Analysis techniques
(applied to non-linear dynamics)

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Chaos detection methods

- Computing/measuring **dynamic aperture** (DA) or particle survival
  
  A. Chao et al., PRL 61, 24, 2752, 1988;  
  F. Willeke, PAC95, 24, 109, 1989.

- Computation of Lyapunov exponents
  
  F. Schmidt, F. Willeke and F. Zimmermann, PA, 35, 249, 1991;  

- Variance of unperturbed action (a la Chirikov)
  
  B. Chirikov, J. Ford and F. Vivaldi, AIP CP-57, 323, 1979  
  J. Tennyson, SSC-155, 1988;  
  J. Irwin, SSC-233, 1989

- Fokker-Planck diffusion coefficient in actions
  
  T. Sen and J.A. Elisson, PRL 77, 1051, 1996

- Frequency map analysis
Dynamic aperture
Dynamic Aperture

- The most direct way to evaluate the non-linear dynamics performance of a ring is the computation of **Dynamic Aperture**
- Particle motion due to multi-pole errors is generally non-bounded, so chaotic particles can **escape to infinity**
- This is not true for all non-linearities (e.g. the beam-beam force)
- Need a **symplectic** tracking code to follow particle trajectories (a lot of initial conditions) for a **number of turns** (depending on the given problem) until the particles start getting lost. This **boundary** defines the **Dynamic aperture**
- As multi-pole errors may not be completely known, one has to track through **several machine models** built by **random distribution** of these errors
- One could start with 4D (only transverse) tracking but certainly needs to simulate 5D (constant energy deviation) and finally 6D (synchrotron motion included)
Dynamic Aperture plots

- Dynamic aperture plots show the maximum initial values of stable trajectories in x-y coordinate space at a particular point in the lattice, for a range of energy errors.
  - The beam size can be shown on the same plot.
  - Generally, the goal is to allow some significant margin in the design - the measured dynamic aperture is often smaller than the predicted dynamic aperture.

![Dynamic Aperture Plots](image)
Including radiation damping and excitation shows that 0.7% of the particles are lost during the damping.

Certain particles seem to damp away from the beam core, on resonance islands.
Min. Dynamic Aperture (DA) with intensity vs crossing angle, for nominal optics (β* = 40 cm) and BCMS beam (2.5 μm emittance), 15 units of chromaticity

For 1.1x10^{11} p

- At θ_c/2 = 185 μrad (~12 σ separation), DA around 6 σ (good lifetime observed)
- At θ_c/2 = 140 μrad (~9 σ separation), DA below 5 σ (reduced lifetime observed)

Improvement for low octupoles, low chromaticity and WP optimisation (observed in operation)
Genetic Algorithms for lattice optimisation

- MOGA – Multi Objective Genetic Algorithms are being recently used to optimise linear but also non-linear dynamics of electron low emittance storage rings
- Use knobs quadrupole strengths, chromaticity sextupoles and correctors with some constraints
- Target ultra-low horizontal emittance, increased lifetime and high dynamic aperture
During LHC design phase, DA target was 2x higher than collimator position, due to statistical fluctuation, finite mesh, linear imperfections, short tracking time, multi-pole time dependence, ripple and a 20% safety margin.

Better knowledge of the model led to good agreement between measurements and simulations for actual LHC.

Necessity to build an accurate magnetic model (from beam based measurements)

E. Mclean, PhD thesis, 2014
DA guiding machine performance

- B1 suffering from lower lifetime in the LHC
- DA simulations predicted the required adjustment
- Fine-tune scan performed and applied in operation, solving B1 lifetime problem

D. Pellegrini et al., 2016
**Reduction of crossing angle at constant luminosity**, reduces **pileup density** (by elongating the luminous region) and **triplet irradiation**

**YP, N. Karastathis and D. Pellegrini et al., 2018**

Relaxed DA

Aggressive DA

![Graph showing the relationship between crossing angle and luminosity](image-url)
Lyapunov exponent
Lyapunov exponent

