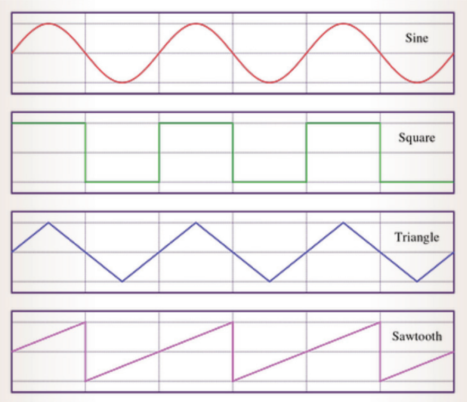
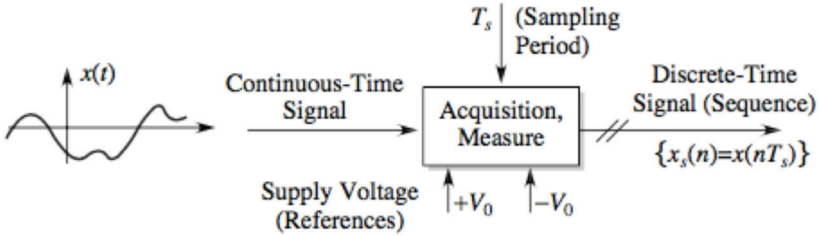
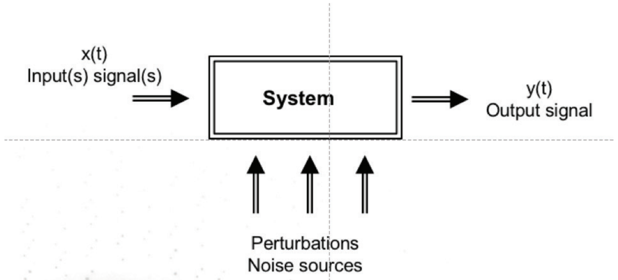
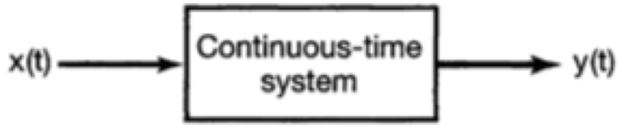


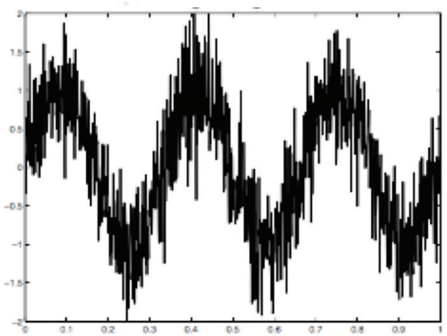


Numerical Methods
Mathematical Background



$$X(f) = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}\,dt$$





Laurent S. Nadolski

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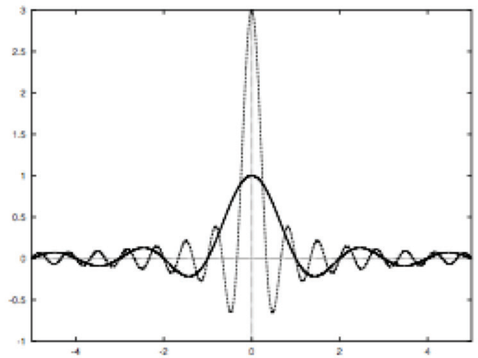
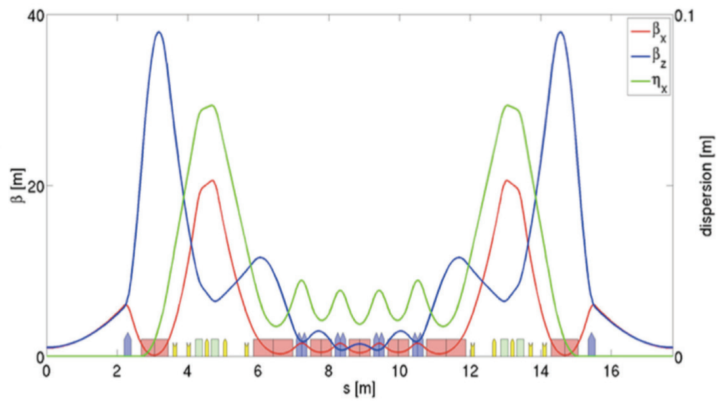


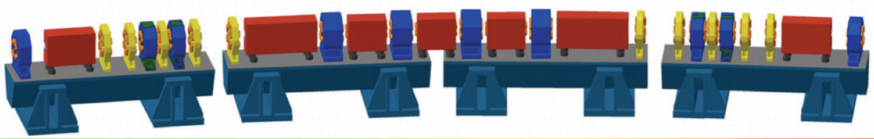
École d'ingénieurs
Télécom Physique Strasbourg

3rd Generation Synchrotron Light Source
27 km South of Paris, France

Accelerator Physicist
Accelerators Coordinator



Forthcoming upgrade



Technical Domains of Expertise

- Non-linear beam dynamics
- Frequency Map Analysis
- Beam-based Measurements
- Tracking Codes
- Control Systems

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Société Française de Physique

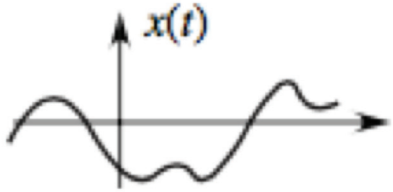
Two staff positions open

Contents for a 2 hour-class

- Introduction: definitions and classifications
- Signal representations
- Toolbox of useful functions and distributions
- Mathematical tools
- Fourier Series, Fourier Transform, Laplace Transform
- Linear Time Invariant (LTI) Filters
- Measurement
- Noise
- Statistics
- References

INTRODUCTION

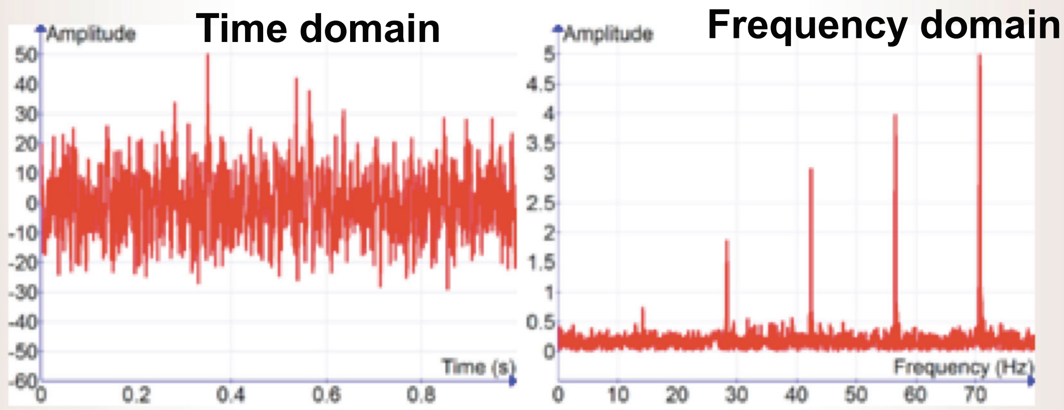
Introduction



- **We will deal with** the concept of *signals* and *systems*.
- The signal x(t) is produced by a measurement device can be seen as a real, **time-varying property**.
- The property represents **physical observables** like voltage (1D), current, temperature, image (2D), pressure (3D), e-m waveform, N-D signal, etc.
- Signals are usually studied in one of the **following domains**:
  1. **time** domain (one-dimensional signals),
  2. **spatial** domain (multidimensional signals),
  3. **frequency** domain,
  4. **autocorrelation** domain,
  5. ***wavelet*** *domain.*
- **Proper mathematical tools shall be chosen wisely**

Signals and Systems: Definitions

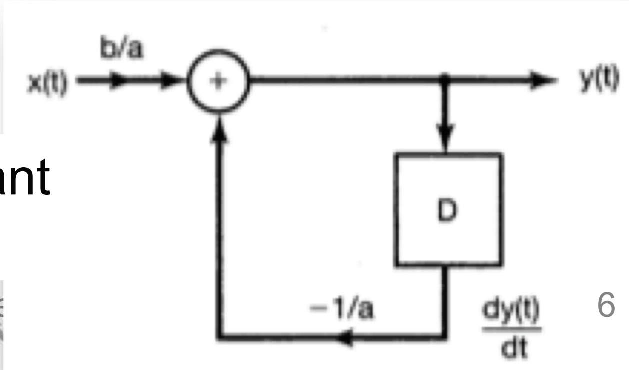
- **Signal**: function of one or several variables that conveys **information** about the behavior or attributes of some physical phenomenon



The signal on the left looks like noise, but the signal processing technique known as the Fourier transform (right) shows that it contains five well defined frequency components.

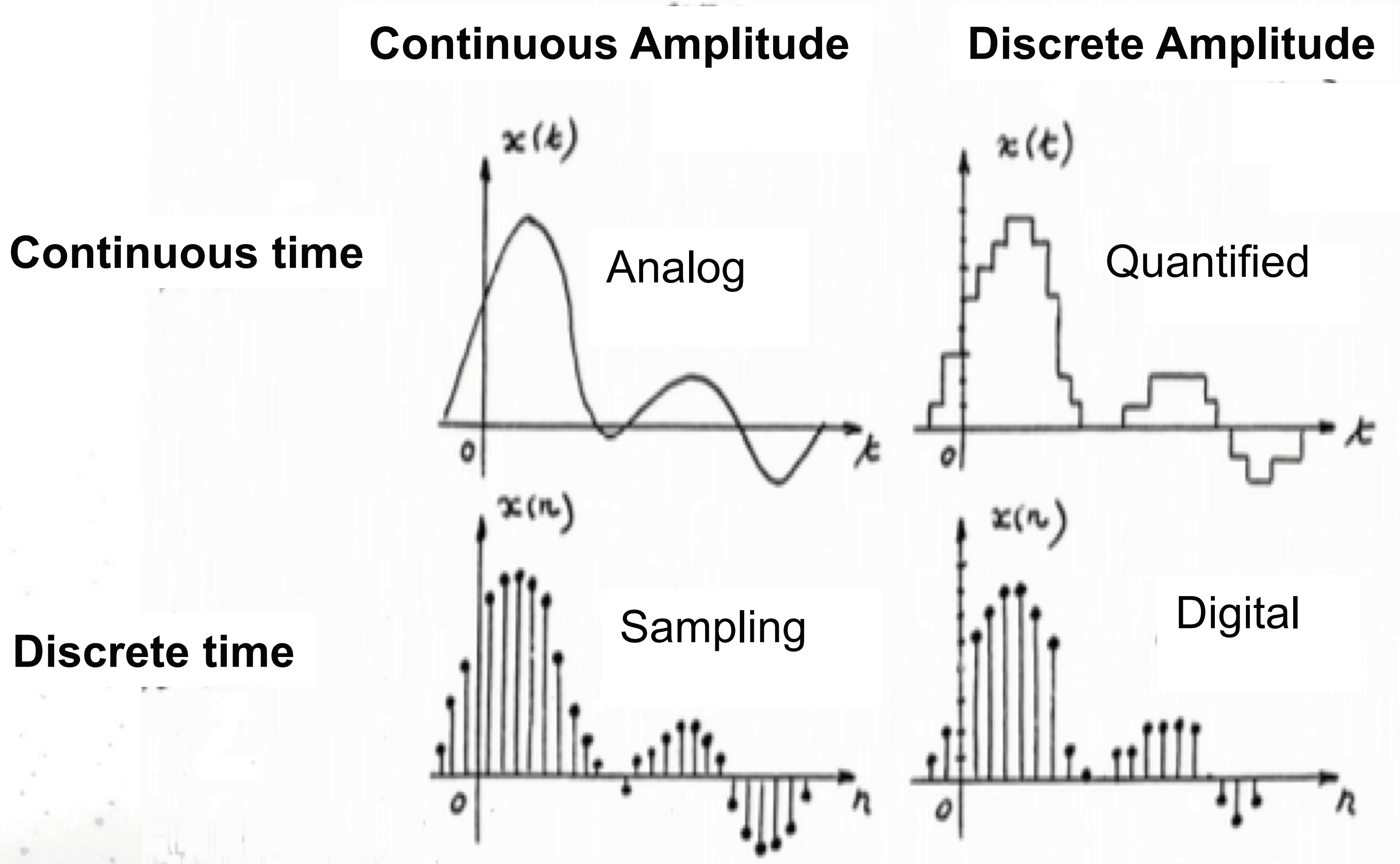
Wikipedia

- **System of Signal Processing**: transforms signals in other signals or parameters in order to extract useful information



A simple time-invariant linear filter

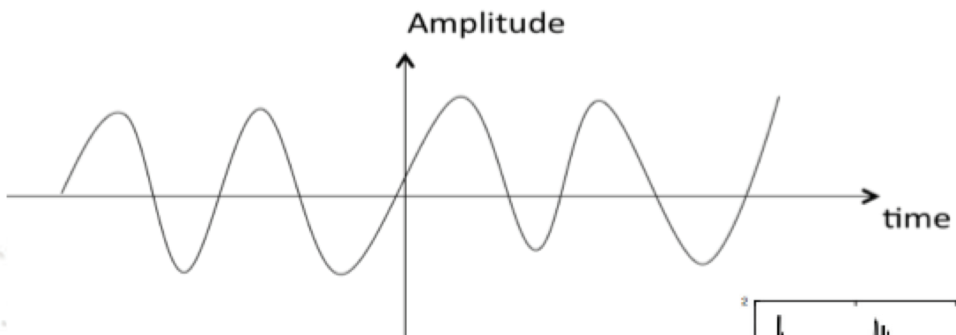
4 types of Signals: Discrete/Continuous



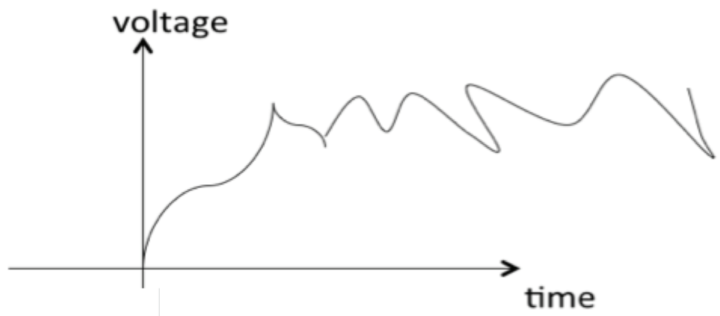
Courtesy of Heitz

Classification: Determinist and Random Signals

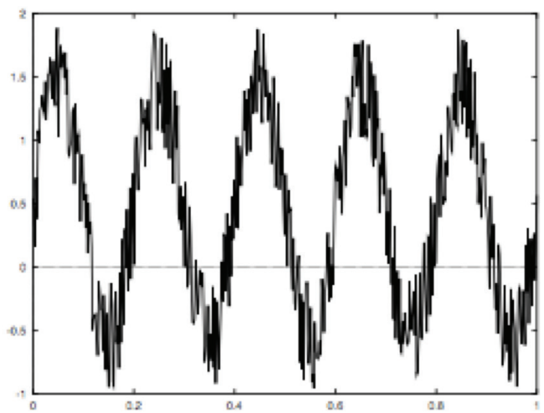
Signal Nature		Math. Representation
Deterministic	Continuous time	Functions, distributions
	Discrete time	Numerical series
Random	Continuous time	Random function
	Discrete time	Random series



Deterministic signal



Random signal



Real life: mixture

Classification: Energy / Average Power

Signals of finite energy

- Transient signals
- Bounded support signals
- All physical signals

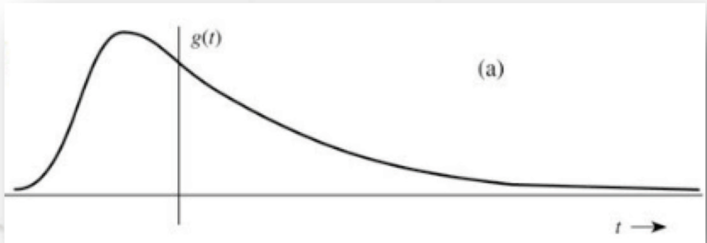
$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt$$

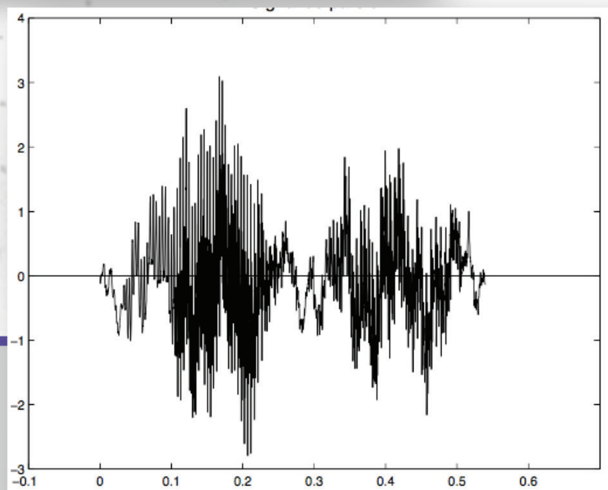
Signals of finite power

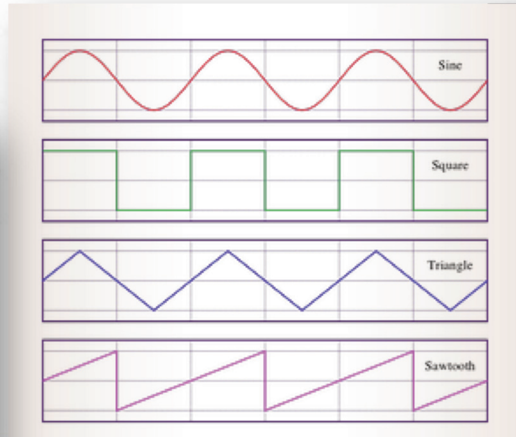
- Ideal Signals out of function generators
- Periodic signals

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 \, dt$$

$$0 < P_x < +\infty \; (\Rightarrow E_x = +\infty)$$

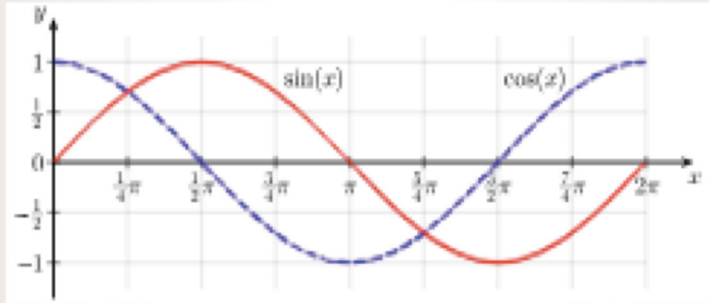






Mathematical Tools to Represent Signals

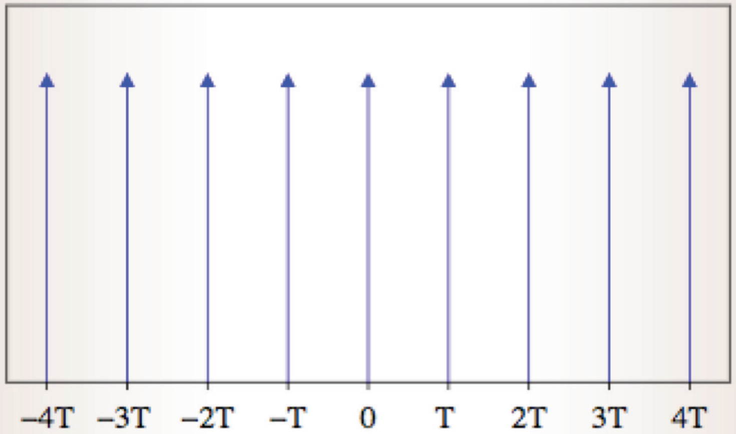
- **Functions**



Graph of two trigonometric functions: sine and cosine.

Wiki

- **Distributions** as an extension of the notion of functions



A Dirac comb is an infinite series of Dirac delta functions spaced at intervals of T

Wiki

Function Properties

- **Real function**: $t \in \mathbb{R}, \quad x(t) \in \mathbb{R}$
- **Complex function**: $x(t) \in \mathbb{C}$
- **Properties**
  - Continuity, derivability
  - **Bounded** function $\exists M \in \mathbb{R}, \quad |x(t)| < M$
  - Integrability

Summable function
Vector space

$$x \in L_1 \quad \text{si} \quad \int_{-\infty}^{+\infty} |x(t)|dt < +\infty$$

Square integrable function
Vector space

$$x \in L_2 \quad \text{si} \quad \int_{-\infty}^{+\infty} |x(t)|^2 dt < +\infty$$

- **Odd/even function**
  - Decomposition

$$x(t) = x(-t)$$
$$x(t) = -x(-t)$$

$$x(t) = \underbrace{\frac{1}{2}\{x(t) + x(-t)\}}_{\text{Even}} + \underbrace{\frac{1}{2}\{x(t) - x(-t)\}}_{\text{Odd}}$$

Real and complex signals

$$x(t) \in \mathbf{R}$$

- Real signal

$$z(t) \in \mathbf{C}$$

- Complex signal

$$j^2 = -1$$

- Complex number

$$z(t) = a(t) + j\ b(t)$$

$$z(t) = |z(t)|\, e^{j \arg z(t)}$$

$$\Re z(t) = a(t)$$

- Real part

$$\Im z(t) = b(t)$$

- Imaginary part

$$\arg(z) = \arctan \frac{b}{a}$$

- Argument

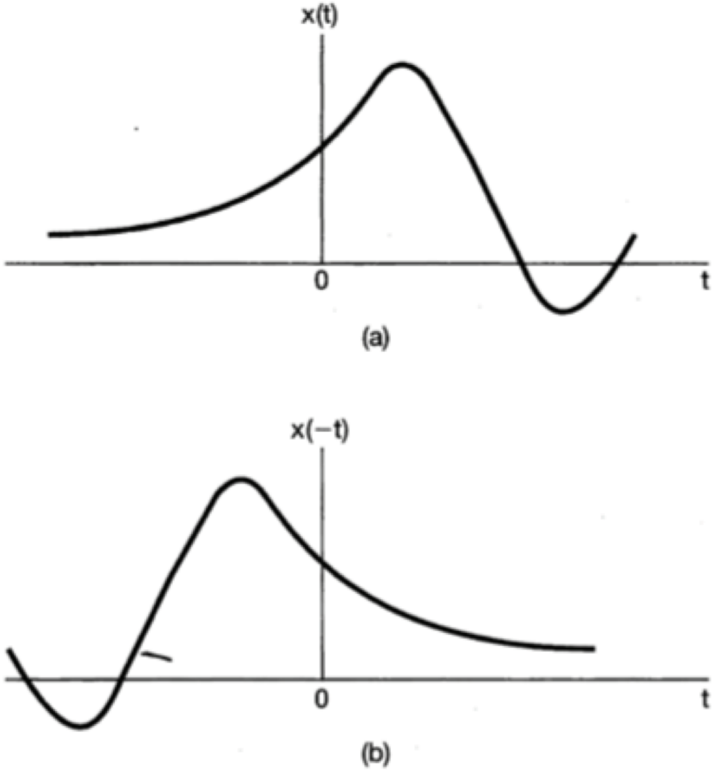
$$|z(t)| = \sqrt{a^2(t) + b^2(t)}$$

- Module

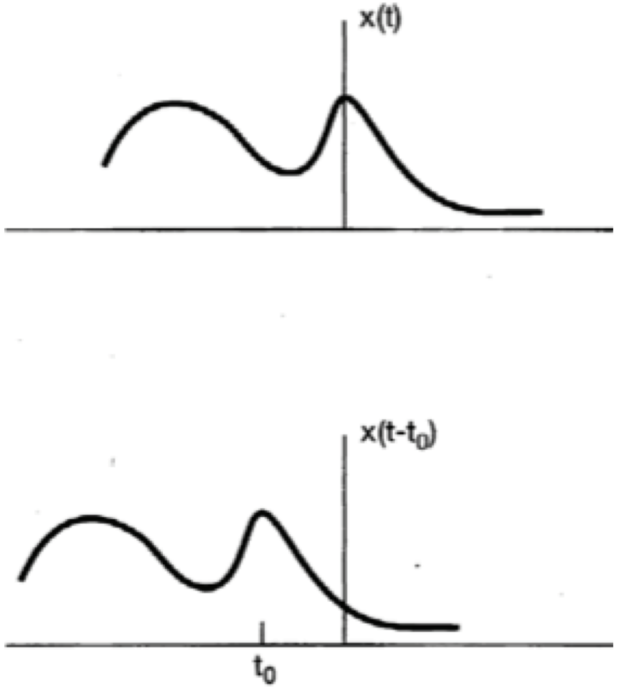
$$z^*(t) = a(t) - j\ b(t)$$

- Complex conjugate

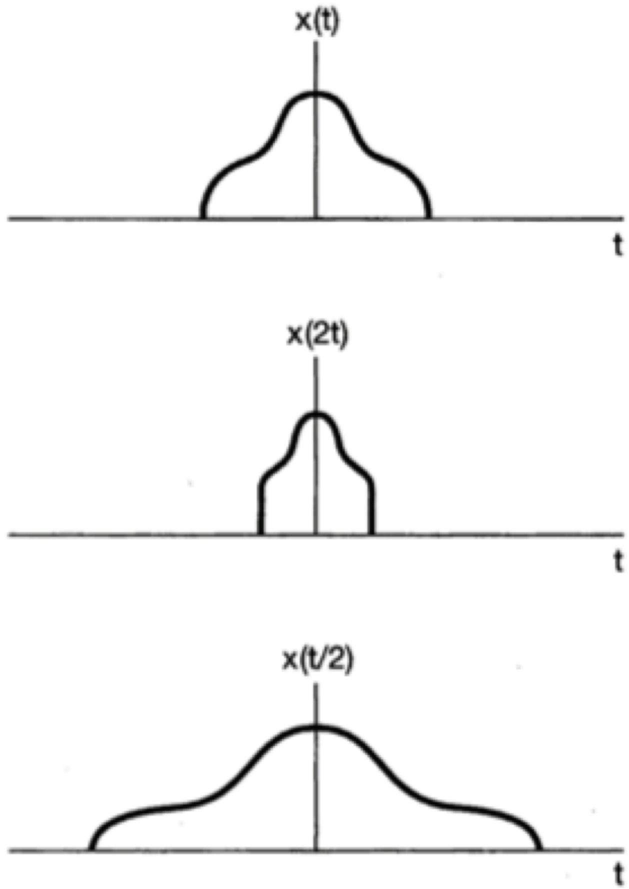
Simple Time-base Transformations



Inversion



Time shifting

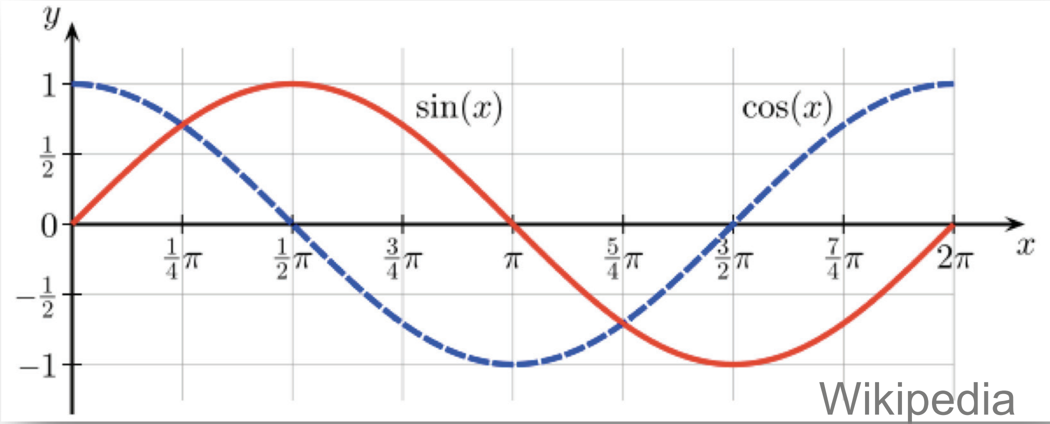


Contraction
Dilatation

Function: Periodicity / Trigonometric Functions

$$x(t) = Ae^{\pm j\omega_0 t}$$

$$x(t) = A\cos(\omega_0 t + \Phi) = A\cos(2\pi f_0 t + \Phi)$$
$$x(t) = A\sin(\omega_0 t + \Phi) = A\sin(2\pi f_0 t + \Phi)$$



