Contrast of Non-Symplectic and Symplectic Integrator

Example: Contrast of Non-Symplectic and Symplectic Advances
Contrast: Numerical and Actual Orbit for a Simple Harmonic Oscillator
use scaled coordinates (max extents unity for analytical solution)
Symplectic Leapfrog Advance:
- 5 steps per period, 100 periods
Non-Symplectic 4th Order Runge-Kutta Advance:
- 10 steps per period, 100 periods

Cosine-type initial conditions
Numerical Orbit

Sine-type initial conditions
Actual Orbit

Example: Contrast of Non-Symplectic and Symplectic Advances (3)
Contrast: Numerical and Actual Orbit for a Simple Harmonic Oscillator
Non-Symplectic 4th Order Runge-Kutta Advance:
- 5 steps per period, 20 periods
- 10 steps per period, 200 periods

Cosine-type initial conditions
Numerical Orbit

Sine-type initial conditions
Actual Orbit

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Courtesy of S. Lund
A Symplectic Multi-Particle Tracking Model (1)

multi-particle Hamiltonian

\[ H = \sum_i p_i^2 / 2 + \frac{1}{2} \sum_i \sum_j q \phi(r_i, r_j) + \sum_i q \psi(r_i) \]

\[
\begin{align*}
\frac{d\mathbf{r}_i}{ds} &= \frac{\partial H}{\partial p_i} \\
\frac{dp_i}{ds} &= -\frac{\partial H}{\partial \mathbf{r}_i}
\end{align*}
\]

space-charge Coulomb potential

external focusing/acceleration

A formal single step solution

\[
\zeta(\tau) = \exp(-\tau : H :) \zeta(0)
\]

\[
\zeta(\tau) = \exp(-\tau : H_1 :) \exp(-\tau : H_2 :) \exp(-\frac{1}{2} \tau : H_1 :) \zeta(0) + O(\tau^3)
\]

\[
\zeta(\tau) = \mathcal{M}(\tau) \zeta(0) = \mathcal{M}_1(\tau/2) \mathcal{M}_2(\tau) \mathcal{M}_1(\tau/2) \zeta(0)
\]

\[ H = H_1 + H_2 \]

\[ \mathcal{M} \] would be symplectic if both \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are symplectic

A Symplectic Multi-Particle Tracking Model (2)

2nd order:
\[ \zeta(\tau) = \mathcal{M}(\tau)\zeta(0) = \mathcal{M}_1(\tau/2)\mathcal{M}_2(\tau)\mathcal{M}_1(\tau/2)\zeta(0) \]

4th order:
\[ \mathcal{M}(\tau) = \mathcal{M}_1\left(\frac{s}{2}\right)\mathcal{M}_2(s)\mathcal{M}_1\left(\frac{\alpha s}{2}\right)\mathcal{M}_2\left((\alpha - 1)s\right)\mathcal{M}_1\left(\frac{\alpha s}{2}\right)\mathcal{M}_2(s)\mathcal{M}_1\left(\frac{s}{2}\right) \]

where \( \alpha = 1 - 2^{1/3} \), and \( s = \tau/(1 + \alpha) \)

higher order:
\[ \mathcal{M}_{2n+2}(\tau) = \mathcal{M}_{2n}(z_0\tau)\mathcal{M}_{2n}(z_1\tau)\mathcal{M}_{2n}(z_0\tau) \]

where \( z_0 = 1/(2 - 2^{1/(2n+1)}) \) and \( z_1 = -2^{1/(2n+1)}/(2 - 2^{1/(2n+1)}) \)

Symplectic condition:
\[ M_i^TJM_i = J \]

where \( J \) denotes the \( 6N \times 6N \) matrix given by
\[ J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \]

and \( I \) is the \( 3N \times 3N \) identity matrix

A Symplectic Multi-Particle Tracking Model (3)

\[ H_1 = \sum_i \frac{p_i^2}{2} + \sum_i q\psi(r_i) \]
\[ M_1 \]

- Symplectic map for \( H_1 \) can be found from charged particle optics method

\[ H_2 = \frac{1}{2} \sum_i \sum_j q\phi(r_i, r_j) \]
\[ M_2 \]

\[ r_i(\tau) = r_i(0) \]
\[ p_i(\tau) = p_i(0) - \frac{\partial H_2(r)}{\partial r_i} \tau \]

\[ M_2 = \begin{pmatrix} I & 0 \\ L & I \end{pmatrix} \]

