Lectures on Partial Differential Equations

Stephan Russenschuck, CAS Thessaloniki
Introduction

- General solution
- Particular solution
- Singular solution
- Eigensolutions
- Exact solution
- Fundamental solution

- You can’t solve differential equations, because if you did, they would name them after yourself; Euler, Laplace, Cauchy, Dirichlet, Bessel, Bernoulli, Poisson, Lagrange, Schroedinger, Hill, anybody?

- We look them up in a book or throw a (FD/FEM) mesh on them; separation of variables, variation of variables, integral transforms, FD, FEM (Galerkin), Runge-Kutta, perturbation theory.

- We then match what is found in books to the given boundary value problems on trivial domains and make sure that the required mathematical structures are compatible with the physical problem at hand.
Episodes

- Affine Spaces and Vector Fields
- Fourier Series
- The Taut String (applications to stretched-wire field measurements)
- Foundations of Vector Analysis
- Maxwell’s Equations in Different Avatars
- Harmonic Fields
- Field Singularities – The Green’s Functions
- Finite-Element Shape Functions
- Numerical Methods for the curl-curl Equation
Vector Fields and their Associated Affine Spaces
Flux Tubes of Mother Earth (or What IS a Magnetic Field)
Different Renderings of the Same Vector Field (ROXIE)

In which way can we declare an algebra or an analysis on this
Vector and Scalar Fields

\[
\begin{align*}
\text{curl } H &= J + \frac{\partial}{\partial t} D, \\
\text{curl } E &= -\frac{\partial}{\partial t} B, \\
\text{div } B &= 0, \\
\text{div } D &= \rho.
\end{align*}
\]
2-Dimensional Trace Space

\[ \mathbf{v} : P_2 \rightarrow \bigcup_{x \in P_2} T_x P_2 : \mathbf{x}(t) \mapsto \mathbf{v}(\mathbf{x}(t)) \]

\[ \mathbf{x}(t) := (x_1(t), x_2(t))^T = \left( \varphi(t), \frac{d\varphi(t)}{dt} \right)^T \]

\[ \frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(t) = \left( \frac{dx_1(t)}{dt}, \frac{dx_2(t)}{dt} \right)^T = \left( \frac{d\varphi(t)}{dt}, \frac{d^2\varphi(t)}{dt^2} \right)^T \]

\[ = \left( x_2(t), -\alpha x_2(t) - \frac{g}{l} \sin x_1(t) \right)^T. \]

Find flow maps:

\[ \phi(t, \mathbf{x}) : P_2 \rightarrow P_2 : \mathbf{x}(0) \mapsto \mathbf{x}(t) \]

\[ \frac{d^2\varphi}{dt^2} + \alpha \frac{d\varphi}{dt} + \frac{g}{l} \sin \varphi = 0 \]
Mathematical Structure of Vector Fields

Affine Space (physical)

1. $P + x \in A$ if $P \in A$ and $x \in V$.
2. $(P + x) + y = P + (x + y)$ for $P \in A$ and $x, y \in V$.
3. There is a unique $x \in V$ such that $P_1 = P_2 + x$ for $P_1, P_2 \in A$.

Vector (Linear) Space (algebraic)

1. For any vectors $a, b, c \in V$ : $(a + b) + c = a + (b + c)$.
2. There is a zero vector $0$ for which $a + 0 = a$ for any vector $a$.
3. For each vector $a \in V$ there is a vector $-a$ in $V$ for which $a + (-a) = 0$.
4. For any vectors $a, b \in V$ : $a + b = b + a$.
5. For any scalar $\lambda \in \mathbb{F}$ and any vectors $a, b \in V$ : $\lambda (a + b) = \lambda a + \lambda b$.
6. For any scalars $\lambda, \mu \in \mathbb{F}$ and any vector $a \in V$ : $(\lambda + \mu)a = \lambda a + \mu a$.
7. For any scalars $\lambda, \mu \in \mathbb{F}$ and any vector $a \in V$ : $(\lambda \mu)a = \lambda (\mu a)$.
8. For the unit scalar $1 \in \mathbb{F}$ and any vector $a \in V$ : $1a = a$.

Inner product space (metric)

1. $\langle a + b, c \rangle = \langle a, c \rangle + \langle b, c \rangle$ and $\langle a, \lambda b + \mu c \rangle = \lambda \langle a, b \rangle + \mu \langle a, c \rangle$.
2. $\langle a, b \rangle = \langle b, a \rangle$.
3. $\langle a, a \rangle > 0$ and $\langle a, a \rangle = 0$ if and only if $a = 0$.

Isomorphism

$P \in A_n \xrightarrow{\text{Origin}} R \in V_n \xrightarrow{\text{Basis}} (x^1, \ldots, x^n) \in \mathbb{R}^n$.

