Non Linear Dynamics - Phenomenology

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Summary

- Phase space dynamics – fixed point analysis
- Poincaré map
- Motion close to a resonance
- Onset of chaos
- Chaos detection methods
  - Dynamic Aperture
  - Lyapunov exponent
  - Frequency map analysis
  - Numerical applications
Phase space dynamics
- Fixed point analysis
Phase space dynamics

- Valuable description when examining trajectories in phase space \((u, p_u)\).
- Existence of integral of motion imposes geometrical constraints on phase flow.
- For the simple harmonic oscillator, the Hamiltonian is:
  \[ H = \frac{1}{2} \left( p_u^2 + \omega_0^2 u^2 \right) \]

Phase space curves are ellipses around the equilibrium point parameterized by the Hamiltonian (energy).
Phase space dynamics

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- Existence of integral of motion imposes geometrical constraints on phase flow
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phase space curves are ellipses around the equilibrium point parameterized by the Hamiltonian (energy)

- By simply changing the sign of the potential in the harmonic oscillator, the phase trajectories become hyperbolas, symmetric around the equilibrium point where two straight lines cross, moving towards and away from it
Non-linear oscillators

- Conservative non-linear oscillators have Hamiltonian

\[ H = E = \frac{1}{2} p_u^2 + V(u) \]

with the potential being a general (polynomial) function of positions

- **Equilibrium points** are associated with extrema of the potential
Non-linear oscillators

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with the potential being a general (polynomial) function of positions.

Equilibrium points are associated with extrema of the potential.

Considering three non-linear oscillators:

- **Quartic** potential (left): two minima and one maximum
- **Cubic** potential (center): one minimum and one maximum
- **Pendulum** (right): periodic minima and maxima
Fixed point analysis

Consider a general second order system

\[
\frac{du}{dt} = f_1(u, p_u)
\]
\[
\frac{dp_u}{dt} = f_2(u, p_u)
\]

Equilibrium or “fixed” points \( f_1(u_0, p_{u0}) = f_2(u_0, p_{u0}) = 0 \) are determinant for topology of trajectories at their vicinity.
Fixed point analysis

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The linearized equations of motion at their vicinity are
\[
\frac{d}{dt} \begin{bmatrix} \delta u \\ \delta p_u \end{bmatrix} = \mathcal{M}_J \begin{bmatrix} \delta u \\ \delta p_u \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1(u_0, p_{u0})}{\partial u} & \frac{\partial f_1(u_0, p_{u0})}{\partial p_u} \\ \frac{\partial f_2(u_0, p_{u0})}{\partial u} & \frac{\partial f_2(u_0, p_{u0})}{\partial p_u} \end{bmatrix} \begin{bmatrix} \delta u \\ \delta p_u \end{bmatrix}
\]

Jacobian matrix

Fixed point nature is revealed by eigenvalues of \( \mathcal{M}_J \), i.e. solutions of the characteristic polynomial \( \det |\mathcal{M}_J - \lambda I| = 0 \).
For **conservative systems** of 1 degree of freedom, the second order characteristic polynomial for any fixed point has two possible solutions:

- Two **complex eigenvalues** with opposite sign, corresponding to **elliptic** fixed points. Phase space flow is described by **ellipses**, with particles evolving clockwise or anti-clockwise.

![Diagram of elliptic fixed point](image)
For **conservative systems** of 1 degree of freedom, the second order characteristic polynomial for any fixed point has two possible solutions:

- Two **complex eigenvalues** with opposite sign, corresponding to **elliptic** fixed points. Phase space flow is described by **ellipses**, with particles evolving clockwise or anti-clockwise.

- Two **real eigenvalues** with opposite sign, corresponding to **hyperbolic** (or saddle) fixed points. Flow described by two lines (or manifolds), incoming (stable) and outgoing (unstable).
Pendulum fixed point analysis

- The “fixed” points for a pendulum can be found at
  \[(\phi_n, p_\phi) = (\pm n\pi, 0), \quad n = 0, 1, 2 \ldots\]

- The Jacobian matrix is
  \[
  \begin{bmatrix}
  0 & 1 \\
  -\frac{g}{L} \cos \phi_n & 0
  \end{bmatrix}
  \]

- The eigenvalues are \(\lambda_{1,2} = \pm i \sqrt{\frac{g}{L} \cos \phi_n}\)
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- Two cases can be distinguished:
  - \[\phi_n = 2n\pi\], for which \[\lambda_{1,2} = \pm i \sqrt{\frac{g}{L}}\]
    corresponding to **elliptic** fixed points
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Two cases can be distinguished:

- \(\phi_n = 2n\pi\), for which \(\lambda_{1,2} = \pm i \sqrt{\frac{g}{L}}\)
  corresponding to **elliptic** fixed points

- \(\phi_n = (2n + 1)\pi\), for which \(\lambda_{1,2} = \pm \sqrt{\frac{g}{L}}\)
  corresponding to **hyperbolic** fixed points

The **separatrix** are the stable and unstable manifolds through the hyperbolic points, separating bounded **librations** and unbounded **rotations**
Consider now a simple harmonic oscillator where the **frequency** is **time-dependent**

\[ H = \frac{1}{2} \left( p_u^2 + \omega_0^2(t) u^2 \right) \]

- Plotting the evolution in phase space, provides trajectories that **intersect** each other
- The phase space has **time** as **extra dimension**
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The phase space has time as extra dimension.

By rescaling the time to become \( \tau = \omega_0 t \) and considering every integer interval of the new “time” variable, the phase space looks like the one of the harmonic oscillator.

