Introduction to Longitudinal Beam Dynamics

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crab nebula, burst of charged particles $E=10^{20}$ eV
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1.) Electrostatic Machines:
(Tandem -) van de Graaff Accelerator (1930 ...)

creating high voltages by mechanical transport of charges

* Terminal Potential: \( U \approx 12 \ldots 28 \text{ MV} \)
using high pressure gas to suppress discharge (\( \text{SF}_6 \))

Problems: *
Particle energy limited by high voltage discharges
* high voltage can only be applied once per particle ... or twice?
Energy Gain

... we have to start again from the basics

Lorentz force \[ \vec{F} = q \times (\vec{E} + \vec{v} \times \vec{B}) \]

in long. direction the B-field creates no force \( \vec{v} \parallel \vec{B} \)

acceleration force is given by the electr. Field

\[ \vec{F} = \frac{d\vec{p}}{dt} = e \vec{E} \]

In relativistic dynamics, energy and momentum satisfy the relation:

\[ E^2 = E_0^2 + p^2 c^2 \quad \text{(} E = E_0 + W \text{)} \]

Hence:

\[ dE = \int Fds = \nu dp \]

and the kinetic energy gained from the field along the z path is:

\[ dW = dE = eE_z ds \quad \Rightarrow \quad W = e \int E_z ds = eV \]
The "Tandem principle": Apply the accelerating voltage twice ... ... by working with negative ions (e.g. $H^-$) and stripping the electrons in the centre of the structure.

\[ dW = dE = eE_z \, ds \quad \Rightarrow \quad W = e \int E_z \, ds = eV \]

*nota bene: all particles are "synchron" with the acceleration potential*
1928, Wideroe: how can the acceleration voltage be applied several times to the particle beam

schematic Layout:

Energy gained after n acceleration gaps

\[ E_n = n \times q \times U_0 \times \sin \psi_s \]

- \( n \) number of gaps between the drift tubes
- \( q \) charge of the particle
- \( U_0 \) Peak voltage of the RF System
- \( \psi_s \) synchronous phase of the particle

* the problem of synchronisation ... between the particles and the rf voltage
* „voltage has to be flipped“ to get the right sign in the second gap
  \( \Rightarrow \) shield the particle in drift tubes during the negative half wave of the RF voltage
Wideroe-Structure: the drift tubes

-屏蔽粒子在 RF 负半波期间的屏蔽

时间跨度的负半波：$\tau_{RF}/2$

长度的漂移管：

$$l_i = v_i \frac{\tau_{rf}}{2}$$

$$\rightarrow v_i = \sqrt{\frac{2E_i}{m}}$$

$$l_i = \frac{1}{v_{rf}} \sqrt{\frac{i* q * U_0 * \sin \psi_s}{2m}}$$

有效于非相对论性粒子 ...

Alvarez-Structure: 1946, surround the whole structure by a rf vessel

能量：$\approx 20 \text{ MeV per Nucleon} \beta \approx 0.04 \ldots 0.6$,  PARTICLES: Protons/Ions
GSI: Unilac, typical Energie $\approx 20$ MeV per Nukleon, $\beta \approx 0.04 \ldots 0.6$, Protons/Ions, $\nu = 110$ MHz

Energy Gain per „Gap“:

$$W = q U_0 \sin \omega_{RF} t$$

Application: until today THE standard proton / ion pre-accelerator CERN Linac 4 is being built at the moment

„synchronisation“ with the acceleration potential is established via the geometry of the machine ... i.e. the drift tube length
4.) The Cyclotron: (Livingston / Lawrence ~1930)

Idea: $B = \text{const}$, $RF = \text{const}$
Synchronisation particle / RF via orbit

Lorentzforce

$$\vec{F} = q \times (v \times B) = q \times v \times B$$

circular orbit

$$q \times v \times B = \frac{m \times v^2}{R} \quad \rightarrow \quad B \times R = \frac{p}{q}$$

revolution frequency

$$\omega_z = \frac{v}{R} = \frac{q \times B_z}{m}$$

the cyclotron (rf-) frequency
is independent of the momentum

rf-frequency = $h \times$ revolution frequency, $h = \text{“harmonic number”}$
Cyclotron:

exact equation for revolution frequency:

\[ \omega_z = \frac{v}{R} = \frac{q}{\gamma m} \ast B_z \]

1.) if \( v \ll c \Rightarrow \gamma \approx 1 \)

2.) \( \gamma \) increases with the energy
   \( \Rightarrow \) no exact synchronisation

“synchronisation” with the acceleration potential is established via the spiraling orbit length