Coordinate map

$\mathbb{R}^n \xrightarrow{[A]} \mathbb{R}^m$.

Mathematical Structure of Vector Fields

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**Norm and Distance**

Length (Norm induced by the scalar product)

\[ \| a \| = \sqrt{\langle a, a \rangle}, \]

Cauchy Schwarz inequality

\[ | \langle a, b \rangle | \leq \| a \| \| b \|, \]

If a basis is present:

\[ \langle a, b \rangle = \sum_{i=1}^{3} \sum_{j=1}^{3} a^i b^j \langle g_i, g_j \rangle \equiv a^i b^j \langle g_i, g_j \rangle =: a^i b^j g_{ij}, \]

(Generalized Pythagoras)

Applications: Calibration of Helmholtz coils, Calibration of 3-axis displacement stages and robots

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Inner and Outer Oriented Surfaces

Outer oriented by the current
Inner and Outer Oriented Surfaces

\[ \int_{\partial \alpha} \mathbf{H} \cdot d\mathbf{s} = \int_{\alpha} \mathbf{J} \cdot d\mathbf{a} \]

Inner oriented because flux is a measure for the voltage that can be generated on the rim.
Discretization on Dual Grids

Faraday Fields are discretized on the primal grid

Ampère-Maxwell Fields are discretized on the dual grid
Consistent inner and outer orientation of a manifold embedded in an encompassing oriented space (requires an origin and a coordinate frame)

09:00 or 15:00?
The Right-Hand Rule or “Magnetic Discussion”

Bruno Touschek (1921-1978)
Framework of our Vectorfields $E_3$ (Euclidean Affine Space)

- $E_3$ has the structure of the affine point space
- It carries the vector (linear) space structure of its associated vector space
- It is equipped with a metric that gives rise to distance and angles

- If an origin and basis is selected, the projection of the position vector on the basis yields the coordinates (in $\mathbb{R}^3$)
- The canonical basis can be made to a basis field by translation
- The components of the field at some point are then the projection on this basis field
Fourier Series
Field Quality

Field map

Good field region
Fourier Series (an Infinite Dimensional Vector Space)

\[ B_r(r_0, \varphi) = \sum_{n=1}^{\infty} (B_n(r_0) \sin n\varphi + A_n(r_0) \cos n\varphi), \]

\[ A_n(r_0) = \frac{1}{\pi} \int_{0}^{2\pi} B_r(r_0, \varphi) \cos n\varphi \, d\varphi, \quad n = 1, 2, 3, \ldots, \]

\[ B_n(r_0) = \frac{1}{\pi} \int_{0}^{2\pi} B_r(r_0, \varphi) \sin n\varphi \, d\varphi, \quad n = 1, 2, 3, \ldots. \]

And on the computer: Discrete setting (don’t bother with the FFT)

\[ A_n(r_0) \approx \frac{2}{N} \sum_{k=0}^{N-1} B_r(r_0, \varphi_k) \cos n\varphi_k, \quad \varphi_k = \frac{2\pi k}{N} \]

\[ B_n(r_0) \approx \frac{2}{N} \sum_{k=0}^{N-1} B_r(r_0, \varphi_k) \sin n\varphi_k, \quad k = 0, 1, 2, \ldots, N - 1. \]
### $E_3$ and $L^2$

<table>
<thead>
<tr>
<th>Structure</th>
<th>Euclidean $E_3$</th>
<th>Hilbert $L^2(\Omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Vector</strong></td>
<td>$x, y$</td>
<td>$f(t), g(t)$</td>
</tr>
<tr>
<td><strong>Basis</strong></td>
<td>${e_1, e_2, e_3}$</td>
<td>${g_n(t)}$</td>
</tr>
<tr>
<td><strong>Scalar product</strong></td>
<td>$\sum_{n=1}^{3} x_n y_n$</td>
<td>$\langle f, g \rangle = \int_{\Omega} f(t) g(t) , dt$</td>
</tr>
<tr>
<td><strong>Norm</strong></td>
<td>$| x | = \sqrt{\sum_{n=1}^{3} x_n^2}$</td>
<td>$| f | = \sqrt{\int_{\Omega}</td>
</tr>
<tr>
<td><strong>Orthonormality</strong></td>
<td>$e_n \cdot e_k = \delta_{nk}$</td>
<td>$\langle g_n, g_k \rangle = \delta_{nk}$</td>
</tr>
<tr>
<td><strong>Expansion</strong></td>
<td>$x = \sum_{n=1}^{3} x_n e_n$</td>
<td>$f(t) = \sum_{n=1}^{\infty} x_n g_n(t)$</td>
</tr>
<tr>
<td><strong>Coefficients</strong></td>
<td>$x_n = x \cdot e_n$</td>
<td>$x_n = \langle g_n, f \rangle$</td>
</tr>
</tbody>
</table>

Hilbert spaces are those in which notation and concepts of ordinary Euclidean geometry hold without any restrictions on the dimension.
The Road Map to Convergence of Fourier Series

- The trigonometric functions are orthogonal
- The Fourier polynomial of grade $n$ is the best approximation of $f$ in $V_n$
- The projections onto the trigonometric functions (scalar product) induces a norm (the RMS error)
- Riemann Lebesgue Lemma: Within this norm, the coefficients converge to zero.