This is the simplest version of a Poincaré surface of section, which is useful for studying geometrically phase space of multi-dimensional systems.
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\[ H = \frac{1}{2} \left( p_u^2 + \omega_0^2(t) u^2 \right) \]

- Plotting the evolution in phase space, provides trajectories that **intersect** each other
- The phase space has **time** as **extra dimension**
- By **rescaling** the **time** to become \( \tau = \omega_0 t \) and considering every integer interval of the **new** \( \tau \) \( \text{“time”} \) variable, the **phase space** looks like the one of the **harmonic oscillator**
- This is the simplest version of a **Poincaré surface of section**, which is useful for studying geometrically phase space of multi-dimensional systems
- The **fixed point** in the surface of section is now a periodic orbit
Poincaré map
First recurrence or Poincaré map (or surface of section) is defined by the intersection of trajectories of a dynamical system, with a fixed surface in phase space.
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For an **autonomous** Hamiltonian system $H(q, p)$ (no *explicit* time dependence), it can be chosen to be any fixed surface in phase space, e.g. $q_i = 0$.

For a **non-autonomous** Hamiltonian system $H(q, p, t)$ (explicit time dependence), which is *periodic*, it can be chosen as the period $t = T_0$.
First recurrence or Poincaré map (or surface of section) is defined by the intersection of trajectories of a dynamical system, with a fixed surface in phase space.

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For a non-autonomous Hamiltonian system $H(q, p, t)$ (explicit time dependence), which is periodic, it can be chosen as the period $t = T_0$.

In a system with $n$ degrees of freedom (or $n + 1$ including time), the phase space has $2n$ (or $2n + 2$) dimensions.

By fixing the value of the Hamiltonian to $H_0$, the motion on a Poincaré map is reduced to $2n - 2$ (or $2n$).
- Particularly useful for a system with 2 degrees of freedom, or 1 degree of freedom + time, as the motion on Poincaré map is described by 2-dimensional curves

- For continuous system, numerical techniques exist to compute the surface exactly (e.g. M. Henon Physica D 5, 1982)
Particularly useful for a system with 2 degrees of freedom, or 1 degree of freedom + time, as the motion on Poincaré map is described by 2-dimensional curves.

For continuous system, numerical techniques exist to compute the surface exactly (e.g. M. Henon Physica D 5, 1982).

Example from Astronomy: the logarithmic galactic potential

\[
H_q(x, y, X, Y) = \frac{1}{2} (X^2 + Y^2) + \ln \left( x^2 + \frac{y}{q^2} + R_c^2 \right)
\]

\[(x, y, X, Y) \mapsto (\phi_x, \phi_y, J_x, J_y)\]

\[y = 0\]

\[\phi_y = 0\]
- Record the particle coordinates at one location in a ring.
- Unperturbed motion lies on a circle in normalized coordinates (simple rotation).

\[ Poincaré\ Section: \]

\[ \Delta \phi_{\text{turn}} \]

\[ U' \]

\[ U \]

\[ 1 \]

\[ 2 \]

\[ 3 \]

\[ \sqrt{2J} \]
- Record the particle coordinates at one location in a ring.

- Unperturbed motion lies on a circle in normalized coordinates (simple rotation).

- Resonance condition corresponds to a periodic orbit or fixed points in phase space.

- For a non-linear kick, the radius will change by \( \delta(\sqrt{2J}) \) and the particles stop lying on circles.
Example: Single Octupole

- Simple map with single octupole kick with integrated strength $k_3$ + rotation with phase advances $\mu_x, \mu_y$

```python
def OctupoleMap(k3, x, px, y, py):
    x1 = x
    px1 = px - k3*(x**3-3*x*y**2)
    y1 = y
    py1 = py - k3*(-3*x**2*y+y**3)
    return x1, px1, y1, py1

def Rotation(mux, muy, x, px, y, py):
    x1 = cos(mux)*x+sin(mux)*px
    px1 = -sin(mux)*x+cos(mux)*px
    y1 = cos(muy)*y+sin(muy)*py
    py1 = -sin(muy)*y+cos(muy)*py
    return x1, px1, y1, py1
```