- **T-periodic**:

$$\exists\ T > 0 \text{ so that } \forall\ t\ \ x(t+T) = x(t)$$

- **Fundamental** period: min(T)

$$T_0 = \frac{2\pi}{\omega_0} \qquad \text{(s)}$$

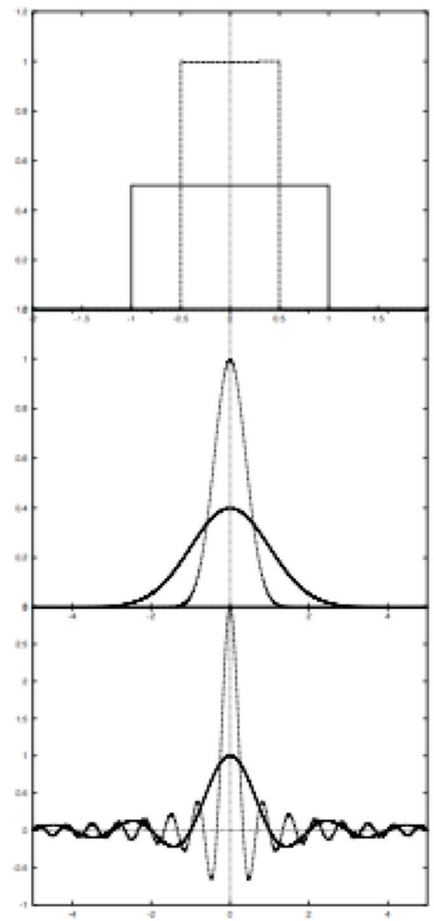
- **Frequency/angular frequency**:

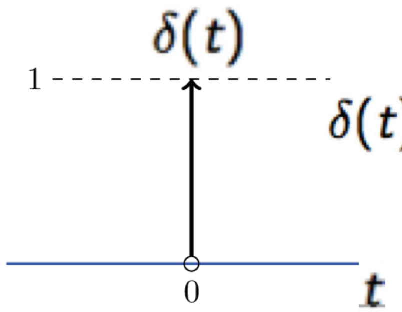
$$f_0 = \frac{1}{T_0} = \frac{\omega_0}{2\pi} \qquad \textbf{(Hz)}$$

$$A\cos(2\pi f_0 t + \Phi) = \frac{A}{2}\left\{e^{j\Phi}e^{j2\pi f_0 t} + e^{-j\Phi}e^{-j2\pi f_0 t}\right\}$$

$$A\sin(2\pi f_0 t + \Phi) = \frac{A}{2j}\left\{e^{j\Phi}e^{j2\pi f_0 t} - e^{-j\Phi}e^{-j2\pi f_0 t}\right\}$$

Distribution as Extension of Functions and the Limit of Usual Functions

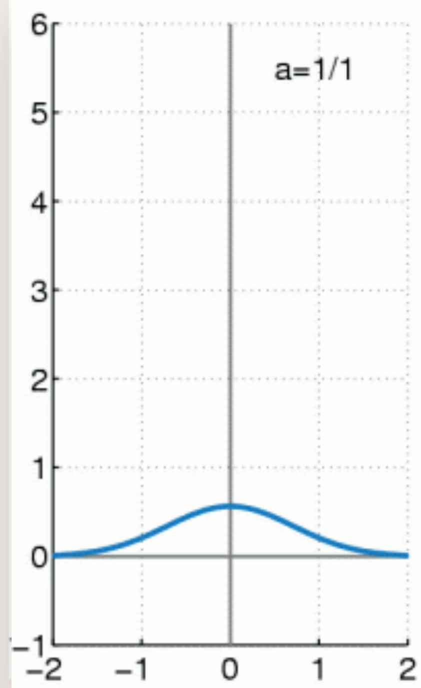




$$\delta(t) = \lim_{k\to\infty} k\,\mathrm{rect}(kx)$$

$$\delta(t) = \lim_{a\to 0} \frac{1}{\sqrt{2\pi}a} e^{-\frac{t^2}{2a^2}}$$

$$\delta(t) = \lim_{B\to\infty} B\frac{\sin(\pi Bt)}{\pi Bt}$$



The Dirac delta function as the limit (in the sense of distributions) of the sequence of zero-centered normal distributions

$$\delta_a(x) = \frac{1}{|a|\sqrt{\pi}} e^{-(x/a)^2}$$

as $a \to 0$.

wiki

Distribution (S) Properties

- **Linear functionals**: Function Space → Scalar Space $< S, \Phi >$

  $\phi(t)$ **Test function: continue, infinity derivable with bounded support**
  Aka a smooth physical function

- **Linearity**

$$< S, \phi_1 + \phi_2 > = < S, \phi_1 > + < S, \phi_2 >$$
$$\forall \lambda \in \mathbb{C} \quad < S, \lambda\phi > = \lambda < S, \phi >$$

- **Continuity**

if $\phi_k \to \phi$ then $< S, \phi_k > \to < S, \phi >$.

- **Regular Distribution**

$$< f, \phi > = \int_{-\infty}^{+\infty} f(t)\phi(t)dt \quad \forall \phi(t)$$

- **Singular Distribution**

$$< \delta, \phi > = \int_{-\infty}^{+\infty} \delta(t)\phi(t)dt = \phi(0)$$

- **Principal Value**
  Cauchy Value

$$< v.p.(\frac{1}{t}), \phi > = \lim_{\epsilon \to 0^-} \int_{-\infty}^{\epsilon} \frac{\phi(t)}{t}dt + \lim_{\epsilon \to 0^+} \int_{\epsilon}^{+\infty} \frac{\phi(t)}{t}dt$$

Distribution: Properties

- **Linearity** $$< S, \phi_1 + \phi_2 > \; = \; < S, \phi_1 > + < S, \phi_2 >$$ $$\forall \lambda \quad < S, \lambda\phi > \; = \; \lambda < S, \phi >$$
- **Additivity** $$< S + T, \Phi > = < S, \Phi > + < T, \Phi >$$
- **Scalar Multiplication** $$< \lambda S, \Phi > = < S, \lambda\Phi > = \lambda < S, \Phi >$$
- **Translation** $$< S(t - a), \Phi(t) > = < S(t), \Phi(t + a) >$$
- **Transposition** $$< S(-t), \Phi(t) > = < S(t), \Phi(-t) >$$
- **Scaling** $$< S(at), \Phi(t) > = \frac{1}{|a|} < S(t), \Phi(\frac{t}{a}) >$$
- **Multiplication by a function infinitely derivable** $$< \psi S, \Phi > = < S, \psi\Phi >$$
- **Derivability** $$< S', \Phi > = - < S, \Phi' >$$
- **Convolution** $$< S * T, \Phi > = < S(t), < T(t'), \Phi(t + t') >>$$

USEFUL FUNCTIONS AND DISTRIBUTIONS

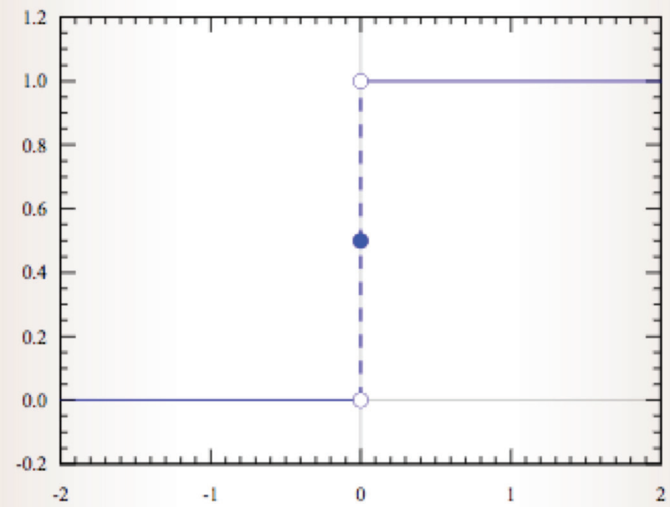
A Set of Useful Functions

- Heaviside function
- Sign function
  - reconstructing a continuous band-limited signal from uniformly spaced samples of that signal.
- Gate or rectangular function
  - Windowing
- Sine cardinal, sampling function
- Triangular function
- Pulse or Dirac function
- Dirac Comb (“Sha”)

Heaviside Function/Unit Step Function H(t), u(t)

Note: Non derivable function

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 0.5 & \text{if } t = 0 \\ 1 & \text{if } t > 0 \end{cases}$$



The Heaviside step function, using the half-maximum convention

Wikipedia

$$H(t) = \frac{d}{dt} max(x(t), 0)$$

$$H(t) = \frac{x(t) + |x(t)|}{2}$$

$$H(t) = \int_{-\infty}^{t} \delta(u)\, du$$

Useful for making causal any signal

$$s(t) = x(t)H(t)$$

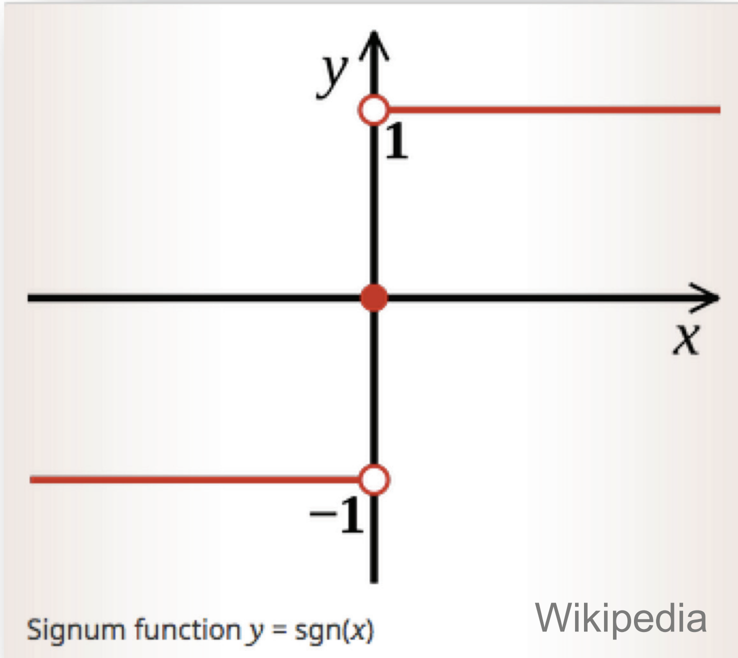
Characterization of a filter

…

Sign Function

Note: Non derivable function

$$sgn(t) = \begin{cases} -1 & \text{if } t < 0 \\ 0 & \text{if } t = 0 \\ 1 & \text{if } t > 0 \end{cases}$$



$$sgn(t) = \frac{d}{dt}|x(t)|$$

$$sgn(t) = 2H(t) - 1$$

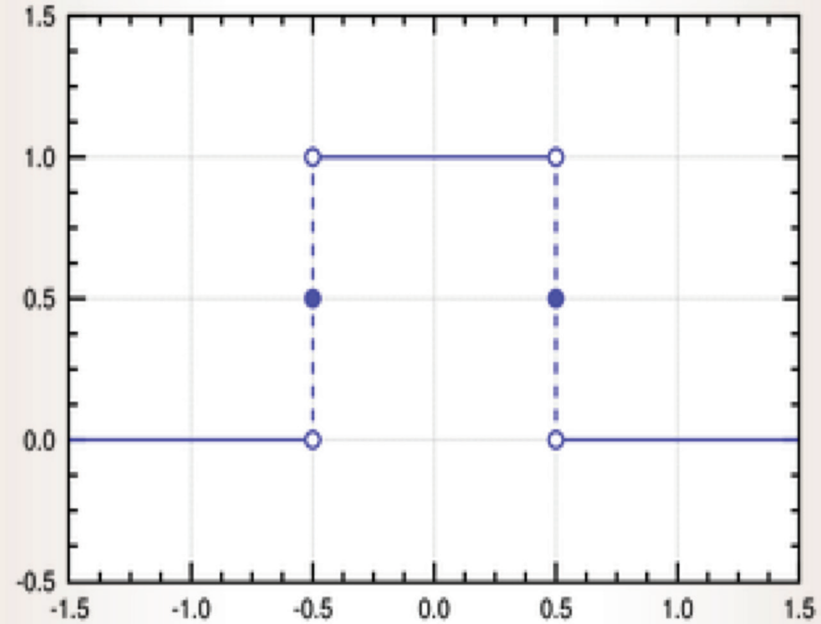
$$\int_{-\infty}^{\infty} \operatorname{sgn}(x) e^{ikx}\, dx = \text{p. v.} \frac{2}{ik},$$

Gate or Rectangular Function

Note: Non derivable function

$$\mathrm{rect}(t) = \Pi(t) = \begin{cases} 0 & \text{if } |t| > \frac{1}{2} \\ \frac{1}{2} & \text{if } |t| = \frac{1}{2} \\ 1 & \text{if } |t| < \frac{1}{2}. \end{cases}$$

$$rect(t) = U(t + \frac{1}{2}) - U(t - \frac{1}{2})$$



Rectangular function Wikipedia

- **Gating of a signal**
- **Bounded Support Signals**
- **Gating of a frequency domain**
- **Zeroth Hold Filter**
- **Top-flat filters**

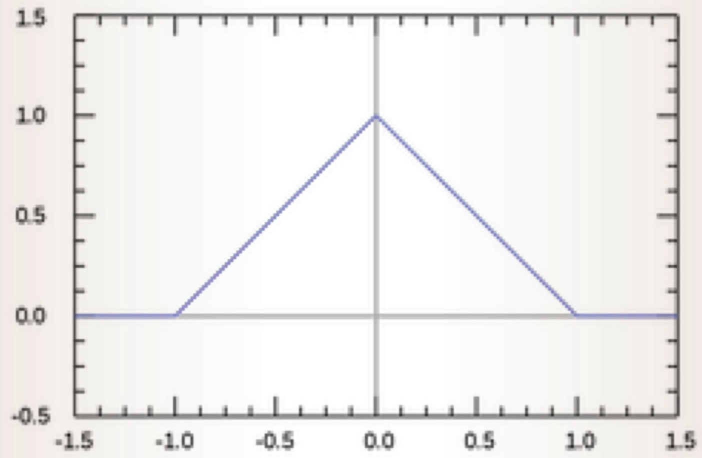
$$\int_{-\infty}^{\infty} \mathrm{rect}(t) \cdot e^{-i2\pi f t}\, dt = \frac{\sin(\pi f)}{\pi f} = \mathrm{sinc}(f).$$

Triangular Function

$$\mathrm{tri}(x) = \Lambda(x) \overset{\mathrm{def}}{=} \max(1 - |x|, 0) = \begin{cases} 1 - |x| & |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{tri}(x) = \Lambda(x) \overset{\mathrm{def}}{=} \max(1 - |x|, 0) = \begin{cases} 1 - |x| & |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$trig(t) = rect(t) * rect(t)$$



Exemplary triangular function

Wikipedia

Autocorrelation of rect(t)

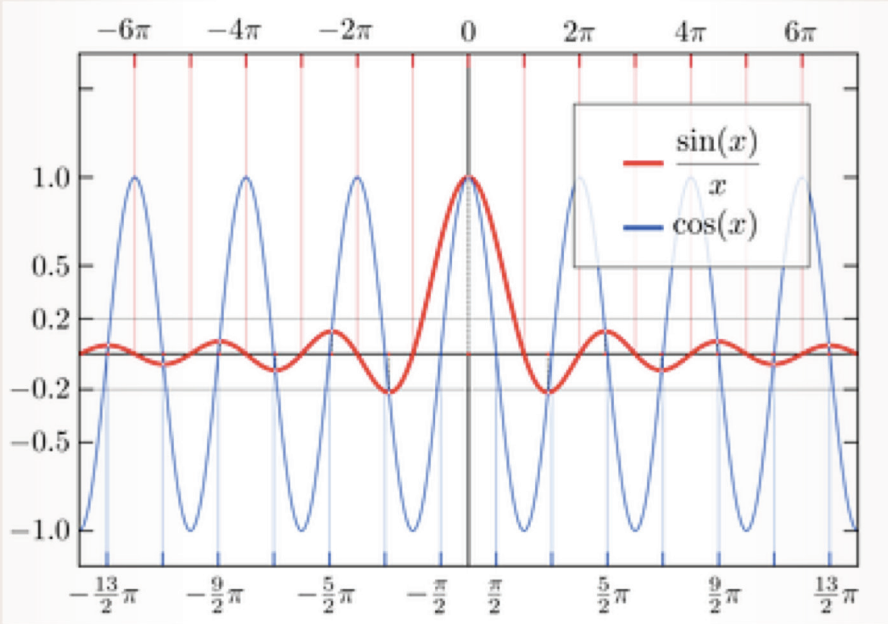
$$\begin{aligned} \mathcal{F}\{\mathrm{tri}(t)\} &= \mathcal{F}\{\mathrm{rect}(t) * \mathrm{rect}(t)\} \\ &= \mathcal{F}\{\mathrm{rect}(t)\} \cdot \mathcal{F}\{\mathrm{rect}(t)\} \\ &= \mathcal{F}\{\mathrm{rect}(t)\}^2 \\ &= \mathrm{sinc}^2(f) \end{aligned}$$

Sine Cardinal Function

$$\frac{d\,\mathrm{sinc}(x)}{dx} = \frac{\cos(x) - \mathrm{sinc}(x)}{x}$$

$$\mathrm{sinc}(x) = \frac{\sin(x)}{x}$$

x	*x*/π	sinc(*x*)	sinc²(*x*)	20 log |sinc(*x*)|
0	0	1	1	0
4,493409	1,430297	−0,217234	0,047190	−13,26
7,725252	2,459024	0,128375	0,016480	−17,83
10,904122	3,470890	−0,091325	0,008340	−20,79
14,066194	4,477409	0,070913	0,005029	−22,99
17,220755	5,481537	−0,057972	0,003361	−24,74
20,371303	6,484387	0,049030	0,002404	−26,19
23,519452	7,486474	−0,042480	0,001805	−27,44
26,666054	8,488069	0,037475	0,001404	−28,53
29,811599	9,489327	−0,033525	0,001124	−29,49
32,956389	10,490344	0,030329	0,000920	−30,36
36,100622	11,491185	−0,027690	0,000767	−31,15
39,244432	12,491891	0,025473	0,000649	−31,88
42,387914	13,492492	−0,023585	0,000556	−32,55



The local maxima and minima (small white dots) of the unnormalized, red sinc function correspond to its intersections with the blue cosine function.

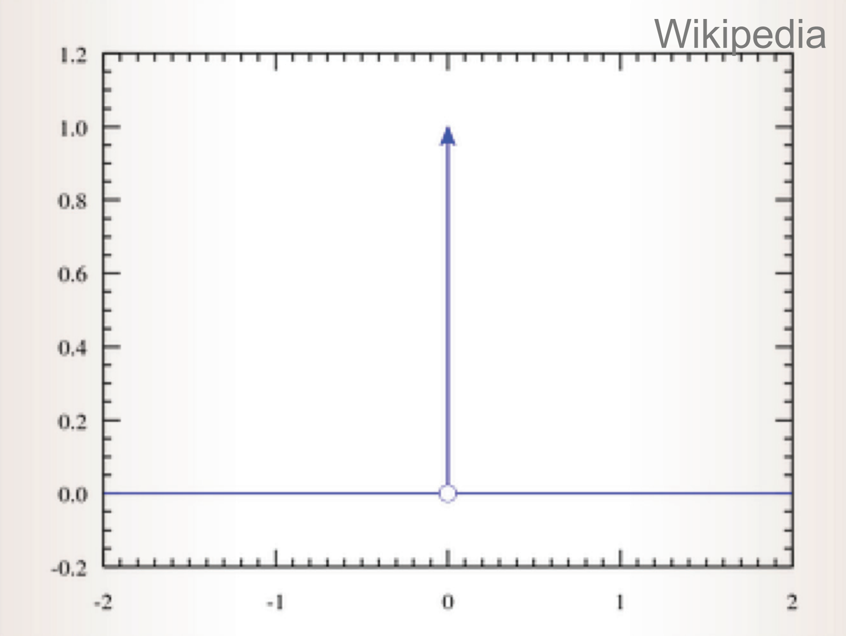
Wikipedia

$$\int_{-\infty}^{\infty} \mathrm{sinc}(t)\, e^{i2\pi ft}\, dt = \mathrm{rect}(f)$$

Pulse, Dirac Delta Function

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$



$$\begin{aligned}
\int_{-\infty}^{+\infty} x(t)\delta(t)dt &= x(0) \\
x(t)\,\delta(t) &= x(0)\,\delta(t) \\
x(t)\,\delta(t-t_0) &= x(t_0)\,\delta(t-t_0) \\
\delta(t-t_0) &= \delta(t_0-t) \\
\delta(\frac{t-t_0}{k}) &= |k|\delta(t-t_0) \\
x(t) * \delta(t-t_0) &= x(t-t_0)
\end{aligned}$$

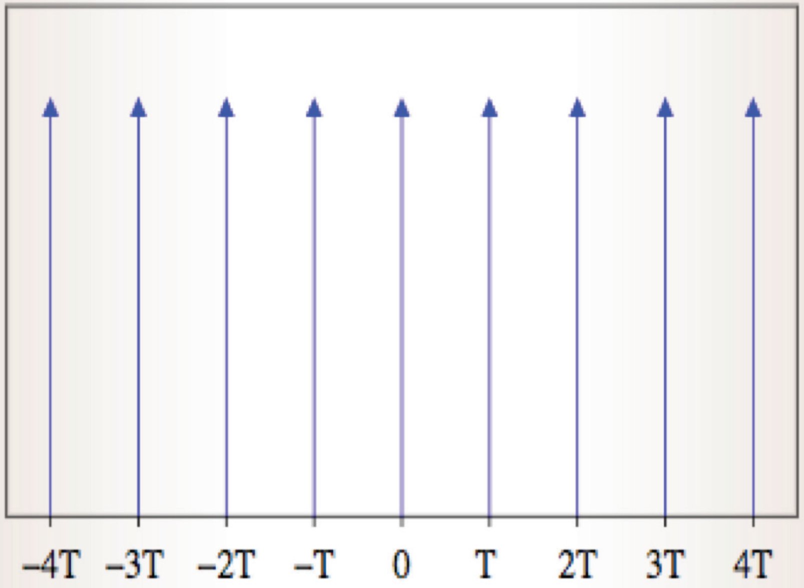
Dirac Comb (Shah Function):

$$\text{Ш}_T(t) \stackrel{\text{def}}{=} \sum_{k=-\infty}^{\infty} \delta(t-kT) = \frac{1}{T}\text{Ш}\left(\frac{t}{T}\right)$$

$$\text{Ш}_T(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{2\pi i n \frac{t}{T}}.$$

$$\text{Ш}_T(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{T}\text{Ш}_{\frac{1}{T}}(f) = \sum_{n=-\infty}^{\infty} e^{-i2\pi f n T}$$

$$(\text{Ш}_T\, x)(t) = \sum_{k=-\infty}^{\infty} x(t)\delta(t-kT) = \sum_{k=-\infty}^{\infty} x(kT)\delta(t-kT).$$



A Dirac comb is an infinite series of Dirac delta functions spaced at intervals of T

Wikipedia

Convolution Product

$$z(t) = x(t) * y(t) = \int_{-\infty}^{+\infty} x(\tau)y(t-\tau)d\tau$$

- **Commutativity**

$$x(t) * y(t) = y(t) * x(t)$$
$$\int_{-\infty}^{+\infty} x(\tau)y(t-\tau)d\tau = \int_{-\infty}^{+\infty} y(\tau)x(t-\tau)d\tau$$

