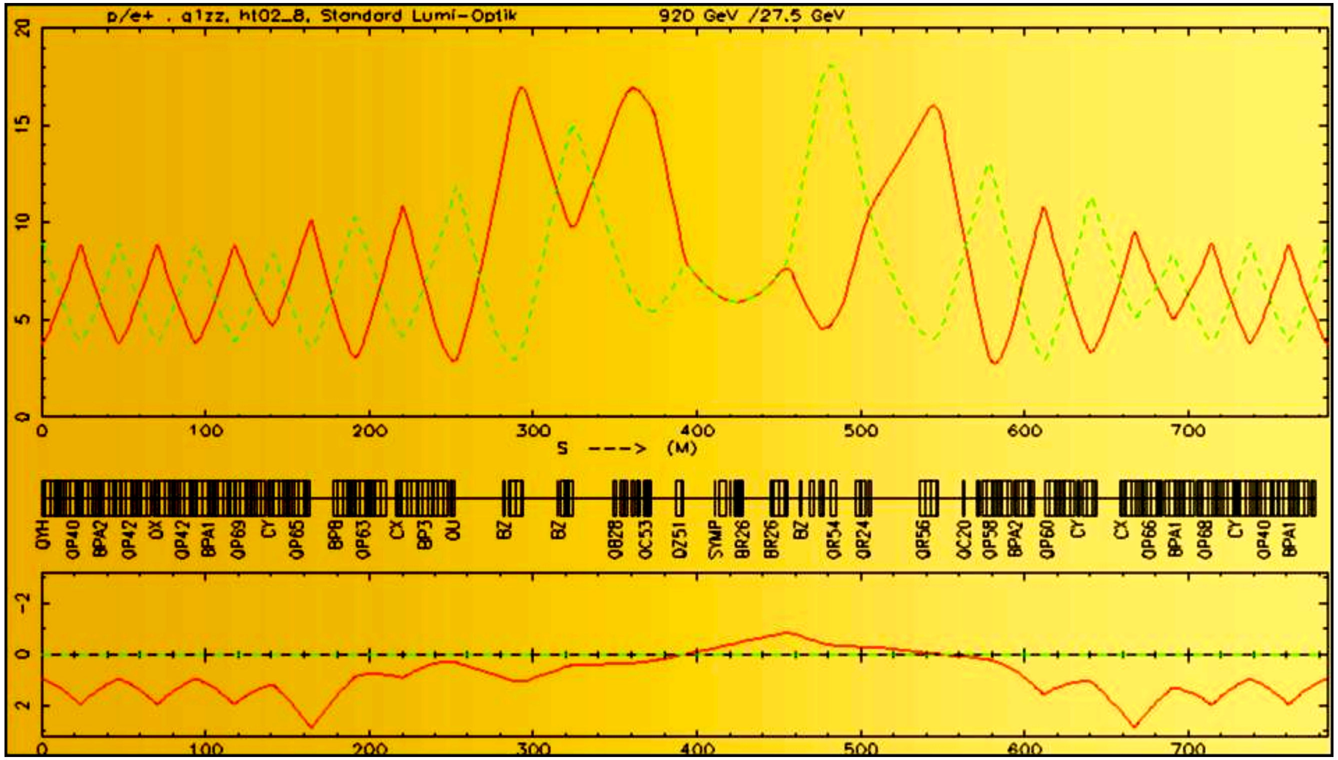
# Lattice Design in Particle Accelerators

## Bernhard Holzer, CERN

$\sqrt{\beta_{x,y}}$

*D*




*1952: Courant, Livingston, Snyder:*

***Theory of strong focusing in particle beams***

# *Lattice Design:* ***„… how to build a storage ring“***

High energy accelerators → circular machines

somewhere in the lattice we need a number of dipole magnets,
that are bending the design orbit to a closed ring

Geometry of the ring:

centrifugal force = Lorentz force

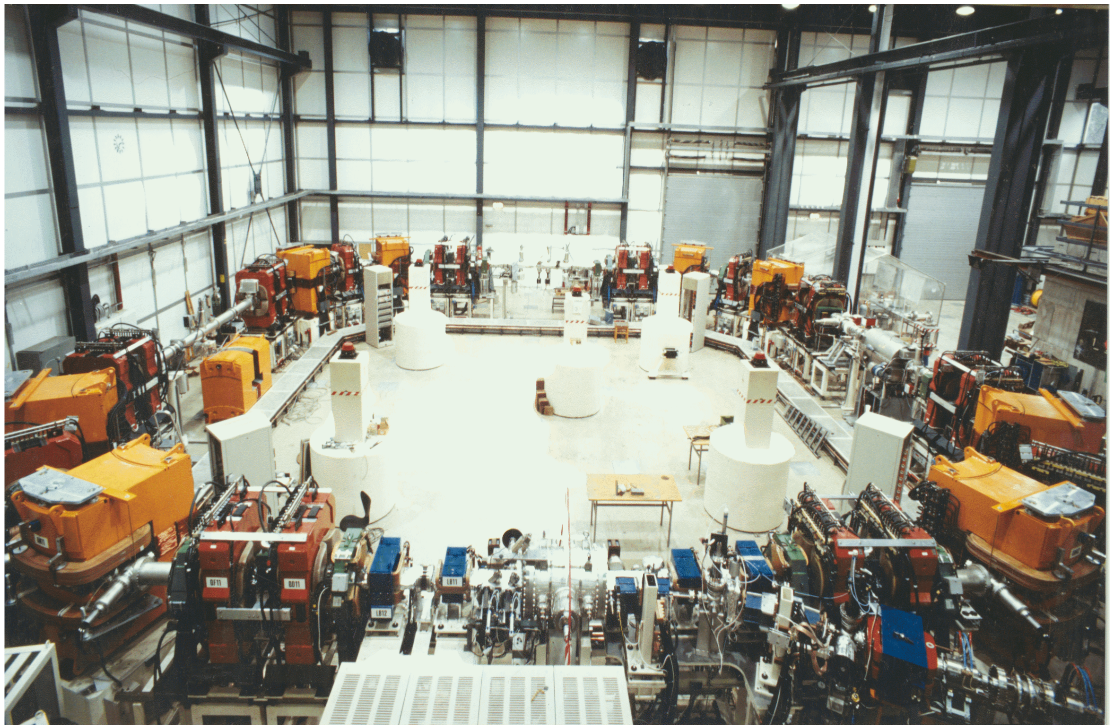
$$e * v * B = \frac{mv^2}{\rho}$$

$$\rightarrow e * B = \frac{mv}{\rho} = p / \rho$$

$$\boxed{\rightarrow B * \rho = p / e}$$

***p = momentum of the particle,***
***ρ = curvature radius***

***Bρ= beam rigidity***



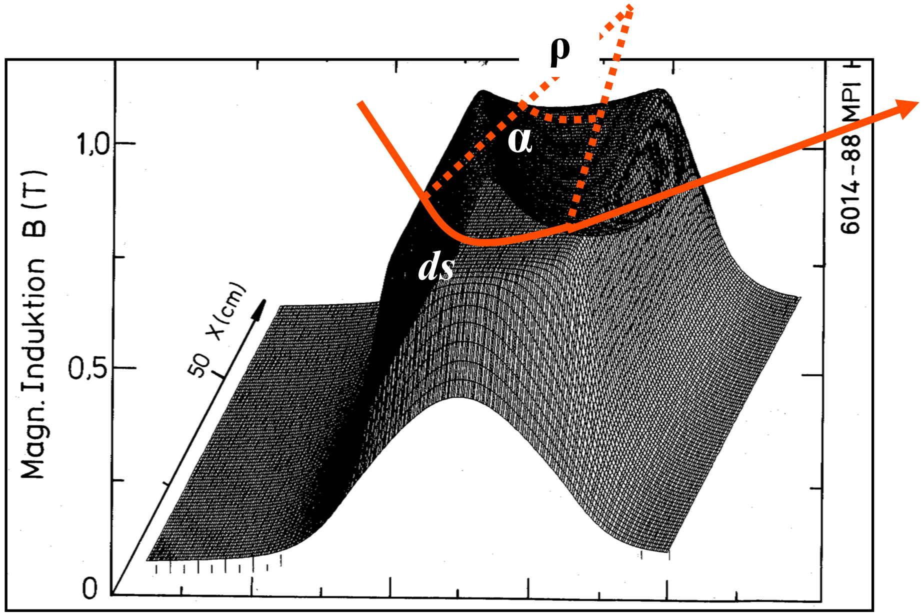
*Example: heavy ion storage ring TSR*
*8 dipole magnets of equal bending strength*

