Longitudinal instabilities

L. Palumbo
Contents:

- Wake Fields and Coupling Impedance
- Short range and long range wake fields
- Potential well distortion
- Microwave instability
- Robinson Instability
- Coupled bunch Instability
Longitudinal Wake Fields and Impedance

$q_1 (z_1, r_1)$: trailing point charge

$q (z, r)$: test point charge

The test charge $q$ can gains or loses energy because of the electromagnetic fields generated behind $q_1$

\[
\Delta z = z_1 - z \quad \Rightarrow \quad U_\parallel (r, r_1; \Delta z) = -\int_{\text{trajectory}} F_\parallel (z, r, z_1, r_1; t) dz
\]

with \( t = (z_1 + \Delta z) / c \)
The energy variation of the test charge $q$, normalized to $q$ and $q_1$ is called longitudinal wake function (green’s function)

$$w_\parallel (\mathbf{r}, \mathbf{r}_1; \Delta z) = \frac{U_\parallel (\mathbf{r}, \mathbf{r}_1; \Delta z)}{qq_1}$$

The energy variation of a test charge inside a bunch, due to the distribution $\rho(z)$, is called longitudinal bunch wake potential

$$W_\parallel (z) = e^2 N_p \int_{-\infty}^{\infty} \rho(z') w_\parallel (z' - z) dz'$$

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DAΦNE wake potential of a 2.5 mm Gaussian bunch.
The longitudinal coupling impedance is the Fourier transform of the wake function

\[ Z_{||}(r, r_1; \omega) = \frac{1}{c} \int_{-\infty}^{\infty} w_{||}(r, r_1; z) \exp \left[ -i \omega \frac{z}{c} \right] dz \]

ДАФНЕ accumulator wake potential of a 2.5 mm Gaussian bunch.

\[ \frac{Z_{||}(\omega)}{n} = \frac{Z_{||}(\omega)}{\omega/\omega_o} \]
Short range wakefield acts over the bunch length

- Vanishes after a distance of few bunch lengths
- Low frequency resolution of Fourier transform and of coupling impedance
- Smoother and broader impedance → broad band impedance mode (e.g. Broad Band Resonator Model)
Long range wakefields acts on many bunches/multi-turn

- Fields oscillating over long distances
- produced by high Q resonant modes
- Determined by only 3 parameters: $Q$, $\omega_r$ and $R_s$
- High peak impedance
In both cases we have

\[
Z_{\parallel}(\omega) = \frac{R_s}{1 + iQ \left( \frac{\omega}{\omega_r} - \frac{\omega_r}{\omega} \right)}
\]

\[
Z_{\parallel}(\omega) \approx i \frac{\omega}{\omega_r} \frac{R_s}{Q}
\]

\[
Z_{\parallel}(\omega) \approx i \frac{\omega_o}{\omega_r} \frac{R_s}{Q}
\]

\[
w_{\parallel} = \frac{\omega_r R_s}{Q} \exp \left( - \frac{\Gamma \Delta z}{c} \right) \left[ \cos \left( \frac{\omega_n \Delta z}{c} \right) - \frac{\omega_r}{2Q \omega_n} \sin \left( \frac{\omega_n \Delta z}{c} \right) \right] H(\Delta z)
\]

where \( \Gamma = \frac{\omega_r}{2Q}, \omega_n^2 = \omega_r^2 - \Gamma^2 \), and \( H(\Delta z) \) is the step function.

Notice that for the short range wake field, the Broand Band Resonator with \( Q \sim 1 \), is only a useful approximation,
Effects on beam dynamics

Short range wakefields:

• Potential well distortion
• Longitudinal emittance growth, microwave instability

Long range wakefields:

• Robinson instabilities (RF fundamental mode)
• Coupled bunch instability
Short range wakefields
Potential well distortion

The motion of a particle in the bunch is confined by the potential due to the RF voltage and to the wake fields

\[ \Psi(z) = \frac{\alpha_c}{L_0} \int_0^z \left[ eV_{RF}(z') - U_0 \right] dz' - \frac{\alpha_c e^2 N_p}{L_0} \int_0^z dz' \int_{-\infty}^\infty \rho(z'') w(z'') |z' - z''| dz'' \]

In the low current regime, with gaussian energy distribution, energy spread \( \sigma_{\varepsilon_0} \), the longitudinal distribution is described by an integral equation known as the Haissinski equation

\[ \rho_0(z) = \bar{\rho} \exp \left[ - \frac{1}{E_0 \alpha_c^2 \sigma_{\varepsilon_0}^2} \Psi(z) \right] \]
Particular solution of Haissinski equation