- Chaotic motion implies sensitivity to initial condition
- Two infinitesimally close chaotic trajectories in phase space with initial difference $\delta Z_0$ will end-up diverging with rate
  \[ |\delta Z(t)| \approx e^{\lambda t} |\delta Z_0| \]
  with the maximum Lyapunov exponent
- There is as many exponents as the phase space dimensions (Lyapunov spectrum)
- The largest one is the Maximal Lyapunov exponent (MLE) is defined as
  \[
  \lambda = \lim_{t \to \infty} \lim_{\delta Z_0 \to 0} \frac{1}{t} \ln \left| \frac{|\delta Z(t)|}{|\delta Z_0|} \right|
  \]
Maximal Lyapunov exponent converges towards a **positive value** for a chaotic orbit.

\[
\lambda = \lim_{t \to \infty} \lim_{\delta Z_0 \to 0} \frac{1}{t} \ln \frac{|\delta Z(t)|}{|\delta Z_0|}
\]
Lyapunov exponent: regular orbit

- Maximum Lyapunov exponent converges towards zero for a chaotic orbit

\[
\lambda = \lim_{t \to \infty} \lim_{\delta Z_0 \to 0} \frac{1}{t} \ln \frac{|\delta Z(t)|}{|\delta Z_0|}
\]
Lyapunov exponent: regular orbit

- Maximum Lyapunov exponent converges more slowly towards zero for a resonant orbit

\[
\lambda = \lim_{t \to \infty} \lim_{\delta Z_0 \to 0} \frac{1}{t} \ln \left( \frac{|\delta Z(t)|}{|\delta Z_0|} \right)
\]
Maximum Lyapunov exponent converges more slowly towards zero for a resonant orbit, in particular close to the separatrix

\[ \lambda = \lim_{t \to \infty} \lim_{\delta Z_0 \to 0} \frac{1}{t} \ln \frac{|\delta Z(t)|}{|\delta Z_0|} \]
Frequency Map Analysis
Frequency Map Analysis (FMA) is a numerical method which springs from the studies of J. Laskar (Paris Observatory) putting in evidence the chaotic motion in the Solar Systems.

FMA was successively applied to several dynamical systems:

- Stability of Earth Obliquity and climate stabilization (Laskar, Robutel, 1993)
- 4D maps (Laskar 1993)
Motion on torus

- Consider an integrable Hamiltonian system of the usual form
  \[ H(J, \varphi, \theta) = H_0(J) \]

- Hamilton’s equations give
  \[ \dot{\phi}_j = \frac{\partial H_0(J)}{\partial J_j} = \omega_j(J) \Rightarrow \phi_j = \omega_j(J)t + \phi_{j0} \]
  \[ \dot{J}_j = -\frac{\partial H_0(J)}{\partial \phi_j} = 0 \Rightarrow J_j = \text{const.} \]

- The actions define the surface of an invariant torus

- In complex coordinates the motion is described by
  \[ \zeta_j(t) = J_j(0)e^{i\omega_j t} = z_{j0}e^{i\omega_j t} \]

- For a non-degenerate system \( \det \left| \frac{\partial \omega(J)}{\partial J} \right| = \det \left| \frac{\partial^2 H_0(J)}{\partial J^2} \right| \neq 0 \)
  there is a one-to-one correspondence between the actions and the frequency, a frequency map can be defined parameterizing the tori in the frequency space

\[ F : (I) \rightarrow (\omega) \]
Quasi-periodic motion

- If a transformation is made to some new variables
  \[ \zeta_j = I_j e^{i\theta_j t} = z_j + \epsilon G_j (z) = z_j + \epsilon \sum_{m} c_m z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n} \]

- The system is still integrable but the tori are distorted
- The motion is then described by
  \[ \zeta_j (t) = z_{j0} e^{i\omega_j t} + \sum_{m} a_m e^{i (m \cdot \omega) t} \]
  i.e.

  a quasi-periodic function of time, with

  \[ a_m = \epsilon \, c_m z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n} \]
  and \( m \cdot \omega = m_1 \omega_1 + m_2 \omega_2 + \cdots + m_n \omega_n \)

- For a non-integrable Hamiltonian, \( H (I, \theta) = H_0 (I) + \epsilon H' (I, \theta) \)
  and especially if the perturbation is small, most tori persist
  (KAM theory)

- In that case, the motion is still quasi-periodic and a frequency map can be built

- The **regularity** (or not) of the map reveals stable (or chaotic) motion
Building the frequency map

When a quasi-periodic function $f(t) = q(t) + ip(t)$ in the complex domain is given numerically, it is possible to recover a quasi-periodic approximation

$$f'(t) = \sum_{k=1}^{N} a'_k e^{i\omega'_k t}$$

in a very precise way over a finite time span $[-T, T]$ several orders of magnitude more precisely than simple Fourier techniques.

