Introduction to Beam Instabilities

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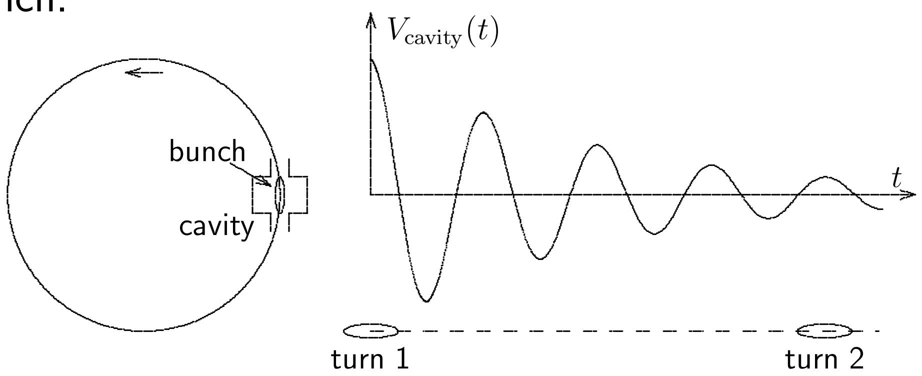
1) Instability mechanisms
2) Impedances and wake functions
3) Robinson instability
4) Potential well bunch lengthening

1) Instability mechanisms

Overview

The motion of a single particle in a storage ring is determined by the external guide fields (dipole and quadrupole magnets, RF-system, etc.), initial conditions and synchrotron radiation. Many particles in a beam may represent a sizable charge and current which act as a source of electromagnetic fields (self fields). They are modified by boundary conditions imposed by the beam surroundings (vacuum chambers, cavities, etc.) and act back on the beam. This can lead to a **frequency shift** (change of the betatron or synchrotron frequency), to an increase of a small disturbance of the beam, i.e. an **instability** or to a **change of the particle distribution**, e.g. bunch lengthening. These phenomena are called **collective effects** being due to a **coherent** or **collective** action of many particles. The role played in this process by the electrical properties of the beam surroundings is expressed by an **impedance**.

As an example we take a bunch in a storage ring going through a cavity where it induces electromagnetic fields which oscillate and decay away slowly. In the next turn the same bunch finds some field left and gets influenced by it. Depending on the phase of the field seen in the next turn, a small initial perturbation is increased or decreased leading to an exponentially growing or decaying oscillation of the bunch.



Multi-turn effects

In the example the induced fields have a memory and the instability is built up over many turns. These self fields are often small compared to the guide fields and their effects is treated as a **perturbation** in 3 steps.

a) We determine the stationary particle distribution given by the guide field, initial condition and synchrotron radiation.

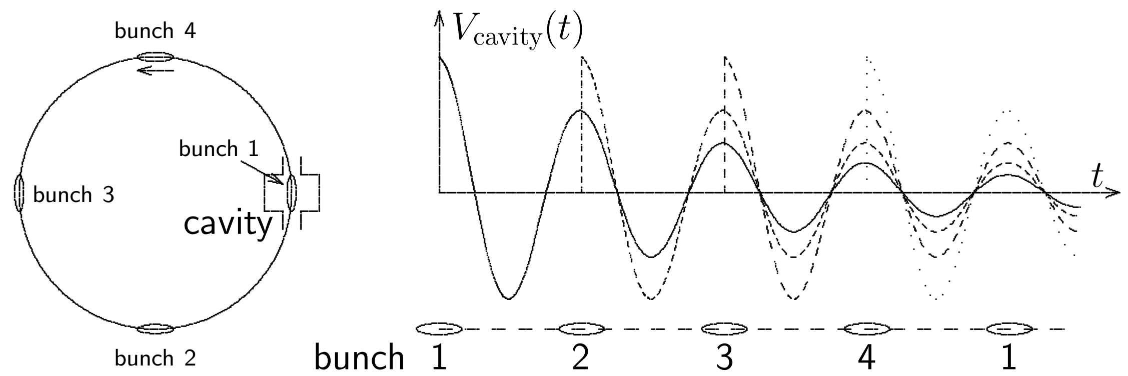
b) We consider small disturbances and calculate the fields they create including the boundary conditions (impedance).

c) We calculate the effect of these fields on the beam to see if the initial disturbance is increased **(instability)** or decreased **(damping)** or the oscillation mode changed **(frequency shift)**.

As disturbances we consider orthogonal (independent) oscillation modes and investigate the stability of each. This works for weak interactions which don't alter the nature of the modes but determine only their exponential growth over many turns. Multi-turn effects are driven by narrow frequency band impedances with memory.

Multi-bunch effects

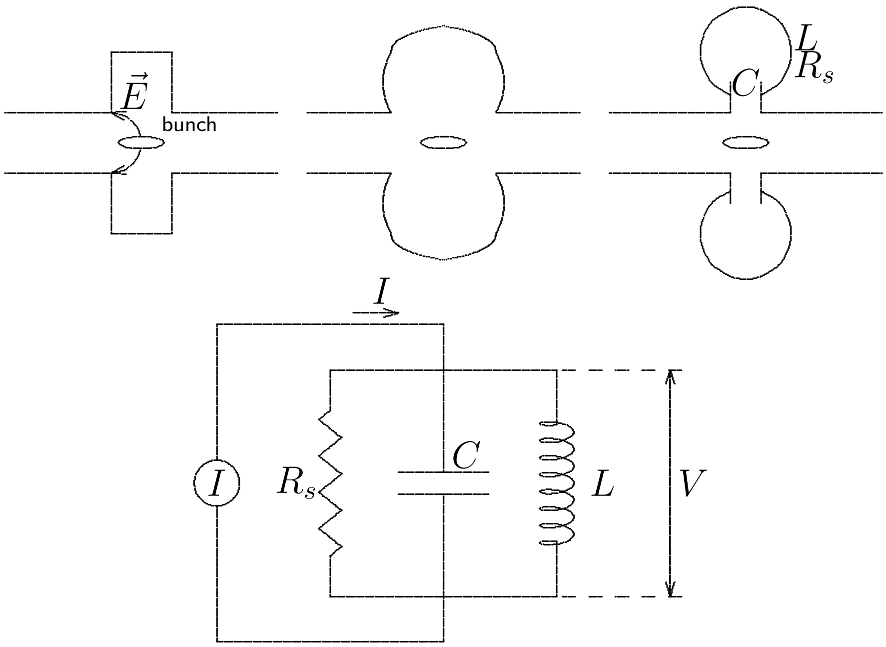
With many circulating bunches their individual oscillations can be coupled by an impedance with a shorter memory bridging just the bunch spacing instead of the revolution time. Multi-turn and multi-bunch instabilities have the same qualitative properties and are called multi-traversal effects.



