

Separate Function Machines: Split the magnets and optimise them according to their job
The Equation of Motion:

Equation for the horizontal motion of a particle inside a storage ring magnet:

\[ x'' + x \left( \frac{1}{\rho^2} - k \right) = 0 \]

Equation for the vertical motion:

\[ \frac{1}{\rho^2} = 0 \quad \text{no dipoles ... in general ...} \]

\[ k \leftrightarrow -k \quad \text{quadrupole field changes sign} \]

\[ y'' + k \; y = 0 \]

A magnet in a synchrotron that acts as hor. focusing lens, has at the same time, a defocusing effect in the vertical plane. *Et vice versa.*
**Solution of Trajectory Equations**

Define ... hor. plane: \( K = \frac{1}{\rho^2} - k \)

... vert. Plane: \( K = k \)

Differential Equation of harmonic oscillator ... with spring constant \( K \)

**Ansatz:** \( x(s) = a_1 \cdot \cos(\omega s) + a_2 \cdot \sin(\omega s) \)

**general solution:** linear combination of two independent solutions

\[
x'(s) = -a_1 \omega \sin(\omega s) + a_2 \omega \cos(\omega s)
\]

\[
x''(s) = -a_1 \omega^2 \cos(\omega s) - a_2 \omega^2 \sin(\omega s) = -\omega^2 x(s) \quad \rightarrow \quad \omega = \sqrt{K}
\]

**general solution:**

\[
x(s) = a_1 \cos(\sqrt{K}s) + a_2 \sin(\sqrt{K}s)
\]
Hor. Focusing Quadrupole $K > 0$:

$$x(s) = x_0 \cdot \cos(\sqrt{|K|}s) + x_0' \cdot \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s)$$

$$x'(s) = -x_0 \cdot \sqrt{|K|} \cdot \sin(\sqrt{|K|}s) + x_0' \cdot \cos(\sqrt{|K|}s)$$

For convenience expressed in matrix formalism:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_1} = \left( M_{\text{foc}} \right) \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

$$M_{\text{foc}} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s) \\ -\sqrt{|K|} \sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}$$

Determine $a_1, a_2$ by boundary conditions:

$$s = 0 \quad \Rightarrow \quad \begin{cases} x(0) = x_0, & a_1 = x_0 \\ x'(0) = x_0', & a_2 = \frac{x_0'}{\sqrt{|K|}} \end{cases}$$
**hor. defocusing quadrupole:**

\[ x'' - K \, x = 0 \]

Remember from school:

\[ f(s) = \cosh(s) \quad , \quad f'(s) = \sinh(s) \]

**Ansatz:**

\[ x(s) = a_1 \cdot \cosh(\omega s) + a_2 \cdot \sinh(\omega s) \]

\[
M_{\text{defoc}} = \begin{pmatrix}
\cosh \sqrt{|K|} l & \frac{1}{\sqrt{|K|}} \sinh \sqrt{|K|} l \\
\sqrt{|K|} \sinh \sqrt{|K|} l & \cosh \sqrt{|K|} l
\end{pmatrix}
\]

**drift space:**

\[ K = 0 \]

\[
M_{\text{drift}} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}
\]

! with the assumptions made, the motion in the horizontal and vertical planes are independent „... the particle motion in \( x \) & \( y \) is uncoupled“
Transformation through a system of lattice elements

combine the single element solutions by multiplication of the matrices

\[
M_{\text{total}} = M_{QF} \times M_D \times M_{QD} \times M_{\text{Bend}} \times M_D^{*} \ldots
\]

\[
\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = M(s_2,s_1) \times \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1}
\]

in each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator „

\[
x(s) = M(s_2,s_1) \times x(s_1)
\]

typical values in a strong foc. machine:
\[x \approx \text{mm}, x' \leq \text{mrad}\]
**Orbit & Tune:**

**Tune:** number of oscillations per turn

- 64.31
- 59.32

**Relevant for beam stability:**

*non integer part*

**LHC revolution frequency:** 11.3 kHz

\[ 0.31 \times 11.3 = 3.5 \text{kHz} \]
Question: what will happen, if the particle performs a second turn?