This approximation is provided by the Numerical Analysis of Fundamental Frequencies – **NAFF** algorithm.

The frequencies $\omega'_k$ and complex amplitudes $a'_k$ are computed through an iterative scheme.
The NAFF algorithm

- The first frequency $\omega_1'$ is found by the location of the maximum of

$$\phi(\sigma) = \langle f(t), e^{i\sigma t} \rangle = \frac{1}{2T} \int_{-T}^{T} f(t) e^{-i\sigma t} \chi(t) dt$$

where $\chi(t)$ is a weight function.

- In most of the cases the Hanning window filter is used

$$\chi_1(t) = 1 + \cos(\pi t / T)$$

- Once the first term $e^{i\omega_1' t}$ is found, its complex amplitude $a_1'$ is obtained and the process is restarted on the remaining part of the function

$$f_1(t) = f(t) - a_1' e^{i\omega_1' t}$$

- The procedure is continued for the number of desired terms, or until a required precision is reached.
The accuracy of a simple FFT even for a simple sinusoidal signal is not better than $|\nu - \nu_T| = \frac{1}{T}$.

Calculating the Fourier integral explicitly:

$$\phi(\omega) = \langle f(t), e^{i\omega t} \rangle = \frac{1}{T} \int_0^T f(t) e^{-i\omega t} dt$$

shows that the maximum lies in between the main peaks of the FFT.

$$|\phi(\omega)| = |\text{sinc} \left( \frac{\nu - \omega}{2} \right)|$$
A more complicated signal with two frequencies

\[ f(t) = a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t} \]

shifts slightly the maximum with respect to its real location.
A window function like the Hanning filter

\[ \chi_1(t) = 1 + \cos\left(\frac{\pi t}{T}\right) \]

kills side-lobes and allows a very accurate determination of the frequency.
**Precision of NAFF**

- For a general window function of order $p$
  
  $$\chi_p(t) = \frac{2^p (p!)^2}{(2p)!} \left(1 + \cos \pi t\right)^p$$

  Laskar (1996) proved a theorem stating that the solution provided by the NAFF algorithm converges asymptotically towards the real KAM quasi-periodic solution with precision

  $$\nu_1 - \nu_1^T \propto \frac{1}{T^{2p+2}}$$

- In particular, for no filter (i.e. $p = 0$) the precision is $\frac{1}{T^2}$, whereas for the Hanning filter ($p$), the precision is of the order of $\frac{1}{T^4}$
Aspects of the frequency map

- In the vicinity of a resonance the system behaves like a **pendulum**
- Passing through the **elliptic point** for a fixed angle, a **fixed frequency** (or rotation number) is observed
- Passing through the **hyperbolic point**, a **frequency jump** is observed
Example: Frequency map for BBLR

- Simple Beam-beam long range (BBLR) kick and a rotation
Example: Frequency map for BBLR

Simple Beam-beam long range (BBLR) kick and a rotation

\[
\begin{pmatrix}
  x \\
  x'
\end{pmatrix}
= \begin{pmatrix}
  \cos \mu & \beta^* \sin \mu \\
  -\sin \mu/\beta^* & \cos \mu
\end{pmatrix}
\begin{pmatrix}
  x + f(x') \\
  x'
\end{pmatrix}
\]

\[
f(x') = K \left[ \frac{1}{x' + \theta_c \left( 1 - e^{-\frac{(x' + \theta_c)^2}{2\sigma^2}} \right)} - \frac{1}{\theta_c \left( 1 - e^{-\frac{\theta_c^2}{2\sigma^2}} \right)} \right]
\]
Diffusion in frequency space

- For a **2 degrees of freedom** Hamiltonian system, the **frequency space** is a **line**, the tori are dots on this lines, and the **chaotic zones** are **confined** by the existing KAM tori.

- For a system with 3 or more degrees of freedom, KAM tori are still represented by dots but do not prevent chaotic trajectories to diffuse.

- This topological possibility of particles diffusing is called **Arnold diffusion**.