- Restrict motion in $(x, p_x)$ plane i.e. $y_0 = p_{y0} = 0$
- Iterate for a number of “turns” (here 1000)
- Simple **map** with single octupole kick with integrated strength $k_3^3$ + rotation with phase advances $(\mu_x, \mu_y)$

```python
def OctupoleMap(k3, x, px, y, py):
x1 = x
px1 = px - k3*(x**3 - 3*x*y**2)
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```

- Restrict motion in $(x, p_x)$ plane i.e. $y_0 = p_{y0} = 0$
- Iterate for a number of “turns” (here 1000)
- **Appearance of invariant curves** ("distorted" circles), where "action" is an integral of motion
- **Resonant islands** with stable and separatrices with unstable fixed points
- **Chaotic motion**
- Electromagnetic fields coming from multi-pole expansions (polynomials) do not bound phase space and chaotic trajectories may eventually escape to infinity (Dynamic Aperture)
- For some fields like beam-beam and space-charge this is not true, i.e. chaotic motion leads to halo formation

**Example: Single Octupole**

\[ \mu_x = 0.22 \]
Motion close to a resonance
Secular perturbation theory

- The vicinity of a resonance $n_1 \omega_1 + n_2 \omega_2 = 0$, can be studied through secular perturbation theory (see appendix) or transforming the 1-turn map (see Etienne’s lectures).

- A canonical transformation is applied such that the new variables are in a frame remaining on top of the resonance.

- If one frequency is slow, one can average the motion and remain only with a 1 degree of freedom Hamiltonian which looks like the one of the pendulum.

- Thereby, one can find the location and nature of the fixed points measure the width of the resonance.
Fixed points for general multi-pole

- For any polynomial perturbation of the form $x^k$ the “resonant” Hamiltonian is written as

\[ \hat{H}_2 = \delta J_2 + \alpha(J_2) + J_2^{k/2} A_{kp} \cos(k \psi_2) \]

- With the distance to the resonance defined as \( \nu = \frac{p}{3} + \delta \), \( \delta << 1 \)

- The non-linear shift of the tune is described by the term \( \alpha(J_2) \)

- The conditions for the fixed points are

\[ \sin(k \psi_2) = 0 \text{ , } \delta + \frac{\partial \alpha(J_2)}{\partial J_2} + \frac{k}{2} J_2^{k/2-1} A_{kp} \cos(k \psi_2) = 0 \]

- There are fixed points for which \( \cos(k \psi_{20}) = -1 \) and the fixed points are stable (elliptic). They are surrounded by ellipses

- There are also fixed points for which \( \cos(k \psi_{20}) = 1 \) and the fixed points are unstable (hyperbolic). The trajectories are hyperbolas
The Hamiltonian for a sextupole close to a third order resonance is
\[ \hat{H}_2 = \delta J_2 + J_2^{3/2} A_{3p} \cos(3\psi_2) \]

Note the absence of the non-linear tune-shift term (in this 1\textsuperscript{st} order approximation!)

By setting the Hamilton’s equations equal to zero, three fixed points can be found at
\[ \psi_{20} = \frac{\pi}{3}, \frac{3\pi}{3}, \frac{5\pi}{3}, \quad J_{20} = \left( \frac{2\delta}{3A_{3p}} \right)^2 \]

For \( \frac{\delta}{A_{3p}} > 0 \) all three points are unstable

Close to the elliptic one at \( \psi_{20} = 0 \) the motion in phase space is described by circles that they get more and more distorted to end up in the “triangular” separatrix uniting the unstable fixed points

The tune separation from the resonance is \( \delta = \frac{3A_{3p}}{2} J_{20}^{1/2} \)
Simple **map** with single **sextupole kick** with integrated strength $k_2$ + rotation with phase advances $(\mu_x, \mu_y)$

```python
def SextupoleMap(k2, x, px, y, py):
    x1 = x
    px1 = px - k2*(x**2-y**2)
    y1 = y
    py1 = py - k2*(-2*x*y)
    return x1, px1, y1, py1
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def Rotation(mux, muy, x, px, y, py):
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```

- Restrict motion in $(x, p_x)$ plane i.e. $y_0 = p_y y_0 = 0$
- Iterate for a number of “turns” (here 1000)
**Example: Single Sextupole**

- Simple **map** with single **sextupole kick** with integrated strength $k_2 + \text{rotation with phase advances} (\mu_x, \mu_y)$

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```

- **Restrict motion in** $(x, p_x)$ **plane i.e.** $y_0 = p_y = 0$
- **Iterate for a number of “turns”** (here 1000)
- Appearance of 3\textsuperscript{rd} order resonance for certain phase advance
- … but also 4\textsuperscript{th} order resonance

\[ \mu_x = 0.38 \]
- Appearance of 3rd order resonance for certain phase advance
- ... but also 4th order resonance

\[ \mu_x = 0.253 \]
- Appearance of 3\textsuperscript{rd} order resonance for certain phase advance
- ... but also 4\textsuperscript{th} order resonance
- ... and 5\textsuperscript{th} order resonance

\[ \mu_x = 0.21 \]
- Appearance of 3\textsuperscript{rd} order resonance for certain phase advance
- … but also 4\textsuperscript{th} order resonance
- … and 5\textsuperscript{th} order resonance
- … and 6\textsuperscript{th} order and 7\textsuperscript{th} order and several higher orders…

\[ \mu_x = 0.18 \]
Fixed points for an octupole

- The resonant Hamiltonian close to the 4th order resonance is written as

\[ \hat{H}_2 = \delta J_2 + cJ_2^2 + J_2^2 A_{4p} \cos(4\psi_2) \]

- The fixed points are found by taking the derivative over the two variables and setting them to zero, i.e.

\[
\begin{align*}
\sin(4\psi_2) &= 0, \\
\delta + 2cJ_2 + 2J_2 A_{kp} \cos(4\psi_2) &= 0
\end{align*}
\]

- The fixed points are at

\[ \psi_{20} = \left( \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi \right) \]

- For half of them, there is a minimum in the potential as \( \cos(4\psi_{20}) = -1 \) and they are elliptic and half of them they are hyperbolic as \( \cos(4\psi_{20}) = 1 \).
Regular motion near the center, with curves getting more deformed towards a rectangular shape

- The separatrix passes through 4 unstable fixed points, but motion seems well contained
- Four stable fixed points exist and they are surrounded by stable motion (islands of stability)
- Question: Can the central fixed point become hyperbolic (answer in the appendix)
Now, if $\mathcal{C} = 0$ the solution for the action is $J_{20} = 0$.

So there is **no minima** in the potential, i.e. the central fixed point is **hyperbolic**.
As for the sextupole, the octupole can excite any resonance

Multi-pole magnets can excite any resonance order

It depends on the tunes, strength of the magnet and particle amplitudes.
Adding a sextupole and an octupole increases the chaotic motion region, when close to the 4th order resonance.
Adding a sextupole and an octupole increases the chaotic motion region, when close to the 4\textsuperscript{th} order resonance.