Single traversal effects
Strong self-fields from broad band impedances change the stationary distribution and modify oscillation modes which are no longer independent. A self consistent solutions is difficult to obtain. The most common such effect is **bunch lengthening**.
Longitudinal and transverse effects
Longitudinal effects involve synchrotron (energy, phase) oscillations and longitudinal impedances. They contain longitudinal instabilities, shift of synchrotron frequencies and bunch lengthening.
Transverse effects involve betatron oscillations and transverse impedances. They contain transverse instabilities and betatron frequency shifts.
In both cases the longitudinal particle distribution (bunch length) is important since it can be "resolved" by the impedance while the transverse distribution is usually not resolved and does not affect the instability.
The most important longitudinal single traversal effects are synchrotron frequency shifts and bunch lengthening. In the transverse case the effect of the chromaticity is important which can lead to head-tail instabilities.

2) Impedances and wake functions

Resonator

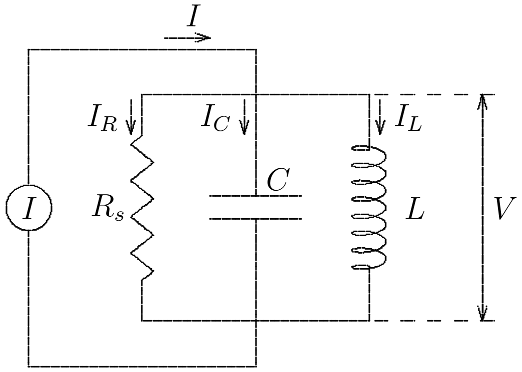


Cavities have narrow band oscillation modes which can drive coupled bunch instabilities. Each resembles an **RCL - circuit** and can, in good approximation, be treated as such. This circuit has a shunt impedance R_s, an inductance L and a capacity C. In a real cavity these parameters cannot easily be separated and we use others which can be measured directly: The **resonance frequency** ω_r, the **quality factor** Q and the **damping rate** α:

$$\omega_r = \frac{1}{\sqrt{LC}} \ , \quad Q = R_s\sqrt{\frac{C}{L}} = \frac{R_s}{L\omega_r} = R_s C \omega_r \ , \quad \alpha = \frac{\omega_r}{2Q}$$

$$L = \frac{R_s}{Q\omega_r} \ , \ C = \frac{Q}{\omega_r R_s}.$$

Driving this circuit with a current I gives the voltages and currents across the elements



$$
\begin{aligned}
V_R &= I_R R_s \\
V_C &= \frac{1}{C}\int I_C dt \\
V_L &= L\frac{dI_L}{dt}
\end{aligned}
$$

$$V_R = V_C = V_L = V \ , \ \ I_R + I_C + I_L = I$$

Differentiating with respect to t gives

$$\dot{I} = \dot{I}_R + \dot{I}_C + \dot{I}_L = \frac{\dot{V}}{R_s} + C\ddot{V} + \frac{V}{L}.$$

Using $L = R_s/(\omega_r Q)$ and $C = Q/(\omega_r R_s)$ gives the differential equation

$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q}\dot{I}$$

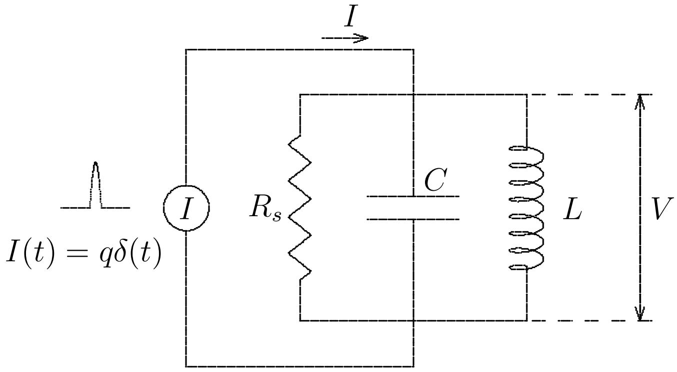
The solution of the homogeneous equation represents a damped oscillation

$$V(t) = \hat{V}\mathrm{e}^{-\alpha\mathrm{t}}\cos\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\mathrm{t}+\phi\right)$$

$$V(t) = \mathrm{e}^{-\alpha\mathrm{t}}\left(\mathrm{A}\cos\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\mathrm{t}\right)+\mathrm{B}\sin\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\mathrm{t}\right)\right)$$

Wake function – Green function

Calculate response of RCL circuit to a delta function pulse



$$\omega_r = \frac{1}{\sqrt{LC}}$$
$$Q = R_s\sqrt{\frac{C}{L}}$$
$$\alpha = \frac{\omega_r}{2Q}$$

$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q}\dot{I}$$

The charge q will charge up the capacity to a voltage

$$V(0^+) = \frac{q}{C} = \frac{\omega_r R_s}{Q} q \text{ using } C = \frac{Q}{\omega_r R_s}$$

Energy stored in capacitor equals energy lost by charge

$$U = \frac{q^2}{2C} = \frac{\omega_r R_s}{2Q} q^2 = \frac{V(0^+)}{2} q = k_{pm} q^2 \text{ with } k_{pm} = \frac{\omega_r R_s}{2Q}$$

with the **parasitic mode loss factor** k_{pm}, measured usually in [V/pC]. The capacitor C discharges first through the resistor R_s

$$\dot{V}(0^+) = -\frac{\dot{q}}{C} = -\frac{I_R}{C} = -\frac{1}{C}\frac{V(0^+)}{R_s} = -\frac{\omega_r^2 R_s}{Q^2} q = -\frac{2\omega_r k_{pm}}{Q} q.$$