- **Distributivity**

$$x(t) * (y_1(t) + y_2(t)) = x(t) * y_1(t) + x(t) * y_2(t)$$

- **Sifting property**

$$x(t) * \delta(t-t_0) = x(t-t_0)$$

$$x(t) * \delta(t) = x(t)$$

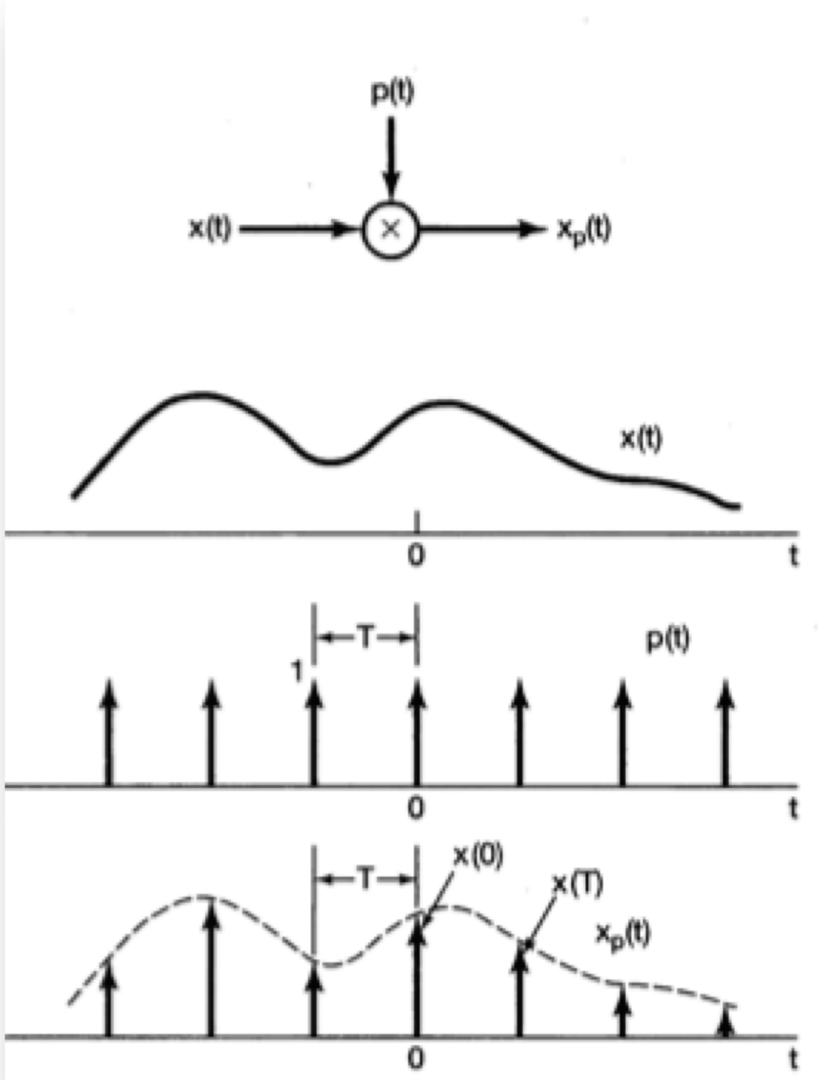
Neutral element

$$x(t) * \delta'(t) = \frac{d}{dt}x(t)$$

Derivation Operator

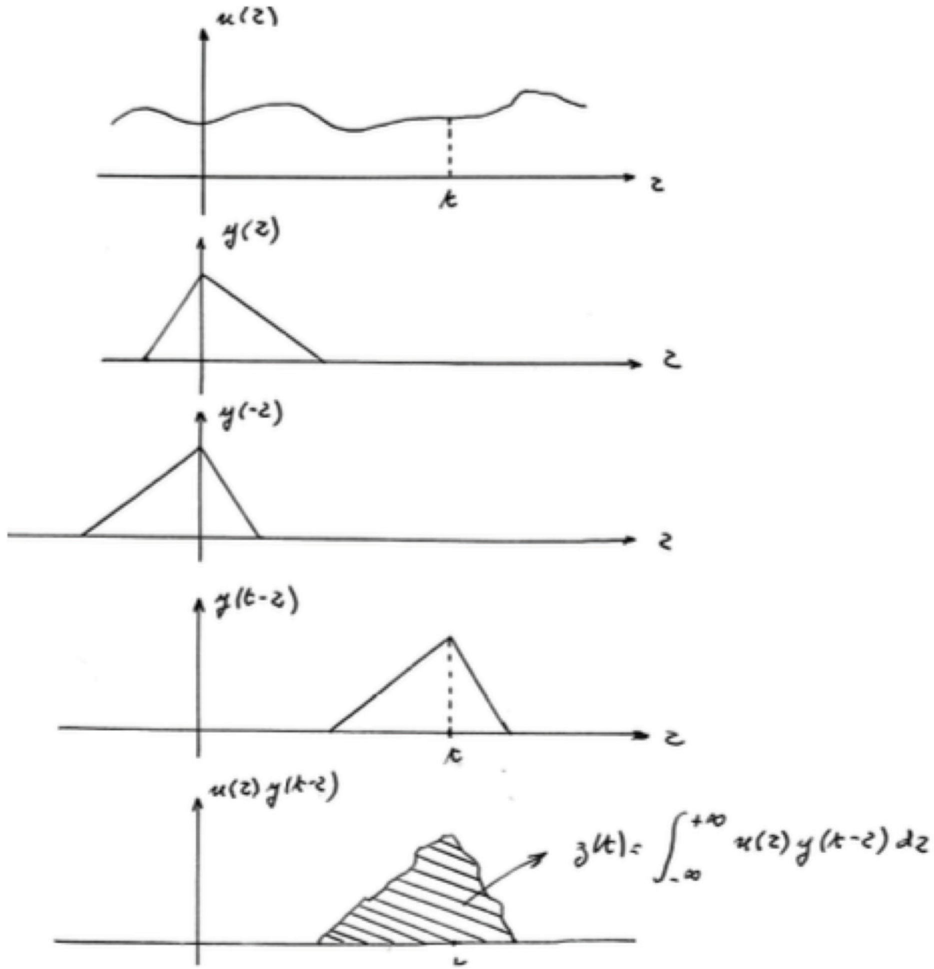
Dirac Comb: the Sampling Operator

$$
\begin{aligned}
& x(t) * Ш_{T_0}(t) \\
= & x(t) * \sum_{k=-\infty}^{+\infty} \delta(t - kT_0) = \sum_{k=-\infty}^{+\infty} x(t - kT_0)
\end{aligned}
$$



Convolution as Graphics

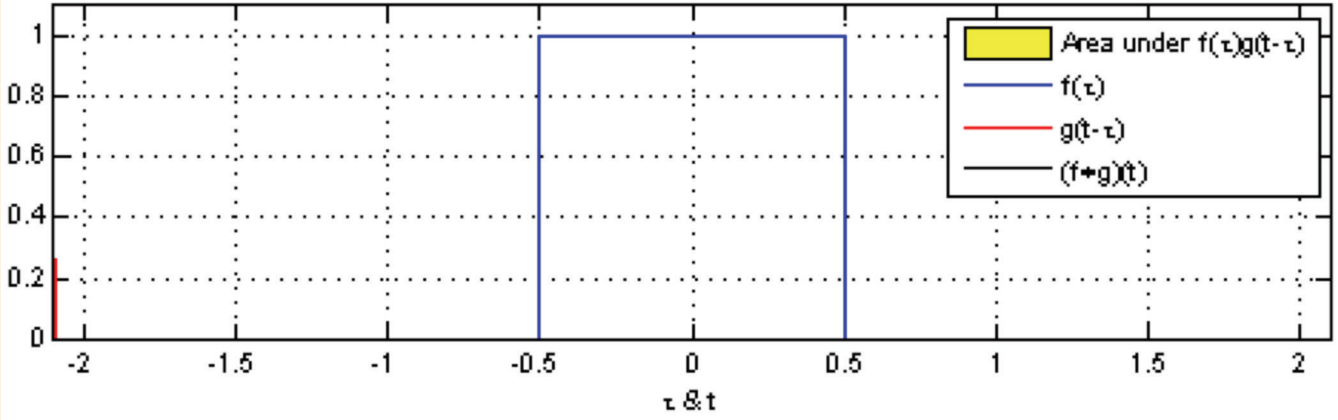
$x(\tau)$
$y(\tau)$
$y(\tau) \longrightarrow y(-\tau)$
$y(-\tau) \longrightarrow y(t-\tau)$
$\int_{-\infty}^{+\infty} x(\tau) y(t-\tau) d\tau$



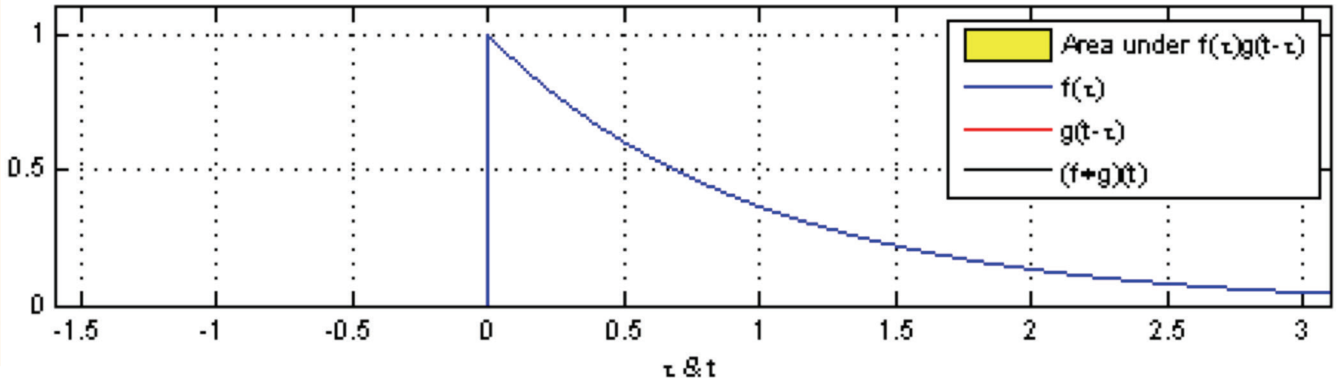
Source: Heitz

Examples of Convolution

In this example, the red-colored "pulse", $g(\tau)$, is an even function ($g(-\tau) = g(\tau)$), so convolution is equivalent to correlation. A snapshot of this "movie" shows functions $g(t-\tau)$ and $f(\tau)$ (in blue) for some value of parameter t, which is arbitrarily defined as the distance from the $\tau = 0$ axis to the center of the red pulse. The amount of yellow is the area of the product $f(\tau) \cdot g(t-\tau)$, computed by the convolution/correlation integral. The movie is created by continuously changing t and recomputing the integral. The result (shown in black) is a function of t, but is plotted on the same axis as τ, for convenience and comparison.



In this depiction, $f(\tau)$ could represent the response of an RC circuit to a narrow pulse that occurs at $\tau = 0$. In other words, if $g(\tau) = \delta(\tau)$, the result of convolution is just $f(t)$. But when $g(\tau)$ is the wider pulse (in red), the response is a "smeared" version of $f(t)$. It begins at $t = -0.5$, because we defined t as the distance from the $\tau = 0$ axis to the center of the wide pulse (instead of the leading edge).



Source: Wikipedia

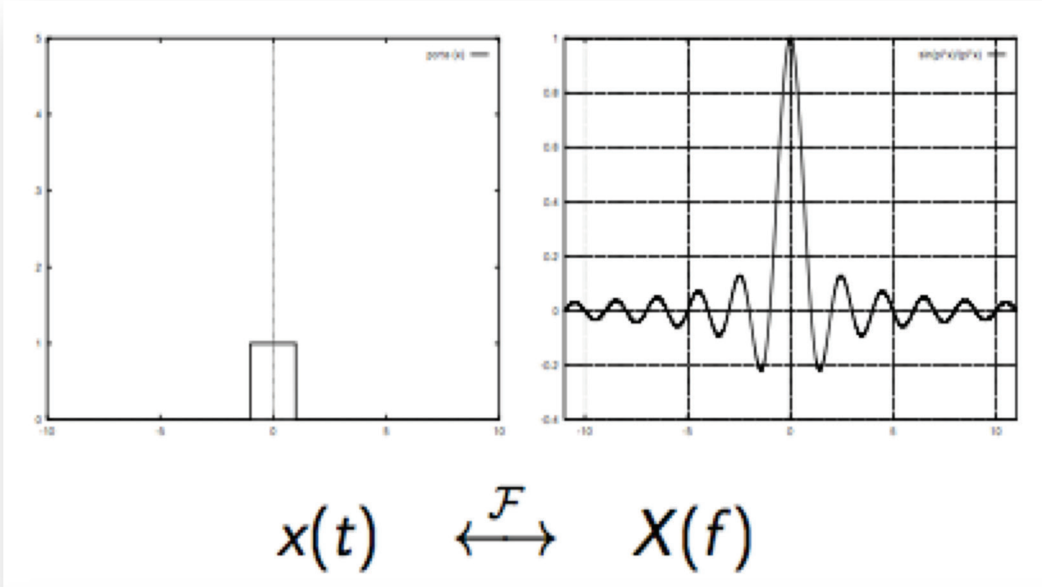
FREQUENCY DOMAIN

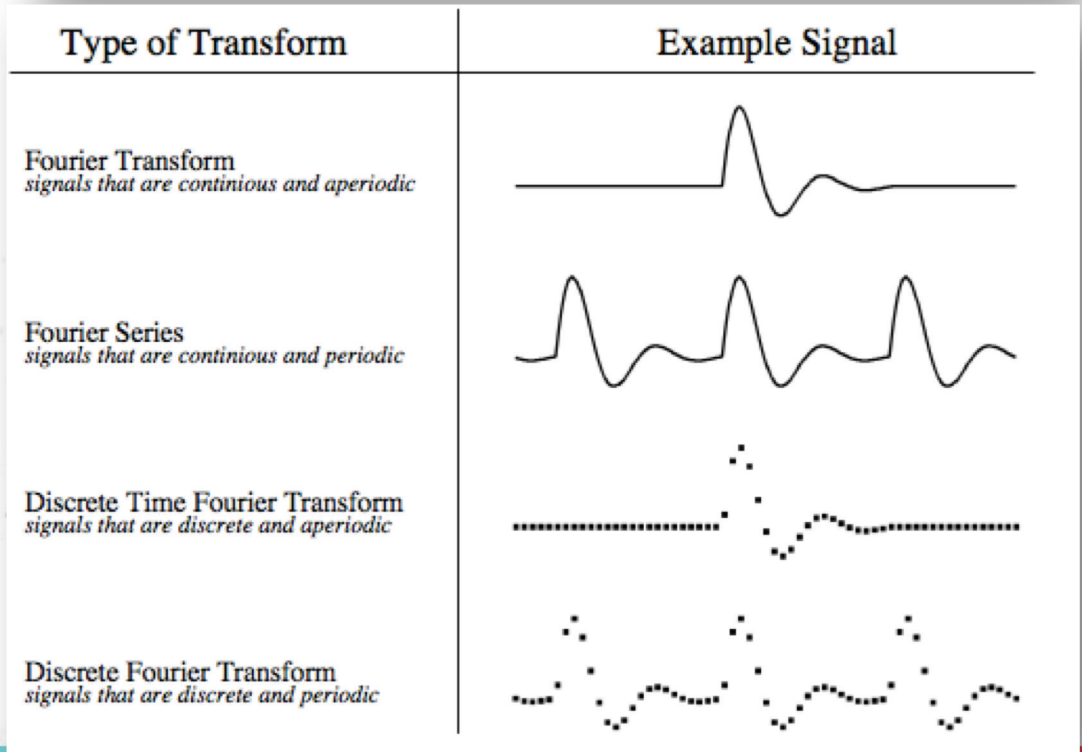
Frequency Analysis

- Two of **equivalent representations** of a signal
  - Temporal contents x(t)
  - Frequency contents X(f)

- Use of a **function basis of variable frequency**

- **Benefits**
  - Powerful tool to analyze and treat a signal
  - Often, **easier to do the treatment in frequency domain**





From Smith

Frequency Analysis: Harmonic Analysis for Analog periodic (continuous) Signal

$$x(t) = \sum_{-\infty}^{+\infty} a_k \, e^{j2\pi k f_0 t}$$

$$\text{with} \quad a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-j2\pi k f_0 t} \, dt$$

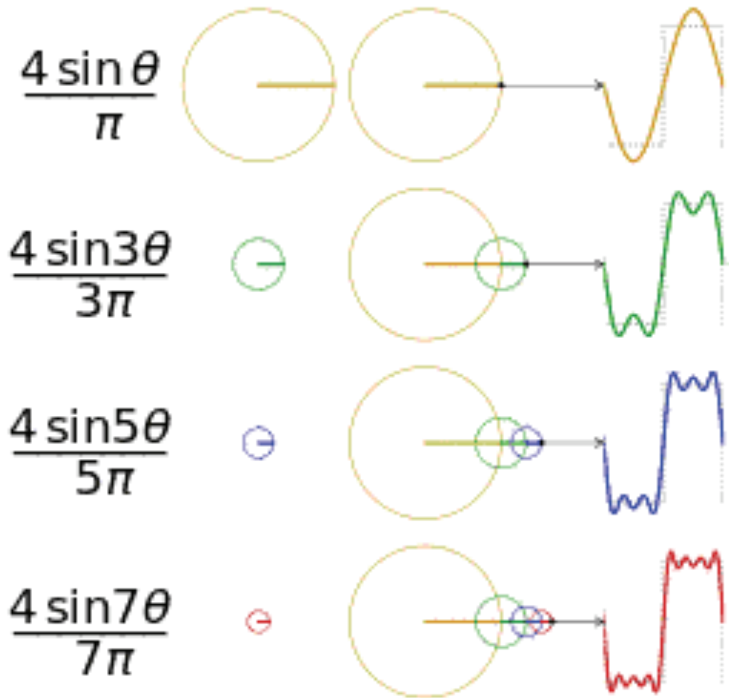
$$x(t) = \frac{\alpha_0}{2} + \sum_{k=1}^{+\infty} [\alpha_k \, \cos 2\pi k f_0 t + \beta_k \, \sin 2\pi k f_0 t]$$

$$\alpha_k = \frac{2}{T_0} \int_{T_0} x(t) \cos 2\pi k f_0 t \, dt$$

$$\beta_k = \frac{2}{T_0} \int_{T_0} x(t) \sin 2\pi k f_0 t \, dt$$

$$\begin{cases} \alpha_k &= a_k + a_{-k} \\ \beta_k &= j\,(a_k - a_{-k}) \end{cases}$$

Odd function: rect(t)



Gibbs oscillations

a_0 Continuous Component
a_1 Fundamental
a_k Harmonics

Fourier Series: Properties

Periodic Signal	Series coefficients	
$x(t)$ (period T_0)	a_k	
$y(t)$ (period T_0)	b_k	
$Ax(t) + By(t)$	$Aa_k + Bb_k$	**Linearity**
$x(t-t_0)$	$a_k e^{-jk(\frac{2\pi}{T_0})t_0}$	**Time-Delay Operator**
$e^{jM(\frac{2\pi}{T_0})t}x(t)$	a_{k-M}	**Frequency-Delay Operator**
$x^*(t)$	a^*_{-k}	
$x(-t)$	a_{-k}	
$x(\alpha t),\ \alpha > 0$ (period $\frac{T_0}{\alpha}$)	a_k	

Fourier Series: Properties

Periodic Signal	Series coefficients
$\frac{1}{T_0}\int_{T_0} x(\tau)y(t-\tau)d\tau$	$a_k b_k$
$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
$\frac{dx(t)}{dt}$	$jk\frac{2\pi}{T_0}a_k$
$\int_{-\infty}^{t} x(\tau)d\tau$ (avec $a_0 = 0$)	$\left(\frac{1}{jk(\frac{2\pi}{T_0})}\right)a_k$
$x(t)$ real	$a_k = a^*_{-k}$
$x(t)$ real and even	a_k real $a_k = a_{-k}$
Parseval formula	$\frac{1}{T_0}\int_{T_0} \lvert x(t)\rvert^2 dt = \sum_{k=-\infty}^{+\infty} \lvert a_k\rvert^2$

Continuous Fourier Series: Properties

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} x(t) = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T} a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{Ev}\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{Od}\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Oppenheim and Willsky, 2017

Discrete Fourier Series (DFT)

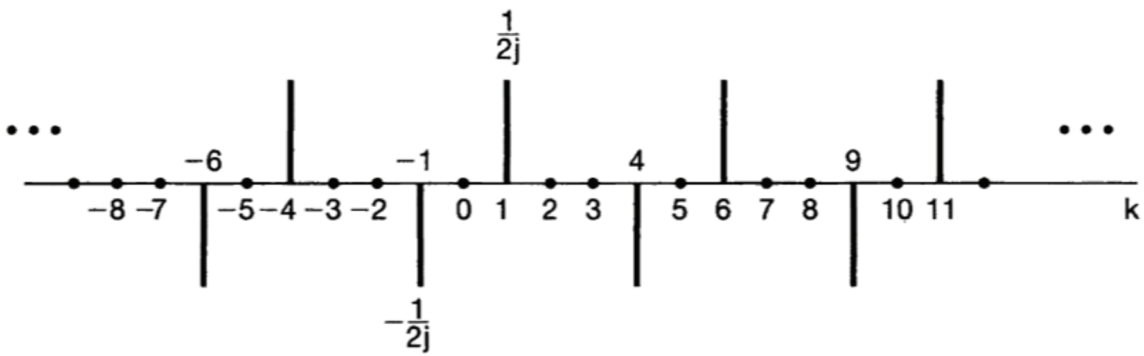
$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

- **N-Periodic Discrete Signal** $x[n] = x[n+N]$.

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n},$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}.$$

<N> = 0..N-1



Fourier coefficients for $x[n] = \sin(2\pi/5)n$.

Oppenheim and Willsky, 2017

Discrete Fourier Series: Properties

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n}x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m}a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{Ev}\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{Od}\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

Oppenheim and Willsky, 2017

Convergence of Fourier Series

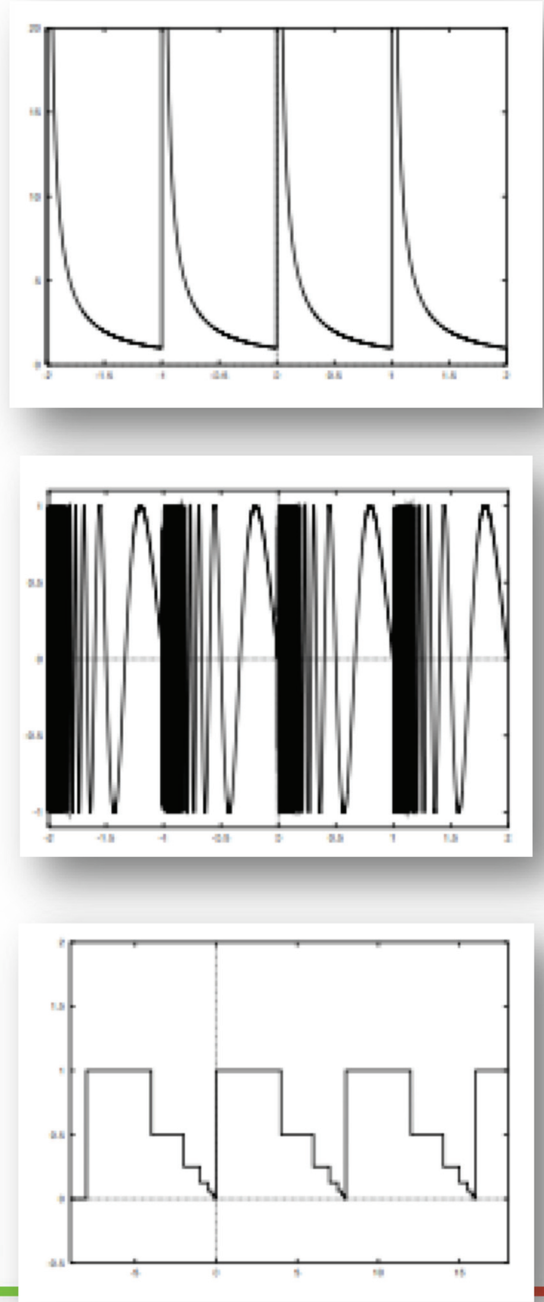
All signals do not have a **convergent Fourier** Series.

Convergence condition

- **Periodic continuous** signals
- **Finite energy** over one period

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt \;<\; +\infty$$

- **Dirichlet Conditions**
  - L1 $\quad \int_{T_0} |x(t)| dt \;<\; +\infty$
  - Finite number of maxima and minima over one period
  - Finite number of discontinuities over any time span



Violation Conditions

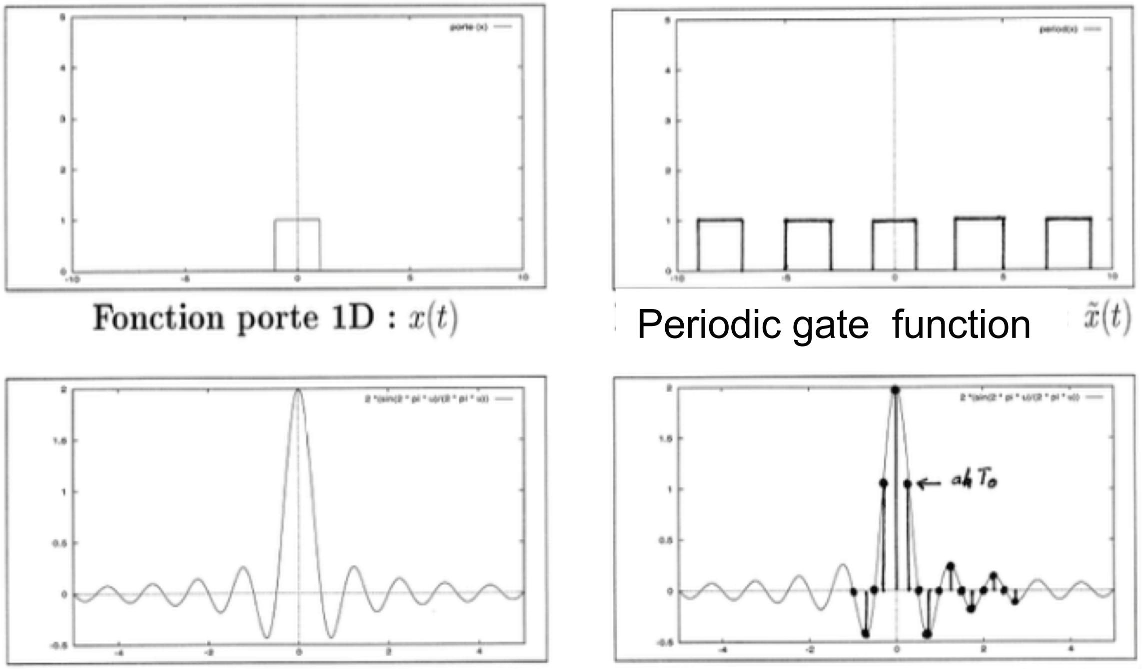
Aperiodic Signals (*infinite period* signals)

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{j2\pi k f_0 t} \xrightarrow{T_0 \to +\infty} x(t) = \int_{-\infty}^{+\infty} X(f) e^{j2\pi ft} df$$

Fourier Transform

$$\begin{cases} X(f) &= \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt \\ x(t) &= \int_{-\infty}^{+\infty} X(f) e^{j2\pi ft} df \end{cases}$$

$$\begin{aligned} a_k &= \frac{1}{T_0} \int_{T_0} \tilde{x}(t) e^{-j2\pi k f_0 t} dt \\ \tilde{x}(t) &= \sum_{k=-\infty}^{+\infty} a_k e^{j2\pi k f_0 t} \end{aligned}$$



$$a_k = \frac{1}{T_0} X(f) \Big|_{f = k f_0}$$

Fourier Transform

- A signal x(t) is projected on a **basis of complex exponential functions** of amplitude $X(f)df$

- Notations:

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f)$$
$$X(f) = \mathcal{F}[x(t)]$$
$$x(t) = \mathcal{F}^{-1}[X(f)]$$

- X(f) is the **frequential component**
- X(0) is the **continuous component**: $X(0) = \int_{-\infty}^{+\infty} x(t)dt$
- **Amplitude** and **Phase Spectrum**

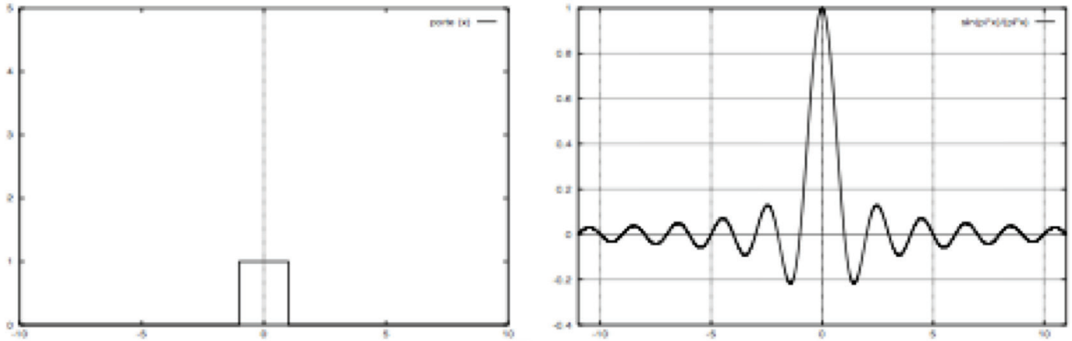
$$X(f) = |X(f)|e^{jArg[X(f)]}$$

$|X(f)|$: Amplitude spectrum

$Arg\,[X(f)]$: Phase spectrum

- **Spectrum Analyzers measure**: $|X(f)|^2$.

