Lattice Design

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The analysis of the cell stability and betatron functions can be done via an algorithmic approach\(^1\) using the method presented yesterday:

1. compute \textit{symbolically} the \(M_{OTM}\),
2. diagonalize it \(M_{OTM} = PDP^{-1}\), with \(\det(P) = -i\) and \(P_{11} = P_{12}\),
3. impose that all the eigenvalues amplitude is 1 to get the stability condition,
4. study \(P\) to get the periodic solution for \(\beta\) and \(\alpha\) at the start of the cell,
5. propagate the solution from the start of the cell along the different lattice element.

We will start considering a \textbf{FODO cell}.

\(^1\) http://cern.ch/go/J8TP
The CERN Large Hadron Collider FODO cell

CERN Large Hadron Collider, Beam 1, Injection Optics 2016, Q1=64.280, Q2=59.310

$1/f = K_1 L \ [m^{-1}]$

CERN Large Hadron Collider, Beam 1, Injection Optics 2016, Q1=64.280, Q2=59.310

$\theta = K_0 L \ [\text{mrad}]$

$\beta_x, \beta_y$

$D_x \ [m]$
Let’s consider a FODO cell of length $L_{\text{cell}}$ in thin lens approximation, where

1. the space of the focusing (F) and defocusing (D) quadrupoles is equal to $L_{\text{cell}}/2$ and
2. the focal length of the F and D quadrupoles equal in module, that is $f_D = -f_F$ with $f_F > 0$.

For convenience we will start and end the FODO cell with half of an F quadrupole (i.e., with focal length $2 \times f_F$) and we will consider, as first step, the horizontal plane.
The FODO $M_{OTM}$ diagonalization

Using symbolic tools (e.g., *sympy*) one can compute\(^2\)

$$M_{OTM} = \begin{bmatrix} \frac{-L_{cell}^2}{8f^2} + 1 & \frac{L_{cell}^2}{4f} + L_{cell} \\ \frac{L_{cell}(L_{cell} - 4f)}{16f^3} & -\frac{L_{cell}^2}{8f^2} + 1 \end{bmatrix}$$

$$D = \begin{bmatrix} \frac{-L_{cell}^2 + L_{cell}}{8f^2} \sqrt{\frac{L_{cell}^2}{2} - 16f^2 + 8f^2} & 0 \\ 0 & \frac{-L_{cell}^2 - L_{cell}}{8f^2} \sqrt{\frac{L_{cell}^2}{2} - 16f^2 + 8f^2} \end{bmatrix}$$

$$P = \begin{bmatrix} \sqrt{f} & \sqrt{f} \\ \sqrt{\frac{-i}{(L_{cell} - 4f) \sqrt{L_{cell}^2 - 16f^2}} (-L_{cell} + 4f)} & \sqrt{\frac{-i}{(L_{cell} - 4f) \sqrt{L_{cell}^2 - 16f^2}} (-L_{cell} + 4f)} \\ \sqrt{\frac{-i}{2 \sqrt{f} \sqrt{(L_{cell} - 4f) \sqrt{L_{cell}^2 - 16f^2}} \sqrt{L_{cell}^2 - 16f^2}}} & \sqrt{\frac{-i}{2 \sqrt{f} \sqrt{(L_{cell} - 4f) \sqrt{L_{cell}^2 - 16f^2}} \sqrt{L_{cell}^2 - 16f^2}}} \end{bmatrix}$$

\(^2\)http://cern.ch/go/J8TP
The stability on the horizontal plane is achieved if $\lambda_1$ and $\lambda_2$ have unitary module.
This implies $-1 < \frac{\lambda_1 + \lambda_2}{2} = \cos \mu < 1$, that is

\[ \frac{L_{\text{cell}}}{4} < f. \]

The stability condition in the vertical plane is exactly equivalent, since $D(f) = D(-f)$.

The stability condition of a FODO lattice (thin lens approximation and no dipoles) imposes an F quadrupole with $f$ larger than $L_{\text{cell}}/4$. 
Remembering that

\[ \mu = \arccos \left( \frac{\lambda_1 + \lambda_2}{2} \right), \]

one gets

\[ \mu = \arccos \left( 1 - \frac{L_{\text{cell}}^2}{8f^2} \right), \]

or, equivalently, from

\[ \sin \left( \frac{\arccos(1 - x)}{2} \right) = \frac{\sqrt{x}}{2} \]

we get

\[ \sin \left( \frac{\mu}{2} \right) = \frac{L_{\text{cell}}}{4f}. \]

