Emittance Preservation in Electron Accelerators

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What’s so special about ‘emittance preservation in electron machines’?

- Theoretically, very little
  - it’s all basically classical mechanics
- Fundamental difference is Synchrotron Radiation
  - Careful choice of lattice for high-energy arcs can mitigate excessive horizontal emittance growth
  - Storage rings: Theoretical Minimum Emittance (TME) lattices
- Electrons are quickly relativistic ($v/c \approx 1$)
  - I will not discuss non-relativistic beams
- Another practical difference: $e\pm$ emittances tend to be significantly smaller (than protons)
  - High-brightness RF guns for linac based light sources
  - SR damping (storage ring light sources, HEP colliders) generate very small vertical emittances (flat beams)
Critical Emittance

• HEP colliders
  – Luminosity
    \[ L \propto \frac{1}{\sqrt{\beta_x^* \beta_y^* \varepsilon_x \varepsilon_y}} \]

• Light sources (storage rings)
  – Brightness
    \[ B \propto \frac{1}{\sqrt{(\varepsilon_x/\beta_x + \sigma_x^2) (\varepsilon_x/\beta_x + \sigma_x^2) (\varepsilon_y/\beta_y + \sigma_y^2) (\varepsilon_y/\beta_y + \sigma_y^2)}} \]

• SASE XFEL
  – e.g. e-beam/photon beam overlap condition
    \[ \varepsilon < \frac{\lambda}{4\pi} \]

radiation emittance: \( \sigma_x \sigma_y = \frac{\lambda}{4\pi} \)
Typical Emittance Numbers

![Graph showing typical emittance numbers for various facilities and energies. The graph plots $\varepsilon_y$ vs. $\varepsilon_x$ with the axes scaled logarithmically. The graph includes markers for different facilities and energies, such as USSLS, NLC, ILC, CLIC, SLS, APS, SOLEIL, BESSY II, and PETRA. The graph also shows lines for different coupling values, such as 10% and 0.1%.]
Typical Emittance Numbers

EURO XFEL @20 GeV
\( \varepsilon_y = 2.5 \times 10^{-11} \, \text{m} \)

ILC @250 GeV
\( \varepsilon_x = 2 \times 10^{-11} \, \text{m} \) \quad \left( \gamma \varepsilon_x = 10^{-5} \, \text{m} \right)
\( \varepsilon_y = 6 \times 10^{-14} \, \text{m} \) \quad \left( \gamma \varepsilon_y = 3 \times 10^{-8} \, \text{m} \right)
Back to Basics:
Emittance Definition

\[ \varepsilon = \oint p_i \, dq_i = cn \tau \]

Liouville’s theorem:
Density in phase space is conserved (under conservative forces)
Back to Basics: Emittance Definition

Statistical Definition

$2^{nd}$-order moments:

$$\sigma = \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle x'^2 \rangle \end{pmatrix}$$

$$\varepsilon = \sqrt{|\sigma|}$$

$$= \begin{pmatrix} \sqrt{\langle x^2 \rangle \langle x'^2 \rangle} \\ \sqrt{\langle xx' \rangle^2 - \langle xx' \rangle^2} \end{pmatrix}$$

RMS emittance is not conserved!
Back to Basics: Emittance Definition

Statistical Definition

Connection to TWISS parameters

\[
\sigma = \begin{pmatrix}
\epsilon \beta \\
-\epsilon \alpha \\
-\epsilon \alpha \\
\epsilon (1 + \alpha^2) / \beta
\end{pmatrix}
\]

\[\sqrt{\sigma} = \epsilon\]
Some Sources of (RMS) Emittance Degradation

- Synchrotron Radiation
- Collective effects
  - Space charge
  - Wakefields (impedance)
- Residual gas scattering
- Accelerator errors:
  - Beam mismatch
    - field errors
  - Spurious dispersion, x-y coupling
    - magnet alignment errors