Fourier Transform: Properties



$$x(t) = \mathrm{rect}(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f) = \mathrm{sinc}(f)$$

- **Real Signal**

$$\begin{aligned} X(-f) &= X^*(f) \\ |X(-f)| &= |X(f)| \\ Arg\, X(-f) &= -Arg\, X(f) \end{aligned}$$

- **Odd or even real Signal**
  - Even x(t): X(f) real and even
  - Odd x(t): X(f) purely imaginary and odd

Warning: Several Definitions for the Fourier Transform

- **Using frequency variable f**

$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt$$

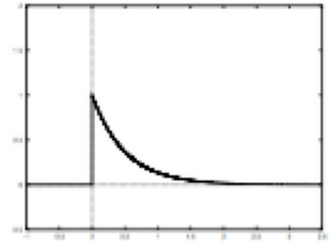
$$x(t) = \int_{-\infty}^{+\infty} X(f) e^{j2\pi ft} df$$

- **Using angular frequency** $\omega = 2\pi f$

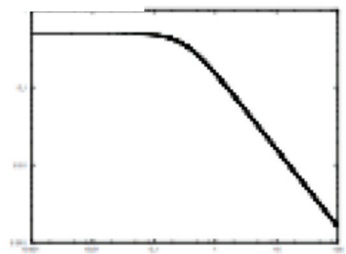
$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$

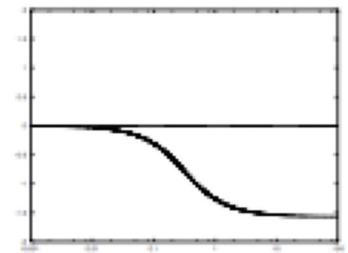
$x(t) = \exp(-2t)\, u(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f) = \frac{1}{2+j2\pi f}$



$20 \log_{10} |X(f)|$ **log scale in f**



$Arg[X(f)]$ **log scale in f**



Convergence of Fourier Transform

- Signals of **finite energy** $\int_{-\infty}^{+\infty} |x(t)|^2 dt < +\infty$

  L2-function

- **Conditions of Dirichlet**
  - L1 $\int_{-\infty}^{+\infty} |x(t)| dt < +\infty$
  - Finite number of maxima and minima over one period
  - Finite number of discontinuities over any time span

- **Examples of signals without FT**
  - x(t) = 1
  - Periodic signals
  - Distributions

Fourier Transform for Distributions

Definition $$< \mathcal{F}[S], \Phi > = < S, \mathcal{F}[\Phi] >$$

$$\delta(t) \stackrel{\mathcal{F}}{\longleftrightarrow} 1$$

$$1 \stackrel{\mathcal{F}}{\longleftrightarrow} \delta(f)$$

$$\delta(t - t_0) \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j2\pi f t_0}$$

$$e^{j2\pi f_0 t} \stackrel{\mathcal{F}}{\longleftrightarrow} \delta(f - f_0)$$

$$\cos 2\pi f_0 t \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]$$

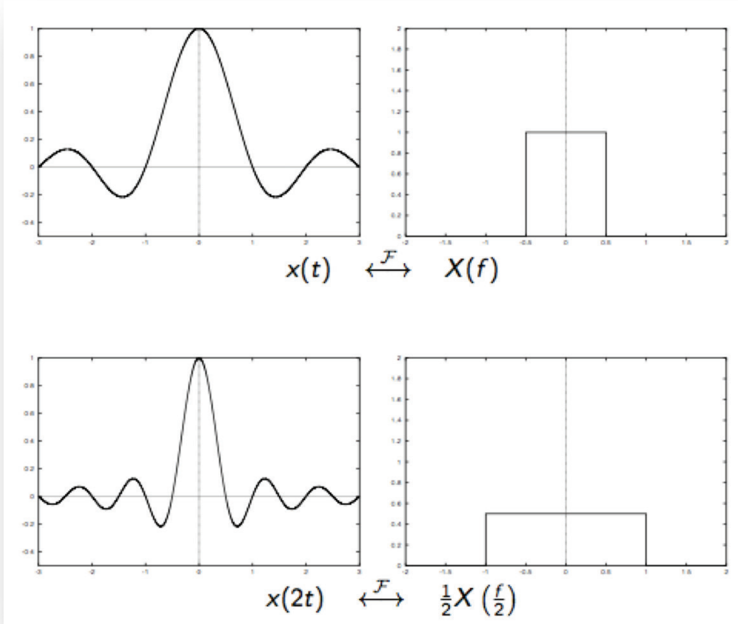
$$sin 2\pi f_0 t \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{2j}[\delta(f - f_0) - \delta(f + f_0)]$$

$$\text{sign}(t) \stackrel{\mathcal{F}}{\longleftrightarrow} v.p.\frac{1}{j\pi f}$$

$$v.p.\frac{1}{t} \stackrel{\mathcal{F}}{\longleftrightarrow} -j\pi \text{sign}(f)$$

$$\text{u}(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}\delta(f) + v.p.\frac{1}{j2\pi f}$$

Fourier Transform: Properties



- **Duality**

$$g(t) \overset{\mathcal{F}}{\longleftrightarrow} h(f)$$
$$h(t) \overset{\mathcal{F}}{\longleftrightarrow} g(-f)$$

- **Translation and dilatation**

$$x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j2\pi f t_0} X(f)$$
$$e^{j2\pi f_0 t} x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f-f_0)$$
$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

- **Derivative, integration**

$$\frac{dx(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j2\pi f X(f)$$
$$-j2\pi t x(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{dX(f)}{df}$$
$$\int_{-\infty}^{t} x(\tau) d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j2\pi f} X(f) + \frac{1}{2} X(0)\delta(f)$$

Fourier Transform: Properties

- **Parseval-Plancherel Formula**
  - Finite Energy Signals: $\int_{-\infty}^{+\infty} x(t)\, y(t)^* \, dt \; = \; \int_{-\infty}^{+\infty} X(f)\, Y(f)^* \, df$
  - Periodic Signals $\frac{1}{T_0} \int_{T_0} x(t)\, y(t)^* \, dt \; = \; \sum_{k=-\infty}^{+\infty} a_k \, b_k^*$

- **Special cases**
  - Finite Energy Signals: $E_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt \; = \; \int_{-\infty}^{+\infty} |X(f)|^2 df$
  - Periodic Signals $P_x = \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt \; = \; \sum_{k=-\infty}^{+\infty} |a_k|^2$

Continuous Fourier Transform: Properties

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t)+by(t)$	$aX(j\omega)+bY(j\omega)$
4.3.2	Time Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t)*y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t}x(t)dt$	$\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$X(j\omega)=X^*(-j\omega)$ $\mathcal{R}e\{X(j\omega)\}=\mathcal{R}e\{X(-j\omega)\}$ $\mathcal{I}m\{X(j\omega)\}=-\mathcal{I}m\{X(-j\omega)\}$ $\|X(j\omega)\|=\|X(-j\omega)\|$ $\sphericalangle X(j\omega)=-\sphericalangle X(-j\omega)$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t)=\mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t)=\mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{R}e\{X(j\omega)\}$ $j\mathcal{I}m\{X(j\omega)\}$

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty}|x(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(j\omega)|^2d\omega$$

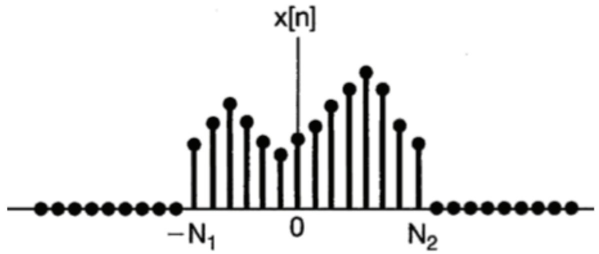
Oppenheim and Willsky, 2017

Fourier Transform of Elementary Functions

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \; k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\,\delta(\omega)$	—

$\delta(t - t_0)$	$e^{-j\omega t_0}$
$e^{-at}u(t), \mathcal{R}e\{a\} > 0$	$\frac{1}{a + j\omega}$
$te^{-at}u(t), \mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$

Discrete Time Fourier Transform DTFT



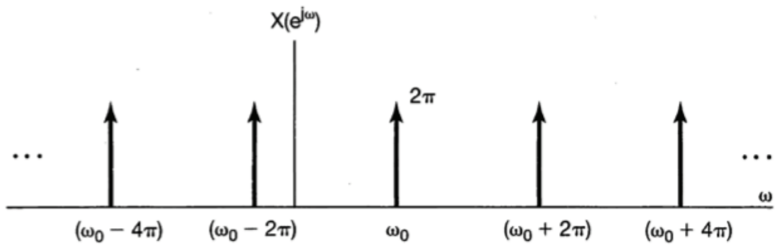
- **Definition**

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega,$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}.$$

- **Periodic Signal** $x[n] = e^{j\omega_0 n}$

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\,\delta(\omega - \omega_0 - 2\pi l)$$



DTFT Properties

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{matrix}X(e^{j\omega})\\ Y(e^{j\omega})\end{matrix}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals		

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

Oppenheim and Willsky

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DTFT of Elementary Functions (I)

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$

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DTFT of Elementary Functions (II)

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \ k \neq 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}, \ k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le W \\ 0, & W < \lvert\omega\rvert \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n+1)a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

Summary of Fourier Series and Transform Expressions

	Continuous time		Discrete time	
	Time domain	**Frequency domain**	**Time domain**	**Frequency domain**
Fourier Series	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ continuous time periodic in time	$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}$ discrete frequency aperiodic in frequency	$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ discrete time periodic in time	$a_k = \frac{1}{N}\sum_{k=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$ discrete frequency periodic in frequency
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$ continuous time aperiodic in time	$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$ continuous frequency aperiodic in frequency	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$ discrete time aperiodic in time	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$ continuous frequency periodic in frequency

duality

duality

duality

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Bounded Support Signals

- **Most of physical signals**
  - Transients
  - Measure signal output of an equipment

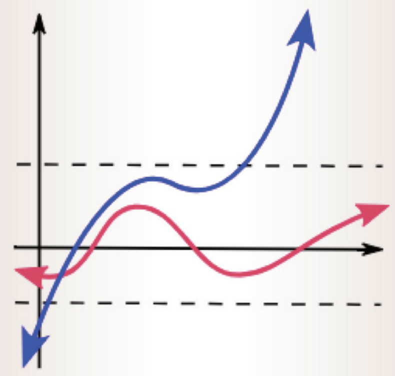
- **Limited bandwidth / Infinite time support**
  - Fast variations no possible

If $|x(t)| < M$ then $\frac{|x(t) - x(t+\tau)|}{M} \ll 1$

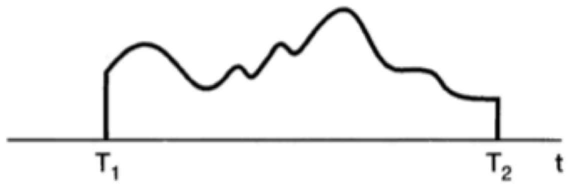
The **rise time** between –M and M has a minimum value (threshold value)

$$\tau \geq \frac{1}{\pi B}$$

- **Limited time support / Infinite bandwidth**
  - Heisenberg Principle



A schematic illustration of a bounded function (red) and an unbounded one (blue). Intuitively, the graph of a bounded function stays within a horizontal band, while the graph of an unbounded function does not. Wiki



Finite Duration Signal

Example: Dirac ←→ Cx Exponential

Bilateral Laplace Transform (BLT)

- Introduced to answer the question of divergence of the FT

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

- **Notations**

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt \qquad s = \sigma + j\omega \in \mathbb{C}.$$

$$X(s) = \mathcal{L}[x(t)]$$

$$x(t) = \mathcal{L}^{-1}[X(s)]$$

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$$

- **TF and BLT**

$$X(\omega) = X(s)\Big|_{s = j\omega}$$

- **Inverse BLT**

$$x(t) = \frac{1}{2\pi j}\int_{\Delta} X(s)e^{st}ds$$

ROC:
- **Region of convergence**

In the complex plane
- Consist of strips parallel to the jω-axis in th s-plane

Section	Property	Signal	Transform	ROC
		$x(t)$	$X(s)$	R
		$x_1(t)$	$X_1(s)$	R_1
		$x_2(t)$	$X_2(s)$	R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\lvert a \rvert}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
9.5.9	Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathcal{R}e\{s\} > 0\}$

Initial- and Final-Value Theorems

9.5.10 If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \longrightarrow \infty$, then

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

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Laplace Transform

of

Elementary functions

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\Re e\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\Re e\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\Re e\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos \omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
12	$[\sin \omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
13	$[e^{-\alpha t}\cos \omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin \omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$

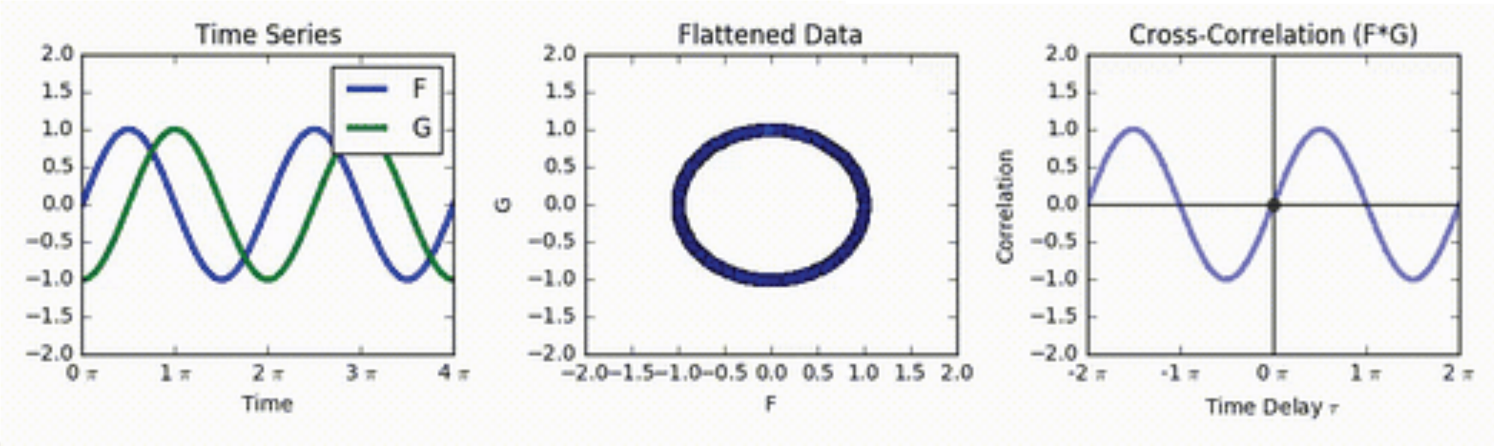
Cross Correlation and Autocorrelation Functions

Cross correlation of x(t) and y(t)

$$R_{xy}(\tau) = \int_{-\infty}^{+\infty} x(t)y^*(t-\tau)dt$$

Autocorrelation of x(t)

$$R_x(\tau) = \int_{-\infty}^{+\infty} x(t)x^*(t-\tau)dt$$



For signals of finite average power

$$R_{xy}(\tau) = \lim_{T\to+\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t)y^*(t-\tau)dt \qquad R_x(\tau) = \lim_{T\to+\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t)x^*(t-\tau)dt$$

Correlation: Properties

- **Convolution**

$$R_{xy}(\tau) \;=\; x(\tau) * y^*(-\tau)$$

- **Symmetry**

$$R_{xy}(-\tau) \;=\; R^*_{yx}(\tau)$$
$$R_x(-\tau) \;=\; R^*_x(\tau)$$

$$R_x(-\tau) \;=\; R_x(\tau)$$

- **Signal Energy**

$$E_x \;=\; R_x(0) \;=\; \int_{-\infty}^{+\infty} |x(t)|^2 dt$$
$$|R_x(\tau)| \;\leq\; R_x(0)$$

Correlation of Periodic Signal

- X(t) of period T_0 and using its Fourier Series

$$R_x(\tau) = \lim_{T\to+\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t)x^*(t-\tau)dt$$
$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{+\frac{T_0}{2}} x(t)x^*(t-\tau)dt = \sum^{+\infty} |a_k|^2 e^{j2\pi k f_0 \tau}$$

- **Autocorrelation function also periodic**. Its value at origin is the average power

$$S_x(f) = \sum_{k=-\infty}^{+\infty} |a_k|^2 \delta(f - kf_0)$$

Spectrum Analyzer

$$P_x = R_x(0) = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{+\frac{T_0}{2}} |x(t)|^2 dt = \int_{-\infty}^{+\infty} S_x(f)df = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Wiener–Khinchin theorem

Power Spectral Density (PSD)

- **PSD is the TF of the autocorrelation function**

$$R_x(\tau) = \int_{-\infty}^{+\infty} x(t)x^*(t-\tau)dt \overset{\mathcal{F}}{\longleftrightarrow} S_x(f) = X(f).X^*(f) = |X(f)|^2$$

$$R_x(\tau) = \lim_{T\to+\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t)x^*(t-\tau)dt \overset{\mathcal{F}}{\longleftrightarrow} S_x(f) = \lim_{T\to+\infty} \frac{1}{T}|X_T(f)|^2$$

- **Parceval-Plancherel Theorem**

$$E_x = R_x(0) = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} S_x(f)df$$

$$P_x = R_x(0) = \lim_{T\to+\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} |x(t)|^2 dt = \int_{-\infty}^{+\infty} S_x(f)df$$

Spectral Resolution

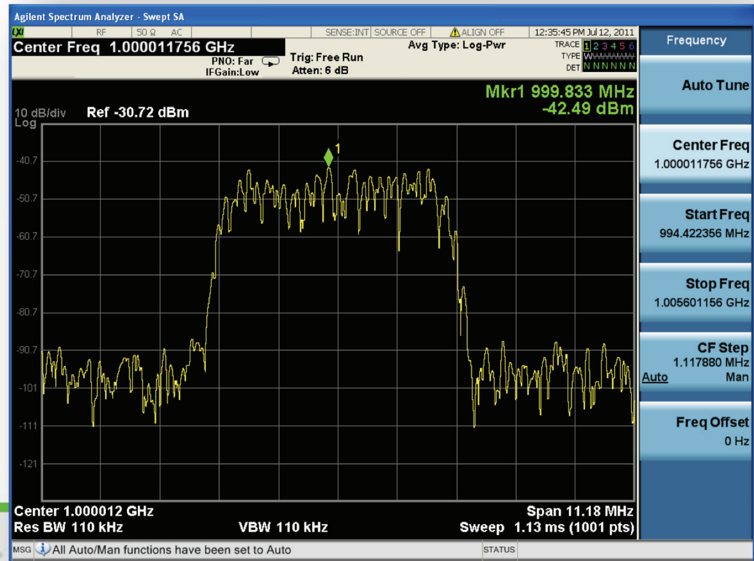
Spectrum analyzers produce an estimation of the PSD of a signal over a finite observation window [0 T]

- Observation window: **truncated signal**

$$x_T(t) = x(t)\,w(t) \qquad \begin{cases} w(t) = 1 & si \quad |t| \leq \frac{T}{2} \\ w(t) = 0 & si \quad |t| > \frac{T}{2} \end{cases}$$

$$X_T(f) = X(f) * \frac{\sin \pi f T}{\pi f}$$

$$S_x(f) \simeq \frac{1}{T} \left| X(f) * \frac{\sin \pi f T}{\pi f} \right|^2$$



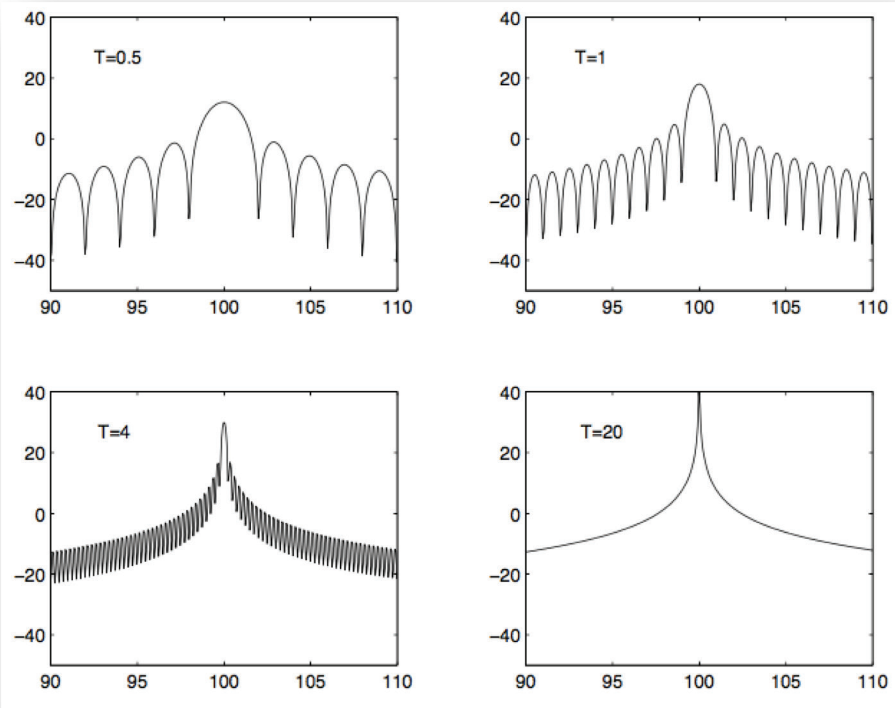
- **Pure frequency** $x(t) = e^{j2\pi f_0 t}$

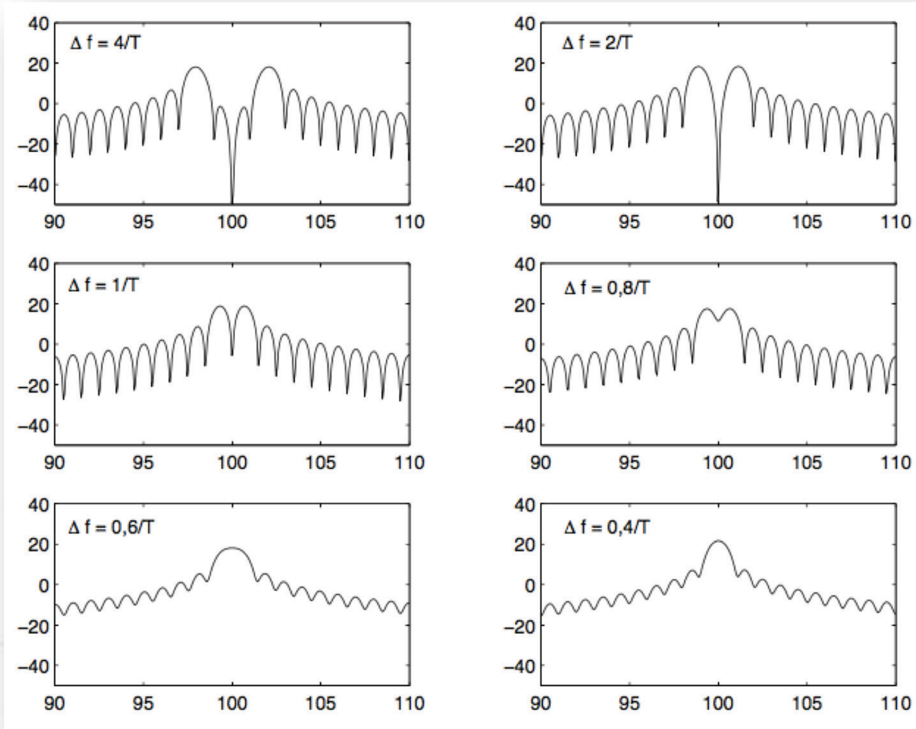
$$S_x(f) \simeq \frac{1}{T} \left| \frac{\sin \pi (f - f_0) T}{\pi (f - f_0)} \right|^2$$

Spectral Resolution

- Convolution of the signal by sinc makes impossible the detection of **two narrow frequencies** in the principal lobe.
  - **Separation is 1/T** where T If the sampling period

Spectrum Signal





Variable Sampling frequency

Variable Frequency splitting

$$cos(2\pi f_0 t)\ ; \qquad f_0 = 100$$

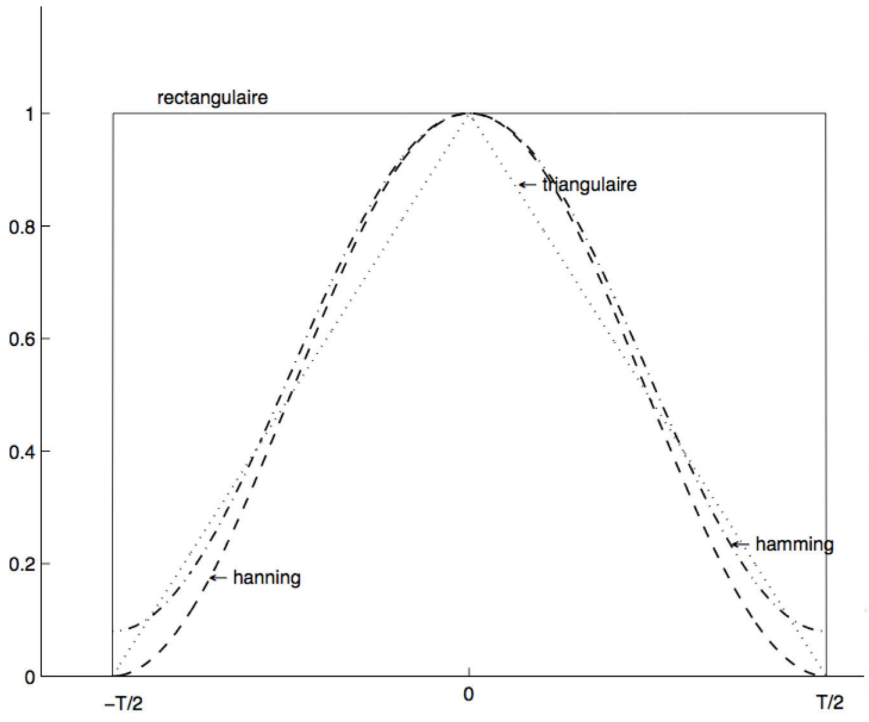
$$cos(2\pi(f_0 - \tfrac{\Delta f}{2})t) \ + \ cos(2\pi(f_0 + \tfrac{\Delta f}{2})t)$$