- 3 Convergence theorems
  - For a $C^1$ function $P_n$ converges uniformly to $f(x)$ in any $x$
  - For “clean jumps” $P_n$ converges pointwise to $0.5 \; (f_+(x) + f_-(x))$
  - The Fourier polynomial converges for every square integrable function in the RMS sense (allows jump discontinuities, e.g., at material boundaries)
Trigonometric Functions as Orthogonal Function Set (from Wikipedia)

\[ \int_{-\pi}^{+\pi} \sin(2x) \sin(1x) \, dx = 0 \]

\[
\int_{-\pi}^{\pi} \cos(mx) \cos(nx) \, dx \\
= \frac{1}{2} \int_{-\pi}^{\pi} \cos((n - m)x) + \cos((n + m)x) \, dx = \pi \delta_{mn},
\]

\[
\int_{-\pi}^{\pi} \sin(mx) \sin(nx) \, dx \\
= \frac{1}{2} \int_{-\pi}^{\pi} \cos((n - m)x) - \cos((n + m)x) \, dx = \pi \delta_{mn},
\]

\[
\int_{-\pi}^{\pi} \cos(mx) \sin(nx) \, dx \\
= \frac{1}{2} \int_{-\pi}^{\pi} \sin((n + m)x) + \sin((n - m)x) \, dx = 0.
\]
The Fourier Polynomial is the best Approximation of $f$ within Polynomial Approximations of order $m$

\[
\| f - \sum_{n=0}^{m} \gamma_{n} g_{n} \|^2 = \langle f - \sum_{n=0}^{m} \gamma_{n} g_{n}, f - \sum_{n=0}^{m} \gamma_{n} g_{n} \rangle
\]

\[
= \langle f, f \rangle - 2 \sum_{n=0}^{m} \gamma_{n} \langle f, g_{n} \rangle + \sum_{n=0}^{m} \gamma_{n}^2
\]

\[
= \| f \|^2 - \sum_{n=0}^{m} |\langle f, g_{n} \rangle|^2 + \sum_{n=0}^{m} |\langle f, g_{n} \rangle - \gamma_{n}|^2
\]

Minimum for $\gamma_{n} = c_{n} = \langle f, g_{n} \rangle$

Projection of the square wave onto the “shape” of the trigonometric functions
The Fourier Polynomial $P_n$ is the best Approximation in $V_n$

Projection of the square wave onto the “shape” of the trigonometric functions
Take any $2\pi$ periodic function and develop according to

$$\frac{C_0}{2} + \sum_{n=1}^{\infty} \left( C_n(r_0) \sin n\varphi + D_n(r_0) \cos n\varphi \right).$$

<table>
<thead>
<tr>
<th>$B_n$</th>
<th>$B_{\varphi}$</th>
<th>$B_x$</th>
<th>$B_y$</th>
<th>$A_z$</th>
<th>$\phi_m$</th>
</tr>
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<td>$B_n = $</td>
<td>$C_n$</td>
<td>$D_n$</td>
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</table>

We can use fields, potentials, fluxes, or wire-oscillation amplitudes as “raw data”. The linear differential operators $\text{grad}$ and $\text{rot}$ transform into simple algebra in the $L^2$ space of Fourier coefficients.

**Method of Superposition**
Bessel Inequality and the Riemann Lebesgue Lemma

The Fourier coefficients tend to zero as $n$ goes to infinity

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2$$

Limits: $10^{-6}$ T, $10^{-8}$ Vs

Always plot your results in logarithmic scale
Episode 3

The Taut String
Oscillating Wire Measurements
Oscillating Wire Measurements

Measure the oscillation amplitudes on $K$ rulings of a cylindrical domain. Develop into a Fourier series.