With the initial conditions $V(0^+)$, $\dot{V}(0^+)$ the general solution

$$V(t) = \mathrm{e}^{-\alpha \mathrm{t}}\left(\mathrm{A}\cos\left(\omega_\mathrm{r}\sqrt{1 - \frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right) + \mathrm{B}\sin\left(\omega_\mathrm{r}\sqrt{1 - \frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right)\right)$$

gives the response of the circuit to a pulse excitation

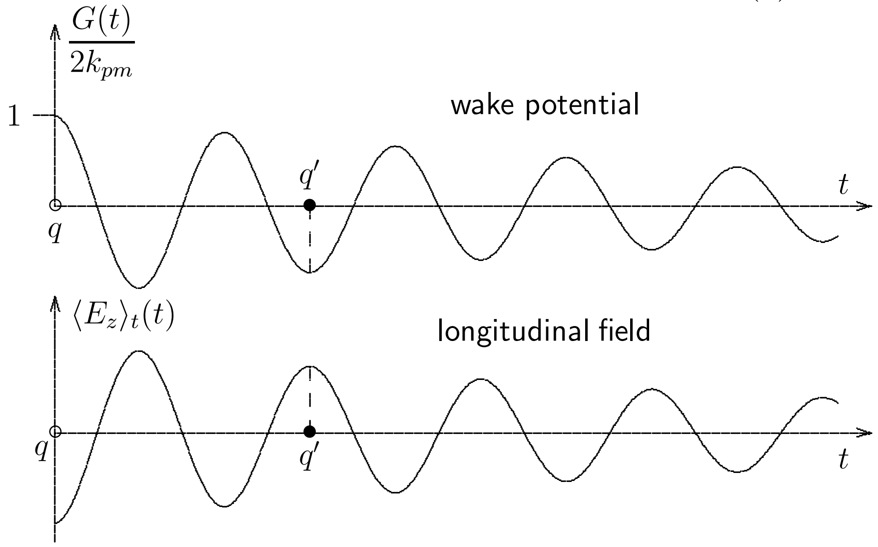
$$V(t) = 2qk_{pm}\mathrm{e}^{-\alpha \mathrm{t}}\left(\cos\left(\omega_\mathrm{r}\sqrt{1 - \frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right) - \frac{\sin\left(\omega_\mathrm{r}\sqrt{1 - \frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right)}{2\mathrm{Q}\sqrt{1 - \frac{1}{4\mathrm{Q}^2}}}\right)$$

Normalized per unit source point charges it is called **Green or wake function** $G(t)$. For our resonator:

$$G(t) = 2k_{pm}\mathrm{e}^{-\alpha \mathrm{t}}\left(\cos\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right) - \frac{\sin\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right)}{2\mathrm{Q}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}}\right)$$

$$G(t) \approx 2k_{pm}\mathrm{e}^{-\alpha \mathrm{t}}\cos(\omega_\mathrm{r}\mathrm{t}) \text{ for } \mathrm{Q} \gg 1,\ \mathrm{k_{pm}} = \frac{\omega_\mathrm{r}\mathrm{R_s}}{2\mathrm{Q}},\ \alpha = \frac{\omega_\mathrm{r}}{2\mathrm{Q}}$$

This voltage induced by charge q at $t = 0$ changes the energy of a second charge q' traversing cavity at t by $U = q'V(t) = qq'G(t)$.



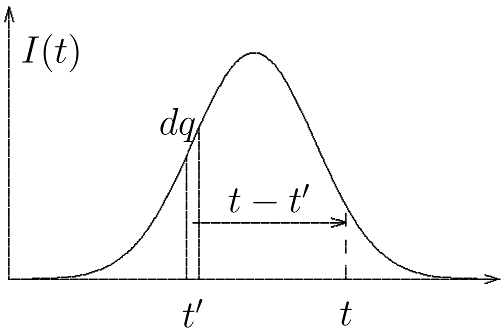
The wake potential is related to the longitudinal field E_z by a field integration which follows the particle with speed $v \approx c$ through the object length while taking the momentary field value

$$V = Gq = -\int_{z_1}^{z_2} E_z(z,t)dz = -f_t\int_{z_1}^{z_2} E_z(z)dz = -\langle E_z\rangle_t \Delta z.$$

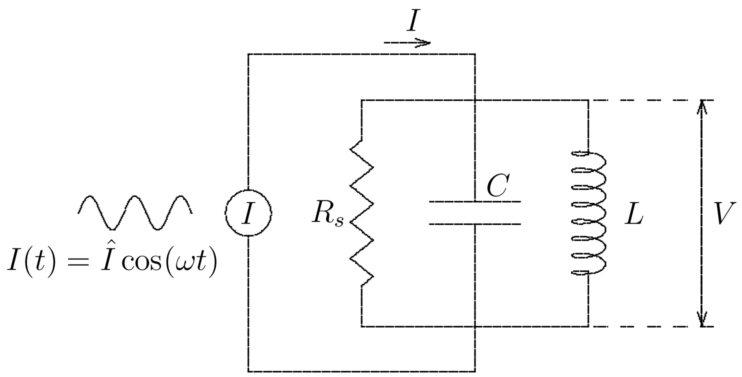
with transit time factor f_t. We use $G(t) > 0$ where energy is lost. A particle inside a bunch of charge q and current $I(t)$ going through a cavity at time t sees the wake function created by all the particles passing at earlier times $t' < t$ resulting in a voltage

$$V(t) = \int_{-\infty}^{t} G(t')dq$$
$$= \int_{-\infty}^{t} I(t')G(t')dt' = qW(t)$$

$W(t) = V(t)/q$ wake potential.