Apodization in Signal Processing

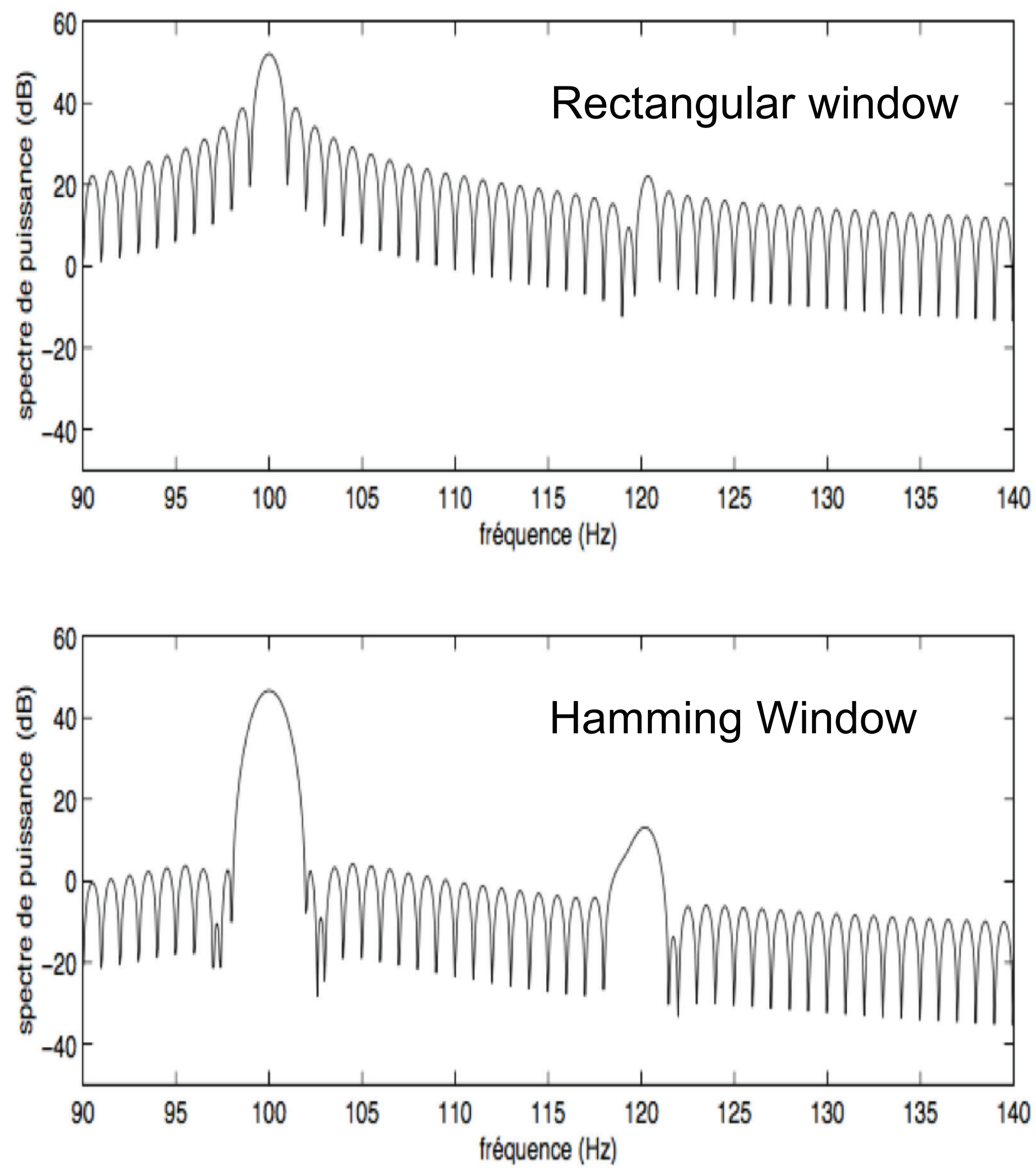
- Secondary sinc lobes mask also small amplitude information
- Solution: **apodization** (making tail flatter) at the prize of An enlargement of the main lobe

Nom	Définition de w(k)	1/2-largeur du lobe principal de W(f)	Niveau des lobes secondaires de W(f)
Rectangulaire	$w(t) = \begin{cases} 1 & \lvert t \rvert \leq \frac{T}{2} \\ 0 & sinon \end{cases}$	$\frac{1}{T}$	- 13 dB
Triangulaire	$w(t) = \begin{cases} 1 - \frac{\lvert t \rvert}{T/2} & \lvert t \rvert \leq \frac{T}{2} \\ 0 & sinon \end{cases}$	$\frac{2}{T}$	- 26 dB
Hanning	$w(t) = \begin{cases} 0,5 + 0,5 \cos \frac{\pi t}{T/2} & \lvert t \rvert \leq \frac{T}{2} \\ 0 & sinon \end{cases}$	$\frac{2}{T}$	- 32 dB
Hamming	$w(t) = \begin{cases} 0,54 + 0,46 \cos \frac{\pi t}{T/2} & \lvert t \rvert \leq \frac{T}{2} \\ 0 & sinon \end{cases}$	$\frac{2}{T}$	- 43 dB

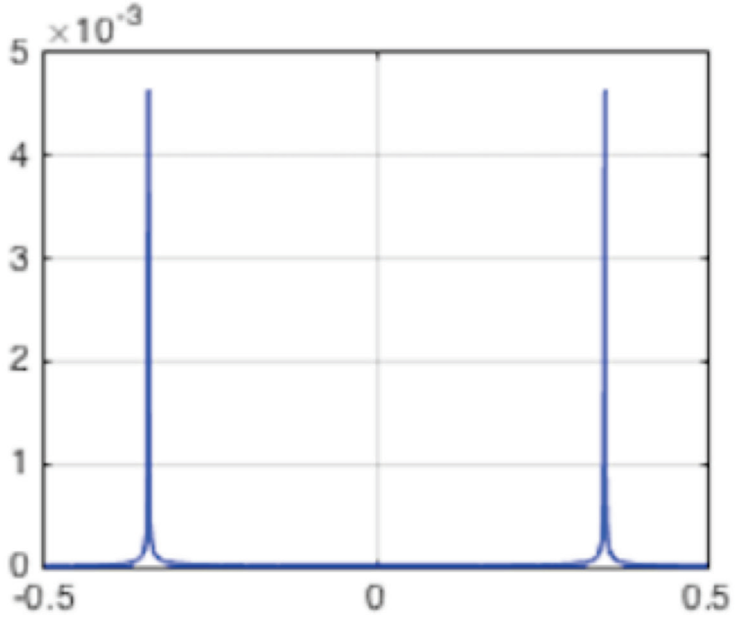
Apodization in Signal Processing (II)

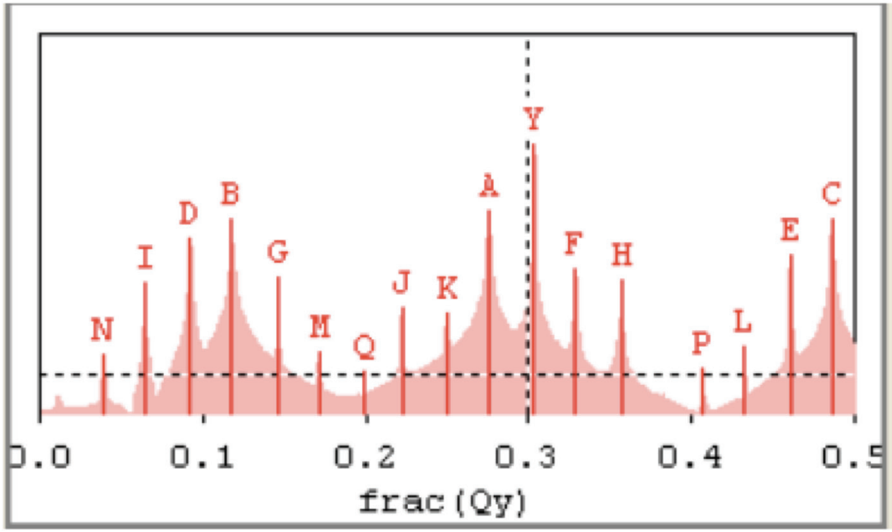


Triangular, Hanning, Hamming windowing



Improved Algorithms: SUSSIX/NAFF (Frequency Map Analysis)





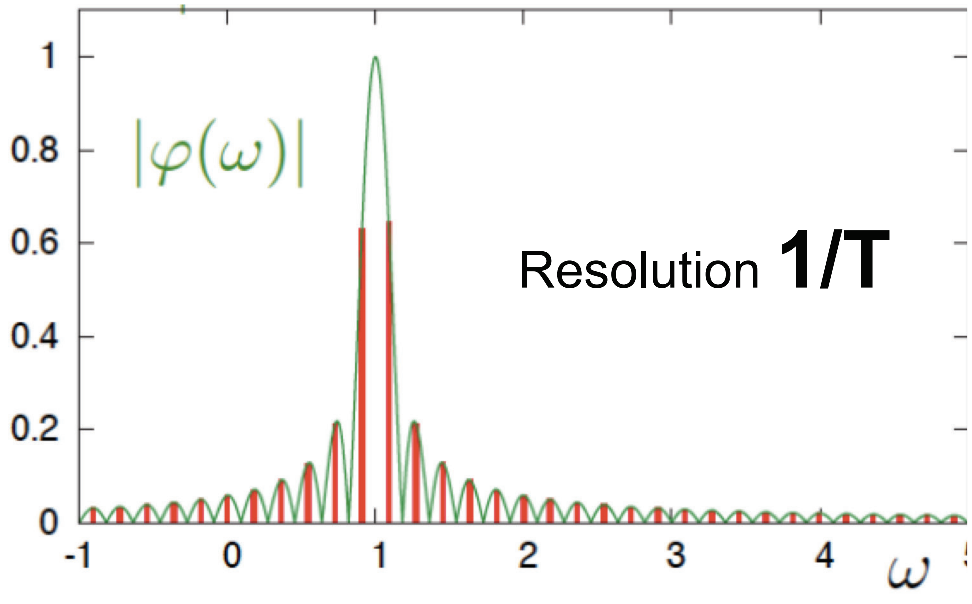
Main Features

- Complex dynamics
- Resonances
- Tuneshift with amplitudes
- Chaos, instabilities

Numerical Techniques

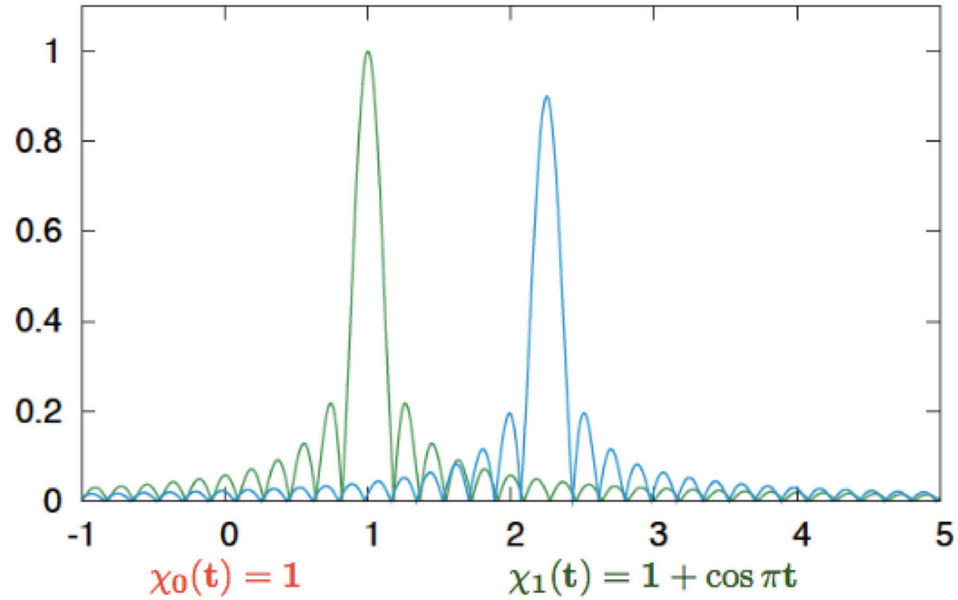
- Complex spectrum
- FFT ($1/T$) or interpolated Fourier analysis, NAFF
- Resolution, Convergence speed ($1/T^2$ or $1/T^4$)

$f(t) = e^{i\nu t}$ on the time Intervalle $[0,\ T]$



$f(t) = a_1 e^{i\nu_1 t} + a_2 e^{i\nu_2 t}$ with $a_1 = 1,\ \nu = 1,\ a_2 = 0.9,\ \nu_2 = 2.25$

$|\phi_1(\omega)|$ $|\phi_2(\omega)|$ $|\phi_1(\omega) + \phi_2(\omega)|$



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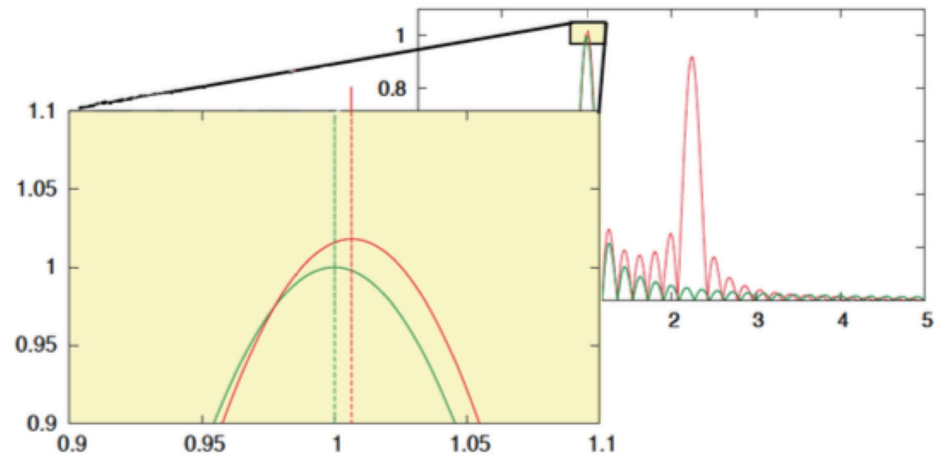
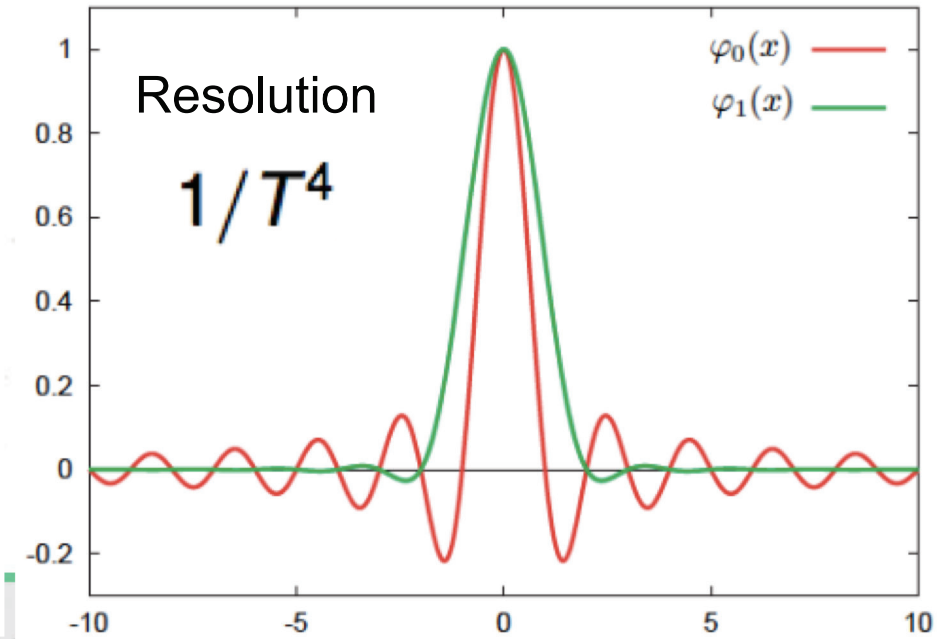


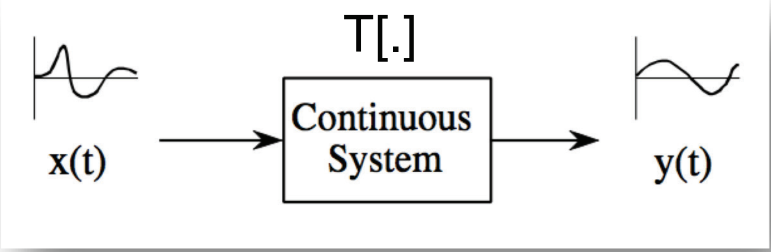
Interpolation to find the maximum

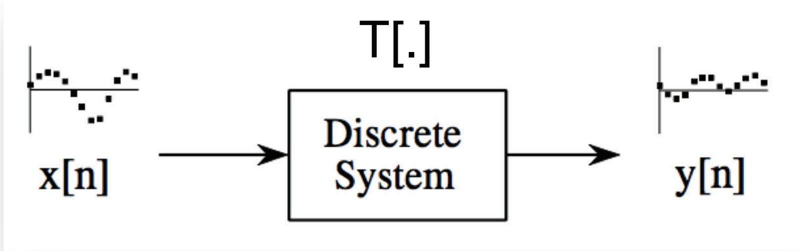


Hanning window (apodization)

APPLICATION TO FILTERS

Linear Time-Invariant Systems (LTI)





- **System:** any process producing an output signal in response to an input signal

$$y(t) = T[x(t)]$$ T like **Transfer Function**

- **Linearity**

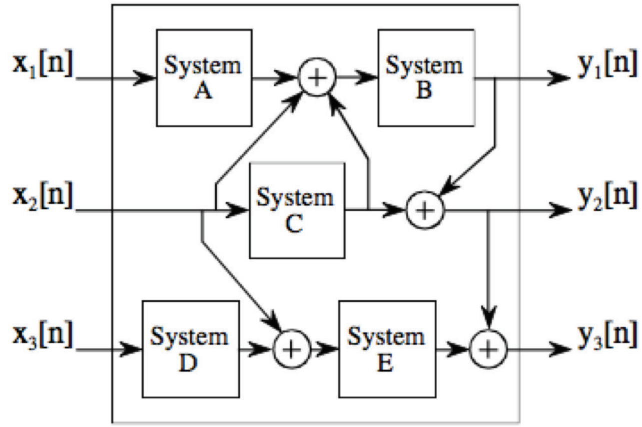
$$\forall\, a,\, b \in \mathbb{C} \quad T[a\, x_1(t) + b\, x_2(t)] = a\, T[x_1(t)] + b\, T[x_2(t)]$$

- **Time-invariance** (stationarity)

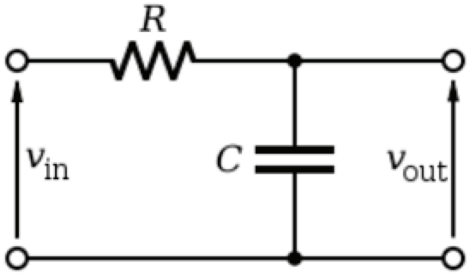
$$\text{if} \quad y(t) = T[x(t)] \quad \text{then} \quad \forall \tau \in \mathbb{R} \quad T[x(t-\tau)] = y(t-\tau)$$

- LTIs are representative of a large class of **physical systems** (at least as a approximation at first order)

Example of LTIs

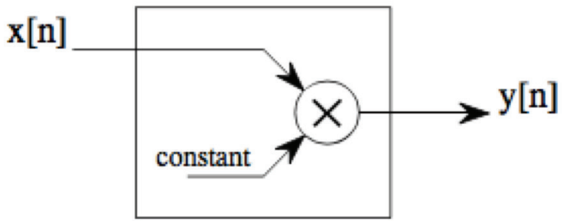


- **Linear Systems**



$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

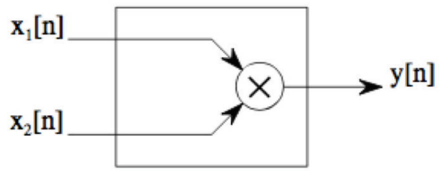
with $y(t_0) = \frac{dy(t_0)}{dt} = \cdots = \frac{d^{N-1}y(t_0)}{dt^{N-1}} = 0$



Linear
a. Multiplication by a constant

- **Non Linear Systems**

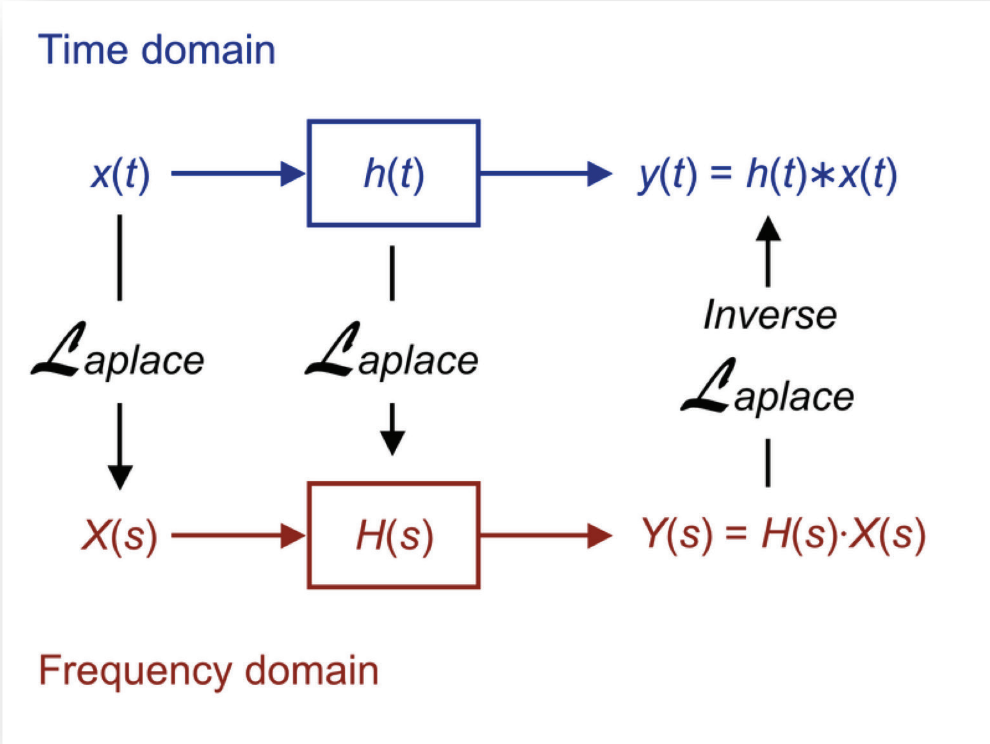
$y(t) = x^2(t)$ $\qquad y(t) = x(t)x(t - \Delta t)$ $\qquad y(t) = e^{x(t)}$



Nonlinear
b. Multiplication of two signals

Filter Definition

- A filter is a **LTI system** that
  - Lets a range of frequencies to be transmitted
  - Attenuates a range of frequencies
- A main function of any filter is to **extract useful component** and **discarding irrelevant, harmful frequency** (noise)
- Best formalism to characterize and study a filter is to look at its **frequency response**



Transfer Function to a Dirac Pulse

- Time domain h(t)
- Frequency domain H(f) or H(ω)
- Laplace (Symbolic) domain H(s)

Impulse Response h(n) or h(t)

- **Definition of h(t):**

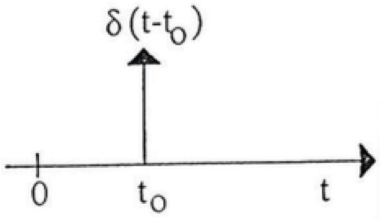
$$h(t) = T[\delta(t)]$$

- **From the knowledge of the impulse response, the general output of the LTI system to any input x(t) is given by:**

$$y(t) = x(t) * h(t) = h(t) * x(t)$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$

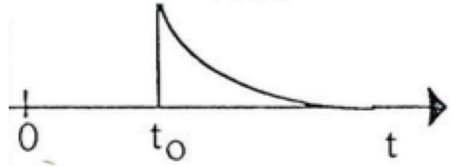
Impulse Response of a LTI System



→ **LTI System** →

For time-invariant systems:

$$h(t, t_0) = h(t - t_0)$$



$$x_0(t)\delta(t - t_0)dt_0 \Rightarrow \text{LTI System} \Rightarrow x(t_0)h(t, t_0)dt_0$$

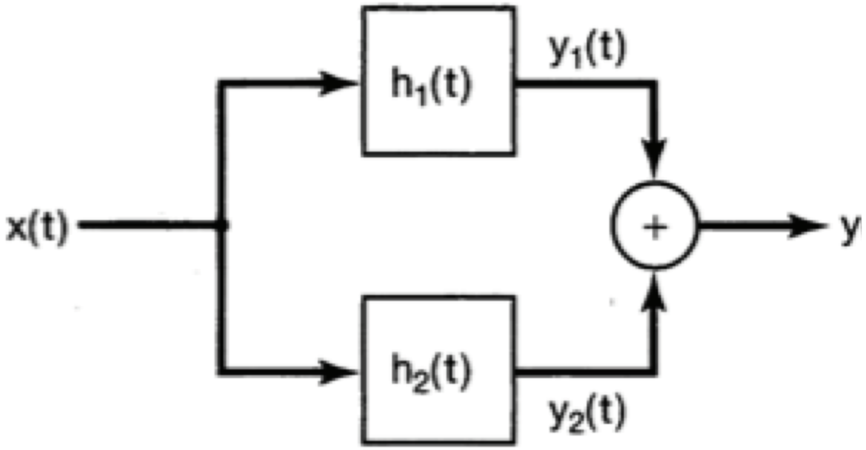
$$\int_{-\infty}^{+\infty} x_0(t)\delta(t - t_0)dt_0 \Rightarrow \text{LTI System} \Rightarrow \int_{-\infty}^{+\infty} x(t_0)h(t, t_0)dt_0$$