- This diffusion is supposed to be extremely small in their vicinity, as tori act as effective barriers (**Nechoroshev theory**).
Building the frequency map

- Choose coordinates \((x_i, y_i)\) with \(p_x\) and \(p_y=0\)
- Numerically integrate the phase trajectories through the lattice for sufficient number of turns
- Compute through NAFF \(Q_x\) and \(Q_y\) after sufficient number of turns
- Plot them in the tune diagram

\[
\mathcal{F}_\tau : p|q=q_0 \rightarrow \mathbb{R}^n
\]
Example: Frequency maps for the LHC

- Frequency maps for the target error table (left) and an increased random skew octupole error in the superconducting dipoles (right)
Diffusion Maps

- Calculate frequencies for two equal and successive time spans and compute frequency diffusion vector:

\[ \mathbf{D}_{t=\tau} = \mathbf{\nu}_{t \in (0, \tau/2]} - \mathbf{\nu}_{t \in (\tau/2, \tau]} \]

- Plot the initial condition space color-coded with the norm of the diffusion vector

- Compute a diffusion quality factor by averaging all diffusion coefficients normalized with the initial conditions radius

\[ D_{QF} = \left\langle \frac{|\mathbf{D}|}{(I_{x0}^2 + I_{y0}^2)^{1/2}} \right\rangle R \]
Example: Diffusion maps for the LHC

Diffusion maps for the target error table (left) and an increased random skew octupole error in the superconducting dipoles (right)
Example: Frequency Map for the ESRF

- All dynamics represented in these two plots
- Regular motion represented by blue colors (close to zero amplitude particles or working point)
- Resonances appear as distorted lines in frequency space (or curves in initial condition space)
- Chaotic motion is represented by red scattered particles and defines dynamic aperture of the machine
Numerical Applications
Correction schemes efficiency

- Comparison of correction schemes for $b_4$ and $b_5$ errors in the LHC dipoles
- Frequency maps, resonance analysis, tune diffusion estimates, survival plots and short term tracking, proved that only half of the correctors are needed
### Beam-Beam interaction

- Long range beam-beam interaction represented by a 4D kick-map

\[ \Delta x = - n_{par} \frac{2r_p N_b}{\gamma} \left[ \frac{x' + \theta_c}{\theta_t^2} \left( 1 - e^{-\frac{\theta_c^2}{2\theta_t^2}} \right) \right] \]

\[ - \frac{1}{\theta_c} \left( 1 - e^{-\frac{\theta_c^2}{2\theta_t^2}} \right) \]

\[ \Delta y = - n_{par} \frac{2r_p N_b}{\gamma} \frac{y'}{\theta_t^2} \left( 1 - e^{-\frac{\theta_c^2}{2\theta_t^2}} \right) \]

with \( \theta_t \equiv \left( (x' + \theta_c)^2 + y'^2 \right)^{1/2} \)

<table>
<thead>
<tr>
<th>Variable</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam energy</td>
<td>( E )</td>
<td>7 TeV</td>
</tr>
<tr>
<td>Particle species</td>
<td>...</td>
<td>protons</td>
</tr>
<tr>
<td>Full crossing angle</td>
<td>( \theta_c )</td>
<td>300 ( \mu )rad</td>
</tr>
<tr>
<td>rms beam divergence</td>
<td>( \sigma'_x )</td>
<td>31.7 ( \mu )rad</td>
</tr>
<tr>
<td>rms beam size</td>
<td>( \sigma_x )</td>
<td>15.9 ( \mu )m</td>
</tr>
<tr>
<td>Normalized transv.</td>
<td>( \gamma e )</td>
<td>3.75 ( \mu )m</td>
</tr>
<tr>
<td>IP beta function</td>
<td>( \beta^* )</td>
<td>0.5 m</td>
</tr>
<tr>
<td>Bunch charge</td>
<td>( N_b )</td>
<td>( 1 \times 10^{11} - 2 \times 10^{12} )</td>
</tr>
<tr>
<td>Betatron tune</td>
<td>( Q_0 )</td>
<td>0.31</td>
</tr>
</tbody>
</table>
- Proved dominant effect of long range beam-beam effect
- Dynamic Aperture (around 6σ) located at the folding of the map (indefinite torsion)
- Experimental effort to compensate beam-beam long range effect with wires (1/r part of the force) or octupoles
In the chaotic region of phase space, the action diffusion coefficient per turn can be estimated by averaging over the quasi-randomly varying betatron phase variable as

\[
D(J) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \ [\Delta J(\phi)]^2
\]
Very good agreement of diffusive aperture boundary (action variance) with frequency variation (loss boundary corresponding to around 1 integer unit change in $10^7$ turns)
Wire compensation