But also allows the appearance of 3\textsuperscript{rd} order resonance stable fixed points.
Onset of chaos
Path to chaos

- When **perturbation** becomes **higher**, motion around the **separatrix** becomes **chaotic** (producing **tongues** or **splitting** of the separatrix)

- **Unstable** fixed points are indeed the **source of chaos** when a perturbation is added
Chaotic motion

- **Poincare-Birkhoff** theorem states that under perturbation of a **resonance** only an **even number** of **fixed points** survives (half stable and the other half unstable)
- **Themselves** get **destroyed** when perturbation gets **higher**, etc. (**self-similar** fixed points)
- **Resonance islands grow** and **resonances** can **overlap** allowing diffusion of particles

\[ \mu_x = 0.22 \]
Resonance overlap criterion

- When perturbation grows, the resonance island width grows
- **Chirikov** (1960, 1979) proposed a **criterion** for the **overlap** of two neighboring **resonances** and the onset of orbit diffusion
  
  The **distance** between two resonances is
  \[ \delta \hat{J}_{1,n,n'} = 2 \left( \frac{1}{n_1 + n_2} - \frac{1}{n_1' + n_2'} \right) \]

  The **simple overlap criterion** is
  \[ \Delta \hat{J}_{n_{\text{max}}} + \Delta \hat{J}_{n'_{\text{max}}} \geq \delta \hat{J}_{n,n'} \]
Resonance overlap criterion

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    - The **distance** between two resonances is \( \delta \hat{J}_{1,n,n'} \).
    - The **simple overlap criterion** is
      \[
      \Delta \hat{J}_n \max + \Delta \hat{J}_{n'} \max \geq \delta \hat{J}_{n,n'}
      \]

- Considering the **width** of chaotic layer and secondary islands, the “two thirds” rule apply
  \[
  \Delta \hat{J}_n \max + \Delta \hat{J}_{n'} \max \geq \frac{2}{3} \delta \hat{J}_{n,n'}
  \]

- **Example:** **Chirikov’s standard map**

\[
p_{n+1} = p_n + K \sin(\theta_n) \quad \theta_{n+1} = \theta_n + p_{n+1}
\]
When perturbation grows, the resonance island width grows

Chirikov (1960, 1979) proposed a criterion for the overlap of two neighboring resonances and the onset of orbit diffusion

The distance between two resonances is

\[ \delta \hat{J}_{1,n,n'} = \frac{2}{\left( \frac{1}{n_1+n_2} - \frac{1}{n_1'+n_2'} \right)} \left| \frac{\partial^2 H_0(\hat{J})}{\partial \hat{J}_1^2} \right|_{\hat{J}_1=\hat{J}_{10}} \]

The simple overlap criterion is

\[ \Delta \hat{J}_{n \max} + \Delta \hat{J}_{n' \max} \geq \delta \hat{J}_{n,n'} \]

Considering the width of chaotic layer and secondary islands, the “two thirds” rule apply

\[ \Delta \hat{J}_{n \max} + \Delta \hat{J}_{n' \max} \geq \frac{2}{3} \delta \hat{J}_{n,n'} \]

The main limitation is the geometrical nature of the criterion (difficulty to be extended for > 2 degrees of freedom)

\[ p_{n+1} = p_n + K \sin(\theta_n) \quad \theta_{n+1} = \theta_n + p_{n+1} \]
Increasing dimensions

- For \((y_0, p_{y0}) \neq (0, 0)\), i.e. by adding another degree of freedom chaotic motion is enhanced

\[
(\mu_x, \mu_y) = (0.22, 0.24)
\]
Increasing dimensions

- For $\left( y_0, p_{y0} \right) \neq (0, 0)$, i.e. by adding another degree of freedom chaotic motion is enhanced.
- At the same time, analysis of phase space on surface of section becomes difficult to interpret, as these are projections of 4D objects on a 2D plane.

$$\left( \mu_x, \mu_y \right) = (0.22, 0.24)$$
Chaos detection methods

- Computing/measuring **dynamic aperture** (DA) or particle survival
  
  A. Chao et al., PRL 61, 24, 2752, 1988;  
  F. Willeke, PAC95, 24, 109, 1989.

- Computation of Lyapunov exponents
  
  F. Schmidt, F. Willeke and F. Zimmermann, PA, 35, 249, 1991;  

- Variance of unperturbed action (a la Chirikov)
  
  B. Chirikov, J. Ford and F. Vivaldi, AIP CP-57, 323, 1979  
  J. Tennyson, SSC-155, 1988;  
  J. Irwin, SSC-233, 1989

- Fokker-Planck diffusion coefficient in actions
  
  T. Sen and J.A. Elisson, PRL 77, 1051, 1996

- Frequency map analysis
Dynamic aperture
Dynamic Aperture

- The most direct way to evaluate the non-linear dynamics performance of a ring is the computation of **Dynamic Aperture**
- Particle motion due to multi-pole errors is generally non-bounded, so chaotic particles can **escape to infinity**
- This is not true for all non-linearities (e.g. the beam-beam force)
- Need a **symplectic** tracking code to follow particle trajectories (a lot of initial conditions) for a **number of turns** (depending on the given problem) until the particles start getting lost. This **boundary** defines the **Dynamic aperture**
- As multi-pole errors may not be completely known, one has to track through **several machine models** built by **random distribution** of these errors
- One could start with 4D (only transverse) tracking but certainly needs to simulate 5D (constant energy deviation) and finally 6D (synchrotron motion included)
Dynamic Aperture plots

- Dynamic aperture plots show the maximum initial values of stable trajectories in x-y coordinate space at a particular point in the lattice, for a range of energy errors.