# *1.) Circular Orbit:*

***„… defining the geometry“***

$$\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho}$$

$$\alpha = \frac{B * dl}{B * \rho}$$




***field map of a storage ring dipole magnet***

The angle swept out in one revolution must be $2\pi$, so

$$\alpha = \frac{\int B dl}{B * \rho} = 2\pi \qquad \rightarrow \qquad \int B dl = 2\pi * \frac{p}{q}$$

… for a full circle

***Nota bene:*** $\frac{\Delta B}{B} \approx 10^{-4}$ ***is usually required !!***

## *Example LHC:*

7000 GeV Proton storage ring

dipole magnets N = 1232

$l = 15$ m

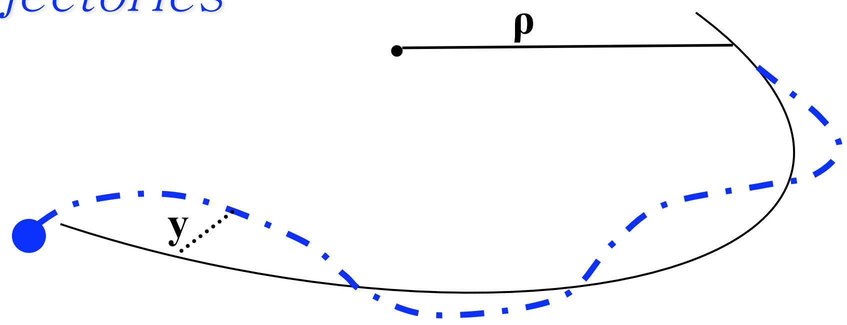
q = +1 e

$$\int B\, dl \approx N\, l\, B = 2\pi\ p / e$$

$$B \approx \frac{2\pi\ \ 7000\ \ 10^{9}\, eV}{1232\ \ 15\, m\ \ 3\ 10^{8}\,\dfrac{m}{s}\ \ e} = 8.3\ Tesla$$

## „ *Focusing forces ··· single particle trajectories*“

$$y'' + K * y = 0$$




$$K = -k + 1/\rho^2$$ **hor. plane**

$$K = k$$ **vert. plane**

**dipole magnet** $\frac{1}{\rho} = \frac{B}{p/q}$

**quadrupole magnet** $k = \frac{g}{p/q}$

**Example: HERA Ring:**

**Bending radius: ρ = 580 m**

**Quadrupol Gradient: g = 110 T/m**

**k = $33.64 * 10^{-3} /m^2$**

**$1/\rho^2 = 2.97 * 10^{-6} /m^2$**

***For estimates in large accelerators the weak focusing term $1/\rho^2$ can in general be neglected***

**Solution for a focusing magnet**

$$y(s) = y_0 * \cos(\sqrt{K} * s) + \frac{y'_0}{\sqrt{K}} * \sin(\sqrt{K} * s)$$

$$y'(s) = -y_0 * \sqrt{K} * \sin(\sqrt{K} * s) + y'_0 * \cos(\sqrt{K} * s)$$

**Or written more convenient in matrix form:**

$$\begin{pmatrix} y \\ y' \end{pmatrix}_s = M * \begin{pmatrix} y \\ y' \end{pmatrix}_0$$

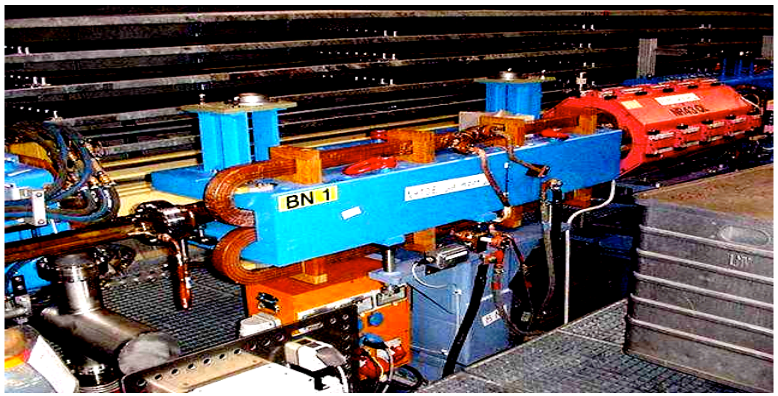
**Hor. focusing Quadrupole Magnet**

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K} * l) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} * l) \\ -\sqrt{K} \sin(\sqrt{K} * l) & \cos(\sqrt{K} * l) \end{pmatrix}$$

**Hor. defocusing Quadrupole Magnet**

$$M_{QD} = \begin{pmatrix} \cosh(\sqrt{K} * l) & \frac{1}{\sqrt{K}} \sinh(\sqrt{K} * l) \\ \sqrt{K} \sinh(\sqrt{K} * l) & \cosh(\sqrt{K} * l) \end{pmatrix}$$

**Drift space** $M_{Drift} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$




$$M_{lattice} = M_{QF1} * M_{D1} * M_{QD} * M_{D1} * M_{QF2} \ldots$$

# *2.) Reminder: Beam Dynamics Language*

## *Transfer Matrix M*

***describes the transformation of amplitude x and angle x' through a number of lattice elements***

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

***... and can be expressed by the optics parameters***

$$M = \begin{pmatrix} \sqrt{\dfrac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s \beta_0}\,\sin\psi_s \\ \dfrac{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\dfrac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$

* ***we can calculate the single particle trajectories between two locations in the ring, if we know the α β γ at these positions.***

* ***and nothing but the α β γ at these positions.***

* ***... !***

# *Periodic Lattices*

**In the case of periodic lattices the transfer matrix can be expressed as a function of a set of periodic parameters α, β, γ**

$$M(s) = \begin{pmatrix} \cos\psi_{period} + \alpha_s \sin\psi_{period} & \beta_s \sin\psi_{period} \\ -\gamma_s \sin\psi_{period} & \cos\psi_{period} - \alpha_s \sin\psi_{period} \end{pmatrix}$$

$$\psi_{period} = \int_s^{s+L} \frac{ds}{\beta(s)}$$

*$\psi$ = phase advance per period:*

**For stability of the motion in periodic lattice structures it is required that**

$$|trace(M)| < 2$$

**In terms of these new periodic parameters the solution of the equation of motion is**

$$y(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\Phi(s) - \delta)$$

$$y'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta}} * \{\sin(\Phi(s) - \delta) + \alpha\cos(\Phi(s) - \delta)\}$$