... or a third one or ... $10^{10}$ turns
Astronomer Hill:

differential equation for motions with periodic focusing properties
„Hill ‘s equation“

Example: particle motion with periodic coefficient

equation of motion:

\[ x''(s) - k(s)x(s) = 0 \]

restoring force ≠ const,  
k(s) = depending on the position s  
k(s+L) = k(s), periodic function

we expect a kind of quasi harmonic oscillation: amplitude & phase will depend on the position s in the ring.
The Beta Function

General solution of Hill’s equation:

\[(i)\quad x(s) = \varepsilon \sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi)\]

\(\varepsilon, \Phi =\) integration constants determined by initial conditions

\(\beta(s)\) periodic function given by focusing properties of the lattice \(\leftrightarrow\) quadrupoles

\[\beta(s + L) = \beta(s)\]

Inserting (i) into the equation of motion ...

\[\psi(s) = \int_0^s \frac{ds}{\beta(s)}\]

\(\Psi(s) = \) „phase advance“ of the oscillation between point „0“ and „s“ in the lattice. For one complete revolution: number of oscillations per turn „Tune“

\[Q = \frac{1}{2\pi} \Phi \frac{ds}{\beta(s)}\]
The Beta Function

Amplitude of a particle trajectory:

\[ x(s) = \sqrt{\varepsilon} \cdot \sqrt{\beta(s)} \cdot \cos(\psi(s) + \varphi) \]

Maximum size of a particle amplitude

\[ \hat{x}(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \]

\( \beta \) determines the beam size ( ... the envelope of all particle trajectories at a given position “s” in the storage ring.

It reflects the periodicity of the magnet structure.
Beam Emittance and Phase Space Ellipse

General solution of Hill equation

\[
\begin{aligned}
(1) \quad x(s) &= \sqrt{\varepsilon} \sqrt{\beta(s)} \cos(\psi(s) + \phi) \\
(2) \quad x'(s) &= -\frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}} \left\{ \alpha(s) \cos(\psi(s) + \phi) + \sin(\psi(s) + \phi) \right\}
\end{aligned}
\]

From (1) we get

\[
\cos(\psi(s) + \phi) = \frac{x(s)}{\sqrt{\varepsilon} \sqrt{\beta(s)}}
\]

Insert into (2) and solve for \( \varepsilon \)

\[
\varepsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'^2(s)
\]

* \( \varepsilon \) is a constant of the motion … it is independent of „s“
* Parametric representation of an ellipse in the \( x \, x' \) space
* Shape and orientation of ellipse are given by \( \alpha, \beta, \gamma \)
**Beam Emittance and Phase Space Ellipse**

\[ \varepsilon = \gamma(s) \ x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s) \ x'^2(s) \]

\[ \text{Liouville: in reasonable storage rings} \]
\[ \text{area in phase space is constant.} \]
\[ A = \pi \varepsilon = \text{const} \]

\[ \varepsilon \text{ beam emittance} = \text{wooziycity of the particle ensemble, intrinsic beam parameter, cannot be changed by the foc. properties.} \]

**Scientifically speaking:** area covered in transverse \( x, x' \) phase space ... and it is constant !!!
Emittance of the Particle Ensemble:
**Phase Space Ellipse**

**particle trajectory:** \[ x(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cos\{\psi(s) + \phi\} \]

**max. Amplitude:** \[ \hat{x}(s) = \sqrt{\varepsilon \beta} \rightarrow x' \text{ at that position} \ldots? \]

... put \( \hat{x}(s) \) into

\[
\varepsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'^2(s)
\]

and solve for \( x' \)

\[
\varepsilon = \gamma \cdot \varepsilon \beta + 2\alpha \sqrt{\varepsilon \beta} \cdot x' + \beta x'^2
\]

\[ x' = -\alpha \cdot \sqrt{\varepsilon / \beta} \]

\* The optical functions determine the shape and orientation of the phase space ellipse.

\* A high \( \beta \)-function means a large beam size and a small beam divergence.

... et vice versa !!!
Emittance of the Particle Ensemble:

\[ x(s) = \sqrt{\epsilon} \sqrt{\beta(s)} \cdot \cos(\Psi(s) + \phi) \]
\[ \hat{x}(s) = \sqrt{\epsilon} \sqrt{\beta(s)} \]

single particle trajectories, \( N \approx 10^{11} \) per bunch

**LHC:**
\[ \beta = 180 \text{ m} \]
\[ \epsilon = 5 \times 10^{-10} \text{ m rad} \]
\[ \sigma = \sqrt{\epsilon \beta} = \sqrt{5 \times 10^{-10} \text{ m} \times 180 \text{ m}} = 0.3 \text{ mm} \]