\[ d_y^k(r_0) = \lambda_y \int_0^L B_x(r_0, \varphi_k) \, dz \]
\[ d_x^k(r_0) = \lambda_x \int_0^L B_y(r_0, \varphi_k) \, dz. \]

\[ \tilde{B}_n(r_0) = \frac{2}{K} \sum_{k=0}^{K-1} d_y^k(r_0) \sin n\varphi_k. \]

\[ b_{n+1}(r_0) = \frac{\tilde{B}_n(r_0)}{\tilde{B}_N(r_0)}. \]
The Taut String

\[ f(z + dz) - f(z) = \left( \frac{\partial f}{\partial z} \right) dz + O(dz)^2 \]

\[ T \sin \beta - T \sin \alpha \approx T \left( \left( \frac{\partial u}{\partial z} \right)_{z+dz} - \left( \frac{\partial u}{\partial z} \right)_{z} \right) \]

\[ \lambda_m \frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} - T \frac{\partial^2 u}{\partial z^2} = -I(t)B_n(z) \]

inertia + damping + spring = excitation force

where \( \lambda_m \) is the mass per unit length \([\lambda_m] = 1 \text{ kg m}^{-1}\), \( \alpha \) the damping coefficient \([\alpha] = 1 \text{ kg m}^{-1} \text{ s}^{-1}\). Note that the physical unit of the coefficient \( T/\lambda_m \) is \([T/\lambda_m] = 1 \text{ s}^{-2}\) and therefore

\[ c := \sqrt{\frac{T}{\lambda_m}} \]

CuBe: 9.7 N, 0.125 mm, 0.85 \(10^{-4}\) kg/m, 340 m/s
The One-dimensional Homogenous Wave Equation (no Damping)

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}
\]

Method of Separation

\[
u(z, t) = U(z) Q(t)
\]

\[
U(z) \frac{\partial^2 Q(t)}{\partial t^2} = c^2 \frac{\partial^2 U(z)}{\partial z^2} Q(t)
\]

\[
\frac{\partial^2 U(z)}{\partial z^2} + \mu U(z) = 0
\]

Eigensolutions

\[
U(z) = A e^{\sqrt{\mu} z} + B e^{-\sqrt{\mu} z},
\]

\[
Q(t) = C e^{c \sqrt{\mu} t} + D e^{-c \sqrt{\mu} t},
\]

for \( \mu > 0 \):

\[
U(z) = A \sin(\sqrt{\mu} z) + B \cos(\sqrt{\mu} z),
\]

\[
Q(t) = C \sin(c \sqrt{\mu} t) + D \cos(c \sqrt{\mu} t),
\]

and for \( \mu = 0 \):

\[
U(z) = A + Bz,
\]

\[
Q(t) = C + Dt.
\]
Boundary Conditions

Mode shape function

\[ u(0, t) = 0 \quad u(L, t) = 0 \]

\[ A \sin(\sqrt{\mu}z) = 0 \quad \sqrt{\mu} = \frac{n\pi}{L} \]

Nodal displacement (in time) function

\[ \frac{\partial^2 Q(t)}{\partial t^2} + c^2 \left( \frac{n\pi}{L} \right)^2 Q(t) = 0 \]

\[ Q_n(t) = A_n \sin(\omega_n t) + B_n \cos(\omega_n t) \]

\[ u(z, t) = \sum_n U_n \sin \left( \frac{n\pi}{L} z \right) \sin(\omega_n t + \varphi_n) \]

Remark: Coefficients are still not know yet. It requires initial conditions – plucked string (Guitar) or struck string (piano)

Normal mode frequency

\[ \omega_n = \frac{n\pi}{L} \sqrt{\frac{T}{\lambda_m}} \]

\[ c = 340 \text{ m/s}, \ 2\text{-m-long}, \ \omega_1 = 534 \text{ Hz} \]
More General Form of the 1D Wave Equation

\[ a^2 \frac{\partial^2 u}{\partial z^2} + F(z, t) = \frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} + ku. \]

spring + excitation = inertia + damping + elasticity

**Method of Separation**

\[ \frac{\partial^2 U(z)}{\partial z^2} + \mu U(z) = 0 \]

\[ \frac{\partial^2 Q(t)}{\partial t^2} + \alpha \frac{\partial Q(t)}{\partial t} + c^2 \mu Q(t) = 0 \]

\[ Q = e^{rt} \]

Sturm-Liouville equation with linear, self-adjoint diff. operator \( F = \langle Lf, g \rangle = \langle f, Lg \rangle \) Functional analysis on infinite dimensional vector-spaces. Preserving the vector space properties under the actions of the operator. **Existence of a converging, orthogonal projection operator (spectral theorem)**

- all the eigenvalues are real,
- to each eigenvalue there is one and only one linearly dependent eigenfunction,
- the eigenfunctions are orthogonal,
- if the function is continuously differentiable on the interval, the function can be expressed as a convergent series of these eigenfunctions.
Eigensolutions of the Sturm-Liouville Problem

\[ r^2 + \alpha r + c^2 \mu = 0 \]

\[ r = -\frac{\alpha}{2} \pm \sqrt{\left(\frac{\alpha}{2}\right)^2 - c^2 \mu} \]

