Landau Damping
part 1

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Landau Damping

a basic mechanism of beam dynamics

- relevant for operation
- active field of development
  (many new papers, conferences)
Landau Damping
A Harmonic Oscillator

With a damping (friction):

\[ x'' + 2\zeta \omega_0 x' + \omega_0^2 x = 0 \]

no damping
damped
critically damped
Landau Damping
Lev Landau (1908-1968)
Institute for Physical Problems, Moscow

Nobel Prize Physics 1962
“Theory of Superfluidity”

Discovery of Collisionless Damping:

Experimental confirmation:
J. Malmberg, C. Wharton,
Phys Rev Lett 13, 184 (1964)

For our damping, “Landau”=“collisionless”=“frictionless”
What kind of oscillations?
Waves
Oscillations: Waves

Water wave

Sound wave

Traveling oscillation in a medium. Very different from the medium particle motion.
Oscillations: Waves

Water wave

- Crest
- Trough
- Wave height
- Wave length
- Wave frequency

Sound wave

- Increased Pressure
- Decreased Pressure
- Atmospheric Pressure

Landau damping:

\[ \text{wave} \leftrightarrow \text{particles} \text{ collisionless interaction.} \]
Oscillations: Waves

Waves can be unstable or damped

The wave frequency is complex:

\[ \omega = \omega_r + i\omega_i \]

The wave physical parameter:

\[ A(t) = A_0 \cos(\omega_r t) e^{\omega_i t} \]
Stability: the basic idea

Stable System

Unstable System

Mechanism to suppress the infinitesimal perturbations (DAMPING)

Mechanism to reinforce the infinitesimal perturbations (INSTABILITY DRIVE)

Here is Landau Damping

small perturbation

perturbation decays

or $|\gamma_{\text{damping}}| > |\gamma_{\text{drive}}|

or $\gamma_{\text{drive}} > |\gamma_{\text{damping}}|$
Landau damping in plasma
Plasma

Plasma is a quasi-neutral gas of unbound ions and electrons.

Waves in plasma: collective propagating oscillations of particles and E-M fields.

Electrons are much lighter: oscillations of the electron density

Some waves can be damped.

“Friction” in plasma is collisions.
A basic plasma oscillation:
Langmuir wave

Wave number $k = 2\pi/\lambda$

The phase velocity $v_{ph} = \omega/k$

There are resonant particles $v_x \approx v_{ph}$

The plasma frequency

$$\omega_p^2 = \frac{n_e e^2}{m_e \varepsilon_0}$$

The dispersion relation

$$\frac{\omega_p^2}{k^2} \int \frac{\partial \tilde{f}_0 / \partial v_x}{v_x - \omega / k} dv_x = 1$$

has a singularity
Landau Damping In Plasma

The wave frequency is complex

\[ \omega = \omega_r + i\omega_i \]

The dispersion relation can be solved, the integral is calculated as \( \text{PV} + \text{residue} \)

\[
\frac{\omega_p^2}{k^2} \left[ \text{PV} \int \frac{\partial \hat{f}_0}{\partial v_x} \frac{dv_x}{v_x - \omega/k} + i\pi \frac{\partial \hat{f}_0}{\partial v_x} \bigg|_{v_x=\frac{\omega}{k}} \right] = 1
\]

\[ \omega_r^2 = \omega_p^2 + 3k^2 v_{th}^2 \]

\[ \omega_i = -\frac{\pi \omega_r}{2} \frac{\omega_p^2 \partial \hat{f}_0}{k^2 \partial v_x} \bigg|_{v_x=\frac{\omega}{k}} \]
Landau Damping In Plasma

spread in particle velocity distribution

particle distribution $f_0(v_x)$

resonant particles: slower faster

plasma

faster slower particles gain energy give energy
gain energy faster slower particles

$-eE_x$ $-eE_x$ $-eE_x$ $-eE_x$

the wave $v_{ph} = \omega/k$

negative $f_0(v_x)$ slope: $N_{gain} > N_{give}$ $\rightarrow$ the wave decays, damping

positive $f_0(v_x)$ slope: $N_{gain} < N_{give}$ $\rightarrow$ the wave grows, instability
Landau Damping In Plasma

Main ingredients of Landau damping:
• wave–particle collisionless interaction. Here this is the electric field
• energy transfer: the wave ↔ the (few) resonant particles

The result is the exponential decay of a small perturbation.