Impedance



We assume now a **harmonic** excitation of the circuit with a current $I = \hat{I}\cos(\omega t)$ and get the differential equation

$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q}\dot{I} = -\frac{\omega_r R_s}{Q}\hat{I}\omega\sin(\omega t)$$

The solution of the homogeneous equation damps down leaving a particular solution $V(t) = A\cos(\omega t) + B\sin(\omega t)$. Inserting this into the differential equation and separating cosine and sine terms gives

$$-(\omega^2 - \omega_r^2)A + \frac{\omega_r\omega}{Q}B = 0 \;\text{ and }\; (\omega^2 - \omega_r^2)B + \frac{\omega_r\omega}{Q}A = \frac{\omega_r\omega R_s}{Q}\hat{I}$$

The voltage induced by the harmonic excitation of the resonator is

$$V(t) = \hat{I}R_s\frac{\cos(\omega t) + Q\frac{\omega^2-\omega_r^2}{\omega_r\omega}\sin(\omega t)}{1 + Q^2\left(\frac{\omega^2-\omega_r^2}{\omega_r\omega}\right)^2}$$

The voltage has a cosine term **in phase** with the exciting current. It absorbs energy and is **resistive**. The sine term is **out of phase**, does not absorb energy and is **reactive**. The ratio between voltage and current is the **impedance**. It is a **function of frequency** ω

$$Z_r(\omega) = R_s\frac{1}{1 + Q^2\left(\frac{\omega_r^2-\omega^2}{\omega_r\omega}\right)^2}\;,\;\; Z_i(\omega) = -R_s\frac{Q\frac{\omega^2-\omega_r^2}{\omega_r\omega}}{1 + Q^2\left(\frac{\omega^2-\omega_r^2}{\omega_r\omega}\right)^2}$$

Its resistive part $Z_r(\omega)$ is always positive while its reactive part $Z_i(\omega)$ positive below and negative above the resonant frequency ω_r.

Complex notation

We have used a harmonic excitation of the form

$$I(t) = \hat{I}\cos(\omega t) = \hat{I}\frac{\mathrm{e}^{j\omega t} + \mathrm{e}^{-j\omega t}}{2} \;\; \text{with} \;\; 0 \leq \omega \leq \infty$$

It is often more convenient to use a complex notation either

$$I(t) \;=\; \text{Re}\;[\hat{I}\mathrm{e}^{j\omega t}] = \text{Re}\;[\hat{I}(\cos(\omega t) + j\sin(\omega t))] = \hat{I}(\cos\omega t)$$

or

$$I(t) \;=\; \hat{I}\mathrm{e}^{j\omega t} \;\; \text{with} \;\; -\infty \leq \omega \leq \infty$$

giving more compact expressions. Using the differential equation

$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q}\dot{I}$$

with $I(t) = \hat{I}\exp(j\omega t)$ and seeking a solution of the form $V(t) = V_0\exp(j\omega t)$, where V_0 is in general complex, one gets

$$\left(-\omega^2\mathrm{e}^{j\omega t} + j\frac{\omega_r\omega}{Q}\mathrm{e}^{j\omega t} + \omega_r^2\mathrm{e}^{j\omega t}\right)V_0 = j\frac{\omega_r\omega R_s}{Q}\hat{I}\mathrm{e}^{j\omega t}$$

and for the impedance which is defined as the ration V/I

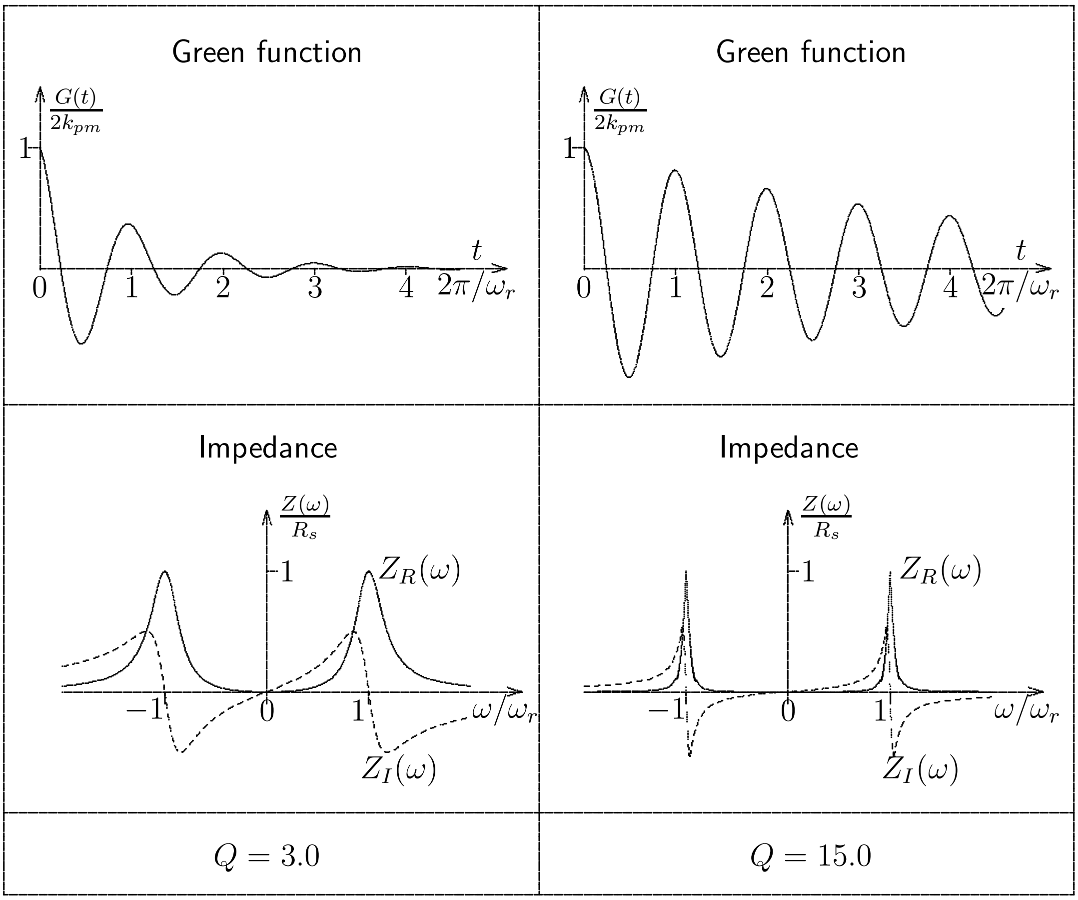
$$Z(\omega) = \frac{V_0}{\hat{I}} = \frac{R_s}{1 + jQ\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)} = R_s\frac{1 - jQ\frac{\omega^2-\omega_r^2}{\omega\omega_r}}{1 + Q^2\left(\frac{\omega^2-\omega_r^2}{\omega\omega_r}\right)^2} = Z_r + jZ_i$$

For $Q \gg 1$ the impedance is only large for $\omega \approx \omega_r$ or for $|\omega - \omega_r|/\omega_r = |\Delta\omega|/\omega_r \ll 1$ and can be simplified

$$Z(\omega) \approx R_s\frac{1 - j2Q\frac{\Delta\omega}{\omega_r}}{1 + 4Q^2\left(\frac{\Delta\omega}{\omega_r}\right)^2}$$

Caution: sometimes $I(t) = \hat{I}\mathrm{e}^{-i\omega t}$ instead of $I(t) = \hat{I}\mathrm{e}^{j\omega t}$ is used, this reverses the sign $Z_i(\omega)$.