Convolution Integral

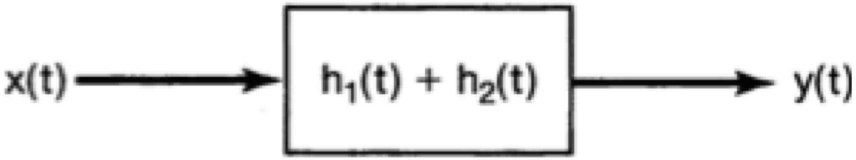
$$x(t) \Rightarrow \text{LTI System} \Rightarrow y(t) = x(t) * h(t)$$

Filter Properties

h(t): Impulse response



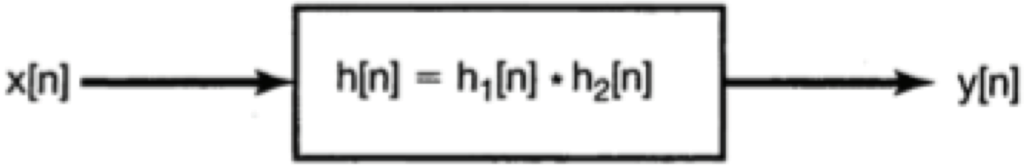
(a)



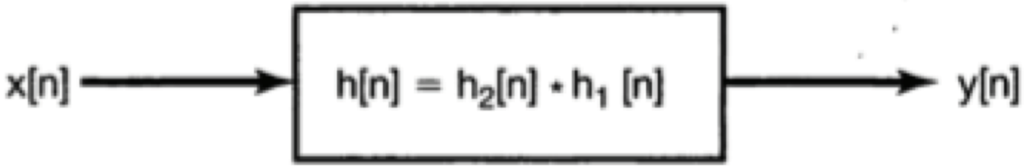
(b)

Distributive Property

(a)



(b)



(c)

(d)

Associative/commutative Property

Principle of Causality: Physical Systems

- **Causality** : the output of a system depends only on the present and past inputs of the system

$$h[n] = 0 \quad \text{for} \quad n < 0$$

Discrete system

$$h(t) = 0, \quad t < 0$$

Continuous system

- **LTI System**

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k]$$

$$y(t) = \int_{0}^{+\infty} x(t_0)h(t-t_0)dt_0$$

- **Paley-Wiener condition**

$$\int_{-\infty}^{+\infty} \frac{|\log|H(f)||}{1+f^2} df < \infty$$

Unite Step Response u(n) or u(t)

Causal signal

$$s[n] = u[n] * h[n]$$

$$s[n] = \sum_{k=-\infty}^{n} h[k]$$

$$s(t) = u(t) * h(t)$$

$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

- The unit impulse response of the accumulator

$$h[n] = s[n] - s[n-1]$$

Discrete

$$h(t) = \frac{ds(t)}{dt}$$

Continuous

Both a continuous and a discrete time system can be fully characterized from the knowledge of its unit impulse response or its unit step response

Transfer Function of a LTI systems H(s)

- **Generalization:**

$$T\left[e^{st}\right] = H(s)e^{st}$$
$$H(s) = \mathcal{L}[h(t)]$$

- **Frequency response**:

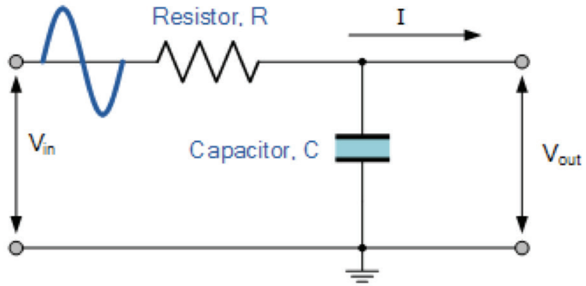
$$s = j\omega = j2\pi f.$$

Input/Output for LTI

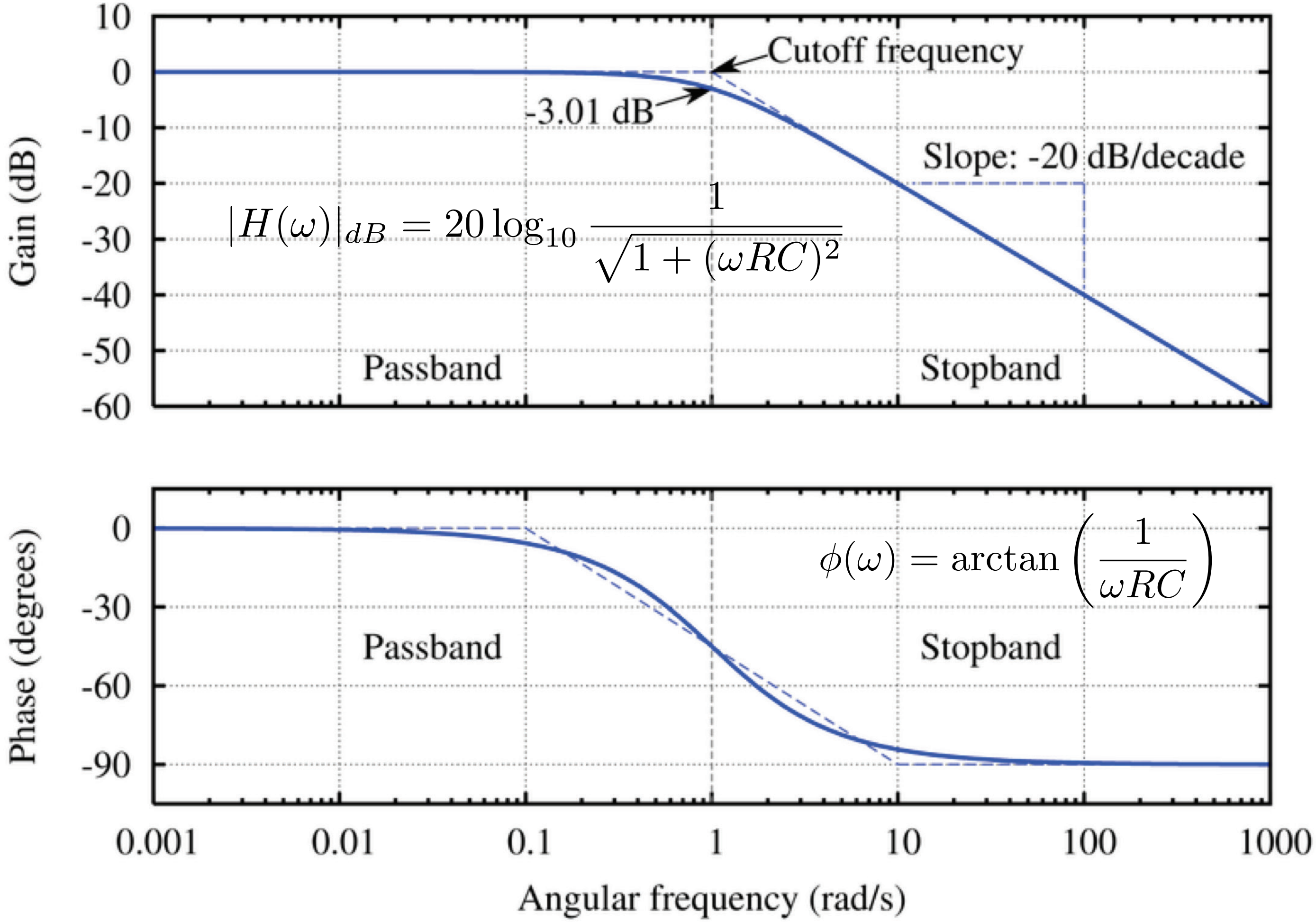
- **Time domain**
- **Frequency Domain**
- **Domain defined by Laplace Transformation**

$$y(t) = h(t) * x(t)$$
$$Y(f) = H(f) . X(f)$$
$$Y(s) = H(s) . X(s)$$

Bode Diagrams:
Low-Pass Filter of first order



$$\frac{V_{out}}{V_{in}} = \frac{1}{1 + j\omega RC}$$



Amplitude

Phase

Asymptotic limits

$\omega_o = 1 / RC$ is the "break frequency" or "-3 dB frequency"

$\omega << \omega_o$ results in a magnitude of 20 log (1/1) = 0 dB

$\omega >> \omega_o$ results in a magnitude of

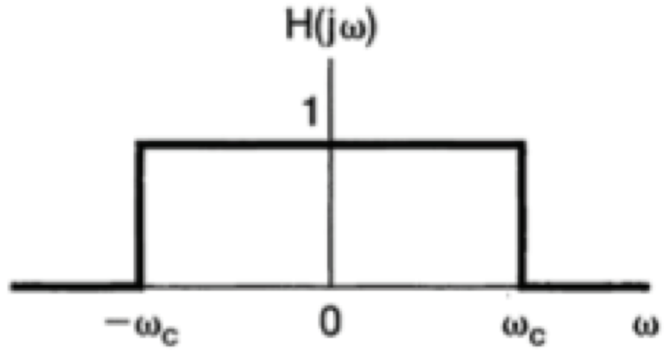
$\omega << \omega_o$ results in a phase of - atan(0) = 0

$\omega >> \omega_o$ results in a phase of - atan(infinity) = - 90°

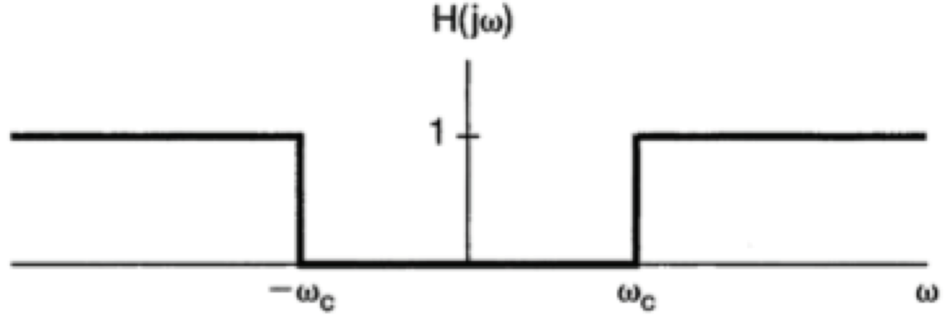
$\omega = \omega_o$ results in a phase of - atan(1) = - 45°

Main classes of Filters

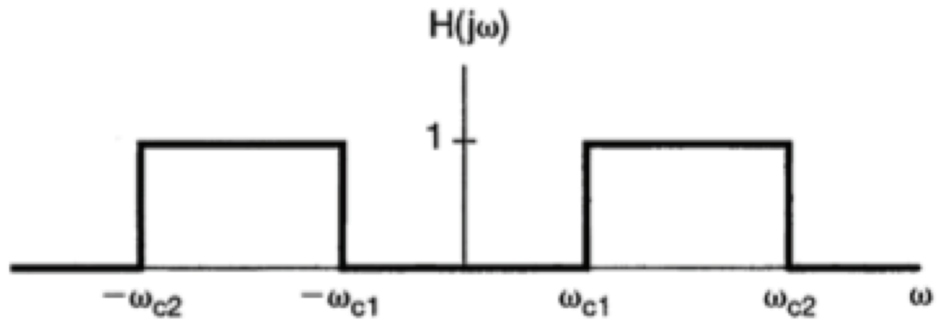
- **Low-Pass Filter**



- **High-Pass Filter**

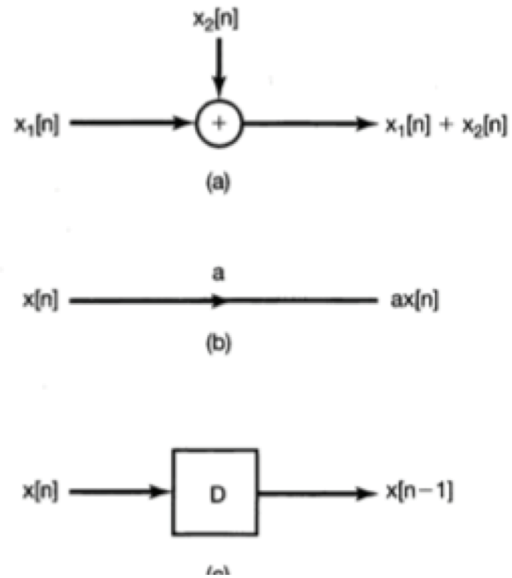


- **Band-Pass Filter**



Linear Time Invariant Systems

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k].$$



Transfer Function

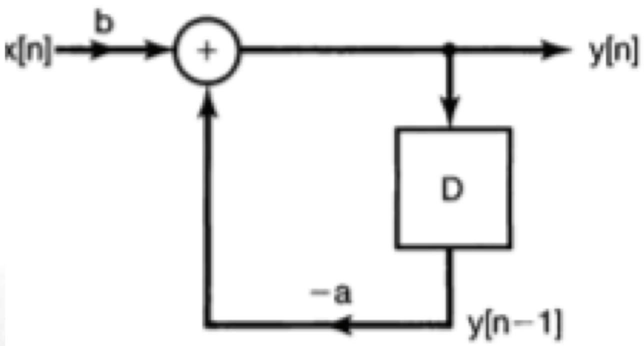
Feedforward

$$H(s) = \frac{b_0 + b_1 s + b_2 s^2 + \cdots + b_N s^N}{a_0 + a_1 s + a_2 s^2 + \cdots + a_M s^M}$$

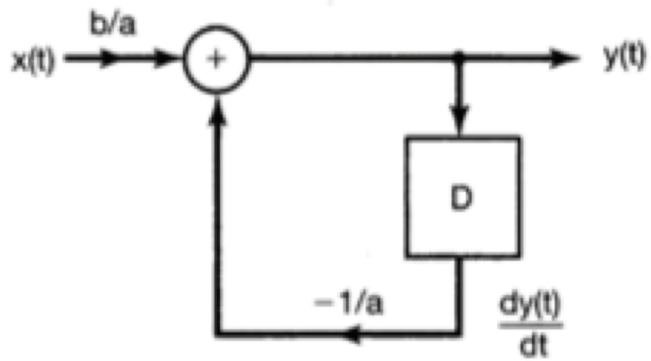
Feedback

Filter order M-N

Practical case (no saturation): **M≥N**



$$y[n] + ay[n-1] = bx[n].$$



$$\frac{dy(t)}{dt} + ay(t) = bx(t).$$

Finite Impulse Response (FIR) Systems

- If N=0, the system simplified as $$y[n] = \sum_{k=0}^{M} \left(\frac{b_k}{a_0}\right) x[n-k].$$
- It is easy to show that the impulse response is of finite duration.
- **Limited Memory filters**

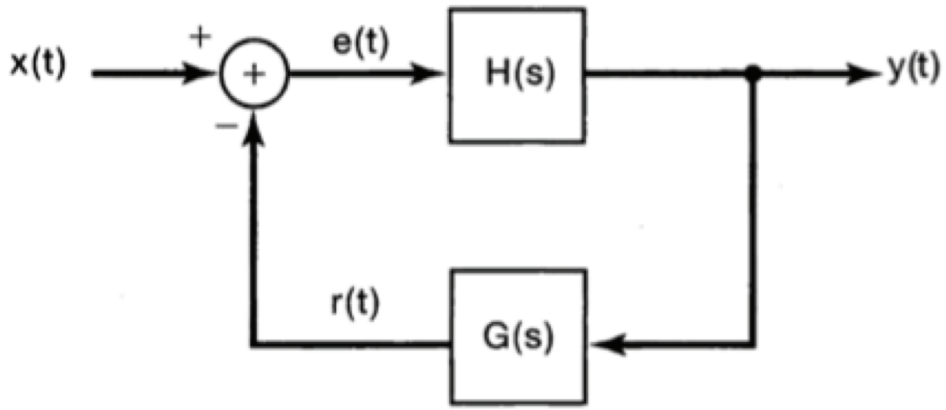
Infinite Impulse Response (IIR) Systems

- If N ≥ 1, recursive solution
- It can be shown that the impulse response is of infinite duration.
- **Infinite Memory Filter** (feedback). **Much more selective filters** than FIRs. Efficiency in implementation, in order to meet a specification in terms of passband, stopband, ripple, and/or roll-off (Gabarit).

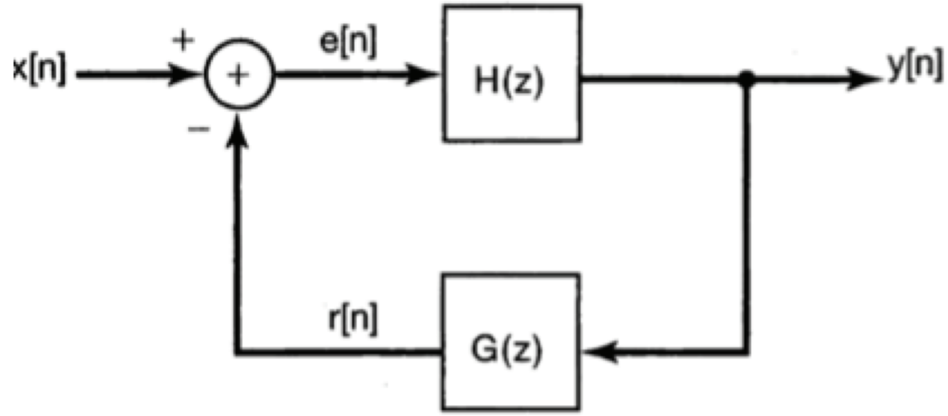
Application to a Linear Feedback Systems Mathematical Representation

$$T(s) = \frac{H(s)}{1 + G(s)H(s)}$$

Error signal



Response signal



(b)

Discrete System (**L-transform**)
Laplace Transform

Discrete System (**z-transform**)

Stability of LTI Systems - BIBO System

- A system is ***stable*** if every bounded input produces a bounded output.

Discrete System

$$y[n] = x[n] * h[n]$$

$$|x[n]| < B \quad \text{for all } n$$

$$|y[n]| \leq B \sum_{k=-\infty}^{+\infty} |h[k]| \quad \text{for all } n.$$

Continuous System

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau.$$

$$|x(t)| < B,$$

$$|y(t)| = \left|\int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau\right| \leq \int_{-\infty}^{+\infty} |h(\tau)||x(t-\tau)|d\tau \leq B\int_{-\infty}^{+\infty} |h(\tau)|d\tau.$$

A system is stable if the impulse response is absolutely summable / integrable

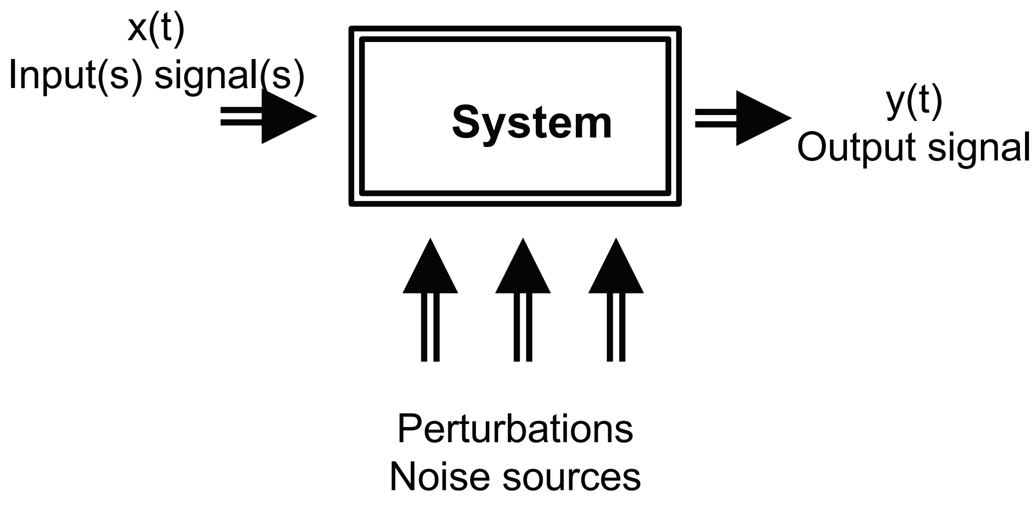
$$\sum_{k=-\infty}^{+\infty} |h[k]| < \infty,$$

$$\int_{-\infty}^{+\infty} |h(\tau)|d\tau < \infty.$$

MEASUREMENT

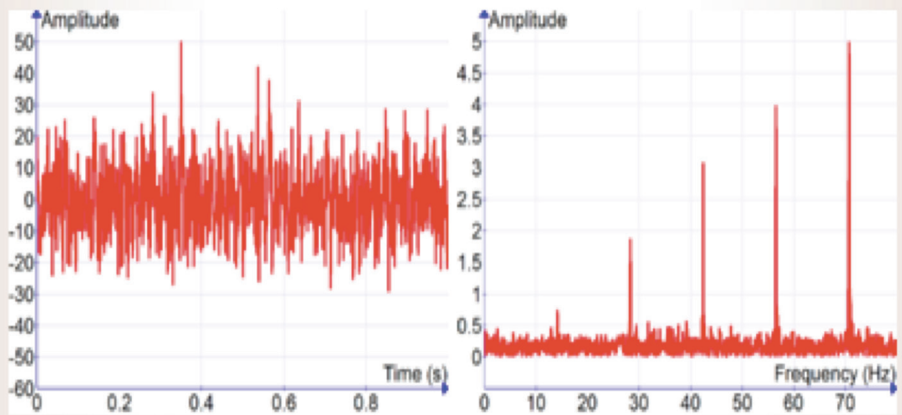
Measurement

- **Signal**:
  - function of one or several variables that conveys **information** about the behavior or attributes of some physical phenomenon
  - description of how one parameter is related to another parameter.



- **Type**
  - Analog/digital
  - Continuous/Discrete Time

- **Purposes**
  - Extract information
  - Verify model vs experiment
  - System often well known
  - Perturbation (noise)
    - Amplitude, phase
    - Time varying
    - Frequency varying



The signal on the left looks like noise, but the signal processing technique known as the Fourier transform (right) shows that it contains five well defined frequency components.

Wikipedia

Error Types

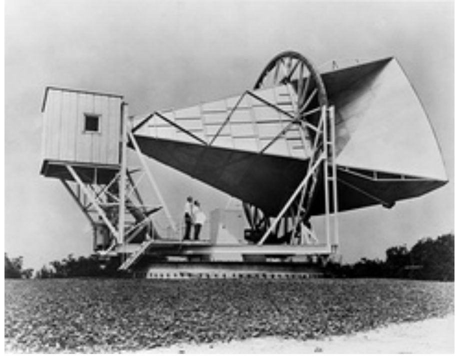
- **Systematic Error**
  - Characteristics of the **measurement device**
    - Offset, gain, amplitude/phase/frequency linearity
  - Can be improved to a certain limit
  - Low limitation given by quantum mechanics
- **Accuracy**

- **Statistical Error**
  - Unforeseen fluctuations
  - Random, stochastics, noise
  - Estimation possible
  - Reduction by averaging, etc.
- Impossible to avoid completely
- **Precision**

NOISE

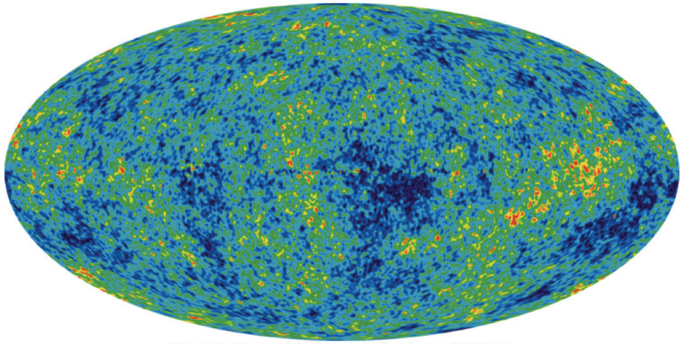
Noise

The Holmdel Horn Antenna on which Penzias and Wilson discovered the cosmic microwave background



- **Definition**
  - **Unwanted signal** that disturbs, distorts the signal to be measured
  - **Perturbation** that makes difficult to perceive the important and useful information of a given signal
  - **What is noise for a measurement can be useful information for another measurement**! Cosmic MicroWave Background Radiation
  - **Sources**: Brownian motions of charges (thermal), Variation of charge number, Quantum effect, Quantification Noise, Electromagnetic compatibility (EMC)

- **Characteristic**
  - Any physical system generates some noise by nature
  - Most of the time random signal
  - Described
    - In frequency domain by if in frequency contents
    - Statistics data (mean value, RMS value, etc.)



All-sky mollweide map of the CMB, created from 9 years of WMAP data

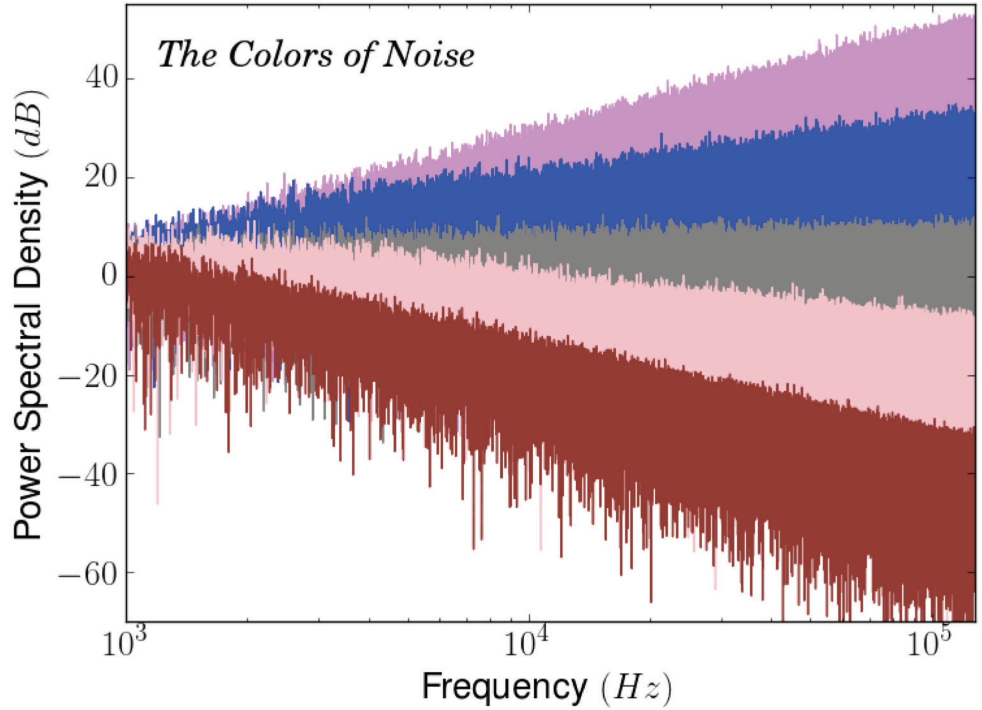
- **Quote by A. Hoffmann**
  - “The art of measurement is always the art of error treatment”
  - Key for good measurement is to reduce the noise source as much as possible

Colors of Noise

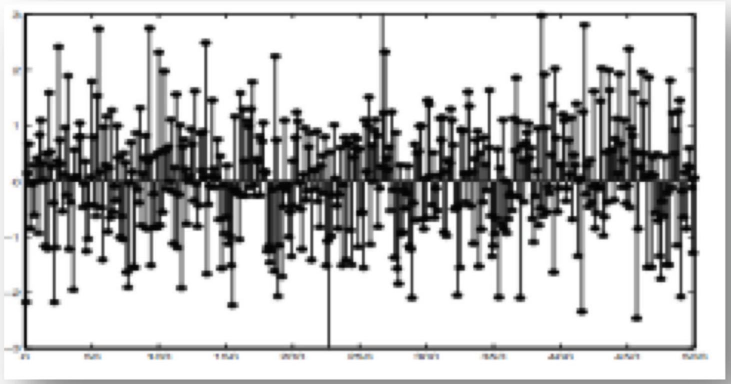
- **Frequency dependence**: Power spectral densities as **a function of frequency** for various colors of noise (violet, blue, white, pink, brown/red).