*Which of these mechanisms result in ‘true’ emittance growth?*
High-Energy Linac

• Simple regular FODO lattice
  – No dipoles
• ‘drifts’ between quadrupoles filled with accelerating structures
• In the following discussions:
  – assume relativistic electrons
    • No space charge
    • No longitudinal motion within the bunch ('synchrotron' motion)
Linearised Equation of Motion in a LINAC

Remember Hill's equation:

\[ y''(s) + K(s)y(s) = 0 \]

Must now include effects of acceleration:

\[ y''(s) + \frac{\gamma'(s)}{\gamma(s)} y'(s) + K(s)y(s) = 0 \]

Include lattice chromaticity (first-order in \( \delta \equiv \Delta p/p_0 \)):

\[ y''(s) + \frac{\gamma'(s)}{\gamma(s)} y'(s) + [1 - \delta(s)] K(s)y(s) = 0 \]

And now we add the errors…
(RMS) Emittance Growth Driving Terms

“Dispersive” effect from quadrupole offsets

\[
y''(s) + \frac{\gamma'(s)}{\gamma(s)} y'(s) + [1 - \delta(s)] K(s) \left( y(s) - y_q(s) \right) = 0
\]

put error source on RHS (driving terms)

\[
y''(s) + \frac{\gamma'(s)}{\gamma(s)} y'(s) + K(s)y(s)
\]

\[
= -K(s)y_q(s) + \delta(s)K(s)y_q(s) + \delta(s)K(s)y(s)
\]

trajectory kicks from offset quads

dispersive kicks from offset quads

dispersive kicks from coherent $\beta$-oscillation
Coherent Oscillation

\[ y''(s) + \frac{\gamma'(s)}{\gamma(s)} y'(s) + K(s)y(s) \]

\[ = -K(s)y_q(s) + \delta(s)K(s)y_q(s) + \delta(s)K(s)y(s) \]

Just chromaticity repackaged
Scenario 1:
Quad offsets, but BPMs aligned

Assuming:
- a BPM adjacent to each quad
- a ‘steerer’ at each quad

simply apply one to one steering to orbit
Scenario 2: Quads aligned, BPMs offset

one-to-one correction BAD!

Resulting orbit not Dispersion Free ⇒ emittance growth

Need to find a steering algorithm which effectively puts BPMs on (some) reference line

real world scenario: some mix of scenarios 1 and 2
Dispersive Emittance Growth

After trajectory correction (one-to-one steering)

\[
\frac{\Delta \varepsilon}{\varepsilon} \propto \frac{\delta_{RMS}^2}{\varepsilon} \frac{1}{\beta_0^2} \frac{1}{1 - \alpha} \left[ \left( \frac{\gamma_f}{\gamma_i} \right)^{2-2\alpha} - 1 \right] \langle y_{BPM}^2 \rangle \beta(s) \propto \gamma^\alpha(s)
\]

Reduction of dispersive emittance growth favours weaker lattice (i.e. larger \(\beta\) functions)
Wakefields and Beam Dynamics

- bunches traversing cavities generate many RF modes.
- higher-order (higher-frequency) modes (HOMs) can act back on the beam and adversely affect it.
- Separate into two time (frequency) domains:
  - long-range, bunch-to-bunch
  - short-range, single bunch effects (head-tail effects)
Long Range Wakefields

\[ V(\omega, t) = I(\omega, t)Z(\omega, t) \]

Bunch ‘current’ generates wake that decelerates trailing bunches.

Bunch current generates transverse deflecting modes when bunches are not on cavity axis.

Fields build up resonantly: latter bunches are kicked transversely

\[ \Rightarrow \text{multi- and single-bunch beam break-up (MBBU, SBBU)} \]

wakefield is the time-domain description of impedance
Transverse HOMs

Wake is sum over modes:  \( W_\perp(t) = \sum_n \frac{2k_n c}{\omega_n} e^{-\omega_n t/2Q_n} \sin(\omega_n t) \)

\( k_n \) is the loss parameter (units \( V/pC/m^2 \)) for the \( n^{th} \) mode

Transverse kick of \( j^{th} \) bunch after traversing one cavity:

\[
\Delta y'_j = \sum_{i=1}^{j-1} \frac{y_i q_i}{E_i} \frac{2k_i c}{\omega_n} e^{-\omega_n i \Delta t / 2Q_n} \sin(\omega_i i \Delta t_b) 
\]

where \( y_i, q_i, \) and \( E_i, \) are the offset wrt the cavity axis, the charge and the energy of the \( i^{th} \) bunch respectively.
Detuning

HOMs can be randomly detuned by a small amount.

