Introduction to Beam Instabilities

CAS 2005, Trieste; Albert Hofmann

1) Instability mechanisms
2) Impedances and wake functions
3) Robinson instability
4) Potential well bunch lengthening
1) Instability mechanisms

Overview
The motion of a single particle in a storage ring is determined by the external guide fields (dipole and quadrupole magnets, RF-system, etc.), initial conditions and synchrotron radiation. Many particles in a beam may represent a sizable charge and current which act as a source of electromagnetic fields (self fields). They are modified by boundary conditions imposed by the beam surroundings (vacuum chambers, cavities, etc.) and act back on the beam. This can lead to a frequency shift (change of the betatron or synchrotron frequency), to an increase of a small disturbance of the beam, i.e. an instability or to a change of the particle distribution, e.g. bunch lengthening. These phenomena are called collective effects being due to a coherent or collective action of many particles. The role played in this process by the electrical properties of the beam surroundings is expressed by an impedance.

As an example we take a bunch in a storage ring going through a cavity where it induces electromagnetic fields which oscillate and decay away slowly. In the next turn the same bunch finds some field left and gets influenced by it. Depending on the phase of the field seen in the next turn, a small initial perturbation is increased or decreased leading to an exponentially growing or decaying oscillation of the bunch.
Multi-turn effects
In the example the induced fields have a memory and the instability is built up over many turns. These self fields are often small compared to the guide fields and their effects is treated as a perturbation in 3 steps.

a) We determine the stationary particle distribution given by the guide field, initial condition and synchrotron radiation.
b) We consider small disturbances and calculate the fields they create including the boundary conditions (impedance).
c) We calculate the effect of these fields on the beam to see if the initial disturbance is increased (instability) or decreased (damping) or the oscillation mode changed (frequency shift).

As disturbances we consider orthogonal (independent) oscillation modes and investigate the stability of each. This works for weak interactions which don’t alter the nature of the modes but determine only their exponential growth over many turns. Multi-turn effects are driven by narrow frequency band impedances with memory.

Multi-bunch effects
With many circulating bunches their individual oscillations can be coupled by an impedance with a shorter memory bridging just the bunch spacing instead of the revolution time. Multi-turn and multi-bunch instabilities have the same qualitative properties and are called multi-traversal effects.
Single traversal effects
Strong self-fields from broad band impedances change the stationary distribution and modify oscillation modes which are no longer independent. A self consistent solutions is difficult to obtain. The most common such effect is **bunch lengthening**.

Longitudinal and transverse effects

**Longitudinal** effects involve synchrotron (energy, phase) oscillations and longitudinal impedances. They contain longitudinal instabilities, shift of synchrotron frequencies and bunch lengthening. **Transverse** effects involve betatron oscillations and transverse impedances. They contain transverse instabilities and betatron frequency shifts.

In both cases the longitudinal particle distribution (bunch length) is important since it can be "resolved" by the impedance while the transverse distribution is usually not resolved and does not affect the instability.

The most important longitudinal single traversal effects are synchrotron frequency shifts and bunch lengthening. In the transverse case the effect of the chromaticity is important which can lead to head-tail instabilities.
2) Impedances and wake functions

Resonator

Cavities have narrow band oscillation modes which can drive coupled bunch instabilities. Each resembles an RCL - circuit and can, in good approximation, be treated as such. This circuit has a shunt impedance $R_s$, an inductance $L$ and a capacity $C$. In a real cavity these parameters cannot easily be separated and we use others which can be measured directly: The **resonance frequency** $\omega_r$, the **quality factor** $Q$ and the **damping rate** $\alpha$:

$$\omega_r = \frac{1}{\sqrt{LC}}$$,
$$Q = R_s \sqrt{\frac{C}{L}} = \frac{R_s}{L\omega_r} = R_s C \omega_r$$,
$$\alpha = \frac{\omega_r}{2Q}$$,

$$L = \frac{R_s}{Q\omega_r}$$,
$$C = \frac{Q}{\omega_r R_s}$$.
Driving this circuit with a current $I$ gives the voltages and currents across the elements

\[ V_R = V_C = V_L = V, \ I_R + I_C + I_L = I \]