To satisfy the symplectic condition:

\[ L = L^T \]

\[ L_{ij} = \frac{\partial p_i(\tau)}{\partial r_j} = -\frac{\partial^2 H_2(r)}{\partial r_i \partial r_j} \tau \]

\( M_2 \) will be symplectic if \( p_i \) is updated from \( H_2 \) analytically
Self-Consistent Space-Charge Transfer Map (1)

\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\frac{\rho}{\epsilon_0}, \]

\[ \phi(x = 0, y) = 0 \]
\[ \phi(x = a, y) = 0 \]
\[ \phi(x, y = 0) = 0 \]
\[ \phi(x, y = b) = 0 \]

\[ \rho(x, y) = \sum_{l=1}^{N_l} \sum_{m=1}^{N_m} \rho^{lm} \sin(\alpha_l x) \sin(\beta_m y) \]

\[ \phi(x, y) = \sum_{l=1}^{N_l} \sum_{m=1}^{N_m} \phi^{lm} \sin(\alpha_l x) \sin(\beta_m y) \]

\[ \rho^{lm} = \frac{4}{ab} \int_0^a \int_0^b \rho(x, y) \sin(\alpha_l x) \sin(\beta_m y) \, dx \, dy \]

\[ \phi^{lm} = \frac{4}{ab} \int_0^a \int_0^b \phi(x, y) \sin(\alpha_l x) \sin(\beta_m y) \, dx \, dy \]

where \( \alpha_l = \frac{l\pi}{a} \) and \( \beta_m = \frac{m\pi}{b} \)

\[ \phi^{lm} = \frac{\rho^{lm}}{\epsilon_0 \gamma^2_{lm}} \]

where \( \gamma^2_{lm} = \alpha_l^2 + \beta_m^2 \)
Self-Consistent Space-Charge Transfer Map (2)

The charge density from macroparticles:

\[
\rho(x, y) = \frac{1}{\Delta x \Delta y N_p} \sum_{j=1}^{N_p} S(x - x_j)S(y - y_j)
\]

The solution of space-charge potential modes:

\[
\phi^{lm} = \frac{4 \pi}{\gamma_{lm}^2 ab N_p} \sum_{j=1}^{N_p} \frac{1}{\Delta x \Delta y} \int_0^a \int_0^b S(x - x_j)S(y - y_j) \\
\times \sin(\alpha_l x) \sin(\beta_m y) \, dx \, dy
\]

The solution of space-charge potential:

\[
\phi(x, y) = 4 \pi \frac{4}{ab N_p} \sum_{j=1}^{N_p} \sum_{l=1}^{N_l} \sum_{m=1}^{N_m} \frac{1}{\gamma_{lm}^2} \sin(\alpha_l x) \sin(\beta_m y) \frac{1}{\Delta x \Delta y} \int_0^a \int_0^b S(\bar{x} - x_j)S(\bar{y} - y_j) \sin(\alpha_l \bar{x}) \sin(\beta_m \bar{y}) \, d\bar{x} \, d\bar{y}.
\]

The space-charge potential on macroparticles:

\[
\phi(x_i, y_i) = \frac{1}{\Delta x \Delta y} \int_0^a \int_0^b \phi(x, y) S(x - x_i)S(y - y_i) \, dx \, dy
\]
Self-Consistent Space-Charge Transfer Map (3)

The interaction potential:

\[ \varphi(x_i, y_i, x_j, y_j) = 4\pi \frac{4}{ab} \frac{1}{N_P} \sum_{l=1}^{N_i} \sum_{m=1}^{N_m} \frac{1}{\gamma_{lm}^2} \int_0^a \int_0^b S(x - x_j) S(y - y_j) \sin(\alpha_l x) \sin(\beta_m y) dxdy \]

\[ \times \frac{1}{\Delta x \Delta y} \int_0^a \int_0^b S(x - x_i) S(y - y_i) \sin(\alpha_l x) \sin(\beta_m y) dxdy. \]