Landau damping is a fundamental mechanism in plasma physics. Extensively studied in experiment, simulations and theory.
Waves in particle beams in accelerators?
Incoherent oscillations

Coherent oscillations

Incoherent tune-shifts

Coherent tune-shifts

Large & important differences

(neighbor lectures)
Incoherent oscillations

No coherent oscillations \( \langle y \rangle = 0 \)

Waves in beams:
- **Coasting beam**: \( L = C, \lambda(z) = \text{const} \), no synchrotron motion, \( \delta p = \text{const} \)

- **Bunched beam**: \( L_{\text{bunch}}, \lambda(z) \) profile, synchrotron oscillations \( Q_s : \delta p - z \)
Waves in Beams

Transverse oscillations in a coasting beam

\[ x(s, t) = x_0 e^{i n s/R - i \Omega t} \]

\( n \) is the mode index.
Wave length: \( C/n \)
Frequencies:
- slow wave \( \Omega_s = (n - Q_\beta) \omega_0 \)
- fast wave \( \Omega_f = (n + Q_\beta) \omega_0 \)

Angular rotation (\( \Omega_{\text{ang}} \)):

\[ \Omega_{\text{ang}} = \left( 1 - \frac{Q_\beta}{n} \right) \omega_0 \]
Waves in Coasting Beams

Experimental observations of the coasting-beam waves

A coasting beam in SIS18. $n=4$, as expected for $Q=3.25$, with correct $\Omega_s$ and $\Omega_{\text{ang}}$

SIS18 synchrotron at GSI Darmstadt

V. Kornilov, O. Boine-Frankenheim, GSI-Acc-Note-2009-008, GSI Darmstadt (2009)
Waves in Bunched Beams

Experimental observations of the waves in bunches

Unstable head-tail modes in ISIS. High-intensity beams, 2 bunches, head-tail mode $k=1$, $\tau=0.1$ ms.

V. Kornilov, et.al, HB2014  East Lansing, MI, USA, Nov 10-14, 2014
Different types of coherent oscillations

- **Transverse, Longitudinal**
  - Dipolar (m=1)
  - Quadrupolar (m=2)
  - Sextupolar (m=3)

Here we consider mostly the dipole transverse oscillations. For the others: the physics and the formalism are similar.
Special waves: Eigenmodes
Eigenmodes: intrinsic orthogonal oscillations of the dynamical system, with the fixed frequencies (eigenfrequencies)

We often talk about the shift:

$$\Delta \Omega = \Omega - \Omega_{\text{eigenfrequency}}$$
Eigenmodes

Transverse eigenmodes in a coasting beam

Eigenmode:
\[ x(s, t) = x_0 e^{ins/R - i\Omega t} \]

Eigenfrequency:
\[ \Omega_s = (n - Q_\beta)\omega_0 \]

Transverse eigenmodes in a bunched beam: Head-Tail Modes

Eigenmode:

Eigenfrequencies:

\[ k=2 \]
Unstable Oscillations

small initial perturbation $\langle x \rangle$

excitation force

produces

$G \propto Z_{\text{ext}} I_0 \langle x \rangle$

reinforcing mechanism

forced oscillations (eigenmode)

the perturbation is amplified $\langle x \rangle \times (1+\Delta)$

Especially concerned about the eigenmode perturbations

The result is $\Delta Q_{\text{coh}}$ and the exponential growth: instability

\[ \langle x \rangle(t) = x_0 e^{\text{Im}(\Omega)t} \]
Wake Fields, Impedances

Dipolar wakes: $F_{x2} \sim \Delta x_1$
(driving)
the same for the whole trailing slice: coherent

Quadrupolar wakes: $F_{x2} \sim \Delta x_2$
(detuning)
different for individual particles: incoherent

Transverse collective instabilities: Dipolar Wakes $W_1(z)$, Impedances $Z_1(\omega)$