Green function and impedance of a resonator



The resonator impedance has some specific properties:

$$\begin{aligned} \text{at } \omega = \omega_r \;&\rightarrow\; Z_r(\omega_r) \text{ has a maximum },\; Z_i(\omega_r) = 0 \\ 0 < \omega < \omega_r \;&\rightarrow\; Z_i(\omega) > 0 \;\; \text{(inductive)} \\ 0 < \omega_r < \omega \;&\rightarrow\; Z_i(\omega) < 0 \;\; \text{(capacitive)} \end{aligned}$$

and any impedance or wake potential has the general properties

$$\begin{aligned} Z_r(\omega) &= Z_r(-\omega) \;\;,\;\; Z_i(\omega) \;=\; -Z_i(-\omega) \\ Z(\omega) &= \int_{-\infty}^{\infty} G(t) \mathrm{e}^{-j\omega t} dt \;\propto\; \text{Fourier transform} \\ G(t) &= 0 \;\; \text{for } t < 0, \;\; \text{no fields before particle arrival.} \end{aligned}$$

3) Robinson instability

Longitudinal dynamics

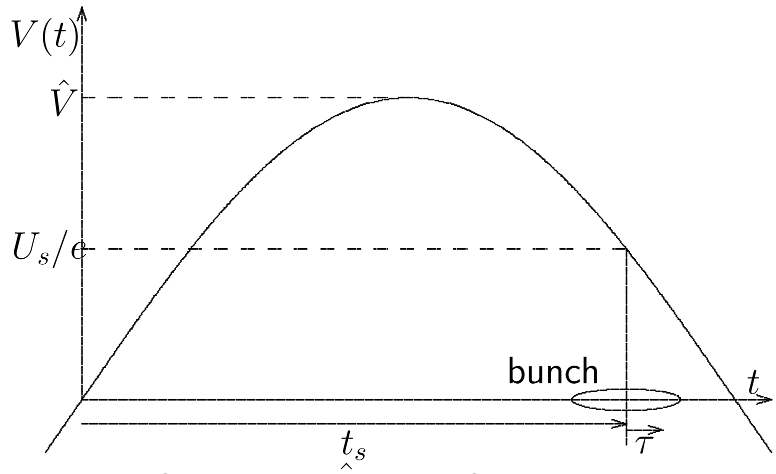
A particle with momentum deviation Δp has a different orbit length L, revolution time T_0 and revolution frequency ω_0

$$\frac{\Delta L}{L} = \alpha_c \frac{\Delta p}{p}\ , \quad \frac{\Delta \omega_0}{\omega_0} = -\frac{\Delta T_0}{T_0} = -\left(\alpha_c - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p} = -\eta_c \frac{\Delta p}{p}$$

with momentum compaction α_c and slip factor $\eta_c = \alpha_c - 1/\gamma^2$. At the transition energy $E_T = m_0 c^2 \gamma_T$ with $\gamma_T = 1/\alpha_c^2$ the revolution frequency dependence on momentum (or energy) changes sign

$$\begin{aligned} E \ &> \ E_T \ \rightarrow \ \frac{1}{\gamma^2} < \alpha_c \ \rightarrow \ \eta_c > 1 \ \rightarrow \ \omega_0 \, \mathrm{decreases\, with}\, \Delta E \\ E \ &< \ E_T \ \rightarrow \ \frac{1}{\gamma^2} > \alpha_c \ \rightarrow \ \eta_c < 1 \ \rightarrow \ \omega_0 \, \mathrm{increases\, with}\, \Delta E\,. \end{aligned}$$

For synchrotron radiation sources the electrons are ultra-relativistic and we approximate $\Delta p/p \approx \Delta E/E = \epsilon$, $\eta_c \approx \alpha_c$.



With an RF cavity of voltage $\hat{V}$ and frequency $\omega_{\mathsf{RF}} = h\omega_0$ and an energy loss per turn U_s due to an impedance, the energy gain or loss δE is

$$\delta E = e\hat{V}\sin(h\omega_0(t_s+\tau)) - U_s\ , \quad \delta T_0 = \eta_c T_0 \Delta p/p$$

with t_s= synchronous arrival time at the cavity, τ= deviation from it and synchronous phase $\phi_s = h\omega_0 t_s$. For $h\omega_0\tau \ll 1$ we develop

$$\delta\epsilon = \delta\left(\frac{\Delta E}{E}\right) = \frac{e\hat{V}\sin(\phi_s)}{E} + \frac{h\omega_0 e\hat{V}\cos\phi_s}{E}\tau - \frac{U_s}{E}.$$

For $\delta\epsilon \ll 1$ we use a smooth approximation

$$\dot{\tau} = \frac{\delta T_0}{T_0} = \eta_c\epsilon\ , \quad \dot{\epsilon} = \frac{\delta\epsilon}{T_0} = \frac{\omega_0 e\hat{V}\sin\phi_s}{2\pi E} + \frac{\omega_0^2 he\hat{V}\cos\phi_s}{2\pi E}\tau - \frac{\omega_0}{2\pi}\frac{U_s}{E}.$$

The energy loss U_s can depend on deviations in energy and time

$$U_s(\epsilon,\tau) \approx U_0 + \frac{\partial U_s}{\partial E}\Delta E + \frac{\partial U_s}{\partial t}\tau.$$

giving for the derivative of the energy loss

$$\dot{\epsilon} = \frac{\omega_0 e\hat{V}\sin\phi_s}{2\pi E} + \frac{\omega_0^2 he\hat{V}\cos\phi_s}{2\pi E}\tau - \frac{\omega_0}{2\pi}\frac{U_0}{E} - \frac{\omega_0}{2\pi}\frac{dU_s}{dE}\epsilon - \frac{\omega_0}{2\pi}\frac{1}{E}\frac{dU_s}{dt}\tau.$$