No wake field contribution: a linear expansion of $V_{RF}$ around $z=0$ gives

$$\rho_0(z) = \bar{\rho} \exp\left[-\frac{z^2}{2\sigma_{z0}^2}\right]$$

$$\bar{\rho} = \frac{1}{\sqrt{2\pi}\sigma_{z0}}$$

$$\sigma_{z0} = \frac{\alpha_e c \sigma_{z0}}{\omega_{s0}}$$

Pure resistive impedance  Pure inductive impedance  Broad band resonator

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Typical measured bunch distributions in the Dafne Rings. The head is to the left.
Longitudinal emittance growth, microwave instability

• Observe energy spread and bunch length as a function of the current.

- $\sigma_\epsilon$ is almost constant up to a threshold current after which it starts to increase with the current according to a given power law (in most cases $1/3$ power).

- $\sigma_z$ starts to increase from the very beginning (potential well distortion), and, after the same threshold current, it grows with the same power law.
Longitudinal emittance growth & microwave instability

Threshold current:

\[ \hat{I} \frac{|Z_{||}/n|}{2\pi\alpha_c (E_0 / e)\sigma_{\varepsilon}^2} \leq 1 \]

\[ \hat{I} = \frac{ceN_p}{\sqrt{2\pi}\sigma_z} \]

\[ n = \frac{\omega}{\omega_0} \]

Above threshold: Boussard criterion

\[ \sigma_{z} = \left( \frac{R^3|Z/n|\xi}{\sqrt{2\pi}} \right)^{1/3} \]

\[ \xi = \frac{I\alpha_c}{v_s^2 E_0 / e} \]

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Longitudinal emittance growth & microwave instability

Chao – Gareyte scaling law:

Assume a power-law behavior of $Z_\parallel (\omega)$

$$\left| \frac{Z}{n} \right| \propto Z_0 \omega^{a-1} \quad \text{then} \quad \sigma_z \propto \left( \xi Z_0 R^3 \right)^{1/(2+a)}$$

For SPEAR $a = 0.68$

From: A. W. Chao, J. Gareyte, Particle Accelerators, Vol. 25, pp. 229-234, 1990
Bunch lengthening in DAFNE

DAFNE Accumulator.
Dots: measurement results
Solid line: numerical simulation.

DAFNE main rings
Circles - measurement results.
Solid line - numerical calculations

NOTICE
Numerical simulations performed before measurements: good impedance model of the machine

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Design strategy: proper design of vacuum chamber

- Single bunch: low broad band impedance $Z/n$

Reduce parasitic loss, taper discontinuities
Impedance budget

<table>
<thead>
<tr>
<th>Element</th>
<th>$\text{Im} \frac{Z_t}{\eta}$ [Ω]</th>
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<tr>
<td>Tapers</td>
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<tr>
<td>Transverse feedback kickers (low frequency)</td>
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<td>Scrapers</td>
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<tr>
<td>Bellows</td>
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<td>Resistive wall (at roll-off frequency)</td>
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<td>Vacuum pump screens</td>
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<td>Injection port</td>
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<tr>
<td>Antechamber slots</td>
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<td>Synchrotron radiation</td>
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<td>Space charge</td>
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<tr>
<td><strong>Total</strong></td>
<td><strong>0.53 Ω</strong></td>
</tr>
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Cures?
- Landau damping

Longitudinal Microwave instability is fast but not destructive
Long range wakefields

A. Hofmann

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Interaction with RF fundamental mode: Robinson Instabilities

Single particle equation of motion (neglecting quantum fluctuation)

\[ \dot{z} = -c \alpha_c \varepsilon \]

\[ \dot{\varepsilon} = \frac{e V_{RF} - U_0}{T_0 E_0} - \frac{D}{T_0} \varepsilon \quad \text{with} \quad D = \frac{2 U_0}{E_0} \] is the damping coefficient

Combined they give a second order differential equation

\[ \ddot{z} + \frac{D}{T_0} \dot{z} + \omega_{s0}^2 z = 0 \quad \text{with} \quad \omega_{s0}^2 = \frac{c^2 \alpha_c 2 \pi \hat{V} \sin(\phi_s)}{L_0^2 E_0} \]

\[ \cos(\phi_s) = \frac{U_0}{e \hat{V}} \quad \left( 0 \leq \phi_s \leq \frac{\pi}{2} \right) \] synchronous phase
Robinson instability …