"Sufficiently" small damping \[ \frac{c^2 \pi}{\alpha L} > 1 \]

\[ (\frac{\alpha}{2})^2 < \min\{c^2 \mu_n\} = \min \left\{ c^2 \left(\frac{n \pi}{L}\right)^2 \right\} \]

\[ Q_n(t) = e^{-\frac{\alpha}{2} t} \left( C_n e^{i\sqrt{c^2 \mu - \left(\frac{\alpha}{2}\right)^2} t} + D_n e^{-i\sqrt{c^2 \mu - \left(\frac{\alpha}{2}\right)^2} t} \right) \]

\[ = e^{-\frac{\alpha}{2} t} \left( (C_n + D_n) \cos \left( c^2 \mu - \left(\frac{\alpha}{2}\right)^2 t \right) + i(C_n - D_n) \sin \left( c^2 \mu - \left(\frac{\alpha}{2}\right)^2 t \right) \right) \]

\[ = e^{-\frac{\alpha}{2} t} \left( A_n \cos \left( c^2 \mu - \left(\frac{\alpha}{2}\right)^2 t \right) + B_n \sin \left( c^2 \mu - \left(\frac{\alpha}{2}\right)^2 t \right) \right) \]

\[ Q_n(t) = U_n e^{-\frac{\alpha}{2} t} \sin(\tilde{\omega}_n t + \varphi_n) \]

\[ \tilde{\omega}_n = \frac{n \pi}{L} \sqrt{\frac{T}{\lambda_m} - \left(\frac{\alpha}{2}\right)^2} \]

\[ u(z, t) = \sum_n e^{-\frac{\alpha}{2} t} \sin \left( \frac{n \pi}{L} z \right) (A_n \sin(\tilde{\omega}_n t) + B_n \cos(\tilde{\omega}_n t)) \]

Nothing is "solved" yet
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Initial Conditions (Again)

\[
u(z, t) = \sum_n e^{-\frac{n^2}{L^2} t} \sin \left(\frac{n\pi z}{L}\right) (A_n \sin(\omega_n t) + B_n \cos(\omega_n t))
\]

\[
u_0(z) = \sum_n B_n \sin \left(\frac{n\pi z}{L}\right)
\]

\[
\int_0^L \sin \left(\frac{m\pi z}{L}\right) \left(\sum_n B_n \sin \left(\frac{n\pi z}{L}\right)\right) \, dz = \int_0^L \sin \left(\frac{m\pi z}{L}\right) u_0(z) \, dz
\]

Unknown

\[
B_n = \frac{2}{L} \int_0^L u_0(z) \sin \left(\frac{n\pi z}{L}\right) \, dz = C_n
\]

Known

Because of orthogonality:

\[
\int_0^L \sin \left(\frac{m\pi z}{L}\right) \sin \left(\frac{n\pi z}{L}\right) \, dz = \frac{L}{2} \quad \text{for} \quad m = n
\]
The Inhomogeneous Equation (Variations of Variables - Guesstimations)

\[ u(x, t) = \sum_n U \sin \left( \frac{n \pi}{L} z \right) \sin(\omega t - \varphi_n) \]

\[ F(z, t) = -B_n(z)I_0 \sin(\omega t) \]

Lorentz Force Term on the Wire
Notice \( n = \) normal

\[ \varphi_m = \arctan \left( \frac{\alpha \omega}{-\lambda_m \omega^2 + T \left( \frac{m \pi}{L} \right)^2} \right) \]

Modal force

\[ u(z, t) = \frac{2I_0}{L} \sum_m \frac{\int_0^L B_n(z) \sin \left( \frac{m \pi}{L} z \right) \, dz}{\sqrt{\left[ T \left( \frac{m \pi}{L} \right)^2 - \lambda_m \omega^2 \right]^2 + (\alpha \omega)^2}} \sin \left( \frac{m \pi}{L} z \right) \sin(\omega t - \varphi_m) \]

Nodal displacement

Mode shape function
Check 2: Numerical simulation (FDTD) and the Steady State Solution

\[
\frac{\partial^2 u}{\partial t^2}(t_i, z_n) \approx \frac{u(t_{i+1}, z_n) - 2u(t_i, z_n) + u(t_{i-1}, z_n)}{(\Delta t)^2} + O((\Delta t)^2),
\]

\[
\frac{\partial^2 u}{\partial z^2}(t_i, z_n) \approx \frac{u(t_i, z_{n+1}) - 2u(t_i, z_n) + u(t_i, z_{n-1})}{(\Delta z)^2} + O((\Delta z)^2).
\]

\[
u(t_i, z_{n+1}) = \sigma^2 u(t_i, z_{n+1}) + 2(1 - \sigma^2)u(t_i, z_n) + \sigma^2 u(t_i, z_{n-1}) - u(t_{i-1}, z_n) - B_n(z_n)I_0(t_i)
\]

\[
\sigma = \frac{c\Delta t}{\Delta z}
\]
Why Have we not Started Numerically

Remember: We wanted to proof that:

\[ d_y^k(r_0) = \lambda_y \int_0^L B_x(r_0, \varphi_k) \, dz \]