A.Chao, Phys. Coll. Beam Instab. in High Energy Acc. 1993
Beam Transfer Function (BTF)

K.Y. Ng, Physics of Intensity Dependent Beam Instabilities, 2006
A. Chao, Phys. Coll. Beam Instab. in High Energy Acc. 1993
Beam Transfer Function

an excitation:
\[ x'' + \omega^2_{\beta_i} x = \hat{G} e^{-i\Omega t} \]

beam forced response:
\[ \langle x \rangle = A e^{-i\Omega t + \Delta \phi} \]

\[ R(\Omega) = 2\omega_{\beta_0} \frac{\langle x(t) \rangle}{\hat{G} e^{-i\Omega t}} \]

kicker

network analyser

exciting signal
\[ \Omega, \phi^\text{in}, A^\text{in} \]

response
\[ \Omega, \phi^\text{out}, A^\text{out} \]

BTF: amplitude(\(\Omega\)), phase(\(\Omega\))

Vladimir Kornilov, CAS, June 9-21, 2019, Denmark
Beam Transfer Function

BTF is:
- Useful diagnostics; gives the tune, $\delta p$, chromaticity, beam distribution
- A fundamental function in the beam dynamics
- Necessary to describe the beam signals and Landau damping

$$R(\Omega) = \text{PV} \int \frac{f(\omega)d\omega}{\omega - \Omega} + i\pi f(\Omega)$$

$$\Delta Q = (\Omega - (m \pm Q_f) f_0) / f_0$$

$$\delta Q_\xi = |m\eta \pm (Q_{f\eta} \eta - Q_0 \xi)| \delta p / p$$

J.Borer, et al, PAC1979
D.Boussard, CAS 1993, CERN 95-06, p.749
A.Chao, Phys. Coll. Beam Instab. in High Energy Acc. 1993
Handbook of Acc. Physics and Eng. 2013, 7.4.17
BTF: a standard measurement with a network analyzer

- Collective response to the excitation
- Observe the incoherent spectrum
- Still, the beam is stable: Landau Damping!

Landau Damping:

Interaction

wave ↔ resonant particles

A. Chao, Phys. Coll. Beam Instab. in High Energy Acc. 1993
K.Y. Ng, Physics of Intensity Dependent Beam Instabilities, 2006
Driven Harmonic Oscillator

The solution = homogeneous solution (pulse response) initial conditions + particular solution (forced oscillations)

Off-resonance ($\Omega \neq \omega_i$) and at resonance ($\Omega = \omega_i$), different particular solutions.
Zero initial conditions.

\[ x'' + \omega_i^2 x = \hat{G} e^{-i\Omega t} \]

\[ x_G(t) = \frac{2\hat{G}}{\omega_i^2 - \Omega^2} \sin\left(\frac{\omega_i - \Omega}{2} t\right) \sin\left(\frac{\omega_i + \Omega}{2} t\right) \]

\[ x_G(t) = \frac{\hat{G}}{2\Omega} t \sin(\Omega t) \]
Driven Harmonic Oscillator

off-resonant beating solution

\[ x_G(t) = \frac{2\hat{G}}{\omega_i^2 - \Omega^2} \sin\left(\frac{\omega_i - \Omega}{2} t\right) \sin\left(\frac{\omega_i + \Omega}{2} t\right) \]

resonant solution

\[ x_G(t) = \frac{\hat{G}}{2\Omega} t \sin(\Omega t) \]

\[ \Delta\omega_i / \Omega = 0.03 \]
\[ \Delta\omega_i / \Omega = 0.01 \]
\[ \omega_i = \Omega \]

wave ↔ particle energy transfer

gain energy

give energy
Landau Damping: Dispersion Relation

A. Chao, Phys. Coll. Beam Instab. in High Energy Acc. 1993
K.Y. Ng, Physics of Intensity Dependent Beam Instabilities, 2006
W. Herr, Introduction to Landau Damping, CAS2013, CERN-2014-009
Coherent Oscillations