Differentiating with respect to $t$ gives

\[ \dot{I} = \dot{I}_R + \dot{I}_C + \dot{I}_L = \frac{\dot{V}}{R_s} + CV + \frac{V}{L}. \]

Using $L = R_s/(\omega_r Q)$ and $C = Q/(\omega_r R_s)$ gives the differential equation

\[ \ddot{V} + \frac{\omega_r}{Q} \dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q} \dot{I}. \]

The solution of the homogeneous equation represents a damped oscillation

\[ V(t) = \hat{V} e^{-\alpha t} \cos \left( \omega_r \sqrt{1 - \frac{1}{4Q^2}} t + \phi \right) \]

\[ V(t) = e^{-\alpha t} \left( A \cos \left( \omega_r \sqrt{1 - \frac{1}{4Q^2}} t \right) + B \sin \left( \omega_r \sqrt{1 - \frac{1}{4Q^2}} t \right) \right) \]
Wake function – Green function

Calculate response of RCL circuit to a delta function pulse

\[
\begin{align*}
\dot{V} + \frac{\omega_r}{Q} \dot{V} + \frac{\omega_r^2 V}{Q} &= \frac{\omega_r R_s}{Q} \dot{I} \\
Q &= R_s \sqrt{\frac{C}{L}} \\
\alpha &= \frac{\omega_r}{2Q}
\end{align*}
\]

The charge \( q \) will charge up the capacity to a voltage

\[
V(0^+) = \frac{q}{C} = \frac{\omega_r R_s}{Q} q \quad \text{using} \quad C = \frac{Q}{\omega_r R_s}
\]

Energy stored in capacitor equals energy lost by charge

\[
U = \frac{q^2}{2C} = \frac{\omega_r R_s}{2Q} q^2 = \frac{V(0^+)}{2} q = k_{pm} q^2 \quad \text{with} \quad k_{pm} = \frac{\omega_r R_s}{2Q}
\]

with the parasitic mode loss factor \( k_{pm} \), measured usually in [V/pC]. The capacitor \( C \) discharges first through the resistor \( R_s \)

\[
\dot{V}(0^+) = -\frac{\dot{q}}{C} = -\frac{I_R}{C} = -\frac{V(0^+)}{C R_s} = -\frac{\omega_r^2 R_s}{Q^2} q = -\frac{2\omega_r k_{pm}}{Q} q.
\]

With the initial conditions \( V(0^+) \), \( \dot{V}(0^+) \) the general solution

\[
V(t) = e^{-\alpha t} \left( A \cos \left( \omega_r \sqrt{1 - \frac{1}{4Q^2}} t \right) + B \sin \left( \omega_r \sqrt{1 - \frac{1}{4Q^2}} t \right) \right)
\]

gives the response of the circuit to a pulse excitation

\[
V(t) = 2q k_{pm} e^{-\alpha t} \left( \cos \left( \omega_r \sqrt{1 - \frac{1}{4Q^2}} t \right) - \frac{\sin \left( \omega_r \sqrt{1 - \frac{1}{4Q^2}} t \right)}{2Q \sqrt{1 - \frac{1}{4Q^2}}} \right)
\]

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Normalized per unit source point charges it is called **Green or wake function** $G(t)$. For our resonator:

$$G(t) = 2k_{pm}e^{-\alpha t} \left( \cos \left( \omega_r \sqrt{1 - \frac{1}{4Q^2}} t \right) - \frac{\sin \left( \omega_r \sqrt{1 - \frac{1}{4Q^2}} t \right)}{2Q \sqrt{1 - \frac{1}{4Q^2}}} \right)$$

$$G(t) \approx 2k_{pm}e^{-\alpha t} \cos (\omega_r t) \text{ for } Q \gg 1, \ k_{pm} = \frac{\omega_r R_s}{2Q}, \ \alpha = \frac{\omega_r}{2Q}$$

This voltage induced by charge $q$ at $t = 0$ changes the energy of a second charge $q'$ traversing cavity at $t$ by $U = q'V(t) = qq'G(t)$.