- Current baring wire can improve DA by 1-2 σ
- Tests in the LHC during 2017-2018

Without correction

With correction

Reduced crossing angle of 450 μrad @ 15cm

S. Fartoukh et al., PRSTAB, 2015

K. Skoufaris et al. 2018
Experimental BBLR compensation

G. Sterbini, A. Poyet, et al. 2017

- Wire current @ 340/190 A and collimator jaw at 5.5 $\sigma_{\text{coll}}$
- **Compensating effect** of the wires visible on beam lifetime
BBLR compensation

\[ \sigma_{EFF} = -\frac{1}{\sum_{IP} L_{IP}} \frac{dN}{dt} \]

- **Compensating effect** of the wires visible on **effective x-section**

G. Sterbini, A. Poyet, et al. 2017
BBLR compensation

- Compensation effect visible also with trains and reduced crossing angle!
SABA\textsubscript{2}C integrator

C. Skokos, YP and J. Laskar, EPAC 2008

- SABA\textsubscript{2}C allows symplectic integration with positive steps
- Several orders of magnitude better precision of SABA\textsubscript{2}C with respect to classical YFR integrator

K. Skoufaris et al. IPAC 2018

1-kick

10-kick

SABA\textsubscript{2}C

LQ (m)

|\;|KQ| (m^{-2})

\phi_{tsa} (CSABA\textsubscript{2}) - \phi_{tsa} (YFR3)

unstable motion
Magnet fringe fields

- Up to now we considered only transverse fields
- Magnet fringe field is the longitudinal dependence of the field at the magnet edges
- Important when magnet aspect ratios and/or emittances are big
Quadrupole fringe field

General field expansion for a quadrupole magnet:

\[ B_x = \sum_{m,n=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^m x^{2n} y^{2m+1}}{(2n)!(2m+1)!} \binom{m}{l} b_{2n+2m+1-2l}[2l] \]

\[ B_y = \sum_{m,n=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^m x^{2n+1} y^{2m}}{(2n+1)!(2m)!} \binom{m}{l} b_{2n+2m+1-2l}[2l] \]

\[ B_z = \sum_{m,n=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^m x^{2n+1} y^{2m+1}}{(2n+1)!(2m+1)!} \binom{m}{l} b_{2n+2m+1-2l}[2l+1] \]

and to leading order

\[ B_x = y \left[ b_1 - \frac{1}{12} (3x^2 + y^2) b_1^{[2]} \right] + O(5) \]

\[ B_y = x \left[ b_1 - \frac{1}{12} (3y^2 + x^2) b_1^{[2]} \right] + O(5) \]

\[ B_z = xy b_1^{[1]} + O(4) \]

The quadrupole fringe to leading order has an octupole-like effect
Magnet fringe fields

From the hard-edge Hamiltonian

\[ H_f = \pm \frac{Q}{12 B \rho (1 + \frac{\delta p}{p})} (y^3 p_y - x^3 p_x + 3 x^2 y p_y - 3 y^2 x p_x), \]

the first order shift of the frequencies with amplitude can be computed analytically

\[
\begin{pmatrix}
\delta \nu_x \\
\delta \nu_y
\end{pmatrix} =
\begin{pmatrix}
a_{hh} & a_{hv} \\
a_{hv} & a_{vv}
\end{pmatrix}
\begin{pmatrix}
2 J_x \\
2 J_y
\end{pmatrix},
\]

with the ”anharmonicity” coefficients (torsion)

\[
a_{hh} = -\frac{1}{16 \pi B \rho} \sum_i \pm Q_i \beta_{xi} \alpha_{xi}
\]
\[
a_{hv} = \frac{1}{16 \pi B \rho} \sum_i \pm Q_i (\beta_{xi} \alpha_{yi} - \beta_{yi} \alpha_{xi})
\]
\[
a_{vv} = \frac{1}{16 \pi B \rho} \sum_i \pm Q_i \beta_{yi} \alpha_{yi}
\]
SNS Working Point \((Q_x,Q_y)=(6.4,6.3)\)

\[\delta p/p = [2\%, -2\%] \text{ at } 480 \pi \text{ mm mrad}\]
Choice of the SNS ring working point