The space-charge Hamiltonian:

\[ H_2 = 4\pi \frac{K}{2ab} \frac{1}{N_P} \sum_{i=1}^{N_p} \sum_{j=1}^{N_p} \sum_{l=1}^{N_l} \sum_{m=1}^{N_m} \frac{1}{\gamma_{lm}^2} \int_0^a \int_0^b S(x - x_j) S(y - y_j) \sin(\alpha_l x) \sin(\beta_m y) dxdy \]

\[ \times \frac{1}{\Delta x \Delta y} \int_0^a \int_0^b S(x - x_i) S(y - y_i) \sin(\alpha_l x) \sin(\beta_m y) dxdy. \]
Symplectic Gridless Symplectic Space-Charge Model

\[ \rho(x, y) = \sum_{j=1}^{N_p} w \delta(x - x_j) \delta(y - y_j) \]

\[ H_2 = \frac{1}{2 \epsilon_0} \frac{4}{ab} w \sum_i \sum_j \sum_l \sum_m \frac{1}{\gamma_{lm}^2} \sin(\alpha_l x_j) \sin(\beta_m y_j) \sin(\alpha_l x_i) \sin(\beta_m y_i) \]

\[ p_{xi}(\tau) = p_{xi}(0) - \tau \frac{1}{\epsilon_0} \frac{4}{ab} w \sum_j \sum_l \sum_m \frac{\alpha_l}{\gamma_{lm}^2} \sin(\alpha_l x_j) \sin(\beta_m y_j) \cos(\alpha_l x_i) \sin(\beta_m y_i) \]

\[ p_{yi}(\tau) = p_{yi}(0) - \tau \frac{1}{\epsilon_0} \frac{4}{ab} w \sum_j \sum_l \sum_m \frac{\beta_m}{\gamma_{lm}^2} \sin(\alpha_l x_j) \sin(\beta_m y_j) \sin(\alpha_l x_i) \cos(\beta_m y_i) \]

\( w \) is the particle charge weight.
Symplectic Particle-In-Cell Model (1)

\[
p_{xi}(\tau) = p_{xi}(0) - \tau 4\pi K \frac{4}{abN_p} \sum_{j=1}^{N_p} \sum_{l=1}^{N_l} \sum_{m=1}^{N_m} \frac{1}{\gamma_{lm}^2} \int_0^a \int_0^b S(x-x_j)S(y-y_j) \sin(\alpha_l x) \sin(\beta_m y) \, dx \, dy \\
\times \frac{1}{\Delta x \Delta y} \int_0^a \int_0^b \frac{\partial S(x-x_i)}{\partial x_i} S(y-y_i) \sin(\alpha_l x) \sin(\beta_m y) \, dx \, dy,
\]

\[
p_{yi}(\tau) = p_{yi}(0) - \tau 4\pi K \frac{4}{abN_p} \sum_{j=1}^{N_p} \sum_{l=1}^{N_l} \sum_{m=1}^{N_m} \frac{1}{\gamma_{lm}^2} \int_0^a \int_0^b S(x-x_j)S(y-y_j) \sin(\alpha_l x) \sin(\beta_m y) \, dx \, dy \\
\times \frac{1}{\Delta x \Delta y} \int_0^a \int_0^b S(x-x_i) \frac{\partial S(y-y_i)}{\partial y_i} \sin(\alpha_l x) \sin(\beta_m y) \, dx \, dy,
\]

\[
p_{xi}(\tau) = p_{xi}(0) - \tau 4\pi K \frac{4}{abN_p} \sum_{j=1}^{N_p} \sum_{l=1}^{N_l} \sum_{m=1}^{N_m} \frac{1}{\gamma_{lm}^2} \sum_{l'=1}^{l'} \sum_{j'=1}^{j'} S(x_{l'}-x_j)S(y_{l'}-y_j) \sin(\alpha_l x_{l'}) \sin(\beta_m y_{l'}) \\
\times \sum_{l} \sum_{j} \frac{\partial S(x_l-x_i)}{\partial x_i} S(y_j-y_i) \sin(\alpha_l x_i) \sin(\beta_m y_j),
\]

\[
p_{yi}(\tau) = p_{yi}(0) - \tau 4\pi K \frac{4}{abN_p} \sum_{j=1}^{N_p} \sum_{l=1}^{N_l} \sum_{m=1}^{N_m} \frac{1}{\gamma_{lm}^2} \sum_{l'=1}^{l'} \sum_{j'=1}^{j'} S(x_{l'}-x_j)S(y_{l'}-y_j) \sin(\alpha_l x_{l'}) \sin(\beta_m y_{l'}) \\
\times \sum_{l} \sum_{j} S(x_l-x_i) \frac{\partial S(y_l-y_i)}{\partial y_i} \sin(\alpha_l x_i) \sin(\beta_m y_j),
\]
Symplectic PIC Model (2)