  - The beam size can be shown on the same plot.
  - Generally, the goal is to allow some significant margin in the design - the measured dynamic aperture is often smaller than the predicted dynamic aperture.

![Dynamic Aperture plots](image-url)
Dynamic aperture including damping

- Including radiation damping and excitation shows that 0.7% of the particles are lost during the damping.
- Certain particles seem to damp away from the beam core, on resonance islands.
- **Min. Dynamic Aperture (DA)** with intensity vs crossing angle, for nominal optics ($\beta^* = 40$ cm) and BCMS beam (2.5 μm emittance), 15 units of chromaticity.

- **For $1.1 \times 10^{11}$ p**
  - At $\theta_c/2 = 185$ μrad (~12 σ separation), DA around 6 σ (good lifetime observed)
  - At $\theta_c/2 = 140$ μrad (~9 σ separation), DA below 5 σ (reduced lifetime observed)

- **Improvement** for low octupoles, low chromaticity and WP optimisation (observed in operation)
MOGA – Multi Objective Genetic Algorithms are being recently used to optimise linear but also non-linear dynamics of electron low emittance storage rings.

- Use knobs quadrupole strengths, chromaticity sextupoles and correctors with some constraints.

- Target ultra-low horizontal emittance, increased lifetime and high dynamic aperture.
Measuring Dynamic Aperture

During LHC design phase, DA target was 2x higher than collimator position, due to statistical fluctuation, finite mesh, linear imperfections, short tracking time, multi-pole time dependence, ripple and a 20% safety margin.

Better knowledge of the model led to good agreement between measurements and simulations for actual LHC.

Necessity to build an accurate magnetic model (from beam based measurements).

E. Mclean, PhD thesis, 2014
DA guiding machine performance

- B1 suffering from lower lifetime in the LHC
- DA simulations predicted the required adjustment
- Fine-tune scan performed and applied in operation, solving B1 lifetime problem

D. Pellegrini et al., 2016
Reduction of **crossing angle** at **constant luminosity**, reduces **pileup density** (by elongating the luminous region) and **triplet irradiation**.

**YP, N. Karastathis and D. Pellegrini et al., 2018**

Relaxed DA

Aggressive DA
Lyapunov exponent
Lyapunov exponent

- Chaotic motion implies **sensitivity to initial condition**

- Two infinitesimally close **chaotic trajectories** in phase space with initial difference $\delta Z_0$ will end-up diverging with rate

  $$ |\delta Z(t)| \approx e^{\lambda t} |\delta Z_0| $$

  with the **maximum Lyapunov exponent**

- There is as many exponents as the phase space dimensions (Lyapunov spectrum)

- The largest one is the **Maximal Lyapunov exponent** (MLE) is defined as

  $$ \lambda = \lim_{t \to \infty} \lim_{\delta Z_0 \to 0} \frac{1}{t} \ln \frac{|\delta Z(t)|}{|\delta Z_0|} $$
Maximum Lyapunov exponent converges towards a **positive value** for a chaotic orbit

\[
\lambda = \lim_{t \to \infty} \lim_{\delta Z_0 \to 0} \frac{1}{t} \ln \frac{|\delta Z(t)|}{|\delta Z_0|}
\]
Maximum Lyapunov exponent converges towards zero for a chaotic orbit

\[ \lambda = \lim_{{t \to \infty}} \lim_{{\delta Z_0 \to 0}} \frac{1}{t} \ln \frac{|\delta Z(t)|}{|\delta Z_0|} \]
Lyapunov exponent: regular orbit

- Maximum Lyapunov exponent converges more slowly towards zero for a resonant orbit

\[ \lambda = \lim_{t \to \infty} \lim_{\delta Z_0 \to 0} \frac{1}{t} \ln \frac{\left| \delta Z(t) \right|}{\left| \delta Z_0 \right|} \]
Maximum Lyapunov exponent converges more slowly towards zero for a resonant orbit, in particular close to the separatrix.

\[ \lambda = \lim_{t \to \infty} \lim_{\delta Z_0 \to 0} \frac{1}{t} \ln \frac{|\delta Z(t)|}{|\delta Z_0|} \]
Frequency Map Analysis
Frequency Map Analysis (FMA) is a numerical method which springs from the studies of J. Laskar (Paris Observatory) putting in evidence the chaotic motion in the Solar Systems.

FMA was successively applied to several dynamical systems:

- Stability of Earth Obliquity and climate stabilization (Laskar, Robutel, 1993)
- 4D maps (Laskar 1993)
Consider an integrable Hamiltonian system of the usual form

\[ H(J, \varphi, \theta) = H_0(J) \]

Hamilton’s equations give

\[ \dot{\varphi}_j = \frac{\partial H_0(J)}{\partial J_j} = \omega_j(J) \Rightarrow \varphi_j = \omega_j(J)t + \varphi_{j0} \]

\[ \dot{J}_j = -\frac{\partial H_0(J)}{\partial \varphi_j} = 0 \Rightarrow J_j = \text{const.} \]

The actions define the surface of an invariant torus

In complex coordinates the motion is described by

\[ \zeta_j(t) = J_j(0)e^{i\omega_j t} = z_{j0}e^{i\omega_j t} \]