Tune Diffusion quality factor

\[ D_{QF} = \left\langle \frac{|D|}{(I_{x0}^2 + I_{y0}^2)^{1/2}} \right\rangle_R \]

Chosen Working Point

- (6.4, 6.3)
- (6.23, 5.24)
- (6.3, 5.8)
- (6.23, 6.20)
Global Working point choice

- Figure of merit for choosing best working point is sum of diffusion rates with a constant added for every lost particle
- Each point is produced after tracking 100 particles
- Nominal working point had to be moved towards “blue” area

\[ e^D = \sqrt{\left(\nu_{x,1} - \nu_{x,2}\right)^2 + \left(\nu_{y,1} - \nu_{y,2}\right)^2} \]

\[ WPS = 0.1N_{\text{lost}} + \sum e^D \]
Sextupole scheme optimization

- Comparing different chromaticity sextupole correction schemes and working point optimization using normal form analysis, frequency maps and finally particle tracking.

- Finding the adequate sextupole strengths through the tune diffusion coefficient.
Frequency Map Analysis with modulation
- Evolution of frequency map over different longitudinal position
- Tunes acquired over each longitudinal period
- Particles with similar longitudinal offset but different amplitudes experience the resonance in different manner
- Particles with different longitudinal offset may experience different resonances

F. Asvesta, et al., 2017
- Quadrupoles of the **inner triplet** right and left of IP1 and IP5, **large beta-functions** increase the sensitivity to non-linear effects

- **Resonance conditions:**

\[
aQ_x + bQ_y + c \frac{f_{\text{modulation}}}{f_{\text{revolution}}} = k \text{ for } a, b, c, k \text{ integers}
\]

- By increasing the modulation depth, sidebands start to appear in the FMAs

S. Kostoglou, et al., 2018
- Quadrupoles of the **inner triplet** right and left of IP1 and IP5, large **beta-functions** increase the sensitivity to non-linear effects

- **Resonance conditions:**

  \[ aQ_x + bQ_y + c \frac{f_{\text{modulation}}}{f_{\text{revolution}}} = k \]
  
  for \( a, b, c, k \) integers

- By increasing the modulation depth, sidebands start to appear in the FMAs

\[ \Delta Q = 1e^{-4} \]
LHC: Power supply ripples

- Scan of different ripple frequencies (50-900 Hz)

$5D, E = 6.5\text{TeV}, I_{\text{oct}} = 510\text{A}, \text{Beam-beam ON, } \varepsilon_n = 2.5\mu\text{m, } \beta^* = 40\text{cm, } q = 15$

$(Q_x, Q_y) = (62.31, 60.32), V_{\text{RF OFF, 5p}} = 27e - 5, 49 \text{ angles, 0.1 - 6.1 } \sigma, \text{ sliding NAFF}$

$f_r = 50.0\text{Hz, } A_r = 10^{-7} \text{ at MQXA.1, MQXA.3, MQXB.A2, MQXB.B2 of IP1, IP5}$

S. Kostoglou, YP et al., 2018
6D FMAs with power supply ripples

6D, $E = 6.5\text{TeV}$, $I_{\text{ct}} = 510\text{A}$. Beam–beam ON, $e_n = 2.5\mu\text{m}$, $\beta^* = 40\text{cm}$, $q = 0$

$(Q_x, Q_y) = (62.31, 60.32)$. $V_{\text{RF}}$ ON, $\delta p = 27\times10^{-5}$, 99 angles, $0.1 - 6.1 \sigma$, sliding NAFF

$f_r = 50\text{Hz}$, $A_r = 10^{-7}$ at MQXA.1, MQXA.3, MQXB.A2, MQXB.B2 of IP1, IP5

S. Kostoglou, YP et al., 2018
Summary

- Appearance of **fixed points** (periodic orbits) determine **topology** of the phase space
- **Perturbation** of unstable (hyperbolic points) opens the path to chaotic motion
- Resonance can overlap enabling the rapid diffusion of orbits
- **Dynamic aperture** by brute force tracking (with symplectic numerical integrators) is the usual quality criterion for evaluating non-linear dynamics performance of a machine
- **Frequency Map Analysis** is a numerical tool that enables to study in a global way the dynamics, by identifying the excited resonances and the extent of chaotic regions
- It can be directly applied to **tracking** and **experimental** data
- A combination of these modern methods enable a thorough analysis of non-linear dynamics and lead to a robust design
Acknowledgments

Appendix
Advanced symplectic integration schemes

- Symplectic integrators with **positive** steps for Hamiltonian systems \( H = A + \epsilon B \) with both \( A \) and \( B \) integrable were proposed by McLachlan (1995).