Define charge density on grid as:

$$\bar{\rho}(x_I', y_J') = \frac{1}{N_p} \sum_{j=1}^{N_p} S(x_I' - x_j)S(y_J' - y_j),$$

Space-charge $M_2$

$$p_{xi}(\tau) = p_{xi}(0) - \tau 4\pi K \sum_{l=1}^{N_l} \sum_{m=1}^{N_m} \frac{1}{\gamma_{lm}^2} \sum_{i'} \sum_{j'} \bar{\rho}(x_I', y_J') \sin(\alpha_{l} x_{i'}) \sin(\beta_{m} y_{j'}) \sin(\alpha_{l} x_{i}) \sin(\beta_{m} y_{j})$$

$$\times \left[ \frac{4}{ab} \sum_{l=1}^{N_l} \sum_{m=1}^{N_m} \frac{1}{\gamma_{lm}^2} \sum_{i'} \sum_{j'} \bar{\rho}(x_I', y_J') \sin(\alpha_{l} x_{i'}) \sin(\beta_{m} y_{j'}) \sin(\alpha_{l} x_{i}) \sin(\beta_{m} y_{j}) \right],$$

$$p_{yi}(\tau) = p_{yi}(0) - \tau 4\pi K \sum_{l=1}^{N_l} \sum_{m=1}^{N_m} S(x_I - x_i) \frac{\partial S(y_I - y_i)}{\partial y_i}$$

$$\times \left[ \frac{4}{ab} \sum_{l=1}^{N_l} \sum_{m=1}^{N_m} \frac{1}{\gamma_{lm}^2} \sum_{i'} \sum_{j'} \bar{\rho}(x_I', y_J') \sin(\alpha_{l} x_{i'}) \sin(\beta_{m} y_{j'}) \sin(\alpha_{l} x_{i}) \sin(\beta_{m} y_{j}) \right].$$
Define potential on grid as:

\[ \phi(x_I, y_J) = \frac{4}{ab} \sum_{l=1}^{N_l} \sum_{m=1}^{N_m} \frac{1}{r_{lm}^2} \sum_{i'} \sum_{j'} \tilde{p}(x_{i'}, y_{j'}) \sin(\alpha_{l}x_{i'}) \sin(\beta_{m}y_{j'}) \sin(\alpha_{l}x_{I}) \sin(\beta_{m}y_{J}). \]

\[ p_{xi}(\tau) = p_{xi}(0) - \tau 4\pi K \sum_{I} \sum_{J} \frac{\partial S(x_I - x_i)}{\partial x_i} S(y_J - y_i) \phi(x_I, y_J) \]

\[ p_{yi}(\tau) = p_{yi}(0) - \tau 4\pi K \sum_{I} \sum_{J} S(x_I - x_i) \frac{\partial S(y_J - y_i)}{\partial y_i} \phi(x_I, y_J) \]

\[ S(x_I - x_i) = \begin{cases} \frac{3}{4} - \left( \frac{x_i - x_I}{\Delta x} \right)^2, & |x_i - x_I| \leq \Delta x/2, \\ \frac{1}{2} \left( \frac{3}{2} - \frac{|x_i - x_I|}{\Delta x} \right)^2, & \Delta x/2 < |x_i - x_I| \leq 3/2\Delta x, \\ 0, & \text{otherwise}, \end{cases} \]

\[ \frac{\partial S(x_I - x_i)}{\partial x_i} = \begin{cases} -2 \left( \frac{x_i - x_I}{\Delta x} \right)/\Delta x, & |x_i - x_I| \leq \Delta x/2, \\ \left( -\frac{3}{2} + \frac{(x_i - x_I)}{\Delta x} \right)/\Delta x, & \Delta x/2 < |x_i - x_I| \leq 3/2\Delta x, x_i > x_I, \\ \left( \frac{3}{2} + \frac{(x_i - x_I)}{\Delta x} \right)/\Delta x, & \Delta x/2 < |x_i - x_I| \leq 3/2\Delta x, x_i \leq x_I, \\ 0, & \text{otherwise}. \end{cases} \]
Non-Symplectic PIC Model