For a non-degenerate system \( \det \left| \frac{\partial \omega(J)}{\partial J} \right| = \det \left| \frac{\partial^2 H_0(J)}{\partial J^2} \right| \neq 0 \)

there is a one-to-one correspondence between the actions and the frequency, a frequency map can be defined parameterizing the tori in the frequency space

\[ F : (I) \rightarrow (\omega) \]
Quasi-periodic motion

- If a transformation is made to some new variables

\[ \zeta_j = I_j e^{i\theta_j t} = z_j + \epsilon G_j(z) = z_j + \epsilon \sum c_m z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n} \]

- The system is still integrable but the tori are distorted
- The motion is then described by

\[ \zeta_j(t) = z_{j0} e^{i\omega_j t} + \sum a_m e^{i(m \cdot \omega) t} \]

i.e. a quasi-periodic function of time, with

\[ a_m = \epsilon c_m z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n} \quad \text{and} \quad m \cdot \omega = m_1 \omega_1 + m_2 \omega_2 + \cdots + m_n \omega_n \]

- For a non-integrable Hamiltonian, \( H(I, \theta) = H_0(I) + \epsilon H'(I, \theta) \)

and especially if the perturbation is small, most tori persist (KAM theory)

- In that case, the motion is still quasi-periodic and a frequency map can be built
- The regularity (or not) of the map reveals stable (or chaotic) motion
Building the frequency map

- When a quasi-periodic function \( f(t) = q(t) + ip(t) \) in the complex domain is given numerically, it is possible to recover a quasi-periodic approximation in a very precise way over a finite time span \([-T, T]\) several orders of magnitude more precisely than simple Fourier techniques

\[
f'(t) = \sum_{k=1}^{N} a_k' e^{i\omega_k' t}
\]

- This approximation is provided by the Numerical Analysis of Fundamental Frequencies – **NAFF** algorithm

- The frequencies \( \omega_k' \) and complex amplitudes \( a_k' \) are computed through an iterative scheme.
The NAFF algorithm

- The first frequency $\omega_1'$ is found by the location of the maximum of

$$\phi(\sigma) = \langle f(t), e^{i\sigma t}\rangle = \frac{1}{2T} \int_{-T}^{T} f(t) e^{-i\sigma t} \chi(t) dt$$

where $\chi(t)$ is a weight function.

- In most of the cases the Hanning window filter is used

$$\chi_1(t) = 1 + \cos(\pi t / T)$$

- Once the first term $e^{i\omega_1' t}$ is found, its complex amplitude $a_1'$ is obtained and the process is restarted on the remaining part of the function

$$f_1(t) = f(t) - a_1' e^{i\omega_1' t}$$

- The procedure is continued for the number of desired terms, or until a required precision is reached.
The accuracy of a simple FFT even for a simple sinusoidal signal is not better than $|\nu - \nu_T| = \frac{1}{T}$

Calculating the Fourier integral explicitly

$$\phi(\omega) = \langle f(t), e^{i\omega t} \rangle = \frac{1}{T} \int_0^T f(t) e^{-i\omega t} \, dt$$

shows that the maximum lies in between the main peaks of the FFT
A more complicated signal with two frequencies

\[ f(t) = a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t} \]

shifts slightly the maximum with respect to its real location
A window function like the Hanning filter
\[ \chi_1(t) = 1 + \cos\left(\frac{\pi t}{T}\right) \]
kills side-lobes and allows a very accurate determination of the frequency.
Precision of NAFF

- For a general window function of order $p$

$$\chi_p(t) = \frac{2^p(p!)^2}{(2p)!} (1 + \cos \pi t)^p$$

Laskar (1996) proved a theorem stating that the solution provided by the NAFF algorithm converges asymptotically towards the real KAM quasi-periodic solution with precision

$$\nu_1 - \nu_1^T \propto \frac{1}{T^{2p+2}}$$

- In particular, for no filter (i.e. $p = 0$) the precision is $\frac{1}{T^2}$, whereas for the Hanning filter ($p$), the precision is of the order of $\frac{1}{T^4}$
Aspects of the frequency map

- In the vicinity of a resonance the system behaves like a pendulum
- Passing through the **elliptic point** for a fixed angle, a **fixed frequency** (or rotation number) is observed
- Passing through the **hyperbolic point**, a **frequency jump** is observed
Example: Frequency map for BBLR

- Simple Beam-beam long range (BBLR) kick and a rotation
Example: Frequency map for BBLR

**Simple Beam-beam long range (BBLR) kick and a rotation**

\[
\begin{align*}
N_b &= 5 \times 10^{11} \\
N_b &= 1 \times 10^{12}
\end{align*}
\]
Diffusion in frequency space

- For a 2 degrees of freedom Hamiltonian system, the frequency space is a line, the tori are dots on this line, and the chaotic zones are confined by the existing KAM tori.

- For a system with 3 or more degrees of freedom, KAM tori are still represented by dots but do not prevent chaotic trajectories to diffuse.

- This topological possibility of particles diffusing is called Arnold diffusion.