- Laskar and Robutel (2001) derived all orders of such integrators.

- Consider the formal solution of the Hamiltonian system written in the Lie representation

\[
\bar{x}(t) = \sum_{n \geq 0} \frac{t^n}{n!} L_H^n \bar{x}(0) = e^{tL_H} \bar{x}(0).
\]

- A symplectic integrator of order \( n \) from \( t \) to \( t + \tau \) consists of approximating the Lie map \( e^{\tau L_H} = e^{\tau (L_A + L_{\epsilon B})} \) by products of \( e^{c_i \tau L_A} \) and \( e^{d_i \tau L_{\epsilon B}} \), \( i = 1, \ldots, n \) which integrate exactly \( A \) and \( B \) over the time-spans \( c_i \tau \) and \( d_i \tau \).

- The constants \( c_i \) and \( d_i \) are chosen to reduce the error.
The $SABA_2$ integrator is written as

$$SABA_2 = e^{c_1 \tau L_A} e^{d_1 \tau L \epsilon_B} e^{c_2 \tau L_A} e^{d_1 \tau L \epsilon_B} e^{c_1 \tau L_A},$$

with $c_1 = \frac{1}{2} \left(1 - \frac{1}{\sqrt{3}}\right)$, $c_2 = \frac{1}{\sqrt{3}}$, $d_1 = \frac{1}{2}$.

When $\{\{A, B\}, B\}$ is integrable, e.g. when $A$ is quadratic in momenta and $B$ depends only in positions, the accuracy of the integrator is improved by two small negative kicks

$$SABA_2 C = e^{-\tau^3 \epsilon^2 \frac{c}{2} L_{\{\{A, B\}, B\}}} (SABA_2) e^{-\tau^3 \epsilon^2 \frac{c}{2} L_{\{\{A, B\}, B\}}}$$

with $c = (2 - \sqrt{3})/24$.

The accuracy of $SABA_2 C$ is one order of magnitude higher than the Forest-Ruth 4th order scheme.

The usual “drift-kick” scheme corresponds to the 2nd order integrator

$$SABA_1 = e^{\frac{\tau}{2} L_A} e^{\tau L \epsilon_B} e^{\frac{\tau}{2} L_A},$$
Experimental methods

- Frequency analysis of turn-by-turn data of beam oscillations produced by a fast kicker magnet and recorded on a Beam Position Monitors

- Reproduction of the non-linear model of the Advanced Light Source storage ring and working point optimization for increasing beam lifetime
Experimental Methods – Tune scans

- Study the resonance behavior around different working points in SPS
- Strength of individual resonance lines can be identified from the beam loss rate, i.e. the derivative of the beam intensity at the moment of crossing the resonance
- Vertical tune is scanned from about 0.45 down to 0.05 during a period of 3s along the flat bottom
- Low intensity 4-5e10 p/b single bunches with small emittance injected
- Horizontal tune is constant during the same period
- Tunes are continuously monitored using tune monitor (tune post-processed with NAFF) and the beam intensity is recorded with a beam current transformer

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Tune Scans from the SPS

- Plot the tunes color-coded with the amount of loss
- Identify the dangerous resonances
- Compare between two different optics
- Try to refine the machine model
Limiting resonances for space charge tune spread: \((H, V) \sim (0.10, \sim 0.19)\)
- Blow-up at integer resonances as expected
- Losses for working point close to the \(Q_x + 2Q_y\) normal sextupole resonance (studied in Fix-line experiment with Q26) and around the the 4\(Q_x = 81\) normal octupole resonance

Identified optimum working point area for vertical tune spread of 0.2
- \(20.16 < Q_x < 20.23, 20.24 < Q_y < 20.33\)
- Losses around 0.5% for 3 s storage time on flat bottom