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\[ ^3 \text{http://cern.ch/go/8qcf} \]
μ vs f and 1/f

\[ f = \frac{1}{2\sqrt{2}}, \mu = 90 \text{ deg} \]

\[ f = \frac{1}{2}, \mu = 60 \text{ deg} \]
\[ K_1 L_Q = 2, \mu = 60 \text{ deg} \]
\[ K_1 L_Q = 2\sqrt{2}, \mu = 90 \text{ deg} \]
Remembering that

\[
P = \begin{pmatrix}
\sqrt{\beta} & \sqrt{\beta} \\
\sqrt{2} & \sqrt{2} \\
-\alpha + i & -\alpha - i \\
\sqrt{2\beta} & \sqrt{2\beta}
\end{pmatrix}
\]

we have

\[
\beta(0) = 2 P_{11}^2 \quad \text{and} \quad \alpha(0) = -P_{11}(P_{21} + P_{22}).
\]
This yields

\[
\beta_x(0) = \frac{2f \sqrt{4f + L_{cell}}}{\sqrt{4f - L_{cell}}} = L_{cell} \frac{1 + \sin(\mu/2)}{\sin(\mu)}
\]

\[
\alpha_x(0) = 0.
\]

With a similar approach, we can compute the \(y\)-plane optical functions by considering \(P(-f)\), getting

\[
\beta_y(0) = \frac{2f \sqrt{4f - L_{cell}}}{\sqrt{4f + L_{cell}}} = L_{cell} \frac{1 - \sin(\mu/2)}{\sin(\mu)}
\]

\[
\alpha_y(0) = 0.
\]
$\beta$-function vs $\mu$

Graph showing $\beta_x(0)$ and $\beta_y(0)$ for varying $\mu$ [degrees].
\( \beta \text{-function vs } \mu \)
The definition of the linear chromaticity is

\[ \xi = \frac{\Delta Q}{\Delta p_{p_0}} = \frac{1}{2\pi} \frac{\Delta \mu}{\Delta p_{p_0}}. \]  

(1)

From the relation

\[ f\left(\frac{\Delta p}{p_0}\right) = f \times \left(1 + \frac{\Delta p}{p_0}\right) \]  

(2)

and from

\[ \sin\left(\frac{\mu}{2}\right) = \frac{L_{cell}}{4f}, \]  

(3)

one can compute the FODO lattice chromaticity

\[ \xi = -\frac{1}{4\pi} \frac{L_{cell}}{f} \frac{1}{\cos(\mu/2)} = \left[ -\frac{1}{\pi} \tan\left(\frac{\mu}{2}\right) \right] \]  

(4)
Chromaticity of a FODO II

![Graph showing the chromaticity of a FODO II lattice design. The graph plots the values of $\xi$ against $\mu$ (degrees). The curve indicates the relationship between the two variables.]
From the FODO lattice we can define at least two additional “flavours”:

1. different focal length in the F and D quadrupoles,
2. uneven distance between quadrupoles.

The stability of the two cases is discussed in http://cern.ch/go/J8TP.
In addition an example on the effect of the dipoles (sector and rectangular bends) and thick quadrupoles is given in http://cern.ch/go/J8TP using MAD-X.
Starting from the FODO we can consider other lattice cells. As an example, by putting back-to-back two OFOD, we have a triplet cell (OFODDOFO).

An example of triplet lattice is presented in http://cern.ch/go/J8TP where the stability condition is discussed.
An stroll along CERN Accelerator Complex

In the following we present few of the CERN Accelerator Complex optics\(^4\).

\(^4\)http://cern.ch/go/MKp7
CERN Proton Synchrotron Booster

CERN Proton Synchrotron Booster, $Q_1=4.172$, $Q_2=4.230$

$1/f=K_1L$ [m$^{-1}$]

$\theta=K_0L$ [rad]

$D_x$ [m]

$\beta_x$, $\beta_y$
CERN Proton Synchrotron Booster

CERN Proton Synchrotron Booster, Q1=4.172, Q2=4.230
CERN Low Energy Ion Ring

The diagram shows the lattice design of the CERN Low Energy Ion Ring, with parameters Q1 = 1.820 and Q2 = 2.720. The graph indicates the behavior of the ring over a range of parameters, including β-functions and Δx values.
CERN Super Proton Synchrotron

CERN Super Proton Synchrotron, Q1=26.130, Q2=26.180

\[ \frac{1}{\beta} = K_1 L \ [m] \]

\[ \theta = K_0 L \ [\text{mrad}] \]

\[ \beta \text{ functions \ [m]} \]

\[ D_x \ [m] \]
CERN Super Proton Synchrotron

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\[ \frac{1}{f} = K_1 L \text{ [m]} \]

\[ \theta = K_0 L \text{ [mrad]} \]

\( \beta_x \), \( \beta_y \)

\( D_x \text{ [m]} \)
CERN Large Hadron Collider, Beam 1, Injection Optics 2016, Q1=64.280, Q2=59.310

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CERN Large Hadron Collider, Beam 1, Injection Optics 2016, Q1=64.280, Q2=59.310

\[ \frac{1}{f} = K_1 L \ [\text{m}] \]

\[ \theta = K_0 L \ [\text{mrad}] \]

functions [m]

\[ \beta_x \]

\[ \beta_y \]

Dx [m]
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CERN Large Hadron Collider, Beam 1, Injection Optics 2016, Q1=64.280, Q2=59.310

\[ \frac{1}{f} = K_1 L [\text{m}] \]

\( \theta = K_0 L [\text{mrad}] \)

\( \beta_x \) and \( \beta_y \) functions [m]

Dx [m]