Over several cavities, wake ‘decoheres’.

Effect of random 0.1% detuning (averaged over 36 cavities).

Still require HOM dampers
Effect of Emittance

vertical beam offset along bunch train
\( (n_b = 2920) \)

Multibunch emittance growth for cavities with 500µm RMS misalignment
Single Bunch Effects

- Completely analogous to low-range wakes
- Wake over a single bunch
- Causality (relativistic bunch): head of bunch affects the tail
- Again must consider
  - Longitudinal: effects energy spread along bunch
  - Transverse: the emittance killer!
- For short-range wakes, tend to consider wake potentials (Greens functions) rather than ‘modes
Transverse Wakefields

dipole mode:
offset bunch – head generates trailing $E$-field which kicks tail of beam

Increase in *projected* emittance
Centroid shift
Transverse Single-Bunch Wakes

When bunch is offset wrt cavity axis, transverse (dipole) wake is excited.

‘kick’ along bunch:

\[
\Delta y'(z) = \frac{q_b}{E(z)} \int_{z'=z}^{\infty} W_\perp(z' - z) \rho(z') y(s; z') dz'
\]

Note: \(y(s; z)\) describes a free betatron oscillation along linac (FODO) lattice (as a function of \(s\))
Effect of coherent betatron oscillation - head *resonantly* drives the tail

head eom (Hill’s equation):

\[ y_1'' + k_\beta^2 y_1 = 0 \]

solution:

\[ y_1(s) = \sqrt{a \beta(s)} \sin(\varphi(s) + \varphi_0) \]

tail eom:

\[ y_2'' + k^2 y_2 = y_1 \frac{q}{2} \frac{2\sigma_z}{E_{beam}} \]

resonantly driven oscillator
BNS Damping

If both macroparticles have an initial offset $y_0$ then particle 1 undergoes a sinusoidal oscillation, $y_1 = y_0 \cos(k_\beta s)$. What happens to particle 2?

$$y_2 = y_0 \left[ \cos(k_\beta s) + s \sin(k_\beta s) \frac{w'_\perp q \sigma_z}{2 k_\beta E_{beam}} \right]$$

Qualitatively: an additional oscillation out-of-phase with the betatron term which grows monotonically with $s$.

How do we beat it? Higher beam energy, stronger focusing, lower charge, shorter bunches, or a damping technique recommended by Balakin, Novokhatski, and Smirnov \textbf{(BNS Damping)}

\textit{curtesy: P. Tenenbaum (SLAC)}
BNS Damping

Imagine that the two macroparticles have different betatron frequencies, represented by different focusing constants $k_{\beta_1}$ and $k_{\beta_2}$.

The second particle now acts like an undamped oscillator driven off its resonant frequency by the wakefield of the first. The difference in trajectory between the two macroparticles is given by:

$$y_2 - y_1 = y_0 \left( 1 - \frac{W' \sigma_z}{E_{beam}} \frac{1}{k_{\beta_2}^2 - k_{\beta_1}^2} \right) \left[ \cos \left( k_{\beta_2} s \right) - \cos \left( k_{\beta_1} s \right) \right]$$

courtesy: P. Tenenbaum (SLAC)
BNS Damping

The wakefield can be locally cancelled (ie, cancelled at all points down the linac) if:

\[
\frac{W'_\perp q\sigma_z}{E_{beam}} \cdot \frac{1}{k_{2\beta} - k_{\beta_1}} = 1
\]

This condition is often known as “autophasing.”