# Transformation of α, β, γ

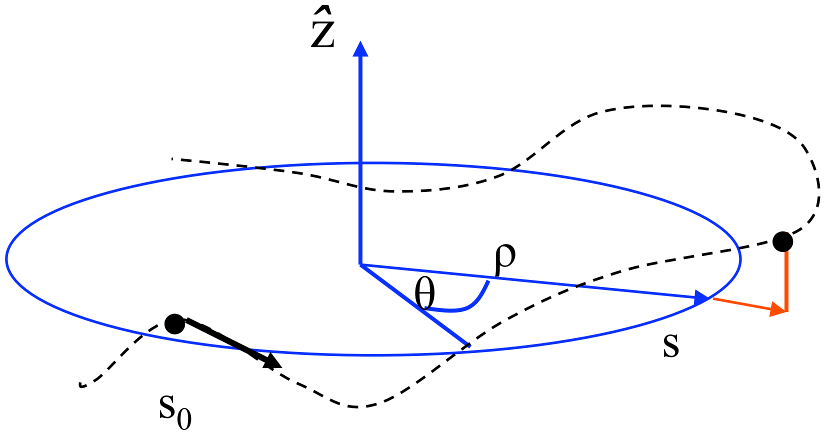
***consider two positions in the storage ring: $s_0$ , s***

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

***since ε = const:***

$$\varepsilon = \beta x'^2 + 2\alpha x x' + \gamma x^2$$

$$\varepsilon = \beta_0 x_0'^2 + 2\alpha_0 x_0 x_0' + \gamma_0 x_0{}^2$$




***express $x_0$, $x'_0$ as a function of x, x′.***

*... remember $W = m_{11} m_{22} - m_{12} m_{21} = 1$*

$$\left.\begin{aligned} &\begin{pmatrix} x \\ x' \end{pmatrix}_0 = M^{-1} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_s \\ &M^{-1} = \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix} \end{aligned}\right\} \rightarrow \quad \begin{aligned} x_0 &= m_{22} x - m_{12} x' \\ x_0' &= -m_{21} x + m_{11} x' \end{aligned}$$

***inserting into ε*** $\quad \varepsilon = \beta x'^2 + 2\alpha x x' + \gamma x^2$

$$\varepsilon = \beta_0 (m_{11} x' - m_{21} x)^2 + 2\alpha_0 (m_{22} x - m_{12} x')(m_{11} x' - m_{21} x) + \gamma_0 (m_{22} x - m_{12} x')^2$$

***sort via x, x′and compare the coefficients to get ....***

**The new parameters α, β, γ can be transformed through the lattice via the lattice matrix elements defined above.**

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s2} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{12}m_{21} + m_{22}m_{11} & -m_{12}m_{22} \\ m_{12}^2 & -2m_{22}m_{21} & m_{22}^2 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s1}$$

*the optical parameters depend on the focusing properties of the lattice,*
*... and can be optimised accordingly !!!*

**... and here starts the lattice design !!!**

# Most simple example: drift space

$$M_{drift} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

**particle coordinates**

$$\begin{pmatrix} x \\ x' \end{pmatrix}_l = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ x' \end{pmatrix}_0 \qquad \rightarrow \qquad \boxed{\begin{aligned} x(l) &= x_0 + l * x_0' \\ x'(l) &= x_0' \end{aligned}}$$

**transformation of twiss parameters:**

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_l = \begin{pmatrix} 1 & -2l & l^2 \\ 0 & 1 & -l \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

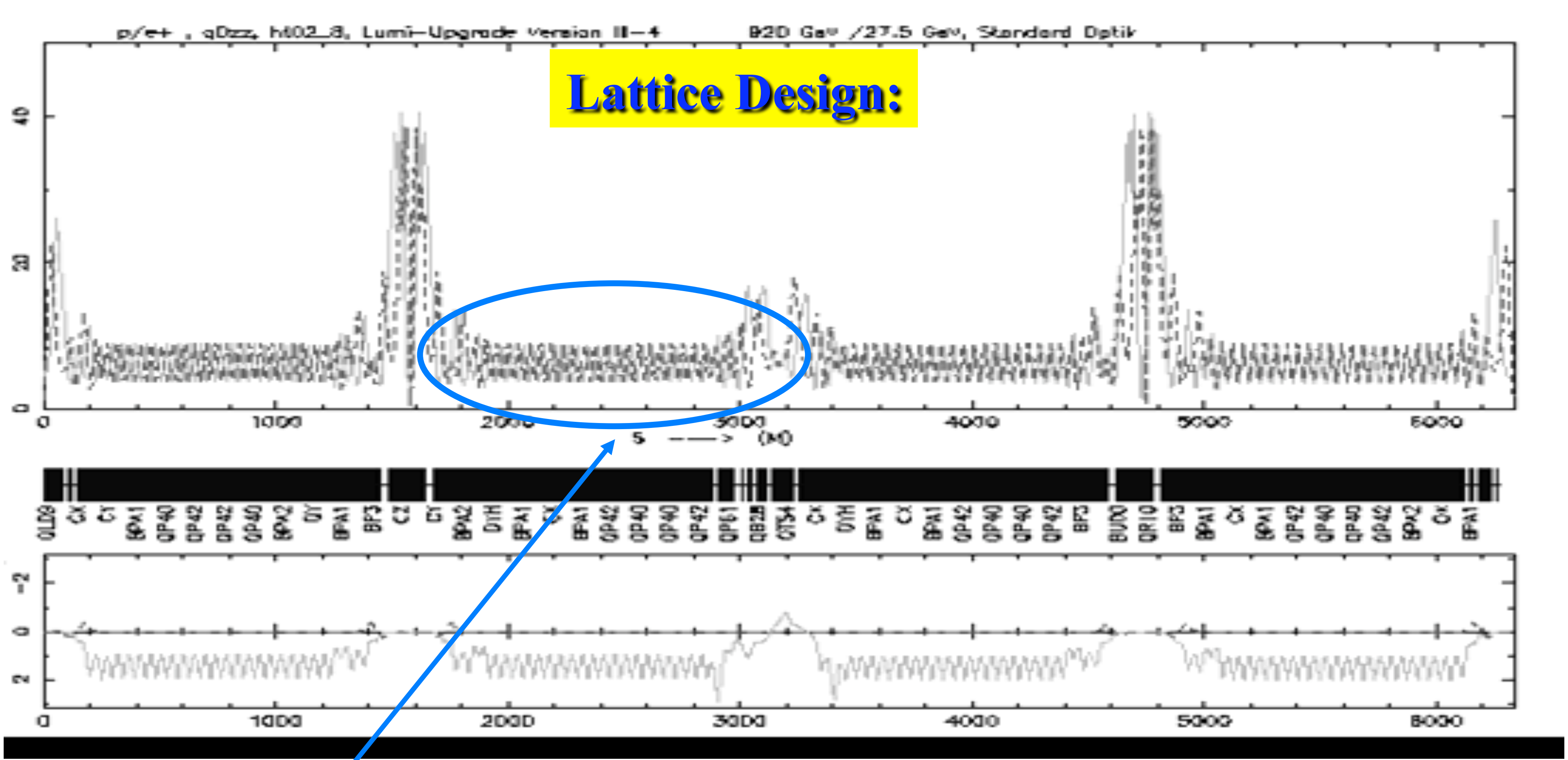
$$\boxed{\beta(s) = \beta_0 - 2l * \alpha_0 + l^2 * \gamma_0}$$