And we can only measure the amplitude at one position \( z_0 \)
Solution of the Wave Equation (Assumptions and Check)

- Uniformity: The string has a constant mass density $\rho$.
- Planar oscillations: The string is suspended at the origin $z = 0$ and $z = L$. The string deflection $u(z, t)$ is caused by the distributed force, which is proportional to the normal field to this plane: $f(z, t) = I(t)B_n(z)$.
- Uniform tension: Each segment of the string pulls on its neighboring segments with the same magnitude of force $T$.
- The only force in the wire is its tension and the Lorentz force. No gravitational, frictional, or other external forces (wind) are considered.
- Small vibrations: The slope $du(z, t)/dz$ remains small in the interval $[0, L]$.
- Steady state oscillations: After an initial setting time, the string oscillates in the form of a standing wave. Then there will be no energy flow along the string and no energy loss in the fixed suspensions.

![Graphs showing wire oscillations](image)
Solution of the Wave Equation (Check 2)

Check: Behavior around the first natural resonance: Are the fit parameters physically meaningful?

\[ f(\omega) = \frac{a}{\sqrt{[b^2 - \omega^2]^2 + (c\omega)^2}} \]

\sim \text{Tension} \quad \sim \text{Damping} \quad \sim \text{Mass density}

![Graphs showing behavior around resonances with frequency on the x-axis and amplitude on the y-axis.](image)
Nonlinearities and Overtones

Nonlinear stress-strain relations in the cable (we tension close to the Hook limit); results in a coupling of the planes of motion.
Foundations of Vector Analysis
Directional Derivative and the Total Differential

Space curve with \( \mathbf{r}(t) = (x(t), y(t), z(t)) \) parametrized such that \( \mathbf{r}(0) = P \).

1-smooth scalar field \( \phi : E_3 \rightarrow \mathbb{R} : \mathbf{r} \mapsto \phi(\mathbf{r}) \) expressed as \( \phi(x, y, z) \), then \( \phi(\mathbf{r}(t)) \) at parameter (time) \( t \).

\[
\frac{\partial \phi}{\partial v} = \frac{d\phi}{dt} = \frac{d}{dt} [\phi(\mathbf{r} + tv)]_{t=0} = \lim_{t \to 0} \frac{\phi(\mathbf{r} + tv) - \phi(\mathbf{r})}{t}
\]

\[
\frac{\partial \phi}{\partial v} = \frac{d}{dt} \phi(\mathbf{r}(t)) = \frac{\partial \phi}{\partial x} \frac{dx}{dt} + \frac{\partial \phi}{\partial y} \frac{dy}{dt} + \frac{\partial \phi}{\partial z} \frac{dz}{dt} = \nabla \phi \cdot \mathbf{v}
\]

\[
\nabla \phi = \frac{\partial \phi}{\partial x} \mathbf{e}_x + \frac{\partial \phi}{\partial y} \mathbf{e}_y + \frac{\partial \phi}{\partial z} \mathbf{e}_z
\]

Best linear approximation of \( \phi \) over displacement distance \( d\mathbf{r} \)

\[
\mathbf{d}r = \mathbf{v} dt = \frac{\mathbf{v}}{v} v dt = \mathbf{T} ds \quad \mathbf{d}a = \mathbf{n} da = \left( \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) dudv \quad \mathbf{d}f = \frac{\partial \mathbf{f}}{\partial x} dx + \frac{\partial \mathbf{f}}{\partial y} dy + \frac{\partial \mathbf{f}}{\partial z} dz
\]
Precondition for the Differential under the Integral

\[
\frac{d}{dt} \int_{\mathcal{A}} \mathbf{B} \cdot d\mathbf{a} = \int_{\mathcal{A}} \left( \frac{\partial \mathbf{B}}{\partial t} + \mathbf{v}_p \text{div} \mathbf{B} \right) \cdot d\mathbf{a} - \int_{\partial \mathcal{A}} (\mathbf{v}_p \times \mathbf{B}) \cdot d\mathbf{r},
\]
Remember the Cauchy Schwarz inequality

Thus for the directional derivative

This implies that the directional derivative takes its maximum when \( \mathbf{v} \) points in the direction of the gradient. Therefore gradient points in the direction of the steepest ascent of \( \Phi \) and is thus normal to the surface of equipotential.