For synchronous particle $\epsilon = 0\,,\ \tau = 0$ we have $U_0 = e\hat{V}\sin\phi_s$

$$\begin{aligned}\dot{\epsilon} &= \omega_0^2\frac{he\hat{V}\cos\phi_s}{2\pi E}\tau - \frac{\omega_0}{2\pi}\frac{dU_s}{dE}\epsilon - \frac{1}{E}\frac{\omega_0}{2\pi}\frac{dU_s}{dt}\tau \\ \dot{\tau} &= \eta_c\epsilon\,.\end{aligned}$$

Combining the two first order equations into a second order one

$$\begin{aligned}
&\ddot{\tau} + \frac{\omega_0}{2\pi}\frac{dU_s}{dE}\dot{\tau} + \left(-\frac{\omega_0^2 h\eta_c e\hat{V}\cos\phi_s}{2\pi E} - \frac{\eta_c\omega_0}{2\pi E}\frac{dU_s}{dt}\right)\tau \\
= \ &\ddot{\tau} + 2\alpha_s\dot{\tau} + \omega_{s0}^2\left(1 - 2\frac{\delta\omega_s}{\omega_{s0}}\right)\tau = 0
\end{aligned}$$

with $\omega_{s0}^2 = -\omega_0^2\frac{h\eta_c e\hat{V}\cos\phi_s}{2\pi E}$, $\alpha_s = \frac{1}{2}\frac{\omega_0}{2\pi}\frac{dU_s}{dE}$, $\delta\omega_s = \frac{1}{2\omega_{s0}}\frac{\eta_c\omega_0}{2\pi E}\frac{dU_s}{dt}$

The solution is a damped oscillation

$$\tau(t) = \hat{\tau}\mathrm{e}^{-\alpha_s t}\cos(\omega_s t - \phi)\ ,\ \epsilon(t) = -\hat{\epsilon}\sin(\omega_{s0}t - \phi)\ ,\ \ \hat{\epsilon} = \hat{\tau}\frac{\omega_{s0}}{\eta_c}$$

with $\omega_s = \omega_{s0}\sqrt{1 - 2\delta\omega/\omega_{s0} - \alpha_s^2/\omega_{s0}^2} \approx \omega_{s0} = Q_{s0}\omega_0$.

We have stability if $\omega_{s0}^2 > 0$

$$E > E_T,\ \eta_c < 0\ \rightarrow \cos\phi_s < 0\ \ ,\ \ E < E_T,\ \eta_c > 0\ \rightarrow \cos\phi_s > 0.$$

and if the loss U_s increases with energy deviation

$$\alpha_s = \frac{1}{2}\frac{\omega_0}{2\pi}\frac{dU_s}{dE} > 0.$$

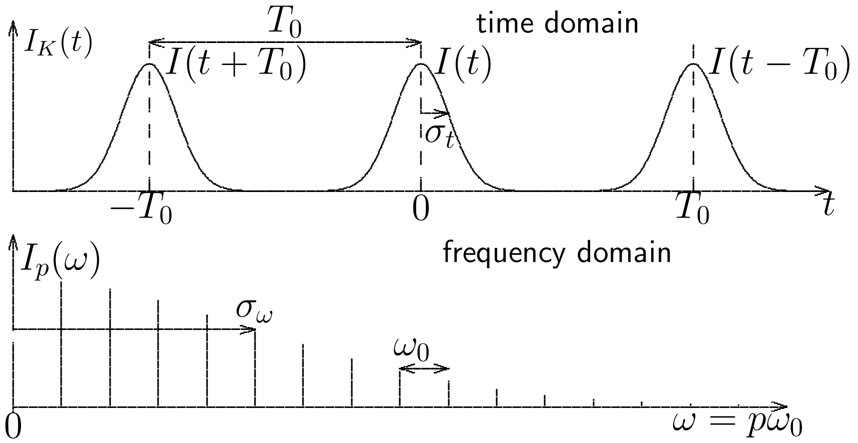
Bunch in time and frequency domain

We take a circulating bunch of charge q, current $I(t)$, symmetry $I(t) = I(-t)$, we use a Gaussian of RMS width σ as example.

Circulating stationary bunch

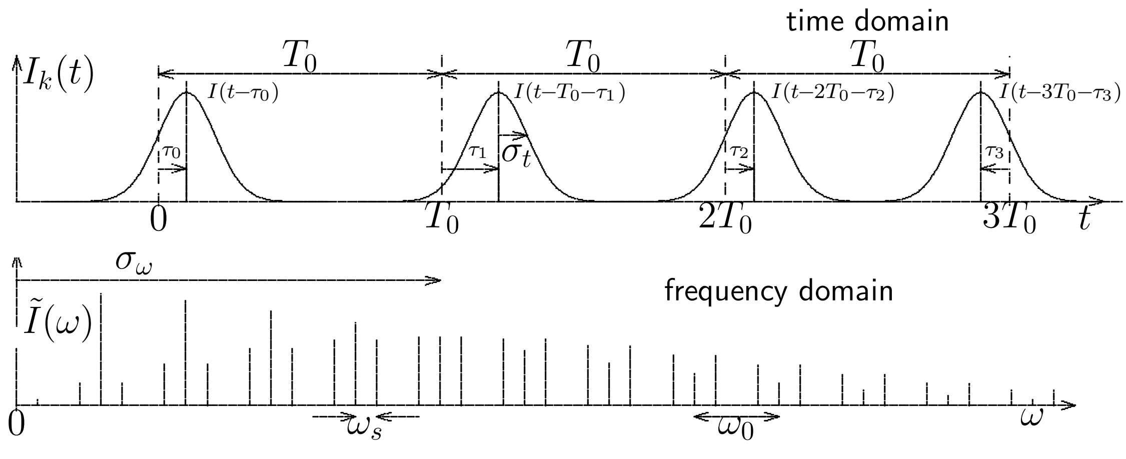
A bunch circulating with revolutions k is periodic of period T_0. Its current, expressed as Fourier series, represents a line spectrum

$$
\begin{aligned}
I_k(t) &= \sum_{k=-\infty}^{k=\infty} I(t - kT_0) = \sum_{p=-\infty}^{\infty} I_p \mathrm{e}^{jp\omega_0 t} = I_0 + 2\sum_{p=1}^{\infty} I_p \cos(p\omega_0 t) \\
I_p &= \frac{1}{T_0}\int_0^{T_0} I(t)\cos(p\omega_0 t)dt = \frac{\omega_0}{\sqrt{2\pi}}\tilde{I}(p\omega_0) = \frac{q}{T_0}\,\mathrm{e}^{-\frac{p\omega_0}{2\sigma_\omega}}.
\end{aligned}
$$