Gauß Particle Distribution:
\[ \rho(x) = \frac{N \cdot e^{-\frac{1}{2} \frac{x^2}{\sigma_x^2}}}{\sqrt{2\pi\sigma_x}} \]

particle at distance 1 \( \sigma \) from centre \rightarrow 68.3 \% of all beam particles

aperture requirements: \( r_0 = 12 \times \sigma \)
13.) Errors in Field and Gradient:
Dispersion: trajectories for $\Delta p / p \neq 0$

Forces acting on the particle

Radial acceleration
\[
a_r = \frac{d^2(\rho + x)}{dt^2} - (\rho + x) \left( \frac{d\theta}{dt} \right)^2
\]

Is counter vailed by the Lorenz force
\[
F = m \frac{d^2}{dt^2}(x + \rho) - \frac{mv^2}{x + \rho} = eB_y v
\]

remember: $x \approx mm, \rho \approx m \rightarrow$ develop for small $x$

\[
m \frac{d^2x}{dt^2} - \frac{mv^2}{\rho} \left( 1 - \frac{x}{\rho} \right) = eB_y v
\]

consider only linear fields, and change independent variable: $t \rightarrow s$

\[
B_y = B_0 + x \frac{\partial B_y}{\partial x}
\]

\[
x'' - \frac{1}{\rho} \left( 1 - \frac{x}{\rho} \right) = \frac{e}{mv} B_0 + \frac{e}{mv} x g
\]

... but now take a small momentum error into account !!!!
Dispersion:

develop for small momentum error \[ \Delta p << p_0 \Rightarrow \frac{1}{p_0 + \Delta p} \approx \frac{1}{p_0} - \frac{\Delta p}{p_0^2} \]

\[
x'' - \frac{1}{\rho} + \frac{x}{\rho^2} \approx \frac{e B_0}{p_0} - \frac{\Delta p}{p_0^2} eB_0 + \frac{x \varepsilon_g}{p_0} - \varepsilon_g \frac{\Delta p}{p_0} \]

\[\underbrace{- \frac{1}{\rho}}_{k \times x} \approx 0\]

\[
x'' + \frac{x}{\rho^2} \approx \frac{\Delta p}{p_0} \cdot (-eB_0) + k \cdot x = \frac{\Delta p}{p_0} \cdot \frac{1}{\rho} + k \cdot x \]

\[\underbrace{\frac{1}{\rho}}_{1}

\[
x'' + \frac{x}{\rho^2} - kx = \frac{\Delta p}{p_0} \cdot \frac{1}{\rho} \quad \Rightarrow \quad x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p_0} \cdot \frac{1}{\rho}
\]

Momentum spread of the beam adds a term on the r.h.s. of the equation of motion.
\rightarrow inhomogeneous differential equation.
**Dispersion:**

\[ x'' + x(\frac{1}{\rho^2} - k) = \frac{\Delta p}{p} \cdot \frac{1}{\rho} \]

**general solution:**

\[ x(s) = x_h(s) + x_i(s) \]

\[
\begin{align*}
  x''_h(s) + K(s) \cdot x_h(s) &= 0 \\
  x''_i(s) + K(s) \cdot x_i(s) &= \frac{1}{\rho} \cdot \frac{\Delta p}{p}
\end{align*}
\]

Normalise with respect to \( \Delta p/p \):

\[ D(s) = \frac{x_i(s)}{\Delta p/p} \]

---

**Dispersion function \( D(s) \)**

* is that special orbit, an ideal particle would have for \( \Delta p/p = 1 \)

* the orbit of any particle is the sum of the well known \( x_\beta \) and the dispersion

* as \( D(s) \) is just another orbit it will be subject to the focusing properties of the lattice
**Dispersion:**

*Example: homogenous dipole field*

\[
\Delta p/p > 0
\]

Matrix formalism:

*Example: for a quadrupole lens:*

\[
M_{foc} = \begin{pmatrix}
\cos(\sqrt{|K|} s) & \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|} s) \\
-\sqrt{|K|} \sin(\sqrt{|K|} s) & \cos(\sqrt{|K|} s)
\end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}
\]

\[
x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}
\]

\[
x(s) = C(s) \cdot x_0 + S(s) \cdot x'_0 + D(s) \cdot \frac{\Delta p}{p}
\]

Amplitude of Orbit oscillation:

- \(x_\beta = 1...2 \text{ mm}\)
- \(D(s) \approx 1...2 \text{ m}\)
- \(\frac{\Delta p}{p} \approx 1 \cdot 10^{-3}\)

Contribution due to dispersion \(\approx\) beam size
16.) Chromaticity:  
A Quadrupole Error for $\Delta p/p \neq 0$