The flux density \( \mathbf{B} \) exits a highly permeable surface in normal direction. Therefore the pole shape of normal conducting magnets can be seen as an equipotential of the magnetic scalar potential.
Grad, Curl and Div in Cartesian Coordinates

\[ \text{grad } \phi := \frac{\partial \phi}{\partial x} \mathbf{e}_x + \frac{\partial \phi}{\partial y} \mathbf{e}_y + \frac{\partial \phi}{\partial z} \mathbf{e}_z \]

\[ \text{curl } \mathbf{g} = \left( \frac{\partial g_z}{\partial y} - \frac{\partial g_y}{\partial z} \right) \mathbf{e}_x + \left( \frac{\partial g_x}{\partial z} - \frac{\partial g_z}{\partial x} \right) \mathbf{e}_y + \left( \frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y} \right) \mathbf{e}_z. \]

\[ \text{div } \mathbf{g} = \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z}. \]
The First Lemma of Poincare

\[
\text{curl } \text{grad } \phi = \text{curl } \left[ \frac{1}{h_1} \frac{\partial \phi}{\partial u^1} \mathbf{e}_u^1 + \frac{1}{h_2} \frac{\partial \phi}{\partial u^2} \mathbf{e}_u^2 + \frac{1}{h_3} \frac{\partial \phi}{\partial u^3} \mathbf{e}_u^3 \right]
\]

\[
= \frac{1}{h_2 h_3} \left( \frac{\partial^2 \phi}{\partial u^2 \partial u^3} - \frac{\partial^2 \phi}{\partial u^3 \partial u^2} \right) \mathbf{e}_u^1
\]

\[
+ \frac{1}{h_3 h_1} \left( \frac{\partial^2 \phi}{\partial u^3 \partial u^1} - \frac{\partial^2 \phi}{\partial u^1 \partial u^3} \right) \mathbf{e}_u^2
\]

\[
+ \frac{1}{h_1 h_2} \left( \frac{\partial^2 \phi}{\partial u^1 \partial u^2} - \frac{\partial^2 \phi}{\partial u^2 \partial u^1} \right) \mathbf{e}_u^3 = 0,
\]

Ugly and not even a universal proof (orthogonality assumed)
Coordinate Free Definition of Grad, Curl, and Div

\[ \int_{P_2} a \cdot dr = \int_{P_2} \text{grad} \phi \cdot dr = \int_{P_2} d\phi = \phi(P_2) - \phi(P_1), \]

\[ n \cdot \text{curl} g = \lim_{a \to 0} \frac{\int_{\partial A} g \cdot dr}{a}, \]

\[ \text{div} g = \lim_{V \to 0} \frac{\int_{\partial V} g \cdot da}{V}, \]

\[ g_z + \frac{\partial g_y}{\partial y} \Delta y, \]

\[ g_y + \frac{\partial g_z}{\partial z} \Delta z, \]

\[ g_y, \]

\[ g_z + \frac{\partial g_x}{\partial x} \Delta x, \]

\[ e_x, e_y, e_z, x, y, z. \]
The Boundary Operator

\[ \partial(\partial \mathcal{V}) = \emptyset, \quad \partial(\partial \mathcal{A}) = \emptyset, \]

\[ \int_{\mathcal{V}} \text{div} \ curl \mathbf{g} \, dV = \int_{\partial \mathcal{V}} \text{curl} \mathbf{g} \cdot d\mathbf{a} = \int_{\partial(\partial \mathcal{V})} \mathbf{g} \cdot d\mathbf{r} = 0, \]

\[ \int_{\mathcal{A}} \text{curl} \ \text{grad} \phi \cdot d\mathbf{a} = \int_{\partial \mathcal{A}} \text{grad} \phi \cdot d\mathbf{r} = \phi \bigg|_{\partial(\partial \mathcal{A})} = 0, \]

Reversal of arguments yields two important statements (next slides):
Much nicer than writing it in coordinates
The second Lemma of Poincare (Contractible Domains)

\[ \text{div } b = 0 \quad \rightarrow \quad b = \text{curl } a. \]

\[ \text{curl } h = 0 \quad \rightarrow \quad h = \text{grad } \phi. \]
Lemmata of Poincare (Non-Contractible Domains)

Toroidal domain $\Omega$ in a cylindrical coordinate system $(r, \varphi, z)$:

$$H_\varphi = \frac{I}{2\pi r}$$

$$\text{curl } \mathbf{H} = \frac{1}{r} \frac{\partial}{\partial r} (rH_\varphi) = 0$$

But $\int_C \mathbf{H} \cdot d\mathbf{s} = I$ and $\Omega$, with $\int_C \text{grad } \phi \cdot d\mathbf{s} = 0$

Domain $\Omega$ between two nested spheres centered at the origin.