It can be achieved by introducing an energy difference between the head and tail of the bunch. When the requirements of discrete focusing (ie, FODO lattices) are included, the autophasing RMS energy spread is given by:

\[
\frac{\sigma_E}{E_{beam}} = \frac{1}{16} \frac{W'_\perp q\sigma_z}{E_{beam}} \cdot \frac{L_{cell}^2}{\sin^2 \left( \pi \nu_\beta \right)}
\]
Wakefields (alignment tolerances)

\[ \frac{\Delta \varepsilon}{\varepsilon} \propto \frac{Q^2 W^2}{\varepsilon} \beta_{lattice} \frac{1}{\alpha} \left[ \left( \frac{\gamma_f}{\gamma_i} \right)^\alpha - 1 \right] \langle \Delta y_{cav}^2 \rangle \beta(s) \propto \gamma^\alpha(s) \]

for wakefield control, prefer stronger focusing (small \( \beta \))
\[ y''(s; z) + \frac{\gamma'(s)}{\gamma(s)} y'(s; z) + K(s) y(s; z) = \]
\[ \begin{align*}
\delta(s; z) K(s) y(s; z) \\
+ K(s) y_q(s) \\
- \delta(s; z) K(s) y_q(s) \\
+ \frac{Q}{\gamma(s)m_0} \left[ 1 - \delta(s; z) \right] \int_{z'=z}^{\infty} W_d(s; z' - z) \lambda(z') y(s; z') dz'
\end{align*} \]

\text{dispersive errors} \quad \text{long. distribution} \quad \text{transverse wake potential} \\
\text{(V C}^{-1} \text{ m}^{-2})
Preservation of RMS emittance

- Transverse
- Longitudinal
- Single-bunch

Wakefields

Dispersion

Quadrupole Alignment

Cavity Alignment

ΔE/E

beam loading

control of
Some Number for ILC

RMS random misalignments to produce 5% vertical emittance growth

<table>
<thead>
<tr>
<th>Component</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>BPM offsets</td>
<td>11 µm</td>
</tr>
<tr>
<td>RF cavity offsets</td>
<td>300 µm</td>
</tr>
<tr>
<td>RF cavity tilts</td>
<td>240 µrad</td>
</tr>
</tbody>
</table>

Impossible to achieve with conventional mechanical alignment and survey techniques

Typical ‘installation’ tolerance: 300 µm RMS ⇒ $5\% \times \left(\frac{300}{11}\right)^2 \approx 3800\%!!$

Use of Beam Based Alignment mandatory
Basics Linear Optics Revisited

thin-lens quad approximation: \( \Delta y' = -KY \)

\[
g_{ij} = \left. \frac{\partial y_i}{\partial y_j'} \right|_{y'_j=0} = R_{34}(i, j)
\]

\[
y_j = \left( -\sum_{i=1}^{j} g_{ij} K_i Y_i \right) - Y_j
\]

linear system: just superimpose oscillations caused by quad kicks.
Introduce matrix notation

Original Equation

\[ y_j = \left( -\sum_{i=1}^{j} g_{ij} K_i Y_i \right) - Y_j \]

Defining Response Matrix \( Q \):

\[ Q = G \cdot \text{diag}(K) + I \]

Hence beam offset becomes

\[ y = -Q \cdot Y \]

\( G \) is lower diagonal:

\[
G = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \cdots \\
g_{21} & 0 & 0 & 0 & 0 & \cdots \\
g_{31} & g_{32} & 0 & 0 & 0 & \cdots \\
g_{41} & g_{42} & g_{43} & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]
Dispersive Emittance Growth

Consider effects of finite energy spread in beam $\delta_{\text{RMS}}$

chromatic response matrix:

$$Q(\delta) = G(\delta) \cdot \text{diag} \left( \frac{K}{1 + \delta} \right) + I$$

lattice chromaticity
dispersive kicks

$$G(\delta) = G(0) + \frac{\partial G}{\partial \delta} \bigg|_{\delta=0}$$

$$R_{34}(\delta) = R_{34}(0) + T_{346} \delta$$

dispersive orbit:

$$\Delta y(\delta) = -[Q(\delta) - Q(0)] \cdot Y$$
What do we measure?