The wake potential is related to the longitudinal field $E_z$ by a field integration which follows the particle with speed $v \approx c$ through the object length while taking the momentary field value

$$V = Gq = -\int_{z_1}^{z_2} E_z(z, t) dz = -f_t \int_{z_1}^{z_2} E_z(z) dz = -\langle E_z \rangle t \Delta z.$$ 

with transit time factor $f_t$. We use $G(t) > 0$ where energy is lost. A particle inside a bunch of charge $q$ and current $I(t)$ going through a cavity at time $t$ sees the wake function created by all the particles passing at earlier times $t' < t$ resulting in a voltage

$$V(t) = \int_{-\infty}^{t} G(t') dq$$

$$= \int_{-\infty}^{t} I(t')G(t') dt' = qW(t)$$

$W(t) = V(t)/q$ wake potential.
Impedance

We assume now a harmonic excitation of the circuit with a current $I = \hat{I} \cos(\omega t)$ and get the differential equation

$$\ddot{V} + \frac{\omega_r}{Q} \dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q} \dot{I} = -\frac{\omega_r R_s}{Q} \hat{I} \omega \sin(\omega t)$$

The solution of the homogeneous equation damps down leaving a particular solution $V(t) = A \cos(\omega t) + B \sin(\omega t)$. Inserting this into the differential equation and separating cosine and sine terms gives

$$-(\omega^2 - \omega_r^2)A + \frac{\omega_r \omega}{Q} B = 0 \quad \text{and} \quad (\omega^2 - \omega_r^2)B + \frac{\omega_r \omega}{Q} A = \frac{\omega_r \omega R_s}{Q} \dot{I}$$

The voltage induced by the harmonic excitation of the resonator is

$$V(t) = \hat{I} R_s \cos(\omega t) + Q \frac{\omega^2 - \omega_r^2}{\omega_r \omega} \sin(\omega t) \frac{1}{1 + Q^2 \left( \frac{\omega^2 - \omega_r^2}{\omega_r \omega} \right)^2}$$

The voltage has a cosine term in phase with the exciting current. It absorbs energy and is resistive. The sine term is out of phase, does not absorb energy and is reactive. The ratio between voltage and current is the impedance. It is a function of frequency $\omega$

$$Z_r(\omega) = R_s \frac{1}{1 + Q^2 \left( \frac{\omega^2 - \omega_r^2}{\omega_r \omega} \right)^2}, \quad Z_i(\omega) = -R_s \frac{Q \omega^2 - \omega_r^2}{\omega_r \omega} \frac{1}{1 + Q^2 \left( \frac{\omega^2 - \omega_r^2}{\omega_r \omega} \right)^2}$$

Its resistive part $Z_r(\omega)$ is always positive while its reactive part $Z_i(\omega)$ positive below and negative above the resonant frequency $\omega_r$. 

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Complex notation

We have used a harmonic excitation of the form

\[ I(t) = \hat{I} \cos(\omega t) = \frac{\hat{I} e^{j\omega t} + e^{-j\omega t}}{2} \quad \text{with} \quad 0 \leq \omega \leq \infty \]

It is often more convenient to use a complex notation either

\[ I(t) = \text{Re} [\hat{I} e^{j\omega t}] = \text{Re} [\hat{I}(\cos(\omega t) + j \sin(\omega t))] = \hat{I}(\cos \omega t) \]

or

\[ I(t) = \hat{I} e^{j\omega t} \quad \text{with} \quad -\infty \leq \omega \leq \infty \]

giving more compact expressions. Using the differential equation

\[ \ddot{V} + \frac{\omega_r}{Q} \dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q} \dot{\hat{I}} \]

with \( I(t) = \hat{I} \exp(j\omega t) \) and seeking a solution of the form \( V(t) = V_0 \exp(j\omega t) \), where \( V_0 \) is in general complex, one gets

\[ \left(-\omega^2 e^{j\omega t} + j \frac{\omega_r \omega}{Q} e^{j\omega t} + \omega_r^2 e^{j\omega t}\right) V_0 = j \frac{\omega_r \omega R_s}{Q} \hat{I} e^{j\omega t} \]

and for the impedance which is defined as the ration \( V/I \)

\[ Z(\omega) = \frac{V_0}{I} = \frac{R_s}{1 + j Q \left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)} = R_s \frac{1 - j Q \frac{\omega^2 - \omega_r^2}{\omega \omega_r}}{1 + Q^2 \left(\frac{\omega^2 - \omega_r^2}{\omega \omega_r}\right)^2} = Z_r + j Z_i \]