\[
\frac{dr_i}{ds} = p_i
\]

\[
\frac{d p_i}{ds} = q(E_i/v_0 - a_z \times B_i)
\]

\[
\mathbf{r}(\tau/2)_i = \mathbf{r}(0)_i + \frac{1}{2}\tau \mathbf{p}_i(0)
\]

\[
E_x(x_I, y_J) = -\sum_{l=1}^{N_l} \sum_{m=1}^{N_m} \alpha_l \phi^{lm} \cos(\alpha_l x) \sin(\beta_m y)
\]

\[
E_y(x_I, y_J) = -\sum_{l=1}^{N_l} \sum_{m=1}^{N_m} \beta_m \phi^{lm} \sin(\alpha_l x) \cos(\beta_m y)
\]

\[
p_{xi}(\tau) = p_{xi}(0) + \tau \left( \frac{qE^e_{x}}{v_0} - qB^e_{y} \right) + \tau 4\pi K \sum_{I} \sum_{J} S(x_I - x_i) S(y_J - y_i) E_x(x_I, y_J)
\]

\[
p_{yi}(\tau) = p_{yi}(0) + \tau \left( \frac{qE^e_{y}}{v_0} + qB^e_{x} \right) + \tau 4\pi K \sum_{I} \sum_{J} S(x_I - x_i) S(y_J - y_i) E_y(x_I, y_J)
\]

\[
\mathbf{r}(\tau)_i = \mathbf{r}(\tau/2)_i + \frac{1}{2}\tau \mathbf{p}_i(\tau)
\]
Benchmark Case 1: FODO Lattice, Below 2\textsuperscript{nd} Order Envelope Instability

- 1 GeV proton beam
- FODO lattice
- 0 current phase advance: 85 degrees
- Initial 4D Gaussian distribution
Significant Difference in Final 4D Emittances Between the Symplectic and the Non-Symplectic Methods (Strong Space-Charge: Phase Advance Change 85 -> 42)

Two symplectic approaches show good agreement.
Final Beam X-Px Phase Spaces Have Similar Shapes
Non-Symplectic Model Has Smaller Area
Final Y-Py Phase Space Show Similar Shapes

- symplectic gridless
- symplectic PIC
- spectral PIC
Horizontal and Vertical Density Profiles from the Symplectic Gridless Model, the Symplectic PIC Model, and the Non-Symplectic Spectral PIC

- Two symplectic solvers produce similar density profiles
- Non-symplectic solver produces larger core density
Finer Step Size Needed for Non-Symplectic PIC (Symplectic PIC vs. Non-Symplectic PIC)
Benchmark Case 2: 1 Turn = 10 FODOs + 1 Sextupole

- 0 current tune 2.417, 30 A current, tune shift 0.113
- sextupole $KL = 10 \text{T/m/m}$
Non-Symplectic PIC Shows Much Less Emittance Growth Compared with Two Symplectic Models (4D Emittance Evolution with Different Currents)
Final Beam X-Px Phase Spaces Have Similar Shapes

- Symplectic gridless
- Symplectic PIC
- Spectral PIC
Final Beam Y-Py Phase Spaces Have Similar Shapes

symplectic gridless

symplectic PIC

spectral PIC
Comparison of Density Profiles

- Two symplectic solvers produce similar density profiles
- Non-symplectic solver produces larger less shoulder
Extra Numerical Emittance Growth with Small Number of Macroparticles

- Little emittance growth in the linear lattice
- Small emittance growth driven by the 3\textsuperscript{rd} order resonance
- Sufficient number of macroparticles needed to suppress numerical emittance growth
Understand the Numerical Emittance Growth from a 1D Model

The smooth and the reconstructed Gaussian distributions from macroparticle sampling with linear, quadratic, and Gaussian kernel deposition

- Much larger mode amplitude fluctuation from the macroparticle depositions than that from the smooth distribution