BPM readings contain additional errors:

- $b_{\text{offset}}$: static offsets of monitors wrt quad centres
- $b_{\text{noise}}$: one-shot measurement noise (resolution $\sigma_{\text{RES}}$)

In principle: all BBA algorithms deal with $b_{\text{offset}}$
BBA using Dispersion Free Steering (DFS)

- Find a set of steerer settings which minimise the dispersive orbit
- in practise, find solution that minimises difference orbit when ‘energy’ is changed
- Energy change:
  • true energy change (adjust linac phase)
  • scale quadrupole strengths
DFS

\[ \Delta y = -\left[ Q\left( \frac{\Delta E}{E} \right) - Q(0) \right] \left( \frac{\Delta E}{E} \right) \cdot Y \]

\[ \equiv M \left( \frac{\Delta E}{E} \right) \cdot Y \]

Note: taking difference orbit \( \Delta y \) removes \( b_{\text{offset}} \)

Solution (trivial): \( Y = M^{-1} \cdot \Delta y \)

Unfortunately, not that easy because of noise sources:

\[ \Delta y = M \cdot Y + b_{\text{noise}} + R \cdot y_0 \]
DFS example

300µm random quadrupole errors

20% $\Delta E/E$

No BPM noise

No beam jitter
Simple solve

$$Y = M^{-1} \cdot \Delta y$$

In the absence of errors, works exactly

Resulting orbit is flat

⇒ Dispersion Free

(perfect BBA)

Now add 1 μm random BPM noise to measured difference orbit
DFS example

Simple solve

\[ Y = M^{-1} \cdot \Delta y \]

Fit is ill-conditioned!

- \[ \text{original quad errors} \]
- \[ \text{fitter quad errors} \]
DFS example

Solution is still Dispersion Free
but several mm off axis!
DFS: Problems

• Fit is ill-conditioned
  – with BPM noise DF orbits have very large unrealistic amplitudes.
  – Need to constrain the absolute orbit

\[
\begin{align*}
\text{minimise} & \quad \frac{\Delta y \cdot \Delta y^T}{2\sigma_{\text{res}}^2} + \frac{y \cdot y^T}{\sigma_{\text{res}}^2 + \sigma_{\text{offset}}^2} \\
\end{align*}
\]

• Sensitive to initial launch conditions \( R \cdot y_0 \)
  (steering, beam jitter)
  – need to be fitted out or averaged away
Minimise

\[
\frac{\Delta y \cdot \Delta y^T}{2\sigma_{res}^2} + \frac{y \cdot y^T}{\sigma_{res}^2 + \sigma_{offset}^2}
\]

absolute orbit now constrained

remember

\(\sigma_{res} = 1\,\mu m\)

\(\sigma_{offset} = 300\,\mu m\)
DFS example

Solutions much better behaved!

Orbit *not quite* Dispersion Free, but very close
DFS practicalities

• Need to align linac in sections (bins), generally overlapping.
• Changing energy by 20%
  – quad scaling: only measures dispersive kicks from quads. Other sources ignored (not measured)
  – Changing energy upstream of section using RF better, but beware of RF steering (see initial launch)
  – dealing with energy mismatched beam may cause problems in practise (apertures)
• Initial launch conditions still a problem
  – coherent $\beta$-oscillation looks like dispersion to algorithm.
  – can be random jitter, or RF steering when energy is changed.
  – need good resolution BPMs to fit out the initial conditions.
• Sensitive to model errors ($M$)
Orbit Bumps

- Localised closed orbit bumps can be used to correct
  - Disperive
  - Wakefields
- “Global” correction (eg. end of linac) can only correct non-filamented part
  - Remaining linear correlation
- Need ‘emittance diagnostic’
  - Beam profile monitors
  - Other signal (eg. luminosity in the ILC)
I’ll Stop Here
Emittance Growth: Chromaticity

Chromatic kick from a thin-lens quadrupole:

\[ \Delta x' = K \delta x; \quad \delta \equiv \Delta p / p \]