- This diffusion is supposed to be extremely small in their vicinity, as tori act as effective barriers (Nechoroshev theory).
Building the frequency map

- Choose coordinates \((x_i, y_i)\) with \(p_x\) and \(p_y = 0\)
- Numerically integrate the phase trajectories through the lattice for sufficient number of turns
- Compute through NAFF \(Q_x\) and \(Q_y\) after sufficient number of turns
- Plot them in the tune diagram

\[
\mathcal{F}_\tau : \mathbb{R}^n \rightarrow \mathbb{R}^n
\]

\[
p|q=q_0
\]

\[
MAP
\]
Example: Frequency maps for the LHC

- Frequency maps for the target error table (left) and an increased random skew octupole error in the superconducting dipoles (right)
Diffusion Maps

- Calculate frequencies for two equal and successive time spans and compute frequency diffusion vector:

\[
D_{t=\tau} = \nu_{t\in(0,\tau/2]} - \nu_{t\in(\tau/2,\tau]}
\]

- Plot the initial condition space color-coded with the norm of the diffusion vector

- Compute a diffusion quality factor by averaging all diffusion coefficients normalized with the initial conditions radius

\[
D_{QF} = \left\langle \frac{|D|}{(I_{x0}^2 + I_{y0}^2)^{1/2}} \right\rangle_R
\]
Example: Diffusion maps for the LHC

Diffusion maps for the target error table (left) and an increased random skew octupole error in the super-conducting dipoles (right)
Example: Frequency Map for the ESRF

- All dynamics represented in these two plots
- Regular motion represented by blue colors (close to zero amplitude particles or working point)

- Resonances appear as distorted lines in frequency space (or curves in initial condition space)
- Chaotic motion is represented by red scattered particles and defines dynamic aperture of the machine
Numerical Applications
Correction schemes efficiency

- Comparison of correction schemes for $b_4$ and $b_5$ errors in the LHC dipoles
- Frequency maps, resonance analysis, tune diffusion estimates, survival plots and short term tracking, proved that only half of the correctors are needed
Beam-Beam interaction

<table>
<thead>
<tr>
<th>Variable</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam energy</td>
<td>$E$</td>
<td>7 TeV</td>
</tr>
<tr>
<td>Particle species</td>
<td>...</td>
<td>protons</td>
</tr>
<tr>
<td>Full crossing angle</td>
<td>$\theta_c$</td>
<td>300 $\mu$rad</td>
</tr>
<tr>
<td>rms beam divergence</td>
<td>$\sigma'_x$</td>
<td>31.7 $\mu$rad</td>
</tr>
<tr>
<td>rms beam size</td>
<td>$\sigma_x$</td>
<td>15.9 $\mu$m</td>
</tr>
<tr>
<td>Normalized transv.</td>
<td>$\gamma_e$</td>
<td>3.75 $\mu$m</td>
</tr>
<tr>
<td>IP beta function</td>
<td>$\beta^*$</td>
<td>0.5 m</td>
</tr>
<tr>
<td>Bunch charge</td>
<td>$N_b$</td>
<td>$(1 \times 10^{11} - 2 \times 10^{12})$</td>
</tr>
<tr>
<td>Betatron tune</td>
<td>$Q_0$</td>
<td>0.31</td>
</tr>
</tbody>
</table>

- Long range beam-beam interaction represented by a 4D kick-map

\[
\Delta x = - n_{par} \frac{2r_p N_b}{\gamma} \left[ \frac{x' + \theta_c}{\theta_t^2} \left( 1 - e^{-\frac{\theta_t^2}{2\theta_x^2,y}} \right) - \frac{1}{\theta_c} \left( 1 - e^{-\frac{\theta_t^2}{2\theta_x^2,y}} \right) \right]
\]

\[
\Delta y = - n_{par} \frac{2r_p N_b}{\gamma} \frac{y'}{\theta_t^2} \left( 1 - e^{-\frac{\theta_t^2}{2\theta_x^2,y}} \right)
\]

with \[
\theta_t \equiv \left( (x' + \theta_c)^2 + y'^2 \right)^{1/2}
\]
- Proved dominant effect of long range beam-beam effect
- Dynamic Aperture (around 6σ) located at the folding of the map (indefinite torsion)
- Experimental effort to compensate beam-beam long range effect with wires (1/r part of the force) or octupoles
In the chaotic region of phase space, the action diffusion coefficient per turn can be estimated by averaging over the quasi-randomly varying betatron phase variable as

\[
D(J) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \ [\Delta J(\phi)]^2
\]
- Very good agreement of diffusive aperture boundary (action variance) with frequency variation (loss boundary corresponding to around 1 integer unit change in $10^7$ turns)
Magnet fringe fields

- Up to now we considered only transverse fields
- Magnet fringe field is the longitudinal dependence of the field at the magnet edges
- Important when magnet aspect ratios and/or emittances are big
Quadrupole fringe field

General field expansion for a quadrupole magnet:

\[ B_x = \sum_{m,n=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^m x^{2n} y^{2m+1}}{(2n)!(2m+1)!} \binom{m}{l} b_{2n+2m+1-2l}^{[2l]} \]

\[ B_y = \sum_{m,n=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^m x^{2n+1} y^{2m}}{(2n+1)!(2m)!} \binom{m}{l} b_{2n+2m+1-2l}^{[2l]} \]

\[ B_z = \sum_{m,n=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^m x^{2n+1} y^{2m+1}}{(2n+1)!(2m+1)!} \binom{m}{l} b_{2n+2m+1-2l}^{[2l+1]} \]

and to leading order

\[ B_x = y \left[ b_1 - \frac{1}{12} (3x^2 + y^2) b_1^{[2]} \right] + O(5) \]

\[ B_y = x \left[ b_1 - \frac{1}{12} (3y^2 + x^2) b_1^{[2]} \right] + O(5) \]

\[ B_z = xy b_1^{[1]} + O(4) \]