- Right Figure :
  - The power spectral densities are arbitrarily normalized such that the value of the spectra are approximately equivalent near 1 kHz.
  - Note the **slope of the power spectral density** for each spectrum provides the context for the respective electromagnetic/color analogy.

- **Amplitude dependence:** Noise can also be proportional to the signal, the square of the signal, etc.



Noise	Freq. law	dB/ decade	Db/ octave	Spectrum
White	Flat			Flat
Pink	1/f	-10	-3	Low Pass.
Brownian (red)	$1/f^2$	-20	-6	Low Pass.
Blue	f	+10	+3	High Pass.
Violet	f^2	+20	+6	High Pass.



White Noise

- Complete **decorrelation**

- **Power Spectral Density (PSD)**
  - Constant, frequency independent

$$\mathbf{E}[X(k)] = 0$$

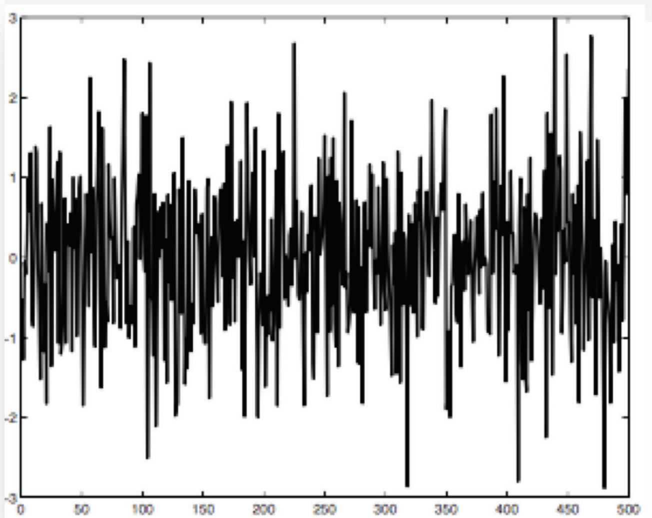
$$R_X(k) = \frac{N_0}{2}\,\delta(k) \quad \text{où } \delta(k) \text{ est l'impulsion unité}$$

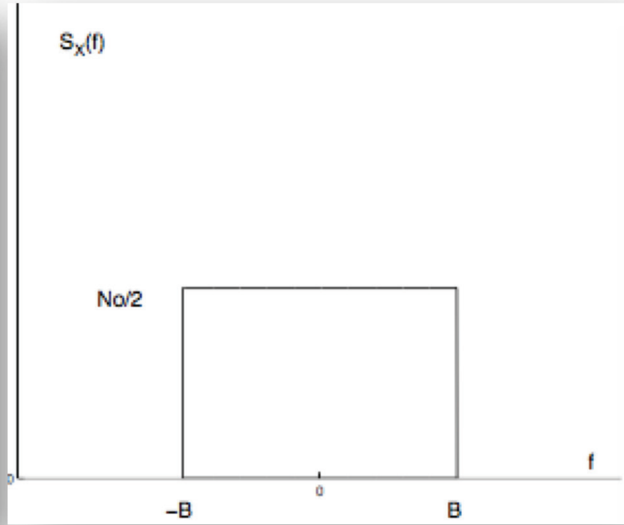
$$S_X(f) = \frac{N_0}{2}$$

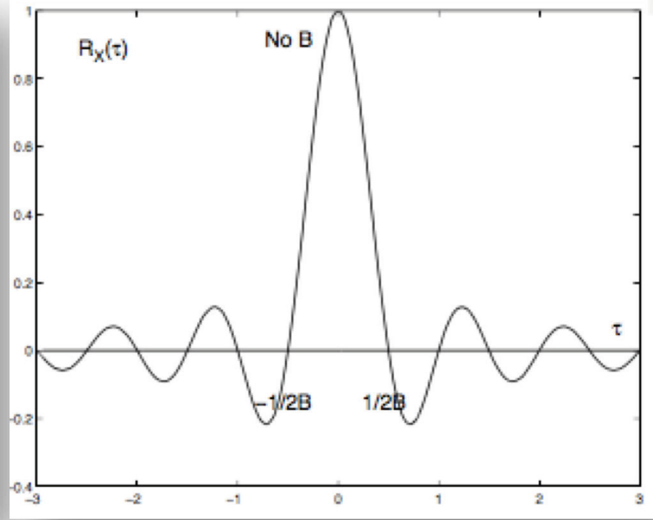
- **Power (variance)**

$$P_X = R_X(0) = \mathbf{E}[X(k)^2] = \sigma_X^2 = \int_{-\frac{1}{2}}^{+\frac{1}{2}} S_X(f)df = \frac{N_0}{2}$$

Physical White Noise







- Noise defined in a **limited bandwidth** [-B B]
- Correlation
- Autocorrelation
- Finite Power

$$\begin{cases} S_X(f) = \frac{N_0}{2} & \text{si } f \in [-B, B] \\ S_X(f) = 0 & \text{sinon} \end{cases}$$

$$R_X(\tau) = \mathcal{F}^{-1}[\frac{N_0}{2}\,\mathrm{rect}_{2B}(f)] = N_0 B \frac{\sin 2\pi B\tau}{2\pi B\tau}$$

$$P_X = R_X(0) = \sigma_X^2 = \int_{-\infty}^{+\infty} S_X(f)\,df = N_0\,B$$

Signal to Noise Ratio (SNR or S/N)

- Comparing the level of a desired signal to the level of background noise
- SNR is defined as the ratio of signal power to the noise power.

$$\mathrm{SNR} = \frac{P_{\mathrm{signal}}}{P_{\mathrm{noise}}} = \left(\frac{A_{\mathrm{signal}}}{A_{\mathrm{noise}}}\right)^2$$

A is the root mean (RMS) amplitude

$$\mathrm{SNR} = \frac{\sigma^2_{\mathrm{signal}}}{\sigma^2_{\mathrm{noise}}}.$$

Power Ratio

Variance Ratio

Because many signals have a very wide dynamic range, signals are often expressed using the **logarithmic decibel (dB) scale** .

$$\mathrm{SNR_{dB}} = 10\log_{10}\left(\frac{P_{\mathrm{signal}}}{P_{\mathrm{noise}}}\right) \qquad P_{\mathrm{signal,dB}} = 10\log_{10}(P_{\mathrm{signal}})$$

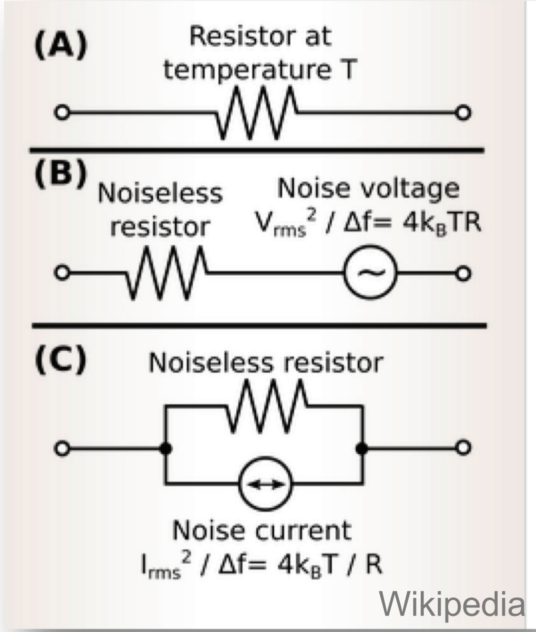
$$\mathrm{SNR_{dB}} = P_{\mathrm{signal,dB}} - P_{\mathrm{noise,dB}}.$$

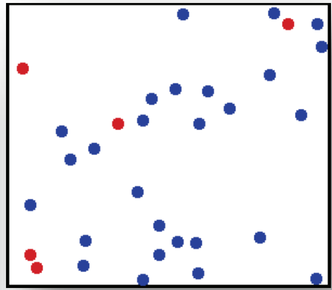
Power Ratio	dB
2	3
10	10
100	20
1000	30

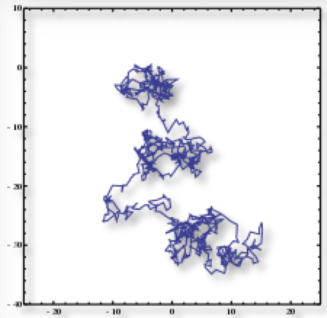
$$\mathrm{SNR_{dB}} = 10\log_{10}\left[\left(\frac{A_{\mathrm{signal}}}{A_{\mathrm{noise}}}\right)^2\right] = 20\log_{10}\left(\frac{A_{\mathrm{signal}}}{A_{\mathrm{noise}}}\right) = (A_{\mathrm{signal,dB}} - A_{\mathrm{noise,dB}}).$$

Johnson(-Nyquist) Noise, Thermal Noise

- **Electronic noise** generated by the thermal agitation of the electrons inside an electrical conductor.
- It happens **regardless of any applied voltage** for T>0K
- **White noise** in ideal resistor
- **Gaussian noise** in a finite bandwidth







Voltage variance (RMS) per Hz

$$\overline{v_n^2} = 4k_B TR$$

$$P = v_n^2/R = Ri_n^2 = u_n i_n$$

$$\sqrt{\overline{v_n^2}} = 0.13\sqrt{R}\,\mathrm{nV}/\sqrt{\mathrm{Hz}}.$$

at room temperature (300 K)

For a bandwidth Δf

$$v_n = \sqrt{\overline{v_n^2}}\sqrt{\Delta f} = \sqrt{4k_B TR\Delta f}$$

$$i_n = \sqrt{\frac{4k_B T\Delta f}{R}}$$

Average Matched Power is **independent of R**: $P_n = \frac{1}{4}P = k_B T\Delta f$

Thermal Noise (II)

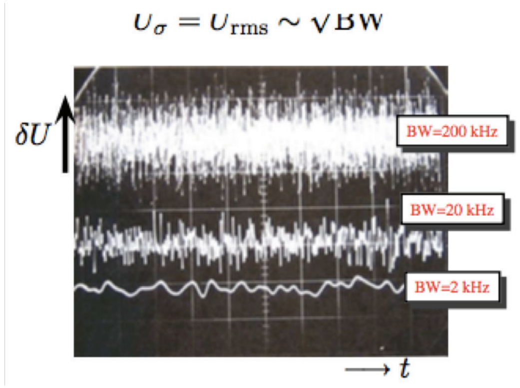
- **Black Body** radiation is governed by the **Plank's law**:

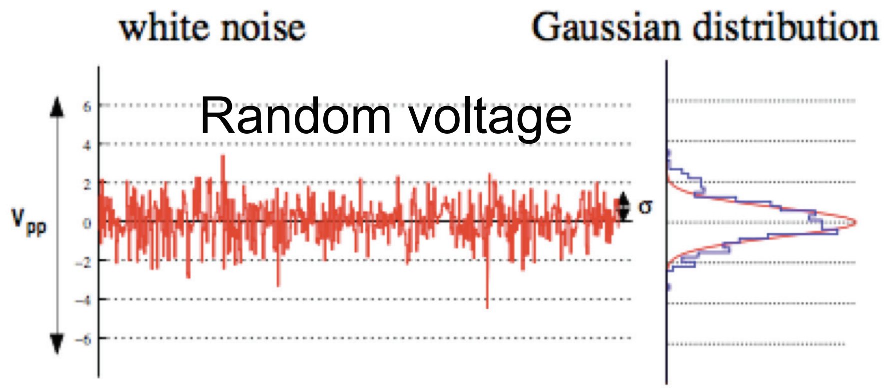
$$P = \int_{\mathrm{BW}} \rho d\nu \qquad \rho = kT \frac{\frac{h\nu}{kT}}{e^{\frac{h\nu}{kT}} - 1} \approx kT$$

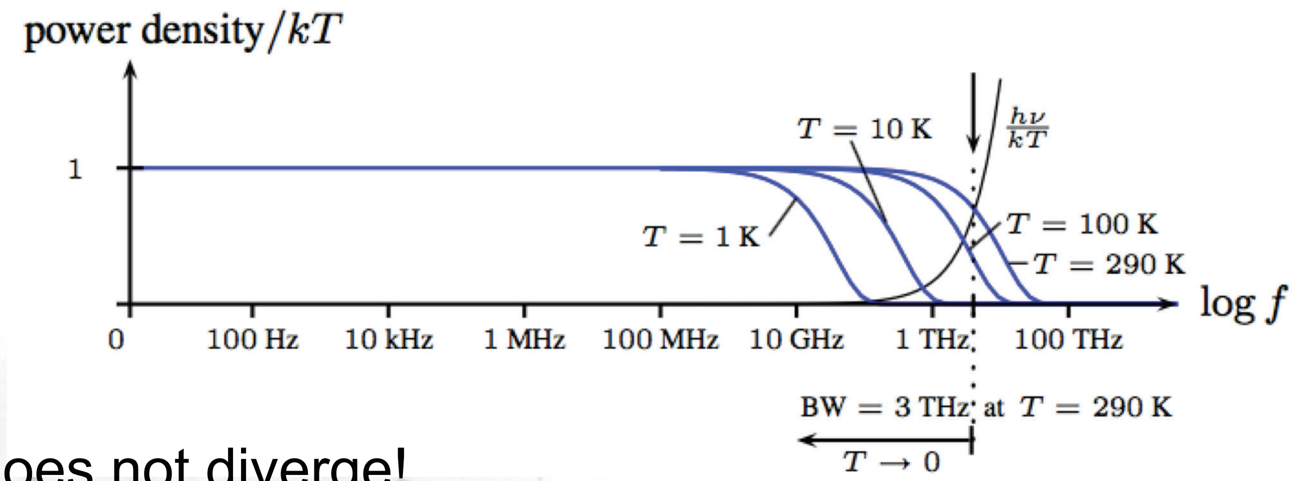
Rayleigh–Jeans

$h \simeq 6,62.10^{-34} \quad J.s$ Planck constant

$k \simeq 1,38.10^{-23} \quad J.K^{-1}$ Boltzmann constant







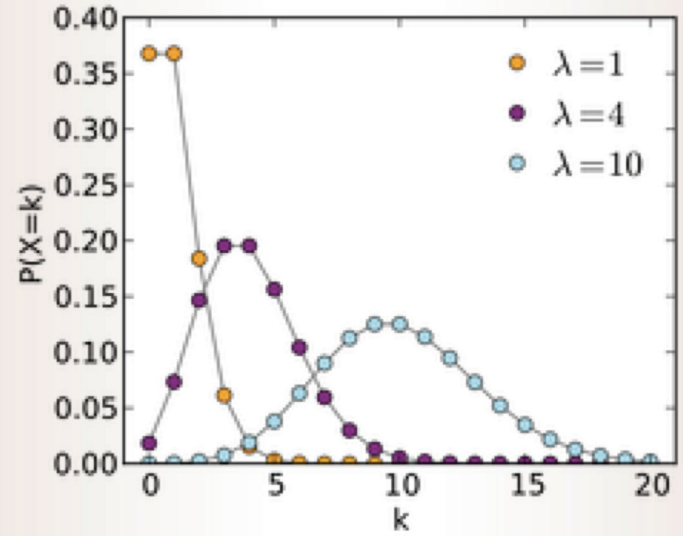
- Power does not diverge!

  Need to use the full formula at large frequency: Cut-off around **50 GHz to 10 THz**

Schottky Noise / "Shot Noise" / Poisson Noise

- Noise resulting from the discrete nature of the electric charge: **variation of the charge density in a conductor**.
- Can also occurrent in photon counting in optical devices

$$\mathrm{SNR} = \frac{N}{\sqrt{N}} = \sqrt{N}.$$



The number of photons that are collected by a given detector varies, and follows a Poisson distribution, depicted here for averages of 1, 4, and 10.

The statistics of e- passage time: **Poisson's Law**

$$\mathbb{E}[i(t)] = e\lambda = I_0$$

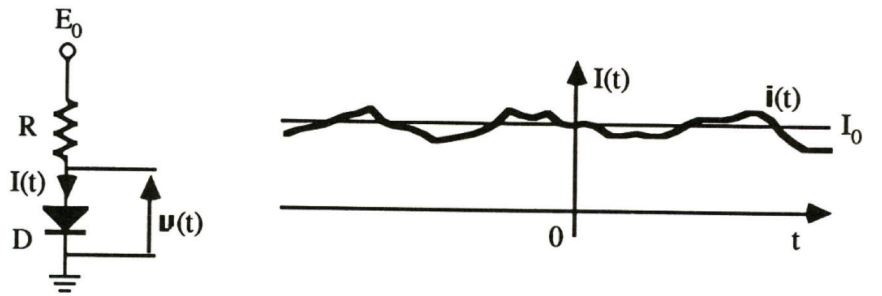
$$q(t) = N(t)\,e \quad \Rightarrow \quad i(t) = e\frac{dN(t)}{dt} = e\sum_{n\geq 1}\delta(t - T_n)$$

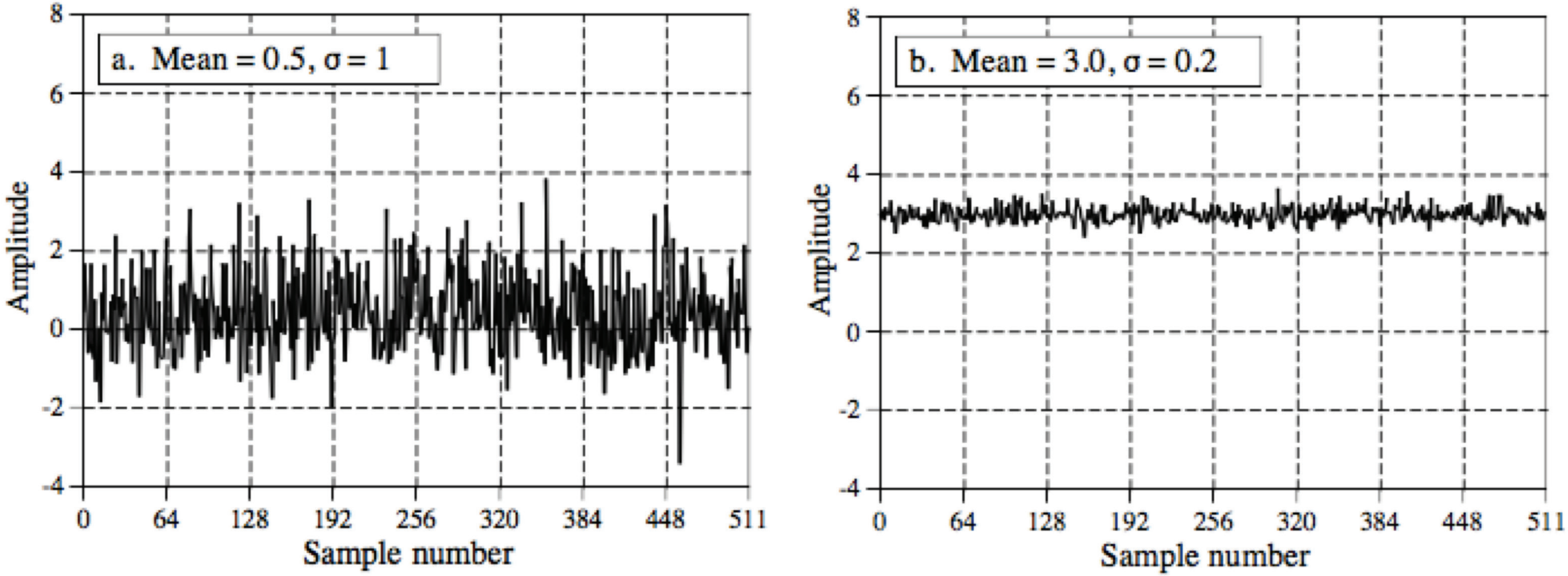
Autocorrelation function:

$$R_i(\tau) = e^2[\lambda^2 + \lambda\delta(\tau)] = I_0^2 + e\,I_0\,\delta(\tau)$$

PSD: **white noise continuous of average I_0**

$$S_i(f) = I_0^2\,\delta(f) + e\,I_0$$




a. Mean = 0.5, σ = 1
Amplitude
Sample number
b. Mean = 3.0, σ = 0.2
Amplitude
Sample number

STATISTICS

Mean and Standard Deviation for N samples

Mean (Expected) Value $\hat{x} := \frac{1}{N}\sum_{i=0}^{N-1} x_i$

Variance $\sigma^2 := \frac{1}{N-1}\sum_{i=0}^{N-1}(x_i - \hat{x})^2 .$ $\sigma_N^2 = \frac{1}{N-1}\left[\underbrace{\sum_{i=0}^{N-1} x_i^2}_{\text{sum of squares}} - \frac{1}{N}\underbrace{\left(\sum_{i=0}^{N-1} x_i\right)^2}_{\text{sum}^2}\right]$

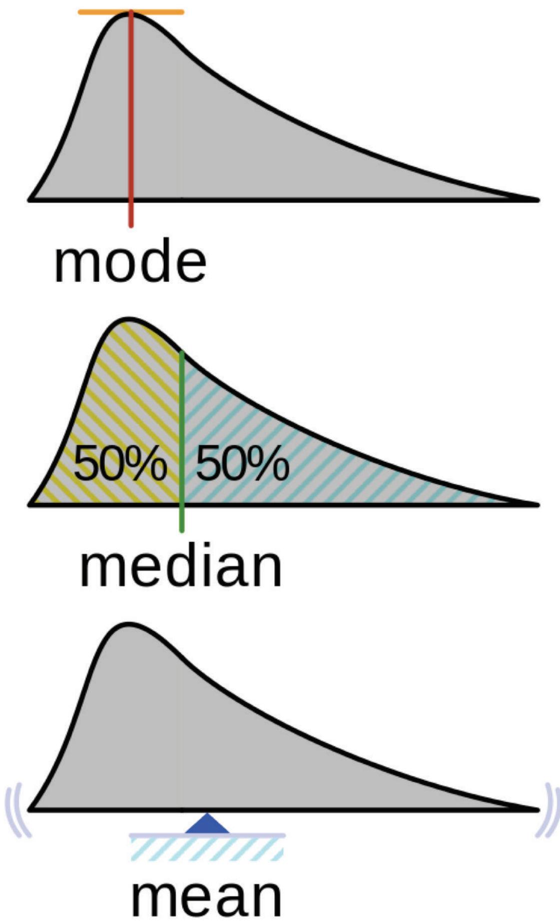
Root Mean Square/ Standard Deviation $x_{\text{rms}} := \sqrt{\frac{1}{N}\sum_{i=0}^{N-1} x_i^2} .$

Variance Coefficient / Relative Standard Deviation (RSD) $\mathrm{CV} = \frac{\sigma}{\hat{x}} \cdot 100\%$

Signal to Noise Ratio $\mathrm{SNR} = \frac{\hat{x}^2}{\sigma^2}$

Definitions

- **Probability Density/Mass Function** (PDF/PMF): f(x)
- **Mode**: The most frequently occurring number
- **Median**: the middle number of the group



- **Mean value** (Moment of Order 1)

$$\mu = \int x f(x)\, dx,$$

- **Variance** (Moment of Order 2)

$$\mathrm{Var}(X) = \sigma^2 = \int (x-\mu)^2 f(x)\, dx$$

- **Moment of order** n

$$\mu_n = \int_{-\infty}^{\infty} (x-c)^n\, f(x)\, \mathrm{d}x.$$

- **Normalized moment** of order n

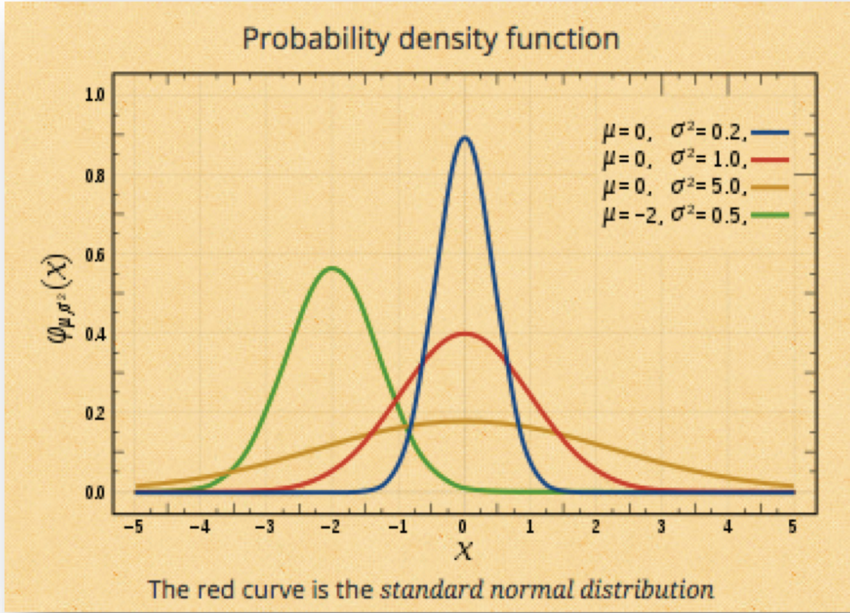
$$\frac{\mu_n}{\sigma^n} = \frac{\mathrm{E}[(X-\mu)^n]}{\sigma^n}$$