For \( Q \gg 1 \) the impedance is only large for \( \omega \approx \omega_r \) or for \( |\omega - \omega_r|/\omega_r = |\Delta \omega|/\omega_r \ll 1 \) and can be simplified

\[ Z(\omega) \approx R_s \frac{1 - j 2 Q \frac{\Delta \omega}{\omega_r}}{1 + 4 Q^2 \left(\frac{\Delta \omega}{\omega_r}\right)^2} \]

Caution: sometimes \( I(t) = \hat{I} e^{-i\omega t} \) instead of \( I(t) = \hat{I} e^{j\omega t} \) is used, this reverses the sign \( Z_i(\omega) \).
The resonator impedance has some specific properties:

\[
\text{at } \omega = \omega_r \rightarrow Z_r(\omega_r) \text{ has a maximum}, \quad Z_i(\omega_r) = 0
\]

\[
0 < \omega < \omega_r \rightarrow Z_i(\omega) > 0 \quad (\text{inductive})
\]

\[
0 < \omega_r < \omega \rightarrow Z_i(\omega) < 0 \quad (\text{capacitive})
\]

and any impedance or wake potential has the general properties

\[
Z_r(\omega) = Z_r(-\omega) , \quad Z_i(\omega) = -Z_i(-\omega)
\]

\[
Z(\omega) = \int_{-\infty}^{\infty} G(t)e^{-j\omega t} dt \propto \text{Fourier transform}
\]

\[
G(t) = 0 \text{ for } t < 0, \quad \text{no fields before particle arrival.}
\]
3) Robinson instability

**Longitudinal dynamics**

A particle with momentum deviation $\Delta p$ has a different orbit length $L$, revolution time $T_0$ and revolution frequency $\omega_0$

$$ \frac{\Delta L}{L} = \alpha_c \frac{\Delta p}{p} \quad \frac{\Delta \omega_0}{\omega_0} = - \frac{\Delta T_0}{T_0} = - \left( \alpha_c - \frac{1}{\gamma^2} \right) \frac{\Delta p}{p} = -\eta_c \frac{\Delta p}{p} $$

with momentum compaction $\alpha_c$ and slip factor $\eta_c = \alpha_c - 1/\gamma^2$. At the transition energy $E_T = m_0c^2 \gamma_T$ with $\gamma_T = 1/\alpha_c^2$ the revolution frequency dependence on momentum (or energy) changes sign

$$ E > E_T \rightarrow \frac{1}{\gamma^2} < \alpha_c \rightarrow \eta_c > 1 \rightarrow \omega_0 \text{ decreases with } \Delta E $$

$$ E < E_T \rightarrow \frac{1}{\gamma^2} > \alpha_c \rightarrow \eta_c < 1 \rightarrow \omega_0 \text{ increases with } \Delta E. $$

For synchrotron radiation sources the electrons are ultra-relativistic and we approximate $\Delta p/p \approx \Delta E/E = \epsilon, \eta_c \approx \alpha_c$. 
With an RF cavity of voltage $\hat{V}$ and frequency $\omega_{RF} = h\omega_0$ and an energy loss per turn $U_s$ due to an impedance, the energy gain or loss $\delta E$ is

$$\delta E = e\hat{V} \sin(h\omega_0(t_s + \tau)) - U_s, \quad \delta T_0 = \eta_c T_0 \Delta p/p$$

with $t_s =$ synchronous arrival time at the cavity, $\tau =$ deviation from it and synchronous phase $\phi_s = h\omega_0 t_s$. For $h\omega_0 \tau \ll 1$ we develop

$$\delta \epsilon = \delta \left( \frac{\Delta E}{E} \right) = \frac{e\hat{V} \sin(\phi_s)}{E} + \frac{h\omega_0 e\hat{V} \cos(\phi_s)}{E} \tau - \frac{U_s}{E}.$$ 