**Stability ...?**

$$trace(M) = 1 + 1 = 2$$

***→A periodic solution doesn't exist in a lattice built exclusively out of drift spaces.***

# Lattice Design:



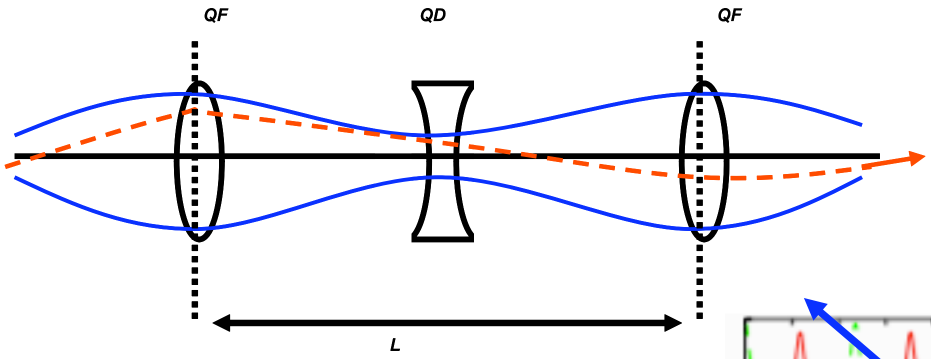

**Arc:** regular (periodic) magnet structure:

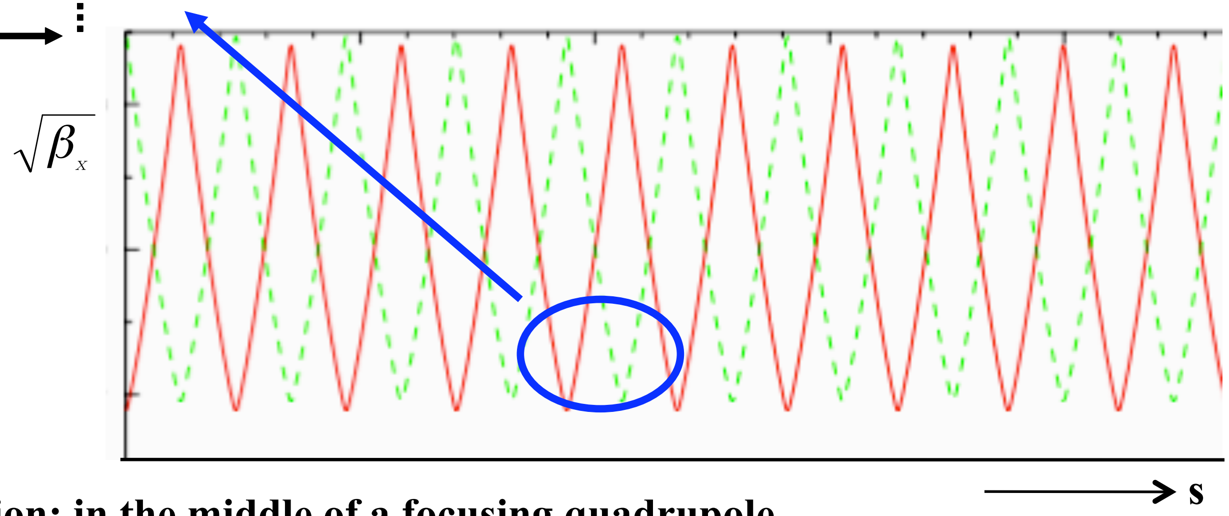
bending magnets → define the energy of the ring
main focusing & tune control, chromaticity correction,
multipoles for higher order corrections

**Straight sections:** drift spaces for injection, dispersion suppressors,
low beta insertions, RF cavities, etc....
... and the high energy experiments if they cannot be avoided

# *3.) The FoDo-Lattice*

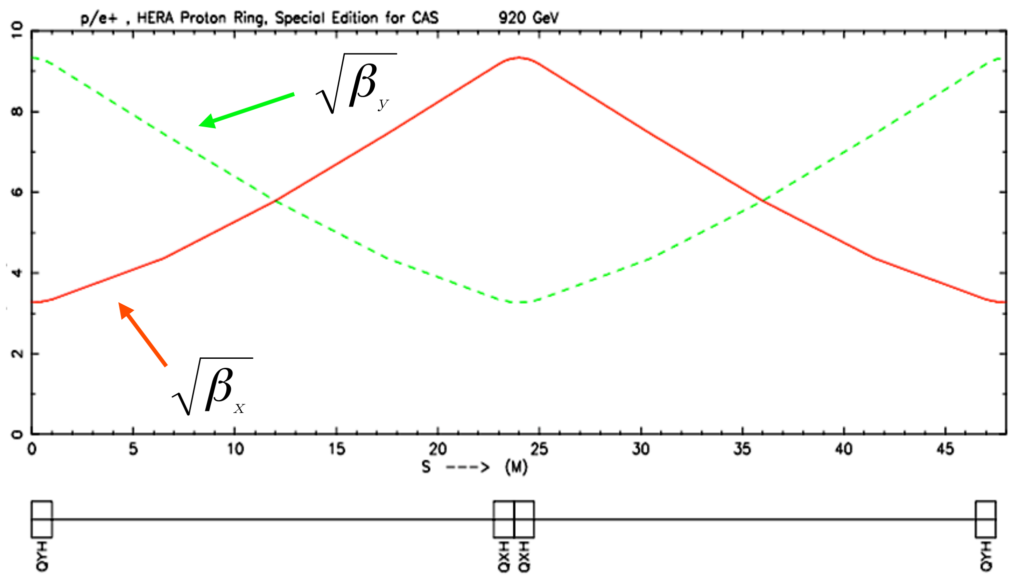
**A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with nothing in between.**
**(Nothing = elements that can be neglected on first sight: drift, bending magnets, RF structures ... and especially experiments...)**




**Starting point for the calculation: in the middle of a focusing quadrupole**
**Phase advance per cell μ = 45°,**
**→ calculate the twiss parameters for a periodic solution**

# *Periodic Solution of a FoDo Cell*




***Output of the optics program:***

| *Nr* | *Type* | *Length* m | *Strength* 1/m2 | $\beta_x$ m | $\alpha_x$ | $\varphi_x$ 1/2π | $\beta_z$ m | $\alpha_z$ | $\varphi_z$ 1/2π |
|---|---|---|---|---|---|---|---|---|---|
| 0 | IP | 0,000 | 0,000 | 11,611 | 0,000 | 0,000 | 5,295 | 0,000 | 0,000 |
| 1 | QFH | 0,250 | -0,541 | 11,228 | 1,514 | 0,004 | 5,488 | -0,781 | 0,007 |
| 2 | QD | 3,251 | 0,541 | 5,488 | -0,781 | 0,070 | 11,228 | 1,514 | 0,066 |
| 3 | QFH | 6,002 | -0,541 | 11,611 | 0,000 | 0,125 | 5,295 | 0,000 | 0,125 |
| 4 | IP | 6,002 | 0,000 | 11,611 | 0,000 | 0,125 | 5,295 | 0,000 | 0,125 |

*QX= 0,125 QZ= 0,125*

**0.125 * 2π = 45⁰**

## *Can we understand what the optics code is doing ?*

**matrices**

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K} * l_q) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} * l_q) \\ -\sqrt{K} \sin(\sqrt{K} * l_q) & \cos(\sqrt{K} * l_q) \end{pmatrix}, \qquad M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1_d \end{pmatrix}$$

**strength and length of the FoDo elements**

***K = +/- 0.54102 m^-2***
***lq = 0.5 m***
***ld = 2.5 m***

**The matrix for the complete cell is obtained by multiplication of the element matrices**

$$M_{FoDo} = M_{qf\,h} * M_{ld} * M_{qd} * M_{ld} * M_{qf}$$

**Putting the numbers in and multiplying out ...**

$$M_{FoDo} = \begin{pmatrix} 0.707 & 8.206 \\ -0.061 & 0.707 \end{pmatrix}$$

**The transfer matrix for 1 period gives us all the information that we need !**

**1.) is the motion stable?**

$$trace(M_{FoDo}) = 1.415 \rightarrow \quad \underline{\underline{< 2}}$$

**2.) Phase advance per cell**

$$M(s) = \begin{pmatrix} \cos\psi_{cell} + \alpha_s \sin\psi_{cell} & \beta_s \sin\psi_{cell} \\ -\gamma_s \sin\psi_{cell} & \cos\psi_{cell} - \alpha_s \sin\psi_{cell} \end{pmatrix}$$

$$\cos\psi_{cell} = \frac{1}{2} trace(M) = 0.707$$

$$\psi_{cell} = \cos^{-1}\left(\frac{1}{2} trace(M)\right) = \underline{\underline{45}}$$