Gaussian / Normal Distribution

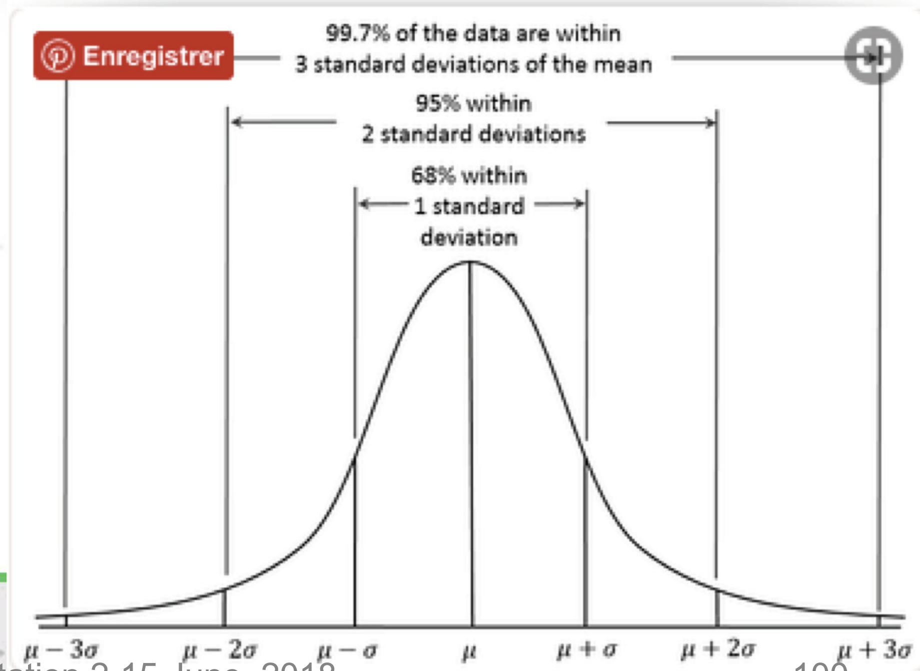
Probability Density Function

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\hat{x})^2}{2\sigma^2}}$$

$$\int_{-\infty}^{+\infty} P(x)dx = 1$$

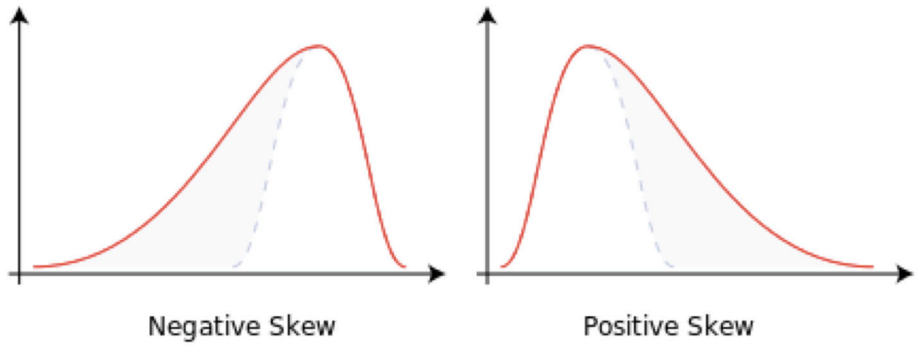


- Mean = Mode = Median $\hat{x}$
- RMS σ
- Skew 0
- Kurtosis 0



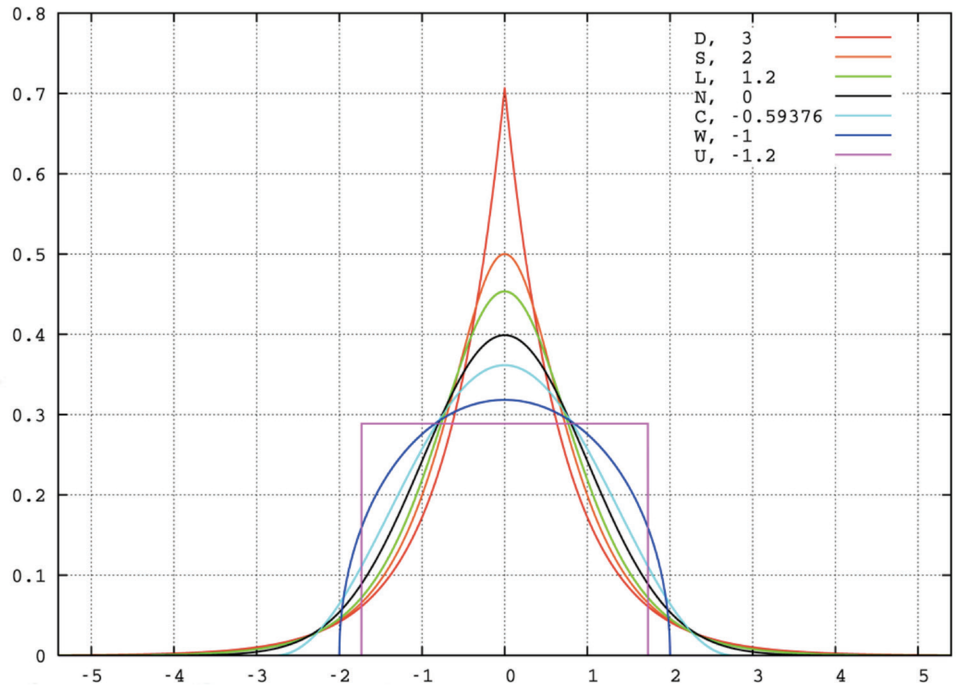
Higher Momenta: Skewness and Kurtosis

- **Skewness** is a measure of the asymmetry of the probability distribution of a real-valued random variable about its mean
- Stand normalized moments



skew[X] $= \mathrm{E}\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{\mu_3}{\sigma^3}$

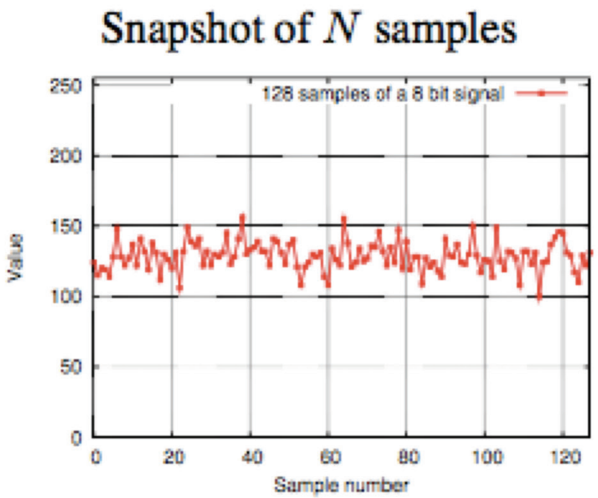
- **Kurtosis** measure of the "tailedness" of the probability distribution of a real-valued random variable

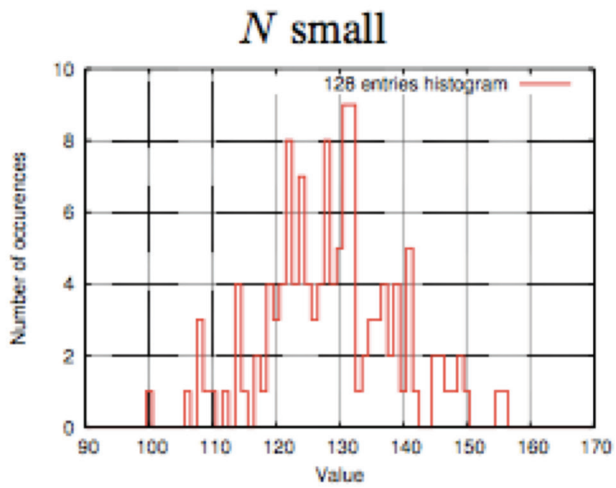


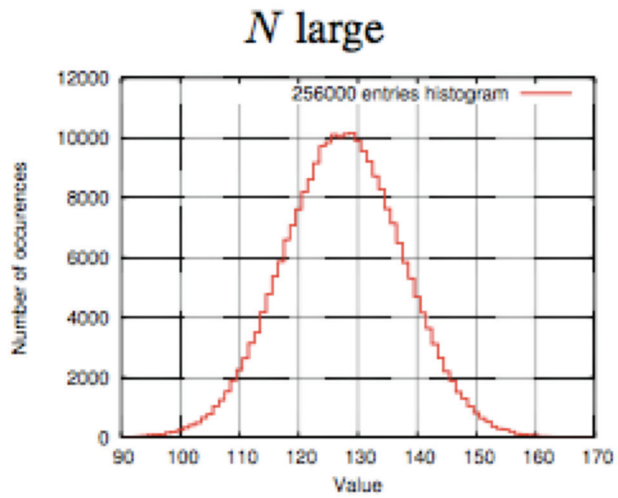
$$\mathrm{Kurt}[X] = \mathrm{E}\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] = \frac{\mu_4}{\sigma^4}$$

Histograms and Probability Density

pdf/pmf







N samples

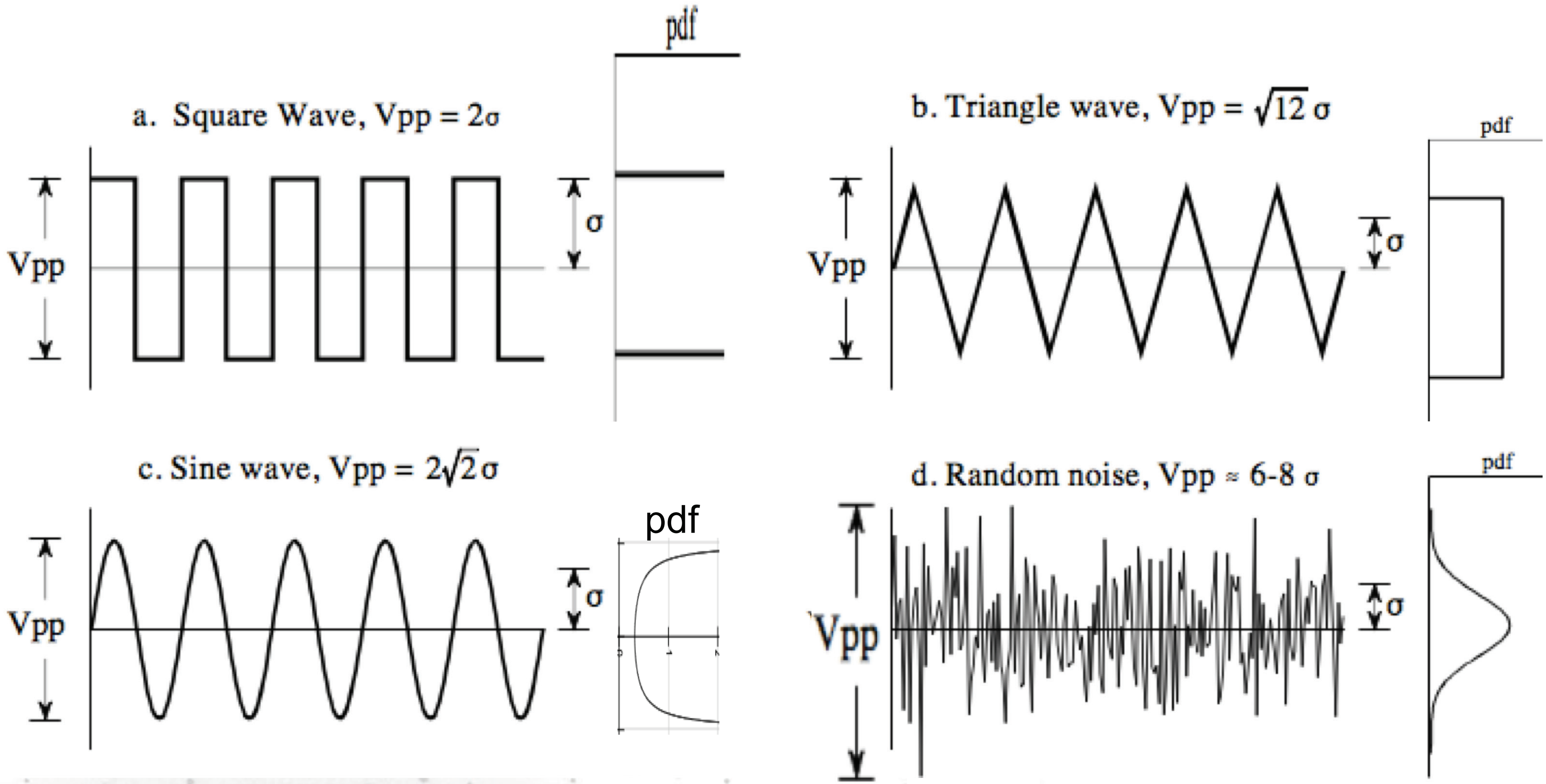
Histogram

Probability mass function
Probability density function

Snapshot of N samples are summed up in **M bins**

$$N = \sum_{i=0}^{M-1} H_i \qquad \hat{x} := \frac{1}{N} \sum_{i=0}^{M-1} i \cdot H_i \; , \qquad \sigma^2 := \frac{1}{N-1} \sum_{i=0}^{M-1} (i - \hat{x})^2 \, H_i$$

Simple Waves



Smith

Error Propagation

- Assume of function depending on n **independent** variables distributed according to a Gaussian law $f = f(\alpha_1, \alpha_2, \ldots, \alpha_n)$

- **What is the Error of the f function?**

$$\Delta f = \sqrt{\sum_i \left(\frac{\partial f}{\partial \alpha_i} \Delta \alpha_i\right)^2}$$

Examples: $x(t) = v \cdot t + x_0$

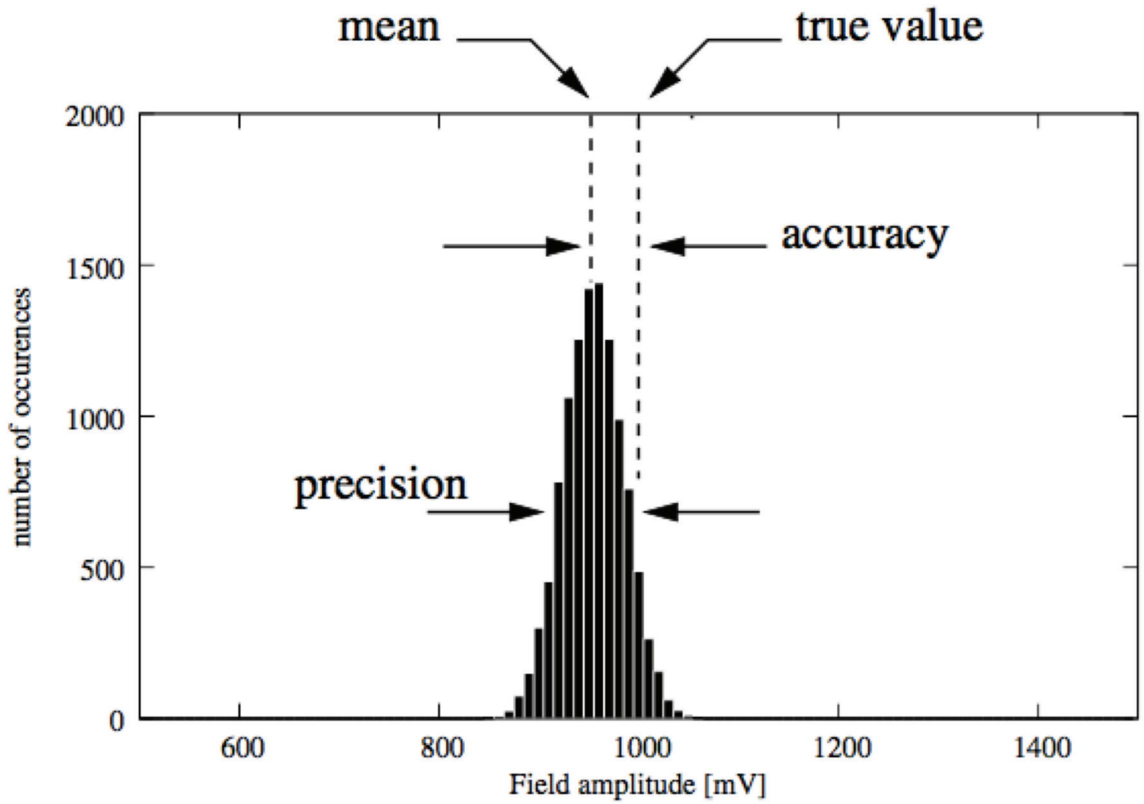
$$\Delta x = \sqrt{\left(\frac{\partial x}{\partial v} \Delta v\right)^2 + \left(\frac{\partial x}{\partial x_0} \Delta x_0\right)^2 + \left(\frac{\partial x}{\partial t} \Delta t\right)^2}$$
$$= \sqrt{(\Delta v \cdot t)^2 + (\Delta x_0)^2} \quad \text{(assuming } \Delta t = 0\text{)}$$

Beta beating in a ring
From N quadrupoles of strength K

$$\frac{\Delta\beta(s)}{\beta(s)} = \frac{[\Delta K l]\beta_0}{2\sin 2\pi\nu} \cos(2|\psi(s) - \psi_0| - 2\pi\nu)$$

$$< \frac{\Delta\beta}{\beta} >_{rms} = \frac{\sqrt{N}\bar{\beta} < \Delta K l >_{rms}}{2\sqrt{2}\sin 2\pi\nu}$$

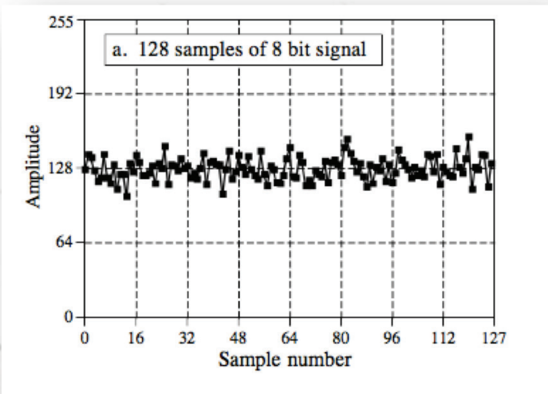
Accuracy / Calibration / Precision Revisited

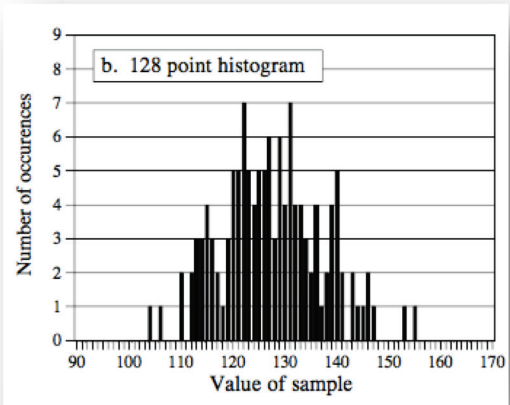


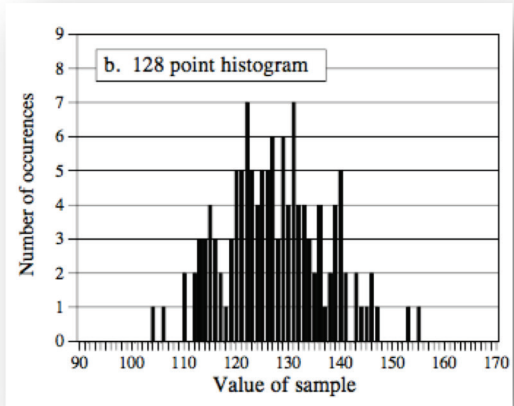
- ***Accuracy*** is a measure of ***calibration***
- ***Precision*** is a measure of ***statistics***
- ***Resolution*** is the bin width

Central Limit Theorem

- ***The sum of independent random numbers (of any distribution) becomes Gaussian distributed***.
- Practical importance of the **central limit theorem** is that the normal distribution can be used as an approximation of some other distributions





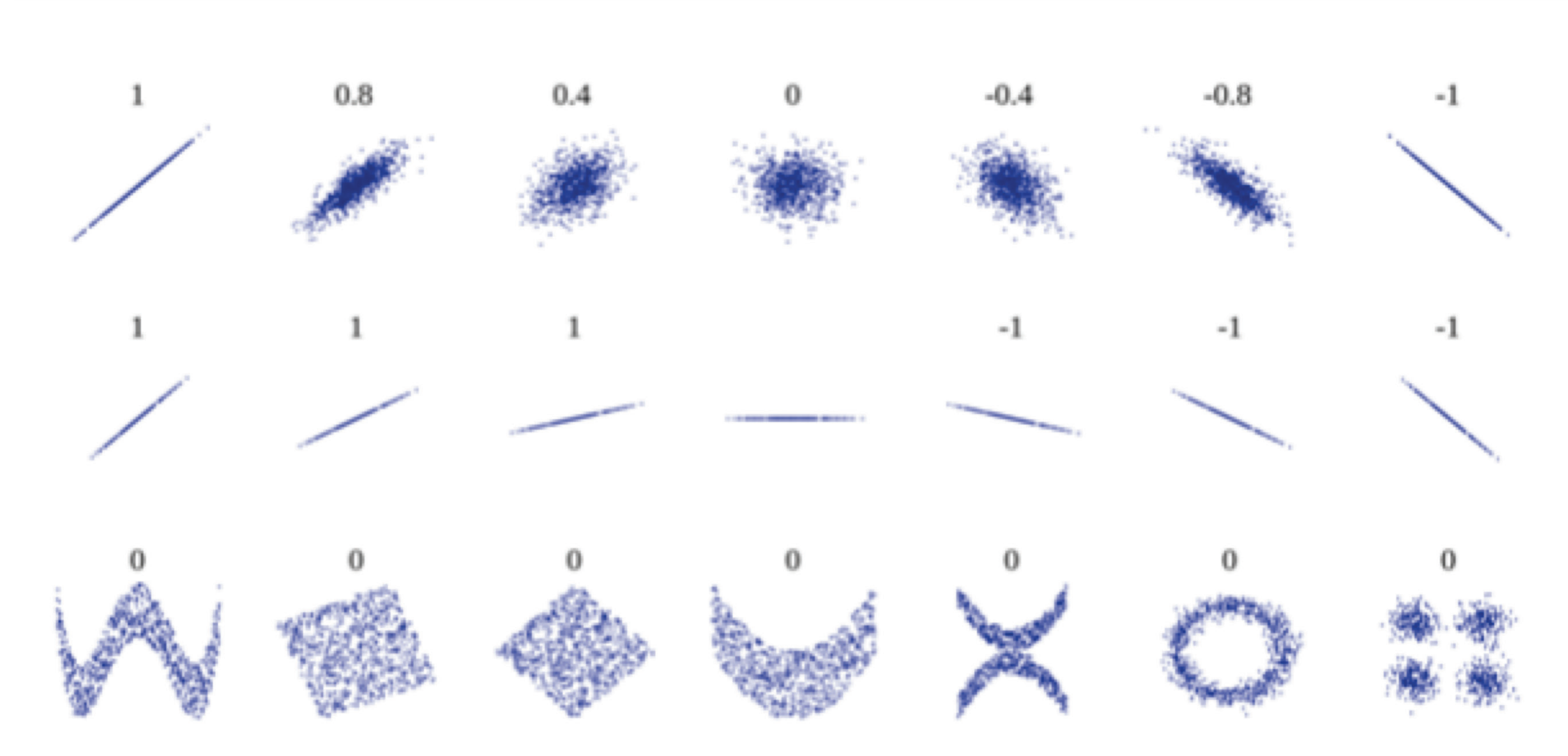


Auto and Cross Correlations Functions

$$R_X(t_1, t_2) = \mathbb{E}[X(t_1)X(t_2)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2 f_{X(t_1),X(t_2)}(x_1, x_2\ ;\ t_1, t_2) dx_1 x_2$$

$$r_X(t_1, t_2) = \frac{\mathbb{E}[(X(t_1) - \mathbb{E}[X(t_1))(X(t_2) - \mathbb{E}[X(t_2))]}{\sigma_X(t_1)\sigma_X(t_2)}$$

$$C_X(t_1, t_2) = \mathbb{E}[(X(t_1) - \mathbb{E}[X(t_1))(X(t_2) - \mathbb{E}[X(t_2))] = R_X(t_1, t_2) - m_X(t_1)m_X(t_2)$$



Stationary Process

- **Invariant by time translation**
  - Average $\mathbb{E}[X(t)] = m_X$
  - Variance $\mathbb{E}[(X(t) - \mathbb{E}(X(t)))^2] = \sigma_X^2$
  - Autocorrelation $R_X(\tau) = \mathbb{E}[X(t)X^*(t-\tau)] = \mathbb{E}[X(t+\tau)X^*(t)]$

    $\tau = t_1 - t_2$
  - Autocovariance

$$
\begin{aligned}
C_X(\tau) &= \mathbb{E}[(X(t) - m_X)(X(t-\tau) - m_X)^*] \\
&= R_X(\tau) - |m_X|^2
\end{aligned}
$$

- Noise is often **not correlated with the measured signal**

Ergodicity

Ergodic process: same behavior averaged over the space of system states and average over time are the same

- **Continuous time**

$$m_X = \mathbb{E}[X(t,\omega)] = \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} X(t,\omega)dt$$

$$R_X(\tau) = \mathbb{E}[X(t,\omega)\,X^*(t-\tau,\omega)] = \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} X(t,\omega)X^*(t-\tau,\omega)dt$$

$$R_{XY}(\tau) = \mathbb{E}[X(t,\omega)\,Y^*(t-\tau,\omega)] = \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} X(t,\omega)Y^*(t-\tau,\omega)dt$$

- **Discrete time**

$$m_X = \mathbb{E}[X(n,\omega)] = \lim_{N\to+\infty} \frac{1}{2N+1}\sum_{n=-N}^{+N} X(n,\omega)$$

$$R_X(k) = \mathbb{E}[X(n,\omega)\,X^*(n-k,\omega)] = \lim_{N\to+\infty} \frac{1}{2N+1}\sum_{n=-N}^{+N} X(n,\omega)X^*(n-k,\omega)$$

$$R_{XY}(k) = \mathbb{E}[X(n,\omega)\,Y^*(n-k,\omega)] = \lim_{N\to+\infty} \frac{1}{2N+1}\sum_{n=-N}^{+N} X(n,\omega)Y^*(n-k,\omega)$$

Contents

- Introduction: definitions and classifications
- Signal representations
- Toolbox of useful functions and distributions
- Mathematical tools
- Fourier Series, Fourier Transform, Laplace Transform
- Linear Time Invariant (LTI) Filters
- Measurement
- Noise
- Statistics
- **References**

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- Blanchet, G., & Charbit, M. Digital Signal and Image Processing using MATLAB, Volume 2: Advances and Applications: The Deterministic Case. John Wiley & Sons**.**
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- S.M. KAY, Modern Spectral Estimation : Theory and Application, Prentice Hall, 1988. T. O'HAVER, A Pragmatic Introduction to Signal Processing with applications in scientific

- **The Scientist and Engineer's Guide to Digital Signal Processing,** By Steven W. Smith, Ph.D. Free of Charge: http://www.dspguide.com/

- Measurement, and with free software to download : http://terpconnect.umd.edu/~toh/spectrum/, Spectral Analysis of Signals, Prentice Hall, 2005. Available to download: http://user.it.uu.se/~ps/SAS-new.pdf

Online resources

- Johns Hopkins University : Signals, Systems, Control (applets)
- http://www.jhu.edu/~signals Analog Signal Processing Applets
- http://cnyack.homestead.com/files/idxpages.htm Signal processing applets
- http://users.ece.gatech.edu/~bonnie/book/applets.html JSyn - Audio Synthesis Software API for Java
- http://www.softsynth.com/jsyn/examples/index.php MATLAB Signal Processing Toolbox
- http://www.mathworks.fr/products/signal/