For $\delta \epsilon \ll 1$ we use a smooth approximation

$$\dot{\tau} = \frac{\delta T_0}{T_0} = \eta_c \epsilon, \quad \dot{\epsilon} = \frac{\delta \epsilon}{T_0} = \frac{\omega_0 e\hat{V} \sin(\phi_s)}{2\pi E} + \frac{\omega_0^2 h e\hat{V} \cos(\phi_s)}{2\pi E} \tau - \frac{\omega_0 U_s}{2\pi E}.$$

The energy loss $U_s$ can depend on deviations in energy and time

$$U_s(\epsilon, \tau) \approx U_0 + \frac{\partial U_s}{\partial \epsilon} \Delta \epsilon + \frac{\partial U_s}{\partial t} \tau.$$

giving for the derivative of the energy loss

$$\dot{\epsilon} = \frac{\omega_0 e\hat{V} \sin(\phi_s)}{2\pi E} + \frac{\omega_0^2 h e\hat{V} \cos(\phi_s)}{2\pi E} \tau - \frac{\omega_0 U_0}{2\pi E} - \frac{\omega_0 dU_s}{2\pi E} \epsilon - \frac{\omega_0}{2\pi E} \frac{1}{d\tau} dU_s.$$

For synchronous particle $\epsilon = 0, \tau = 0$ we have $U_0 = e\hat{V} \sin(\phi_s)$

$$\dot{\epsilon} = \frac{\omega_0^2 h e\hat{V} \cos(\phi_s)}{2\pi E} \tau - \frac{\omega_0 dU_s}{2\pi E} \epsilon - \frac{\omega_0}{2\pi E} \frac{1}{d\tau} dU_s,$$

$$\dot{\tau} = \eta_c \epsilon.$$
Combining the two first order equations into a second order one

\[
\ddot{\tau} + \frac{\omega_0}{2\pi} \frac{dU_s}{dE} \dot{\tau} + \left( -\frac{\omega_0^2 h\eta c e\hat{V} \cos \phi_s}{2\pi E} - \frac{\eta c \omega_0 dU_s}{2\pi E \ dt} \right) \tau
\]

\[= \ddot{\tau} + 2\alpha_s \dot{\tau} + \omega_{s0}^2 \left( 1 - \frac{2\delta\omega_s}{\omega_{s0}} \right) \tau = 0\]

with \(\omega_{s0}^2 = -\frac{2h\eta c e\hat{V} \cos \phi_s}{2\pi E}\), \(\alpha_s = \frac{1}{2} \frac{\omega_0 dU_s}{2\pi dE}\), \(\delta\omega_s = \frac{1}{2\omega_{s0}} \frac{\eta c \omega_0 dU_s}{2\pi E \ dt}\)

The solution is a damped oscillation

\[
\tau(t) = \hat{\tau} e^{-\alpha_s t} \cos(\omega_s t - \phi), \ \epsilon(t) = -\hat{\epsilon} \sin(\omega_{s0} t - \phi), \ \hat{\epsilon} = \frac{\hat{\tau}\omega_{s0}}{\eta c}
\]

with \(\omega_s = \omega_{s0} \sqrt{1 - 2\delta\omega/\omega_{s0} - \alpha_s^2/\omega_{s0}^2} \approx \omega_{s0} = Q_{s0} \omega_0\).

We have stability if \(\omega_{s0}^2 > 0\)

\(E > E_T, \ \eta_c < 0 \rightarrow \cos \phi_s < 0\), \(E < E_T, \ \eta_c > 0 \rightarrow \cos \phi_s > 0\).

and if the loss \(U_s\) increases with energy deviation

\[\alpha_s = \frac{1}{2} \frac{\omega_0 dU_s}{2\pi dE} > 0.\]
Bunch in time and frequency domain
We take a circulating bunch of charge $q$, current $I(t)$, symmetry $I(t) = I(-t)$, we use a Gaussian of RMS width $\sigma$ as example.