**3.) hor β-function**

$$\beta = \frac{m_{12}}{\sin\psi_{cell}} = \underline{\underline{11.611\, m}}$$

**4.) hor α-function**

$$\alpha = \frac{m_{11} - \cos\psi_{cell}}{\sin\psi_{cell}} = \underline{\underline{0}}$$

***Can we do a bit easier ?***

***We can ... in thin lens approximation !***

**Matrix of a focusing quadrupole magnet:**

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K} * l) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} * l) \\ -\sqrt{K} \sin(\sqrt{K} * l) & \cos(\sqrt{K} * l) \end{pmatrix}$$

**If the focal length *f* is much larger than the length of the quadrupole magnet,**

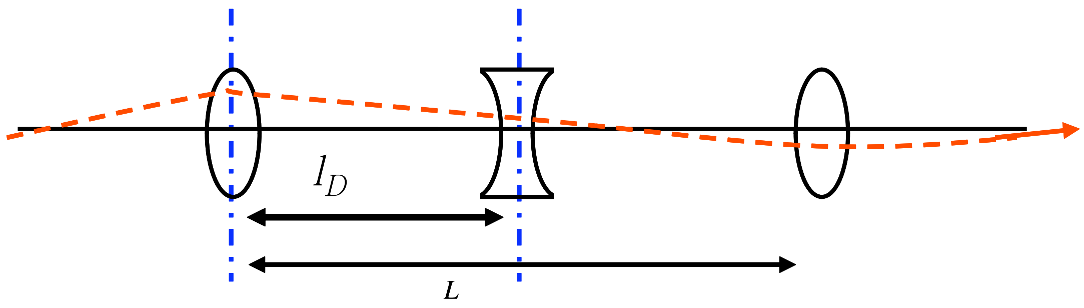
$$f = 1/kl_Q >> l_Q$$

**the transfer matrix can be aproximated using ♪**

$$kl_q = const, \ \ l_q \rightarrow 0$$

$$M = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$

# 4.) FoDo in thin lens approximation




$l_D = L/2,$

$\tilde{f} = 2f$

***Calculate the matrix for a half cell, starting in the middle of a foc. quadrupole:***

$$M_{half\ Cell} = M_{QD/2} * M_{lD} * M_{QF/2}$$

$$M_{half\ Cell} = \begin{pmatrix} 1 & 0 \\ 1/\tilde{f} & 1 \end{pmatrix} * \begin{pmatrix} 1 & l_D \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ -1/\tilde{f} & 1 \end{pmatrix}$$

*note: $\tilde{f}$ denotes the focusing strength of half a quadrupole, so $\tilde{f} = 2f$*

$$M_{half\ Cell} = \begin{pmatrix} 1 - l_D/\tilde{f} & l_D \\ -l_D/\tilde{f}^2 & 1 + l_D/\tilde{f} \end{pmatrix}$$

*for the second half cell set f → -f*

# FoDo in thin lens approximation

**Matrix for the complete FoDo cell**

$$M = \begin{pmatrix} 1+{l_D}/{\tilde{f}} & l_D \\ -{l_D}/{\tilde{f}^2} & 1-{l_D}/{\tilde{f}} \end{pmatrix} * \begin{pmatrix} 1-{l_D}/{\tilde{f}} & l_D \\ -{l_D}/{\tilde{f}^2} & 1+{l_D}/{\tilde{f}} \end{pmatrix}$$

$$M = \begin{pmatrix} 1-\frac{2l_D^2}{\tilde{f}^2} & 2l_D(1+\frac{l_D}{\tilde{f}}) \\ 2(\frac{l_D^2}{\tilde{f}^3}-\frac{l_D}{\tilde{f}^2}) & 1-2\frac{l_D^2}{\tilde{f}^2} \end{pmatrix}$$

**Now we know, that the phase advance is related to the transfer matrix by**

$$\cos\psi_{cell} = \frac{1}{2}trace(M) = \frac{1}{2}*(2-\frac{4l_d^2}{\tilde{f}^2}) = 1-\frac{2l_d^2}{\tilde{f}^2}$$

**After some beer and with a little bit of trigonometric gymnastics**

$$\cos(x) = \cos^2({x}/{2}) - \sin^2(x/2) = 1-2\sin^2({x}/{2})$$

*we can calculate the phase advance as a function of the FoDo parameter …*

$$\cos\psi_{cell} = 1 - 2\sin^2(\psi_{cell}/2) = 1 - \frac{2l_d^2}{\tilde{f}^2}$$

$$\sin(\psi_{cell}/2) = l_d / \tilde{f} = \frac{L_{cell}}{2\tilde{f}}$$

$$\sin(\psi_{cell}/2) = \frac{L_{cell}}{4f}$$

*Example:*
*45-degree Cell*

$$L_{Cell} = l_{QF} + l_D + l_{QD} + l_D = 0.5m+2.5m+0.5m+2.5m = 6m$$

$$1/f = k*l_Q = 0.5m*0.541\ m^{-2} = 0.27\ m^{-1}$$

$$\sin(\psi_{cell}/2) = \frac{L_{cell}}{4f} = 0.405$$

$$\rightarrow \quad \psi_{cell} = 47.8^\circ$$

$$\rightarrow \quad \beta = 11.4\ m$$

*Remember:*
*Exact calculation yields:*

$$\rightarrow \quad \psi_{cell} = 45^\circ$$

$$\rightarrow \quad \beta = 11.6\ m$$

# Stability in a FoDo structure


*SPS Lattice*

$$M_{FoDo} = \begin{pmatrix} 1-\dfrac{2l_D^2}{\tilde{f}^2} & 2l_D(1+\dfrac{l_D}{\tilde{f}}) \\ 2(\dfrac{l_D^2}{\tilde{f}^3}-\dfrac{l_D}{\tilde{f}^2}) & 1-2\dfrac{l_D^2}{\tilde{f}^2} \end{pmatrix}$$

***Stability requires:***

$$\left|\boldsymbol{Trace}(\boldsymbol{M})\right| < 2$$

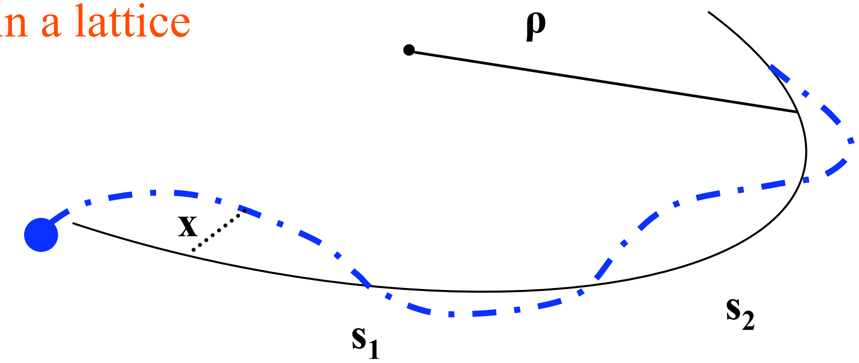
$$\left|\boldsymbol{Trace}(\boldsymbol{M})\right| = \left|2-\frac{4\boldsymbol{l}_d^2}{\tilde{\boldsymbol{f}}^2}\right| < 2$$

$$\rightarrow f > \frac{L_{cell}}{4}$$

**For stability the focal length has to be larger than a quarter of the cell length**
**... don't focus to strong !**

## *Transformation Matrix in Terms of the Twiss Parameters*

Transformation of the coordinate vector (x,x´) in a lattice




$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = M_{s1,s2} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