2\textsuperscript{nd}-order moments:

\[ \langle x^2 \rangle \rightarrow \langle x'^2 \rangle \]
\[ \langle xx' \rangle \rightarrow \langle xx' \rangle \]
\[ \langle x'^2 \rangle \rightarrow \langle x'^2 \rangle + K^2 \langle \delta^2 \rangle \langle x^2 \rangle \]
Emittance Growth: Chromaticity

Chromatic kick from a thin-lens quadrupole:

\[ \Delta x' = K \delta x; \quad \delta \equiv \Delta p / p \]

RMS emittance:

\[ \varepsilon_x^2 = \left\langle x^2 \right\rangle \left\langle x'^2 \right\rangle - \left\langle xx' \right\rangle \]

\[ = \varepsilon_{x,0}^2 + K^2 \left\langle \delta^2 \right\rangle \left\langle x^2 \right\rangle^2 \]

\[ = \varepsilon_{x,0}^2 + K^2 \delta_{RMS}^2 \beta_x^4 \varepsilon_{x,0}^4 \]

\[ \frac{\Delta \varepsilon_x}{\varepsilon_{x,0}} \approx \frac{1}{2} K^2 \delta_{RMS}^2 \beta_x^4 \varepsilon_{x,0}^2 \]
Synchrotron Radiation
Synchrotron Radiation
Synchrotron Radiation
Synchrotron Radiation

\[ \gamma (\Delta E / E = -\hbar \omega / E \equiv \delta u) \]
Synchrotron Radiation

\[ \gamma \quad (\Delta E / E = -\hbar \omega / E \equiv \delta u) \]
Synchrotron Radiation

\[ \gamma \quad (\Delta E / E = -\hbar \omega / E \equiv \delta u) \]

\[ \Delta x_i(L) = R_{16}(s)\delta u_i \]

\[ \langle \Delta x_i^2 \rangle_\gamma = R_{16}^2(s)\langle \delta u_i^2 \rangle \]

\[ \langle \delta u^2 \rangle \approx 4.13 \times 10^{-11} \frac{E^5[GeV]}{\rho^3[m]} \Delta s[m] \]
Synchrotron Radiation

\[ \langle \Delta x^2 \rangle = C_\gamma E^5 \int_{s=0}^{L} \frac{R_{16}^2(s)}{\rho^3(s)} \, ds \quad \text{(phase space due to quantum excitation)} \]

\[ \langle \Delta x'^2 \rangle = C_\gamma E^5 \int_{s=0}^{L} \frac{R_{26}^2(s)}{\rho^3(s)} \, ds \]

\[ \langle \Delta x \Delta x' \rangle = C_\gamma E^5 \int_{s=0}^{L} \frac{R_{16}(s)R_{26}(s)}{\rho^3(s)} \, ds \]

\[ \Delta \varepsilon_\gamma = C_\gamma E^5 \left[ \int_{0}^{L} \frac{R_{16}^2(s)}{\rho^3(s)} \, ds \int_{0}^{L} \frac{R_{26}^2(s)}{\rho^3(s)} \, ds - \left( \int_{0}^{L} \frac{R_{16}(s)R_{26}(s)}{\rho^3(s)} \, ds \right)^2 \right]^{1/2} \]

We have ignored the mean energy loss
(assumed to be small, or we have taken some suitable average)
What is the additional emittance when our initial beam has a finite emittance?

$$\varepsilon = \varepsilon_0 + \Delta \varepsilon_\gamma$$

Quantum emission is uncorrelated, so we can add 2nd-order moments.

When quantum induced phase space and original beam phase space are ‘geometrically similar’, just add emittances.
Synchrotron Radiation

What is the additional emittance when our initial beam has a finite emittance?

\[ \varepsilon = \varepsilon_0 + \Delta \varepsilon \gamma \]

- ellipse shape

\[ \sigma = \varepsilon_0 \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} + \Delta \varepsilon \gamma \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} \]

\[ = (\varepsilon_0 + \Delta \varepsilon \gamma) \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} \]
Synchrotron Radiation

When quantum induced phase space and initial beam phase space are *dissimilar*, an additional (cross) term must be included.