$$D_R = \frac{Q}{4\pi R^2 e_R}$$

$$\text{div } \mathbf{D} = \frac{1}{r^2} \frac{\partial}{\partial R} (R^2 D_R) = 0$$

But $\oint_a \mathbf{D} \cdot d\mathbf{a} = Q$ and $\oint_a \text{curl } \mathbf{A} \cdot d\mathbf{a} = 0$
Kelvin-Stokes Theorem

Smooth vector fields, smooth surfaces with simply connected, closed, piecewise-smooth and consistently oriented boundaries, and volumes with piecewise-smooth, closed and consistently oriented surfaces.

\[
\int_{\partial A} \mathbf{g} \cdot d\mathbf{r} = \int_{\partial A_1} \mathbf{g} \cdot d\mathbf{r} + \int_{\partial A_2} \mathbf{g} \cdot d\mathbf{r} = \int_{\partial A_{11}} \mathbf{g} \cdot d\mathbf{r} + \int_{\partial A_{22}} \mathbf{g} \cdot d\mathbf{r},
\]

No jump discontinuities (for example, co-moving shielding devices)

\[
\int_{\partial A} \mathbf{g} \cdot d\mathbf{r} = \lim_{I \to \infty} \sum_{i=1}^{I} \int_{\partial A_i} \mathbf{g} \cdot d\mathbf{r} = \lim_{I \to \infty} \sum_{i=1}^{I} \Delta a_i \frac{1}{\Delta a_i} \int_{\partial A_i} \mathbf{g} \cdot d\mathbf{r}
\]

\[
= \lim_{I \to \infty} \sum_{i=1}^{I} (\text{curl} \mathbf{g})_i \cdot \mathbf{n} \Delta a_i = \int_{A} \text{curl} \mathbf{g} \cdot d\mathbf{a}.
\]
Gauss’ Theorem

Smooth vector fields, smooth surfaces with simply connected, closed, piecewise-smooth and consistently oriented boundaries, and volumes with piecewise-smooth, closed and consistently oriented surfaces.

\[
\int_{\partial \mathcal{V}} \mathbf{g} \cdot d\mathbf{a} = \lim_{I \to \infty} \sum_{i=1}^{I} \int_{\partial \mathcal{V}_i} \mathbf{g} \cdot d\mathbf{a} = \lim_{I \to \infty} \sum_{i=1}^{I} \Delta V_i \frac{1}{\Delta V_i} \int_{\partial \mathcal{V}_i} \mathbf{g} \cdot d\mathbf{a}
\]

\[
= \lim_{I \to \infty} \sum_{i=1}^{I} (\text{div } \mathbf{g})_i \Delta V_i = \int_{\mathcal{V}} \text{div } \mathbf{g} \, dV.
\]
More Integral Theorems

Green’s First
\[ \int_{\mathcal{V}} \left( \text{grad} \, \phi \cdot \text{grad} \, \psi + \phi \nabla^2 \psi \right) \, dV = \int_{\partial \mathcal{V}} \phi \, \partial_n \psi \, da. \]

Green’s Second
\[ \int_{\Omega} \left( \phi \nabla^2 \psi - \psi \nabla^2 \phi \right) \, dV = \int_{\Gamma} \left( \phi \partial_n \psi - \psi \partial_n \phi \right) \, da \]

Vector Form of Green’s Second
\[ \int_{\mathcal{V}} \mathbf{a} \cdot \text{curl} \, \mathbf{b} \, dV = \int_{\mathcal{V}} \mathbf{b} \cdot \text{curl} \, \mathbf{a} \, dV - \int_{\partial \mathcal{V}} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{n}) \, da. \]

Generalization of the Integration by Parts Rule
\[ -\int_{\mathcal{V}} \mathbf{a} \cdot \text{grad} \, \phi \, dV = \int_{\mathcal{V}} \phi \, \text{div} \, \mathbf{a} \, dV - \int_{\partial \mathcal{V}} \phi (\mathbf{a} \cdot \mathbf{n}) \, da. \]

Stratton #1 and #2
\[ \int_{\mathcal{V}} \text{div} (\mathbf{a} \times \text{curl} \, \mathbf{b}) \, dV = \int_{\partial \mathcal{V}} (\mathbf{a} \times \text{curl} \, \mathbf{b}) \cdot \mathbf{n} \, da \]
\[ \int_{\mathcal{V}} (\mathbf{a} \text{curl} \, \text{curl} \, \mathbf{b} - \mathbf{b} \text{curl} \, \text{curl} \, \mathbf{a}) \, dV = \int_{\partial \mathcal{V}} (\mathbf{b} \times \text{curl} \, \mathbf{a} - \mathbf{a} \times \text{curl} \, \mathbf{b}) \cdot \mathbf{n} \, da. \]