General solution of the equation of motion

$$x(s) = \sqrt{\varepsilon * \beta(s)} * \cos(\psi(s) + \varphi)$$

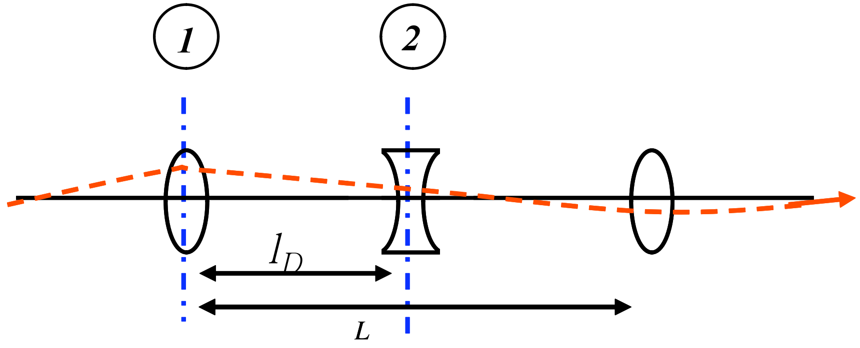
$$x'(s) = \sqrt{\varepsilon / \beta(s)} * \{ \alpha(s)\cos(\psi(s) + \varphi) + \sin(\psi(s) + \varphi) \}$$

Transformation of the coordinate vector (x,x´)
expressed as a function of the twiss parameters

$$M_{1\to 2} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\psi_{12} + \alpha_1 \sin\psi_{12}) & \sqrt{\beta_1\beta_2}\sin\psi_{12} \\ \frac{(\alpha_1 - \alpha_2)\cos\psi_{12} - (1 + \alpha_1\alpha_2)\sin\psi_{12}}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\psi_{12} - \alpha_2 \sin\psi_{12}) \end{pmatrix}$$

## Transfer Matrix for half a FoDo cell:

$$M_{half\ cell} = \begin{pmatrix} 1 - \frac{l_D}{\tilde{f}} & l_D \\ -\frac{l_D}{\tilde{f}^2} & 1 + \frac{l_D}{\tilde{f}} \end{pmatrix}$$




**Compare to the twiss parameter form of M**

$$M_{1\to 2} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\psi_{12} + \alpha_1 \sin\psi_{12}) & \sqrt{\beta_1\beta_2}\sin\psi_{12} \\ \frac{(\alpha_1 - \alpha_2)\cos\psi_{12} - (1 + \alpha_1\alpha_2)\sin\psi_{12}}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\psi_{12} - \alpha_2 \sin\psi_{12}) \end{pmatrix}$$

**In the middle of a foc (defoc) quadrupole of the FoDo we allways have $\alpha = 0$, and the half cell will lead us from $\beta_{max}$ to $\beta_{min}$**

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\check{\beta}}{\hat{\beta}}}\cos\frac{\psi_{cell}}{2} & \sqrt{\check{\beta}\hat{\beta}}\sin\frac{\psi_{cell}}{2} \\ \frac{-1}{\sqrt{\hat{\beta}\check{\beta}}}\sin\frac{\psi_{cell}}{2} & \sqrt{\frac{\hat{\beta}}{\check{\beta}}}\cos\frac{\psi_{cell}}{2} \end{pmatrix}$$

***Solving for*** $\beta_{max}$ ***and*** $\beta_{min}$ ***and remembering that ….*** $\sin\frac{\psi_{cell}}{2} = \frac{l_d}{\tilde{f}} = \frac{L}{4f}$

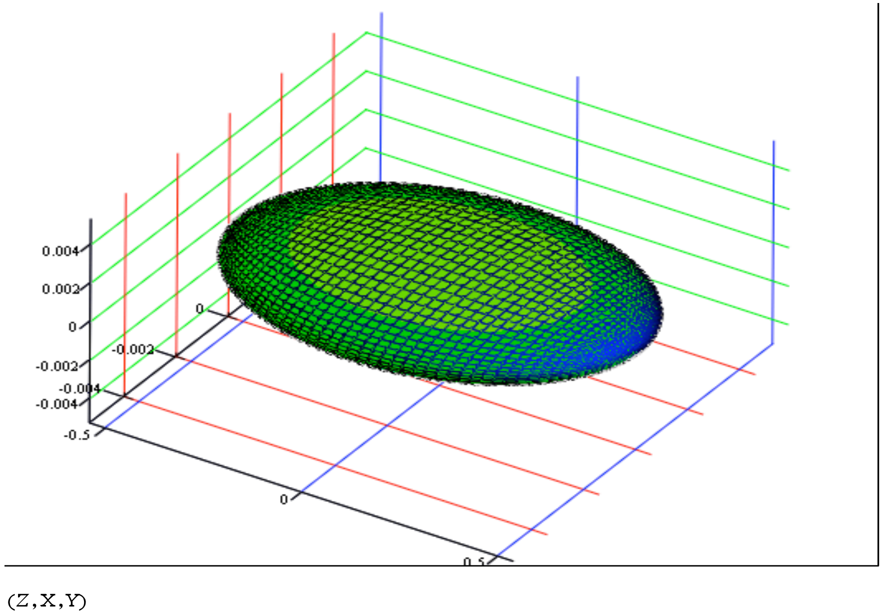
$$\frac{m_{22}}{m_{11}} = \frac{\hat{\beta}}{\breve{\beta}} = \frac{1 + l_d/\tilde{f}}{1 - l_d/\tilde{f}} = \frac{1+\sin(\psi_{cell}/2)}{1-\sin(\psi_{cell}/2)}$$

$$\frac{m_{12}}{m_{21}} = \hat{\beta}\breve{\beta} = \tilde{f}^2 = \frac{l_d^2}{\sin^2(\psi_{cell}/2)}$$

$$\rightarrow$$

$$\hat{\beta} = \frac{(1+\sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}} \quad !$$

$$\breve{\beta} = \frac{(1-\sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}} \quad !$$




***The maximum and minimum values of the β-function are solely determined by the phase advance and the length of the cell.***