The mode amplitude of the smooth and the reconstructed Gaussian distributions from macroparticle sampling with linear, quadratic, and Gaussian kernel deposition.
Quantify the Mode Amplitude Fluctuation with Standard Deviation

\[ \rho^l = \frac{1}{N_p N_g \Delta x} \sum_i \sum_I S(x_I - x_i) \sin(\alpha_l x_i) \]

\[ \text{var}(\rho^l) = \frac{1}{N_p} \text{var}(\frac{2}{N_g \Delta x} \sum_I S(x_I - x_i) \sin(\alpha_l x_i)) \]

\[ \text{var}(\frac{2}{N_g \Delta x} \sum_I S(x_I - x_i) \sin(\alpha_l x_i)) \approx \frac{1}{N_p} \left(\frac{2}{N_g \Delta x}\right)^2 \sum_i \left(\sum_I S(x_I - x_i) \sin(\alpha_l x_i)\right)^2 - (\rho^l)^2 \]

- Higher order macroparticle deposition scheme leads to smaller fluctuation
Mode Amplitude Fluctuation Decreases with the Increase of Macroparticle Number

- Fluctuation standard deviation $\sim \frac{1}{\sqrt{N_p}}$
Mode Amplitude Fluctuation Increases with the Increase of Grid Number

- Grid number mainly affects mode number > 10
- Larger grid number results in larger fluctuation
Numerical Errors of in the Charge Density Distribution from Macroparticles Results in Numerical Emittance Growth

\[ \Delta \epsilon \approx (\langle x^2 \rangle \langle x' \delta F \rangle - \langle xx' \rangle \langle x \delta F \rangle) \tau / \epsilon + \frac{1}{2} (\langle x^2 \rangle \langle (\delta F)^2 \rangle - \langle x \delta F \rangle^2) \tau^2 / \epsilon \]

\[ \frac{\Delta \epsilon}{\tau} \approx \frac{1}{2} \langle x^2 \rangle \langle (\delta F)^2 \rangle / \tau / \epsilon \]

- Numerical emittance growth scales close to \(1/Np\) as expected
- Numerical emittance growth scales close to \(1/\sqrt{Np}\)
- The growth mechanism is more complicated

sextupole KL = 0, 64x64 modes

sextupole KL = 10, 64x64 modes
Removing Small Amplitude Fluctuation Modes Using Relative Amplitude Threshold (1)

Spectral amplitude of a 2D Gaussian density (64x64 mode)

Spectral amplitude of a 2D Gaussian density with 1% threshold
Removing Small Amplitude Fluctuation Modes Using Relative Amplitude Threshold (2)

Spectral amplitude of a 2D Gaussian density with 2 sigma threshold

Spectral amplitude of a 2D Gaussian density with 4 sigma threshold
Mitigate the Numerical Emittance Growth by Removing High Frequency Modes in Linear Lattice

- Sextupole \( KL = 0 \), current = 30 A, 25 k macroparticles

- Both numerical filters work well
- Numerical emittance growth is mainly due high frequency errors

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**Graph 1:** Brute force cut-off

**Graph 2:** Threshold filtering

- Emittance growth (%)
- 1000 turns
Mitigate the Numerical Emittance Growth through Threshold Filtering in Nonlinear Lattice

- Direct brute force cut-off filtering is not efficient
- Numerical emittance growth can be mitigated with threshold filtering
- The numerical growth is mainly due low frequency errors

sextupole KL = 10, current = 30 A, 25 k macroparticles
Predefined Maximum Fraction and Four Sigma Threshold Filtering Yields Similar Emittance Growth

sextupole $KL = 10$, current = 30 A, 25 k macroparticles

**Maximum Fraction**
- **Pro** – easy to calculate the threshold value
- **Con** – another hyperparameter

**Standard Deviation**
- **Pro** – calculate the threshold value dynamically
- **Con** – computationally expensive
Computational Complexity

• Symplectic PIC/Spectral PIC: $O(N_p) + O(N_g \log(N_g))$, parallelization can be a challenge
• Symplectic gridless particle: $O(N_m N_p)$, easy parallelization

Summary

- Symplectic space-charge model will help improve the accuracy of simulation for long-term simulation.

- Numerical emittance growth from finite macroparticle sampling can be mitigated using threshold filtering in frequency domain.

- For small number of modes and particles used, the symplectic gridless particle model can be computationally efficient; otherwise, the symplectic PIC model would be more efficient.