By including also the beam loading effect we have (see A. Hofmann lecture)

\[ \ddot{z} + \left( \frac{D}{T_0} - \alpha_r \right) \dot{z} + \omega_s^2 z = 0 \]

\[ \alpha_r = \frac{eN_p \alpha_c h\omega_0}{\omega_s (E_0/e)T_0^2} \text{Re}[\Delta Z] \]

\[ \text{Re}[\Delta Z] = \text{Re}[Z(n\omega_0 + \omega_s) - Z(n\omega_0 - \omega_s)] \]

\[ z = A_0 \exp\left[-\frac{D}{T_0} + \alpha_r \right] \cos[\omega_s t + \theta_0] \]
Robinson instability ……

Example of stability

Oscillation amplitude

Exponential decay

Damped synchrotron oscillation

number of turns
Interaction with HOMs: Coupled bunch instability
(Macroparticle model)

The equations of motion are the same of the single particle. The difference is in the voltage induced by other bunches in the HOMs

\[
\dot{z}_n = -c\alpha_c \varepsilon_n \\
\ddot{\varepsilon}_n = \frac{eV_{RF}(z_n) - U_0}{T_0 E_0} - \frac{eV^n_w}{T_0 E_0} - \frac{D}{T_0} \varepsilon_n
\]

\(V^n_w\) is the voltage seen by the \(n^{th}\) bunch and induced by the long range wake fields.

\[
\ddot{z}_n + \frac{D}{T_0} \dot{z}_n = -\frac{c\alpha_c}{T_0 E_0} \left[ eV_{RF}(z_n) - U_0 - eV^n_w \right]
\]
By linearizing the RF voltage and the HOM induced wake fields with respect to $z_n$ we obtain three terms

1) A term independent on $z$ that modifies synchronous phase

$$e \hat{V} \cos(\phi_{sn}) = U_0 + e \sum_{h=0}^{N_b-1} \sum_{q=-\infty}^{\infty} Q_h w_{||} \left[ \left( q - \frac{h}{N_b} + \frac{n}{N_b} \right) L_0 \right]$$

2) A term dependent on $z_n(t)$ that modifies synchronous frequency

$$\omega_{sn}^2 = \frac{c^2 \alpha_c e}{L_0 E_0} \left[ \frac{2 \pi h \hat{V} \sin(\phi_{sn})}{L_0} + \sum_{h=0}^{N_b-1} \sum_{q=-\infty}^{\infty} Q_h \frac{dw_{||}}{dz} \left( q - \frac{h}{N_b} + \frac{n}{N_b} \right) L_0 \right]$$

3) Other terms dependent on $z_h$ at previous passages that are seen as ‘external coupling forces’
\[ \ddot{z}_n + \frac{D}{T_0} \dot{z}_n + \omega_{sn}^2 z_n = \frac{c \alpha_c e}{T_0 E_0} \]

\[ \sum_{h=0}^{N_b-1} \sum_{q=-\infty}^{\infty} Q_h \frac{dw}{dz} \left( q - \frac{h}{N_b} + \frac{n}{N_b} \right) L_0 \]
\[ z_h \left( t - qT_0 + \frac{h}{N_b} T_0 - \frac{n}{N_b} T_0 \right) \]

To solve the equation system we seek a solution of the kind
\[ z_n(t) = a_n \exp[i \Omega t] \]
and obtain
\[ \lambda(\Omega^{(\mu)}) = -i \frac{c^2 \alpha_c e^2 N_p N_b}{L_0^2 E_0} \sum_{q=-\infty}^{\infty} [\Omega^{(\mu)} + (qN_b + \mu)\omega_0] Z_{||} [\Omega^{(\mu)} + (qN_b + \mu)\omega_0] \]

\[ \lambda(\Omega) = \Omega^2 - i \frac{D}{T_0} \Omega - \omega_{sn}^2 \quad \mu = 0, \ldots, N_b - 1 \]
Example with 2 bunches

\[
\ddot{z}_1 + \frac{D}{T_0} \dot{z}_1 + \omega_s^2 z_1 = \frac{c\alpha_c eQ}{T_0 E_0} \sum_{q=-\infty}^{\infty} \left\{ \frac{dw}{dz} \bigg|_{qL_0} z_1(t - qT_0) + \frac{dw}{dz} \bigg|_{(q-\frac{1}{2})L_0} z_2 \left( t - qT_0 - \frac{1}{2}T_0 \right) \right\}
\]