***Longer cells lead to larger β***

*typical shape of a proton bunch in a FoDo Cell*

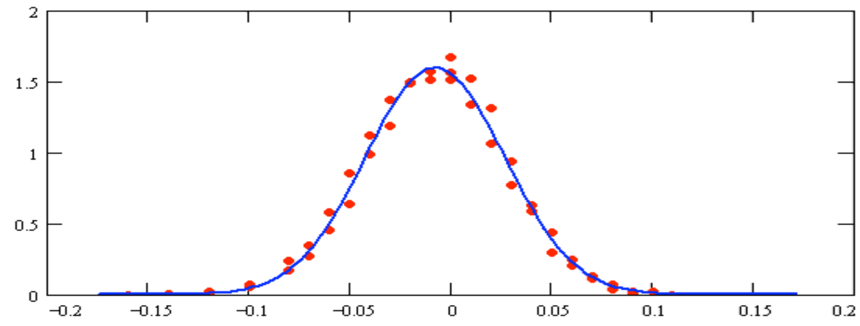
# *5.) Beam dimension:*

## *Optimisation of the FoDo Phase advance:*

**In both planes a gaussian particle distribution is assumed, given by the beam emittance ε and the β-function**

$$\sigma = \sqrt{\varepsilon\beta}$$

*HERA beam size*

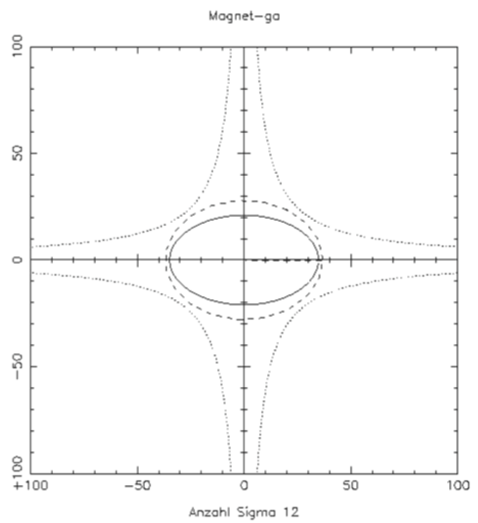



**In general proton beams are „*round*“ in the sense that**

$$\varepsilon_x \approx \varepsilon_y$$

**So for highest aperture we have to minimise the β-function in both planes:**

$$r^2 = \varepsilon_x\beta_x + \varepsilon_y\beta_y$$




***typical beam envelope, vacuum chamber and pole shape in a foc. Quadrupole lens in HERA***

## *Optimising the FoDo phase advance*

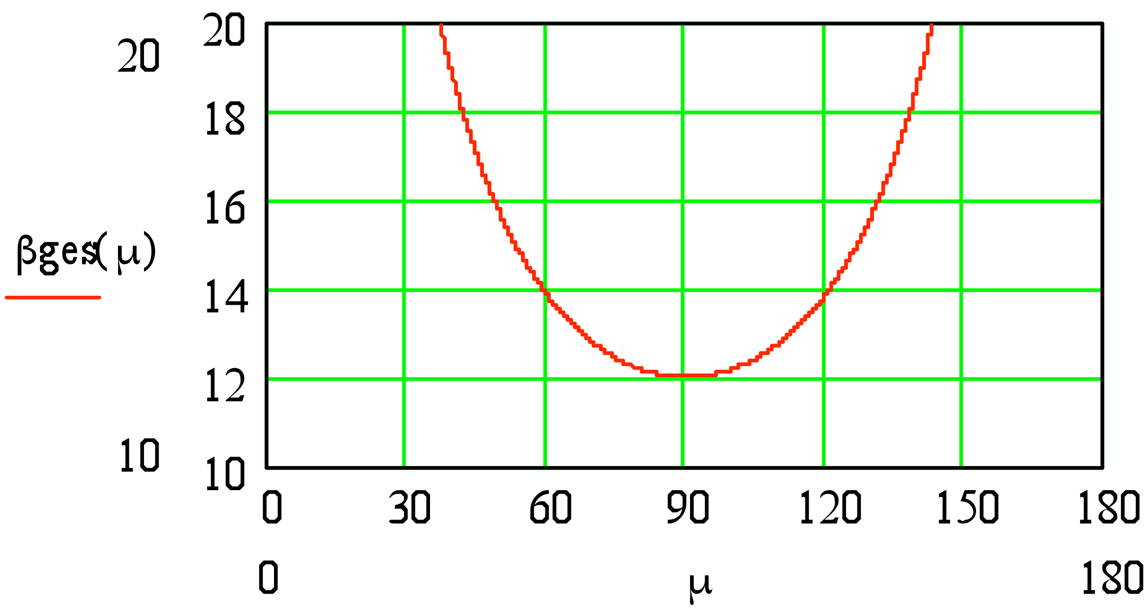
$$r^2 = \varepsilon_x \beta_x + \varepsilon_y \beta_y$$

***search for the phase advance μ that results in a minimum of the sum of the beta's***

$$\hat{\beta} + \breve{\beta} = \frac{(1+\sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}} + \frac{(1-\sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}}$$

$$\hat{\beta} + \breve{\beta} = \frac{2L}{\sin\psi_{cell}}$$

$$\frac{d}{d\psi_{cell}}\left(\frac{2L}{\sin\psi_{cell}}\right) = 0$$



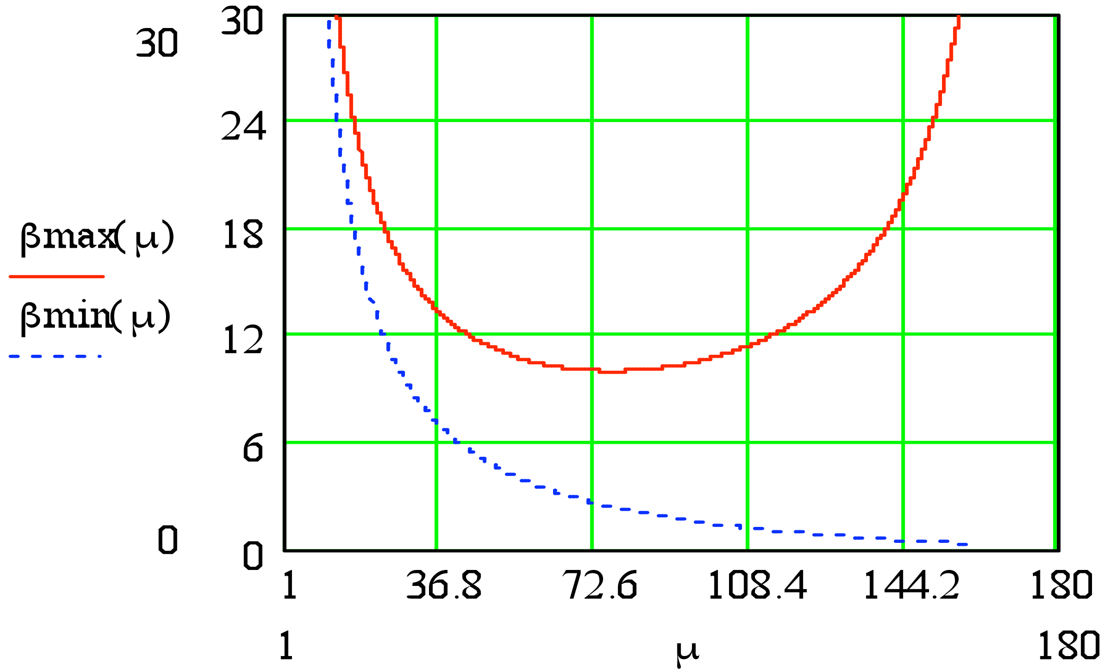

$$\frac{L}{\sin^2\psi_{cell}} * \cos\psi_{cell} = 0 \quad \rightarrow \quad \psi_{cell} = 90^\circ$$

# Electrons are different

***electron beams are usually flat,*** $\varepsilon_y \approx 2 - 10\ \%\ \varepsilon_x$

***→ optimise only*** $\boldsymbol{\beta_{hor}}$

$$\frac{d}{d\psi_{cell}}(\hat{\beta}) = \frac{d}{d\psi_{cell}} \frac{L(1+\sin\frac{\psi_{cell}}{2})}{\sin\psi_{cell}} = 0 \quad \rightarrow \quad \psi_{cell} = 76^{\circ}$$