Influence of external fields on the beam: prop. to magn. field & prop. zu 1/p

**dipole magnet**  
$$\alpha = \frac{\int B \, dl}{p/e}$$

**focusing lens**  
$$k = \frac{g}{p/e}$$

$$x_D(s) = D(s) \frac{\Delta p}{p}$$

*particle having ... to high energy to low energy ideal energy*
Chromaticity: $Q'$

$$k = \frac{g}{p/e}$$

$$p = p_0 + \Delta p$$

_in case of a momentum spread:_

$$k = \frac{eg}{p_0 + \Delta p} \approx \frac{e}{p_0} \left(1 - \frac{\Delta p}{p_0}\right) g = k_0 + \Delta k$$

$$\Delta k = -\frac{\Delta p}{p_0} k_0$$

... which acts like a quadrupole error in the machine and leads to a tune spread:

$$\Delta Q = -\frac{1}{4\pi} \frac{\Delta p}{p_0} k_0 \beta(s) ds$$

_definition of chromaticity:_

$$\Delta Q = Q' \frac{\Delta p}{p} ; \quad Q' = -\frac{1}{4\pi} \int k(s) \beta(s) ds$$
... what is wrong about Chromaticity:

Problem: chromaticity is generated by the lattice itself!!

$Q'$ is a number indicating the size of the tune spot in the working diagram, $Q'$ is always created if the beam is focussed → it is determined by the focusing strength $k$ of all quadrupoles

*Example: LHC*

$Q' = 250$

Δ $p/p = +/- 0.2 \times 10^{-3}$

$Δ Q = 0.256 \ldots 0.36$

→ Some particles get very close to resonances and are lost in other words: the tune is not a point it is a pancake

Tune signal for a nearly uncompensated chromaticity ($Q' \approx 20$)

Ideal situation: chromaticity well corrected, ($Q' \approx 1$)
Correction of Chromaticity

1.) sort the particles according to their momentum  \[ x_D(s) = D(s) \frac{\Delta p}{p} \]

2.) apply a magnetic field that rises quadratically with \( x \) (sextupole field)

\[
B_x = \tilde{g} x z \\
B_z = \frac{1}{2} \tilde{g} (x^2 - z^2)
\]

\[
\frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x} = \tilde{g} x
\]

linear rising „gradient“:

Sextupole Magnets:

normalised quadrupole strength:

\[
k_{\text{sext}} = \frac{\tilde{g} x}{p / e} = \frac{m_{\text{sext}} x}{e}
\]

\[
k_{\text{sext}} = m_{\text{sext}} D \frac{\Delta p}{p}
\]

corrected chromaticity:

\[
Q' = -\frac{1}{4\pi} \oint \{ K(s) - mD(s) \} \beta(s) ds
\]
**Resume:**

The geometry and the maximum momentum of the particles is defined by the dipole strength

\[ \frac{p}{e} = B \rho \]

Strong focusing quadrupole lenses lead to a transverse oscillation of the particles

\[ x'' + x \left( \frac{1}{\rho^2} - k \right) = 0 \]

**Focusing properties of a magnet**

Foc. & defoc. lenses have to be combined to lead to an overall focusing scheme in both planes.

\[ M_{\text{foc}} = \begin{pmatrix} \cos(Ks) & \frac{1}{\sqrt{|K|}} \sin(Ks) \\ -\sqrt{|K|} \sin(Ks) & \cos(Ks) \end{pmatrix} \]

\[ M_{\text{defoc}} = \begin{pmatrix} \cosh|K| & \frac{1}{\sqrt{|K|}} \sinh|K| \\ \sqrt{|K|} \sinh|K| & \cosh|K| \end{pmatrix} \]

The $\beta$-function defines an envelope enclosing the single particle trajectories and together with the emittance it defines the beam size

\[ \hat{x}(s) = \sqrt{\epsilon \beta} \]

Number of oscillations per turn (the tune) depends on the overall focusing fields in the ring

\[ Q = \frac{1}{2\pi} \int \frac{ds}{\beta(s)} \]

The beam emittance is a intrinsic beam property and describes the quality of the particle distribution. It corresponds to the area of an ellipse in $x, x'$ phase space and is constant (for a given energy).
**Resume:**

**Dispersion:** effect of a momentum error (spread) on the particle orbit

**Chromaticity:** ... on the focusing properties (tune)

\[ x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p} \]

\[ \Delta Q = Q' \cdot \frac{\Delta p}{p} \]
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