\[
\ddot{z}_2 + \frac{D}{T_0} \dot{z}_2 + \omega_s^2 z_2 = \frac{c\alpha_c eQ}{T_0 E_0} \sum_{q=-\infty}^{\infty} \left\{ \frac{dw}{dz} \bigg|_{qL_0} z_2(t - qT_0) + \frac{dw}{dz} \bigg|_{(q-\frac{1}{2})L_0} z_1 \left( t - qT_0 - \frac{1}{2}T_0 \right) \right\}
\]

Seek for a solution of the kind
\[ z_1(t) = a_1 \exp[i\Omega t] \text{ and } z_2(t) = a_2 \exp[i\Omega t] \text{ and obtain} \]

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\[
\left( \Omega^2 - i \frac{D}{T_0} \Omega - \omega_s^2 \right) a_1 = -i \frac{c^2 \alpha_c eQ}{L_0^2 E_0} \sum_{q=-\infty}^{\infty} (\Omega + q \omega_0) Z_{\|} (\Omega + q \omega_0) (a_1 + a_2 e^{i\pi q})
\]

\[
\left( \Omega^2 - i \frac{D}{T_0} \Omega - \omega_s^2 \right) a_2 = -i \frac{c^2 \alpha_c eQ}{L_0^2 E_0} \sum_{q=-\infty}^{\infty} (\Omega + q \omega_0) Z_{\|} (\Omega + q \omega_0) (a_1 e^{i\pi q} + a_2)
\]

- Homogeneous system of two equations.
- Non trivial solution the matrix determinant must be zero.
- Consider a single narrow band HOM:

\[
\left( \Omega^2 - i \frac{D}{T_0} \Omega - \omega_s^2 \right) = -i \frac{2c^2 \alpha_c e^2 Q}{L_0^2 E_0} \left[(q\omega_0 + \Omega) Z_{\|} (\Omega + q\omega_0) - (q\omega_0 - \Omega) Z_{\|} (\Omega - q\omega_0) \right]
\]
That can be further simplified ($\Omega \approx \omega_s$)

$$\Omega = \omega_s + i \frac{D}{2T_0} - i \frac{c^2 \alpha_0 e Q}{L_0^2 E_0 \omega_s} \left[ (q\omega_0 + \omega_s)Z_{||}(\omega_s + q\omega_0) - (q\omega_0 - \omega_s)Z_{||}(\omega_s - q\omega_0) \right]$$

Looks similar to Robinson …..
Notice: even q’s corresponds to the two bunches oscillating in phase

odd q’s corresponds to the two bunches oscillating with $\pi$ phase shift
Design strategy: proper design of resonant devices

- Reduce HOM's, low Rs / Q and Q
Cures

- Longitudinal feedbacks
- Landau damping
In general, if we suppose a single narrow band HOM and $\Omega^{(\mu)} \approx \omega_s$ then only two (different) oscillation modes are excited, and we obtain

$$\Omega^{(\mu+)} = \omega_s + i \frac{D}{2T_0} - i \frac{c^2 \alpha_c e^2 N_p N_b}{2L_0^2 E_0 \omega_s} [(q_1 N_b + \mu_+)\omega_0 + \omega_s]Z_||[(q_1 N_b + \mu_+)\omega_0 + \omega_s]$$

$$\Omega^{(\mu-)} = \omega_s + i \frac{D}{2T_0} + i \frac{c^2 \alpha_c e^2 N_p N_b}{2L_0^2 E_0 \omega_s} [(q_2 N_b - \mu_-)\omega_0 - \omega_s]Z_||[-(q_2 N_b - \mu_-)\omega_0 + \omega_s]$$

(q_1 and q_2 > 0)

$\mu_+$ (positive synchrotron sideband) is the unstable oscillation mode

$\mu_-$ (negative synchrotron sideband) is the stable oscillation mode
Conclusions

• The Longitudinal Instability mechanisms are well understood;

• With an accurate model of the machine impedance one can predict the single bunch and multibunch dynamics;

• Single bunch instabilities are not destructive but lead to beam heating (increase of energy spread and bunch length)

• Multibunch instabilities are destructive and require the installation of a fast feedback system on the ring.

• Necessary an accurate design of the vacuum chamber and RF devices