Circulating stationary bunch
A bunch circulating with revolutions $k$ is periodic of period $T_0$. Its current, expressed as Fourier series, represents a line spectrum

$$I_k(t) = \sum_{k=-\infty}^{\infty} I(t - kT_0) = \sum_{p=-\infty}^{\infty} I_p e^{ip\omega_0 t} = I_0 + 2 \sum_{p=1}^{\infty} I_p \cos(p\omega_0 t)$$

$$I_p = \frac{1}{T_0} \int_0^{T_0} I(t) \cos(p\omega_0 t) dt = \frac{\omega_0}{\sqrt{2\pi}} \tilde{I}(p\omega_0) = \frac{q}{T_0} e^{-\frac{p\omega_0}{2\sigma\omega}}.$$

![Time domain](image1)

![Frequency domain](image2)

Circulating oscillating bunch

$$I_k(t) = \sum_{-\infty}^{\infty} I(t - kT_0 - \tau_k), \quad \tau_k = \hat{\tau} \cos(2\pi Q_s k) \approx \tau_k = \hat{\tau} \cos(\omega_s t)$$

$$= 2 \sum_{\omega > 0} \left[ I_p \cos(p\omega_0 t) + \frac{p\omega_0 \hat{\tau}}{2} \left( I_{p+} \sin(\omega_{p+} t) + I_{p-} \sin(\omega_{p-} t) \right) \right]$$

$$\omega_{p\pm} = \omega_0(p \pm Q_s), \quad I_{p\pm} = \frac{\omega_0}{\sqrt{2\pi}} \tilde{I}(\omega_{p\pm}) \approx \frac{\omega_0}{\sqrt{2\pi}} \tilde{I}(p\omega_0) = I_p.$$

![Time domain](image3)

![Frequency domain](image4)

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Qualitative treatment of the Robinson instability

The most important longitudinal instability is an interaction between a bunch and a narrow band cavity with memory, called Robinson instability. As a qualitative treatment we consider a single circulating bunch interacting with a cavity of resonance frequency $\omega_r$ and impedance $Z(\omega)$ of which we consider only its resistive part $Z_r$. Its band-width is sufficiently narrow that only a single revolution frequency harmonics $p\omega_0$ interacts with it.

\[ Z_R(\omega) \]

\[ Z_R(\omega) \]

\[ \omega_r \]

\[ p\omega_0 \]

\[ \omega \]

\[ \omega_0 \]

\[ \frac{\Delta\omega_0}{\omega_0} = -\eta_c \Delta E \frac{E}{E} = -\eta_c \hat{\epsilon} \sin(\omega_s t) \].

The revolution frequency $\omega_0$ depends on energy deviation $\Delta E$

While the bunch is executing a coherent dipole mode oscillation $\epsilon(t) = \hat{\epsilon} \cos(\omega_s t)$ its energy and revolution frequency are modulated. **Above transition** $\omega_0$ is small when the energy is high and $\omega_0$ is large when the energy is small. If the cavity is tuned to a resonant frequency slightly smaller than the RF frequency $\omega_r < p\omega_0$ the bunch sees a higher impedance and loses more energy when it has an energy excess and it loses less energy when it has a lack of energy. This leads to a damping of the oscillation. If $\omega_r > p\omega_0$ this is reversed and leads to an instability. Below transition energy the dependence of the revolution frequency is reversed which changes the stability criterion.
Effect of the impedance at the synchrotron side-bands

Instead of the $\omega_0$-variation we consider the spectrum of an oscillating bunch with carriers $p\omega_0$ and side-bands $\pm \omega_s$. The current of the oscillating bunch $I_k(t)$ is split into a stationary one $I(t)$ and a periodic perturbation $I_1(t)$. The voltage induced by this perturbation in an impedance $Z_r(\omega_r)$ for $\omega_r = (p \pm Q_s)\omega_0$ makes an energy change in the next turn as shown for the case $p = 2$ and $Q_s = 0.25$. Its effect is seen in a time-energy $(\tau, \epsilon)$ phase space diagram of the synchrotron oscillation. For $\gamma > \gamma_T$ the voltage induced by the upper sideband enhances the oscillation, the one from the lower sideband reduces it. Below transition the situation is reversed.