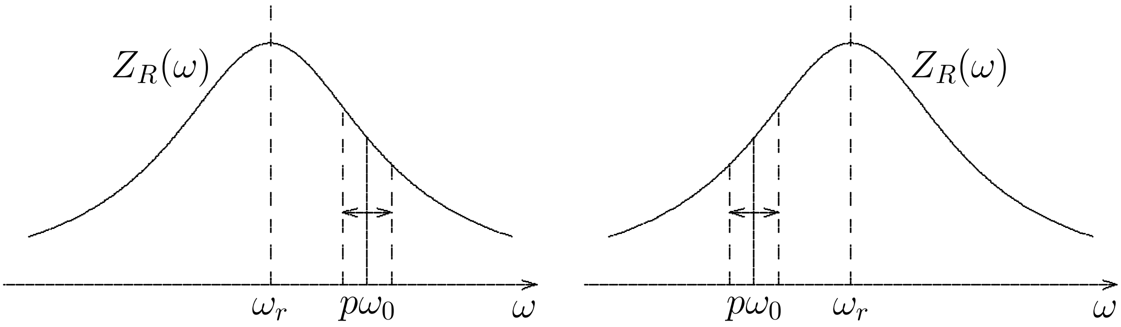
Circulating oscillating bunch

$$
\begin{aligned}
I_k(t) &= \sum_{-\infty}^{\infty} I(t - kT_0 - \tau_k)\ ,\ \tau_k = \hat{\tau}\cos(2\pi Q_s k) \approx \tau_k = \hat{\tau}\cos(\omega_s t) \\
&= 2\sum_{\omega>0}\left[I_p\cos(p\omega_0 t) + \frac{p\omega_0\hat{\tau}}{2}\left(I_{p+}\sin(\omega_{p+}t) + I_{p-}\sin(\omega_{p-}t)\right)\right] \\
\omega_{p\pm} &= \omega_0(p \pm Q_s),\quad I_{p\pm} = \frac{\omega_0}{\sqrt{2\pi}}\tilde{I}(\omega_{p\pm}) \approx \frac{\omega_0}{\sqrt{2\pi}}\tilde{I}(p\omega_0) = I_p.
\end{aligned}
$$



Qualitative treatment of the Robinson instability

The most important longitudinal instability is an interaction between a bunch and a narrow band cavity with memory, called **Robinson** instability. As a qualitative treatment we consider a single circulating bunch interacting with a cavity of resonance frequency ω_r and impedance $Z(\omega)$ of which we consider only its resistive part Z_r. Its band-width is sufficiently narrow that only a single revolution frequency harmonics $p\omega_0$ interacts with it.



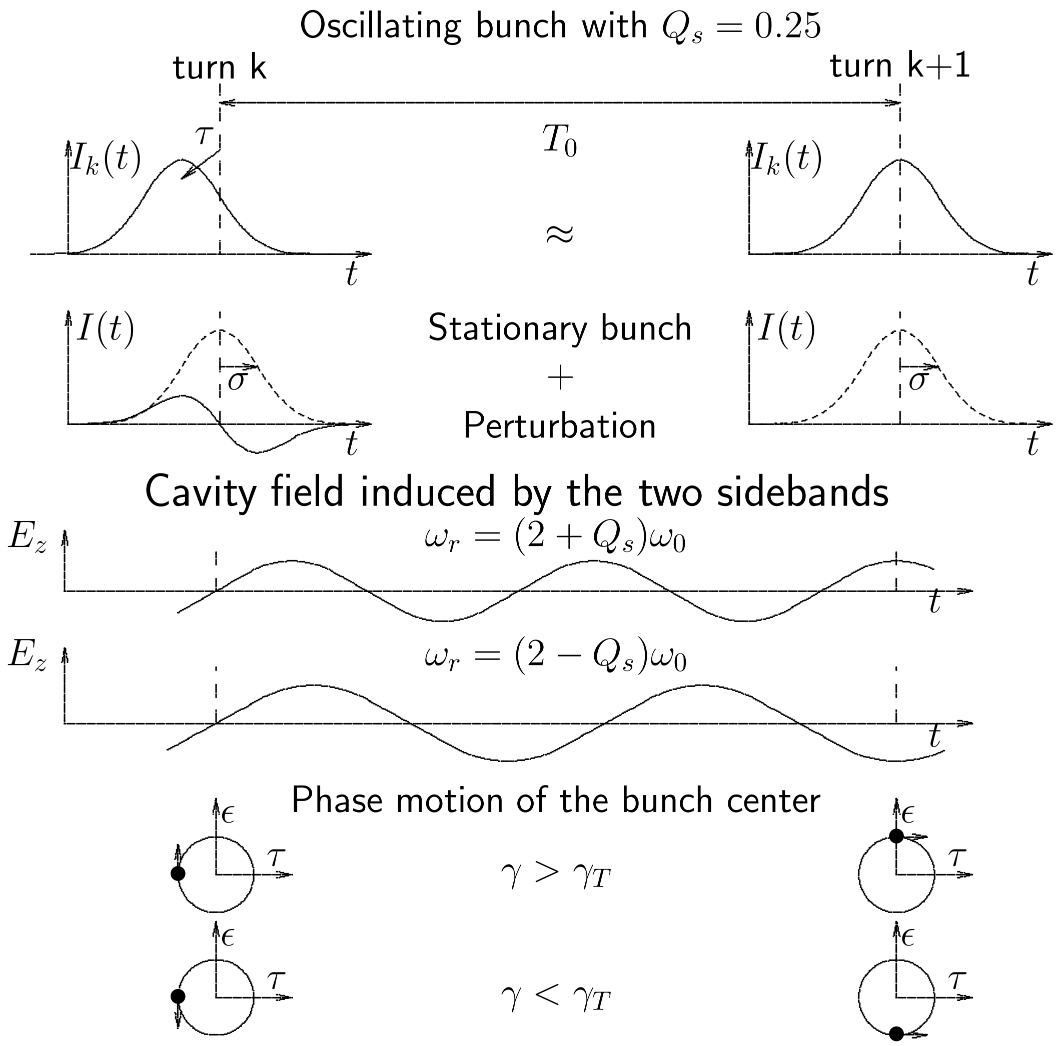
The revolution frequency ω_0 depends on energy deviation ΔE

$$\frac{\Delta\omega_0}{\omega_0} = -\eta_c \frac{\Delta E}{E} = -\eta_c \hat{\epsilon} \sin(\omega_{s0} t).$$