An easy derivation of the dispersion relation

the external drive is $\text{INTENSITY} \times \text{IMPEDEANCE} \times \text{PERTURBATION}$

$$G = \frac{\langle F_x \rangle}{m \gamma} = \frac{q \beta}{m \gamma c} i Z_{\text{ext}}^\perp I_0 \langle x \rangle$$

the no-damping complex coherent tune shift is

$\text{INTENSITY} \times \text{IMPEDEANCE}$

$$\Delta Q_{\text{coh}} = \frac{I_0 q_{\text{ion}}}{4 \pi \gamma m c Q_0 \omega_0} i Z_{\text{ext}}^\perp$$

thus, the external drive is

$$G = 2 \omega_{\beta_0} \omega_0 \Delta Q_{\text{coh}} \langle x \rangle$$

only the dipole impedance here, no incoherent effects
Dispersion Relation

An easy derivation of the dispersion relation

the external drive is IMPEDANCE TUNE SHIFT $\times$ PERTURBATION

$$G = 2\omega_{\beta_0}\omega_0\Delta Q_{\text{coh}} \langle x \rangle$$

the beam response is the BTF

$$\langle x \rangle = \frac{G}{2\omega_{\beta_0}\sigma_\omega} R(u)$$

combined: the DISPERSION RELATION

$$\Delta Q_{\text{coh}} R(\Omega) = 1$$

provides the resulting $\Omega$ for the given impedance and beam

Vladimir Kornilov, CAS, June 9-21, 2019, Denmark
Stability Diagram

\[ \Delta Q_{\text{coh}} R(\Omega) = 1 \]
\[ \Delta Q_{\text{coh}} \omega_0 \int \frac{f(\omega) d\omega}{\omega - \Omega} = 1 \]

Re(Z)>0: the slow wave
\[ \omega_s = (n - Q_0) \omega_0 \]
\[ \delta Q \xi = \left| \eta (n - Q_0) + Q_0 \xi \right| \delta_p \]


Vladimir Kornilov, CAS, June 9-21, 2019, Denmark
Stability Diagram

the resulting $\Omega$ for the given impedance and beam

\[
\Delta Q_{\text{coh}} R(\Omega) = 1
\]

\[
\left| \frac{\Delta Q_{\text{coh}}}{\delta Q_\xi} \right| = 1
\]

\[
\delta Q_\xi = \left| \eta(n - Q_0) + Q_0 \xi \right| \delta_p
\]

Strength of Landau Damping is proportional to the tune spread

Tune spread provides Landau Damping
Resonances: \( kQ_x + mQ_y = n \) 
- 2\(^{nd}\) order (quadrupole)
- 3\(^{rd}\) order (sextupole)
- 4\(^{th}\) order (octupole)

SIS100 (FAIR@GSI Darmstadt)
nominal tune: \( Q_x = 18.84, Q_y = 18.73 \)

Green area: tune spread due to the chromaticity \( \xi \) (only!)

Tune spread provides Landau Damping
Beam Transfer Function

BTF provides a direct measure of Landau Damping

\[ \Delta Q_{coh} R(\Omega) = 1 \]

Measured BTF in SIS18

Resulting Stability Diagram

Longitudinal Stability

Coasting Beam:
Spread in the revolution frequency

\[ A I_0 \frac{Z_{\|}(\Omega_{\|})}{n} \int \frac{\partial f(\omega_0)/\partial \omega_0}{\omega_0 - \Omega_{\|}/n} d\omega_0 = 1 \]

\[ \left| \frac{Z}{n} \right| \leq 0.6 \frac{2\pi \beta^2 E_0 \eta(\Delta p/p)^2}{\varepsilon I_0} \]

Bunched beams:

\[ \Delta \omega_s^{\text{coh}} \int \frac{f(\omega_s) d\omega_s}{\Omega_{\|} - \omega_s} = 1 \]

the physics and the formalism are similar

K.Y. Ng, Physics of Intensity Dependent Beam Instabilities, 2006
Landau Damping

Incomplete (!) mechanism of Landau Damping in beams for the end of the first part

Main ingredients of Landau damping:
- wave–particle collisionless interaction: Impedance driving field
- energy transfer: the wave ↔ the (few) resonant particles
Landau Damping

End of part 1