The quadrupole fringe field has an octupole-like effect
Magnet fringe fields

From the hard-edge Hamiltonian

\[ H_f = \frac{\pm Q}{12 B \rho (1 + \delta \rho \rho)} (y^3 p_x - x^3 p_y + 3 x^2 y p_y - 3 y^2 x p_x), \]

the first order shift of the frequencies with amplitude can be computed analytically

\[
\begin{pmatrix}
\delta \nu_x \\
\delta \nu_y
\end{pmatrix} = \begin{pmatrix}
a_{hh} & a_{hv} \\
a_{hv} & a_{vv}
\end{pmatrix} \begin{pmatrix} 2 J_x \\ 2 J_y \end{pmatrix},
\]

with the "anharmonicity" coefficients (torsion)

\[
a_{hh} = \frac{-1}{16 \pi B \rho} \sum_i \pm Q_i \beta_x \alpha_x i
\]

\[
a_{hv} = \frac{1}{16 \pi B \rho} \sum_i \pm Q_i (\beta_x \alpha_y i - \beta_y \alpha_x i)
\]

\[
a_{vv} = \frac{1}{16 \pi B \rho} \sum_i \pm Q_i \beta_y \alpha_y i
\]

Tune footprint for the SNS based on hard-edge (red) and realistic (blue) quadrupole fringe-field
SNS Working Point \((Q_x, Q_y) = (6.4, 6.3)\)

\[\delta p/p = [2\%, -2\%] \text{ at } 480 \pi \, \text{mm mrad}\]
Choice of the SNS ring working point

Tune Diffusion quality factor

\[ D_{QF} = \left\langle \frac{|D|}{(I_{x0}^2 + I_{y0}^2)^{1/2}} \right\rangle_R \]

Chosen Working Point

- (6.4, 6.3)
- (6.23, 5.24)
- (6.3, 5.8)
- (6.23, 6.20)
Global Working point choice

- Figure of merit for choosing best working point is sum of diffusion rates with a constant added for every lost particle.
- Each point is produced after tracking 100 particles.
- Nominal working point had to be moved towards “blue” area.

\[ e^D = \sqrt{\frac{(\nu_{x,1} - \nu_{x,2})^2 + (\nu_{y,1} - \nu_{y,2})^2}{N/2}} \]

\[ WPS = 0.1N_{\text{lost}} + \sum e^D \]
Comparing different chromaticity sextupole correction schemes and working point optimization using normal form analysis, frequency maps and finally particle tracking

Finding the adequate sextupole strengths through the tune diffusion coefficient
Frequency Map Analysis with modulation
- Evolution of frequency map over different longitudinal position
- Tunes acquired over each longitudinal period
- Particles with similar longitudinal offset but different amplitudes experience the resonance in different manner
- Particles with different longitudinal offset may experience different resonances

F. Asvesta, et al., 2017
- Quadrupoles of the **inner triplet** right and left of **IP1 and IP5**, **large beta-functions** increase the sensitivity to non-linear effects

- **Resonance conditions:**

\[ aQ_x + bQ_y + c \frac{f_{\text{modulation}}}{f_{\text{revolution}}} = k \text{ for } a, b, c, k \text{ integers} \]

S. Kostoglou, et al., 2018

- **By increasing the modulation depth, sidebands start to appear in the FMAs**
- Quadrupoles of the **inner triplet** right and left of **IP1 and IP5**, **large beta-functions** increase the sensitivity to non-linear effects

- **Resonance conditions:**

$$aQ_x + bQ_y + c \frac{f_{modulation}}{f_{revolution}} = k$$ for $$a, b, c, k$$ integers

- By increasing the modulation depth, sidebands start to appear in the FMAs

$$\Delta Q=1e^{-4}$$
LHC: Power supply ripples

- Scan of different ripple frequencies (50-900 Hz)

5D, $E = 6.5\text{TeV}$, $I_{\text{opt}} = 510\text{A}$, Beam - beam ON, $\varepsilon_n = 2.5\mu\text{m}$, $\beta^* = 40\text{cm}$, $q = 15$

$(Q_x, Q_y) = (62.31, 60.32)$, $V_{\text{RF}}$ OFF, $\delta p = 27e - 5$, 49 angles, $0.1 - 6.1\sigma$, sliding NAFF

$f_r = 50.0\text{Hz}$, $A_r = 10^{-7}$ at MQXA.1, MQXA.3, MQXB.A2, MQXB.B2 of IP1, IP5

S. Kostoglou, YP et al., 2018
6D, E = 6.5 TeV, $I_{on} = 510$A. Beam - beam ON, $e_n = 2.5$μm, $\beta^* = 40$cm, $q = 0$

$(Q_x, Q_y) = (62.31, 60.32)$, $V_{RF}$ ON, $\delta_p = 2.7 \times 10^{-5}$, 99 angles. 0.1 - 6.1 $\sigma$, sliding NAFF

$f_r = 50$Hz, $A_r = 10^{-7}$ at MQXA.1, MQXA.3, MQXB.A2, MQXB.B2 of IP1, IP5

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S. Kostoglou, YP et al., 2018
Summary

- Appearance of **fixed points** (periodic orbits) determine **topology** of the phase space
- **Perturbation** of unstable (hyperbolic points) opens the path to chaotic motion
- Resonance can overlap enabling the rapid diffusion of orbits
- **Dynamic aperture** by brute force tracking (with symplectic numerical integrators) is the usual quality criterion for evaluating non-linear dynamics performance of a machine
- **Frequency Map Analysis** is a numerical tool that enables to study in a global way the dynamics, by identifying the excited resonances and the extent of chaotic regions
- It can be directly applied to tracking and experimental data
- A combination of these modern methods enable a thorough analysis of non-linear dynamics and lead to a robust design
Acknowledgments