While the bunch is executing a coherent dipole mode oscillation $\epsilon(t) = \hat{\epsilon}\cos(\omega_s t)$ its energy and revolution frequency are modulated. **Above transition** ω_0 is **small** when the **energy is high** and ω_0 is **large** when the **energy is small**. If the cavity is tuned to a resonant frequency slightly smaller than the RF frequency $\omega_r < p\omega_0$ the bunch sees a higher impedance and **loses more energy** when it has an **energy excess** and it **loses less energy** when it has a **lack of energy**. This leads to a **damping** of the oscillation. If $\omega_r > p\omega_0$ this is reversed and leads to an **instability**. Below transition energy the dependence of the revolution frequency is reversed which changes the stability criterion.

Effect of the impedance at the synchrotron side-bands

Instead of the ω_0-variation we consider the spectrum of an oscillating bunch with carriers $p\omega_0$ and side-bands $\pm\omega_s$. The current of the oscillating bunch $I_k(t)$ is split into a stationary one $I(t)$ and a periodic perturbation $I_1(t)$. The voltage induced by this perturbation in an impedance $Z_r(\omega_r)$ for $\omega_r = (p \pm Q_s)\omega_0$ makes an energy change in the next turn as shown for the case $p = 2$ and $Q_s = 0.25$. Its effect is seen in a time-energy (τ, ϵ) phase space diagram of the synchrotron oscillation. For $\gamma > \gamma_T$ the voltage induced by the upper sideband enhances the oscillation, the one from the lower sideband reduces it. Below transition the situation is reversed.



Damping rate is given by the side-band impedance difference

$$\alpha_s = \frac{\omega_{s0} p I_p^2 (Z_{pr}^+ - Z_{pr}^-)}{2 I_0 h \hat{V} \cos\phi_s}.$$

4) Potential well bunch lengthening

A ring impedance has many resonances with frequency ω_r, shunt impedance R_s and quality factor Q. At low frequencies, $\omega < \omega_r$, their impedance can be approximated by an inductance L

$$Z(\omega) = R_s \frac{1 - jQ\frac{\omega^2-\omega_r^2}{\omega\omega_r}}{1 + \left(Q\frac{\omega^2-\omega_r^2}{\omega\omega_r}\right)^2} \approx j\frac{R_s\omega}{Q\omega_r} + ... = jL\omega + ...$$

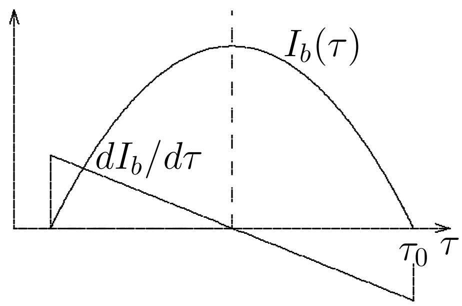
The sum impedance at low frequencies of all these resonances divided by the mode number $n = \omega/\omega_0$ is called

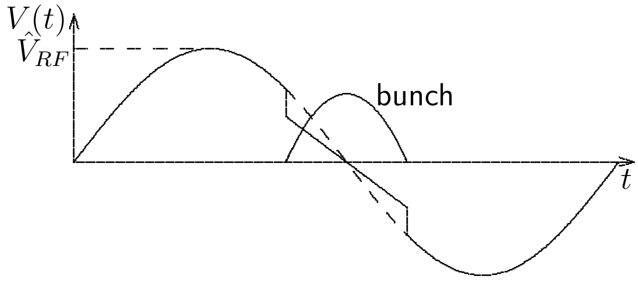
$$\left|\frac{Z}{n}\right|_0 = \sum_k \frac{R_{sk}\omega_0}{Q_k\omega_{rk}} = L\omega_0.$$

We ad the voltage $V_i = -LdI_b/dt$, induced by a bunch current $I_b(t)$ of parabolic shape, to the RF-voltage using $h\omega_0 t = \phi_s + h\omega_0\tau$

$$V(t) = \hat{V}_{RF}\sin(h\omega_0 t) - L\frac{dI_b}{dt} \approx \hat{V}_{RF}(\sin\phi_s + h\omega_0\cos\phi_s\tau) - L\frac{dI_b}{dt}.$$

$$I_b(\tau) = \hat{I}\left(1 - \frac{\tau^2}{\tau_0^2}\right) = \frac{3\pi I_0}{2\omega_0\tau_0}\left(1 - \frac{\tau^2}{\tau_0^2}\right),\ \frac{dI_b}{d\tau} = -\frac{3\pi I_0\tau}{\omega_0\tau_0^3},\ I_0 = \langle I_b\rangle.$$





The voltage V_t seen **inside** the bunch gives an incoherent ω_s

$$V_t = \hat{V}_{RF}\sin\phi_s + \hat{V}\cos\phi_s h\omega_0\left(1 + \frac{3\pi|Z/n|_0 I_0}{h\hat{V}\cos\phi_s(\omega_0\tau_0)^3}\right)\tau.$$

$$\omega_s^2 = \omega_{s0}^2\left(1 + \frac{3\pi|Z/n|_0 I_0}{2h\hat{V}_{RF}\cos\phi_s(\omega_0\tau_0)^3}\right),\ \frac{\Delta\omega_s}{\omega_s} \approx \frac{3\pi|Z/n|_0 I_0}{2h\hat{V}\cos\phi_s(\omega_0\tau_0)^3}$$

$\Delta\omega_s = \omega_s - \omega_{s0}$ is the shift from the unperturbed frequency ω_{s0}. It is negative above transition and gives bunch lengthening.