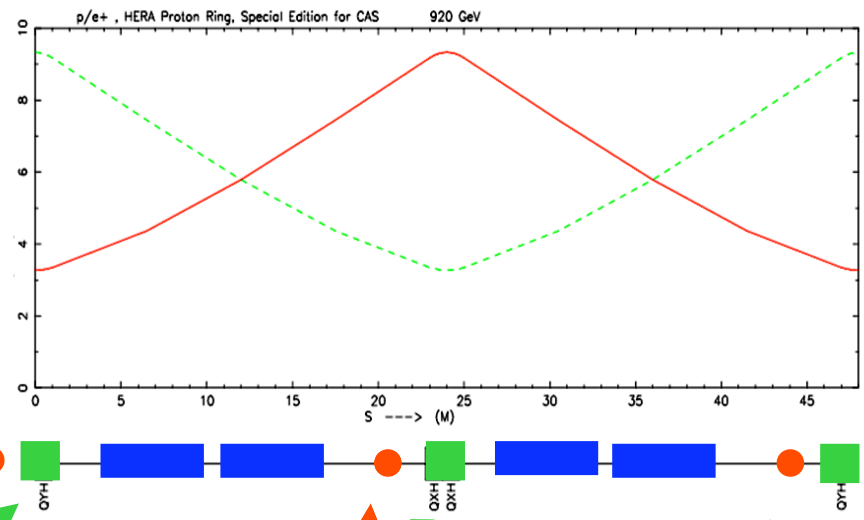



***red curve:*** $\boldsymbol{\beta_{max}}$

***blue curve:*** $\beta_{min}$

***as a function of the phase advance*** $\psi$

# *Orbit distortions in a periodic lattice*




**field error of a dipole/distorted quadrupole**

$$\rightarrow \delta(mrad) = \frac{ds}{\rho} = \frac{\int B ds}{p/e}$$

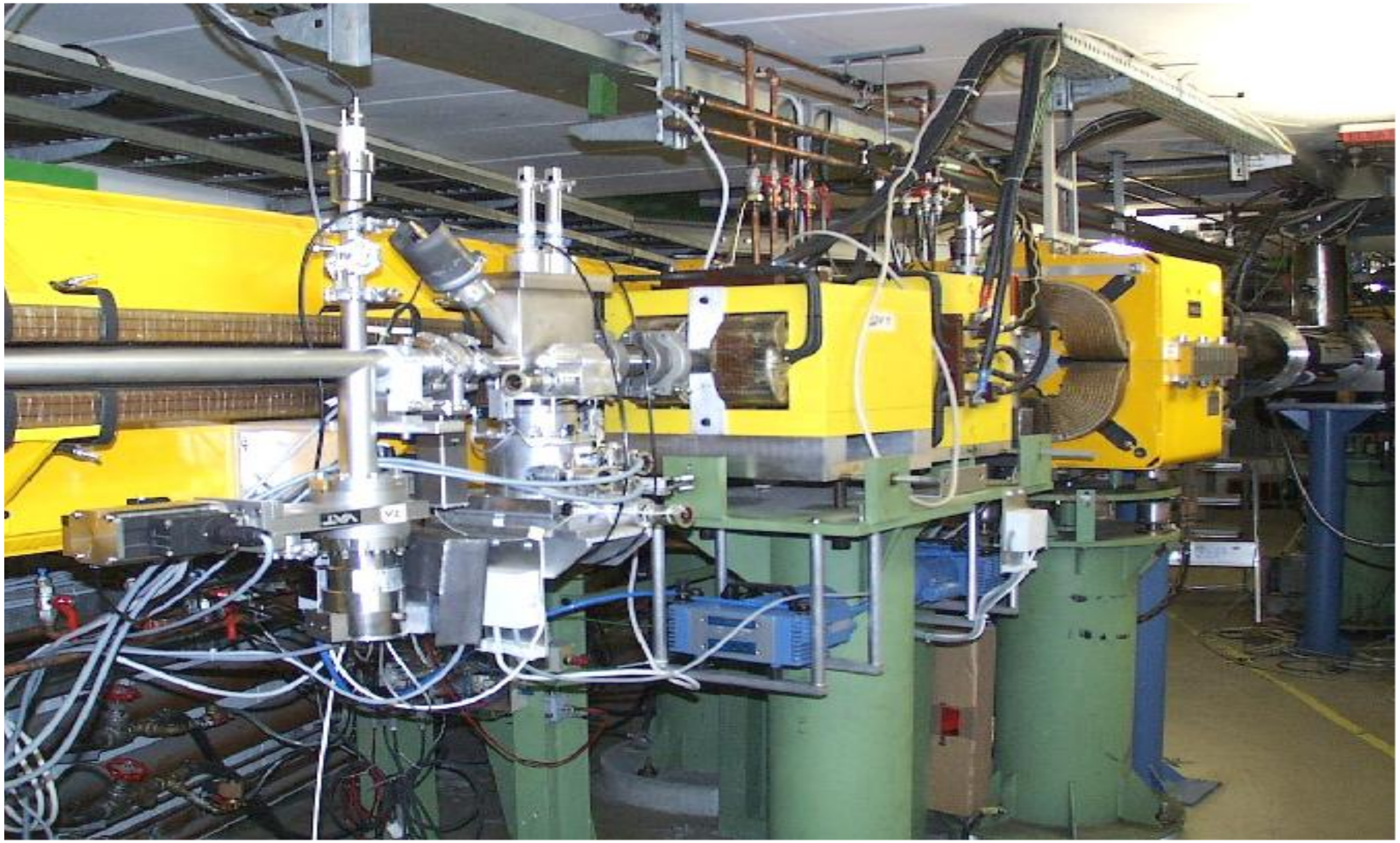
**the particle will follow a new closed trajectory, the distorted orbit:**

$$x(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi Q)} * \oint \sqrt{\beta(\tilde{s})}\, \frac{1}{\rho(\tilde{s})} \cos(|\psi(\tilde{s}) - \psi(s)| - \pi Q) d\tilde{s}$$

* **the orbit amplitude will be large if the β function at the location of the kick is large**
$\beta(\tilde{s})$ **indicates the sensitivity of the beam → here orbit correctors should be placed in the lattice**

* **the orbit amplitude will be large at places where in the lattice β(s) is large → here beam position monitors should be installed**

## *Orbit Correction and Beam Instrumentation in a storage ring*



*Elsa ring, Bonn*

*

## Resumé:

1.) Dipole strength

$$\int B ds = N * B_0 * l_{eff} = 2\pi \frac{p}{q}$$

**$l_{eff}$ effective magnet length, N number of magnets**

2.) Stability condition

$$Trace(M) < 2$$

**for periodic structures within the lattice / at least for the transfer matrix of the complete circular machine**

3.) Transfer matrix for periodic cell

$$M(s) = \begin{pmatrix} \cos\psi_{cell} + \alpha_s \sin\psi_{cell} & \beta_s \sin\psi_{cell} \\ -\gamma_s \sin\psi_{cell} & \cos\psi_{cell} - \alpha_s \sin\psi_{cell} \end{pmatrix}$$

**α,β,γ depend on the position s in the ring, μ (phase advance) is independent of s**

4.) Thin lens approximation

$$M_{QF} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f_Q} & 1 \end{pmatrix} \qquad f_Q = \frac{1}{k_Q l_Q}$$

**focal length of the quadrupole magnet** $f_Q = 1/(k_Q l_Q) >> l_Q$

## 5.) Tune (rough estimate)

$$\psi_{period} = \int_{s}^{s+L} \frac{ds}{\beta(s)}$$

*Tune = phase advance in units of 2π*

$$Q = N * \frac{\psi_{period}}{2\pi} = \frac{1}{2\pi} * \oint \frac{ds}{\beta(s)} \approx \frac{1}{2\pi} * \frac{2\pi \bar{R}}{\bar{\beta}} = \frac{\bar{R}}{\bar{\beta}}$$

*$\bar{R}, \bar{\beta}$ average radius and β-function*

$$Q \approx \frac{\bar{R}}{\bar{\beta}}$$

## 6.) Phase advance per FoDo cell

*(thin lens appro*

$$\sin\frac{\psi_{cell}}{2} = \frac{l_d}{\tilde{f}} = \frac{L_{cell}}{4 f_Q}$$

**$L_{Cell}$ length of the complete FoDo cell, $f_Q$ focal length of the quadrupole, $\mu$ phase advance per cell**

## 7.) Stability in a FoDo cell

*(thin lens approx)*

$$f_Q > \frac{L_{Cell}}{4}$$

## 8.) Beta functions in a FoDo cell

*(thin lens approx)*

$$\hat{\beta} = \frac{(1 + \sin\frac{\psi_{cell}}{2}) L_{cell}}{\sin\psi_{cell}} \qquad \breve{\beta} = \frac{(1 - \sin\frac{\psi_{cell}}{2}) L_{cell}}{\sin\psi_{cell}}$$

**$L_{Cell}$ length of the complete FoDo cell, $\mu$ phase advance per cell**