Oscillating bunch with $Q_s = 0.25$

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<table>
<thead>
<tr>
<th>Turn k</th>
<th>Turn k+1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_k(t)$</td>
<td>$I_k(t)$</td>
</tr>
<tr>
<td>$I(t)$</td>
<td>$I(t)$</td>
</tr>
<tr>
<td>$E_z$</td>
<td>$E_z$</td>
</tr>
</tbody>
</table>
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Stationary bunch + Perturbation

Cavity field induced by the two sidebands

$$\omega_r = (2 + Q_s)\omega_0$$

$$\omega_r = (2 - Q_s)\omega_0$$

Phase motion of the bunch center

$$\gamma > \gamma_T$$

$$\gamma < \gamma_T$$

Damping rate is given by the side-band impedance difference

$$\alpha_s = \frac{\omega_0 p I_p^2 (Z^+ - Z^-)}{2 I_0 \hbar V \cos \phi_s}.$$
4) Potential well bunch lengthening

A ring impedance has many resonances with frequency $\omega_r$, shunt impedance $R_s$ and quality factor $Q$. At low frequencies, $\omega < \omega_r$, their impedance can be approximated by an inductance $L$

$$Z(\omega) = R_s \frac{1 - jQ\frac{\omega^2 - \omega_r^2}{\omega_r^2}}{1 + \left(Q\frac{\omega^2 - \omega_r^2}{\omega_r^2}\right)^2} \approx j\frac{R_s\omega}{Q\omega_r} + \ldots = jL\omega + \ldots$$

The sum impedance at low frequencies of all these resonances divided by the mode number $n = \omega/\omega_0$ is called

$$\frac{|Z|}{n}_0 = \sum_k \frac{R_s k \omega_0}{Q_k \omega_r} = L\omega_0.$$  

We ad the voltage $V_i = -LdI_b/dt$, induced by a bunch current $I_b(t)$ of parabolic shape, to the RF-voltage using $h\omega_0 t = \phi_s + h\omega_0 \tau$

$$V(t) = \hat{V}_{RF} \sin(h\omega_0 t) - L \frac{dI_b}{dt} \approx \hat{V}_{RF}(\sin \phi_s + h\omega_0 \cos \phi_s \tau) - L \frac{dI_b}{dt}.$$  

$$I_b(\tau) = \hat{I} \left(1 - \frac{\tau^2}{\tau_0^2}\right) = \frac{3\pi I_0}{2\omega_0 \tau_0} \left(1 - \frac{\tau^2}{\tau_0^2}\right), \quad \frac{dI_b}{d\tau} = -\frac{3\pi I_0 \tau}{\omega_0 \tau_0^3}, \quad I_0 = \langle I_b \rangle.$$  

The voltage $V_i$ seen inside the bunch gives an incoherent $\omega_s$

$$V_t = \hat{V}_{RF} \sin \phi_s + \hat{V} \cos \phi_s h\omega_0 \left(1 + \frac{3\pi |Z/n|_0 I_0}{h\hat{V} \cos \phi_s (\omega_0 \tau_0)^3}\right) \tau.$$  

$$\omega_s^2 = \omega_s^2 \left(1 + \frac{3\pi |Z/n|_0 I_0}{2h\hat{V}_{RF} \cos \phi_s (\omega_0 \tau_0)^3}\right), \quad \frac{\Delta \omega_s}{\omega_s} \approx \frac{3\pi |Z/n|_0 I_0}{2h\hat{V} \cos \phi_s (\omega_0 \tau_0)^3}$$  

$\Delta \omega_s = \omega_s - \omega_s^0$ is the shift from the unperturbed frequency $\omega_s^0$. It is negative above transition and gives bunch lengthening.