\[ \sigma = \varepsilon_0 \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} + \Delta \varepsilon_\gamma \begin{pmatrix} \beta_\gamma & -\alpha_\gamma \\ -\alpha_\gamma & \gamma_\gamma \end{pmatrix} \]

\[ \varepsilon^2 = |\sigma| = \varepsilon_0^2 + \Delta \varepsilon_\gamma^2 + 2\varepsilon_0 \Delta \varepsilon_\gamma \left[ \frac{1}{2} (\gamma_\beta - 2\alpha_\gamma \alpha + \gamma_\gamma \beta) \right] \]
Beam Mismatch (Filamentation)

\[ v \equiv \frac{\alpha x + \beta x'}{\sqrt{\beta}} \]

Matched beam (normalised to unit circle)

Mismatched beam (\( \beta \)-mismatch) rotates with nominal phase advance along beamline

\[ u \equiv \frac{x}{\sqrt{\beta}} \]

\( \Rightarrow \beta \)-beat along machine (but emittance remains constant)
Beam Mismatch (Filamentation)

Mismatched beam ($\beta$-mismatch) rotates with nominal phase advance along beamline

\[ v \equiv \frac{\alpha x + \beta x'}{\sqrt{\beta}} \]

\[ u \equiv \frac{x}{\sqrt{\beta}} \]

Mismatched beam ($\beta$-mismatch) rotates with nominal phase advance along beamline

\[ \Rightarrow \beta \text{-beat along machine (but emittance remains constant)} \]

Finite energy spread in beam + lattice chromaticity causes mismatch to “filament”

\[ \Rightarrow \text{Emittance growth} \]
Beam Mismatch (Filamentation)

\[ M = \frac{1}{\sqrt{\beta_0}} \begin{pmatrix} 1 & 0 \\ \alpha_0 & \beta_0 \end{pmatrix} \]

Normalisation matrix (matched beam)

\[ \sigma = \varepsilon_0 \begin{pmatrix} \beta & -\alpha \\ -\alpha & \frac{1 + \alpha^2}{\beta} \end{pmatrix} \]

Mismatched beam

\[ R(\varphi) = \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix} \]

Phase space rotation

\[ \sigma_{fil} = \frac{1}{\pi} \int_{0}^{\pi} R(\varphi) \cdot M \cdot \sigma \cdot M^T \cdot R^T(\varphi) d\varphi \]

Fully filamented beam

\[ \sqrt{\sigma_{fil}} = \frac{1}{2} \left[ \gamma_0 \beta + \gamma \beta_0 - 2\alpha \alpha_0 \right] \varepsilon_0 \]

\[ \beta = 2\beta_0; \alpha = \alpha_0 = 0; \varepsilon_{fil} = 1.2\varepsilon_0 \]
Longitudinal Wake

Consider the TESLA wake potential \( W_{||}(z = ct) \)

\[
W_{||}(z) \approx -38.1 \left[ \frac{V}{\text{pC} \cdot \text{m}} \right] 1.165 \exp \left( -\sqrt{\frac{s}{3.65 \times 10^{-3} \text{[m]}}} \right) - 0.165
\]

wake over bunch given by convolution: \( (\rho(z) = \text{long. charge dist.}) \)

\[
W_{||,\text{bunch}}(z) = \int_{z'}^{\infty} W_{||}(z' - z) \rho(z')dz'
\]

average energy loss:

\[
\langle \Delta E \rangle = q_b \int_{-\infty}^{\infty} W_{||,\text{bunch}}(z) \rho(z)dz
\]

For TESLA LC: \( \langle \Delta E \rangle \approx -46 \text{ kV/m} \)
RMS Energy Spread

accelerating field along bunch:

\[ E(z) = q_b W_{||,bunch}(z) + E_0 \cos(2\pi z / \lambda_{RF} + \phi) \]

Minimum energy spread along bunch achieved when bunch rides ahead of crest on RF.

Negative slope of RF compensates wakefield.

For TESLA LC, minimum at about \( \phi \sim +6^\circ \)
RMS Energy Spread

![Graph showing RMS energy spread for different RF phases and energy settings.](image-url)