B = constant
\( \gamma \omega_{RF} = \) constant
\( \omega_{RF} \) decreases with time

\[ \omega_s(t) = \omega_{rf}(t) = \frac{q}{\gamma(t) \ast m_0} \ast B \]

keep the synchronisation condition by varying the rf frequency
**RF Cavities, Acceleration and Energy Gain**

\[ dW = dE = eE_z ds \quad \Rightarrow \quad W = e \int E_z ds = eV \]

RF acceleration: \( V \neq \text{const} \)

In this case the electric field is oscillating. So it is for the potential. The energy gain will depend on the RF phase experienced by the particle.

\[ \int \hat{E}_z \, dz = \hat{V} \]

\[ E_z = \hat{E}_z \cos \omega_{RF} t = \hat{E}_z \cos \Phi(t) \]

Neglecting the transit time in the gap.

\[ W = e \hat{V} \cos \Phi \]
Energy Gain in RF structures:

Transit Time Factor

Oscillating field at frequency $\omega$ (amplitude is assumed to be constant all along the gap)

$$E_z = E_0 \cos \omega t = \frac{V}{g} \cos \omega t$$

Consider a particle passing through the middle of the gap at time $t=0:\: z=vt$

The total energy gain is:

$$\Delta W = \frac{eV}{g} \int_{-g/2}^{g/2} \cos \omega \frac{z}{v} dz$$

$$T = \frac{\sin \theta / 2}{\theta / 2}$$

transit time factor ($0 < T < 1$)

$$\theta = \frac{\omega g}{v}$$

transit angle

ideal case: $T = \frac{\sin \theta / 2}{\theta / 2} \rightarrow 1 \quad \Leftrightarrow \quad \theta / 2 \rightarrow 0$

el. static accelerators $\omega \rightarrow 0$

minimise acc. gap $g \rightarrow 0$
The Synchrotron (Mac Millan, Veksler, 1945)

The synchrotron: Ring Accelerator of const. R where the increase in momentum (i.e. B-field) is automatically synchronised with the correct synchronous phase of the particle in the rf cavities.

Energy gain per turn

\[ eV \]

Synchronous particle

\[ \Phi = \Phi_s = cte \]

RF synchronism

\[ \omega_{RF} = h\omega_r \]

Constant orbit

\[ \rho = cte \quad R = cte \]

Variable magnetic field

\[ B\rho = \frac{P}{e} \Rightarrow B \]
**Momentum Compaction Factor: \( \alpha_p \)**

A particle with a displacement \( x \) to the design orbit → path length \( dl \) ...

\[
\frac{dl}{ds} = \frac{\rho + x}{\rho}
\]

→ \( dl = \left(1 + \frac{x}{\rho(s)}\right) ds \)

Circumference of an off-energy closed orbit

\[
l_{\Delta E} = \int dl = \int \left(1 + \frac{x_{\Delta E}}{\rho(s)}\right) ds
\]

Remember: \( x_{\Delta E}(s) = D(s) \frac{\Delta p}{p} \)

\[
\delta l_{\Delta E} = \frac{\Delta p}{p} \int \left(\frac{D(s)}{\rho(s)}\right) ds
\]

* The lengthening of the orbit for off-momentum particles is given by the dispersion function and the bending radius.
For first estimates assume: \( \frac{1}{\rho} = \text{const.} \)

\[
\int D(s) \, ds \approx l_{\Sigma(dipoles)} \cdot \langle D \rangle_{\text{dipole}}
\]

\[
\alpha_p = \frac{1}{L} \cdot \frac{l_{\Sigma(dipoles)} \cdot \langle D \rangle}{\rho} = \frac{2\pi \rho \cdot \langle D \rangle}{L} \quad \rightarrow \quad \alpha_p \approx \frac{2\pi}{L} \langle D \rangle \approx \frac{\langle D \rangle}{R}
\]

\( \nu \approx c \)

\[
\frac{\delta T}{T} = \frac{\delta l_e}{L} = \alpha_p \frac{\Delta p}{p}
\]
Dispersion Effects in a Synchrotron

If a particle is slightly shifted in momentum it will have a different orbit:

$$\alpha = \frac{p}{R} \frac{dR}{dp}$$

This is the “momentum compaction” generated by the bending field.

If the particle is shifted in momentum it will have also a different velocity. As a result of both effects the revolution frequency changes:

$$\eta = \frac{p}{f_r} \frac{df_r}{dp}$$

p=particle momentum
R=synchrotron physical radius
f_r=revolution frequency
Dispersion Effects in a Synchrotron

\[ \eta = \frac{p}{f_r} \frac{df_r}{dp} \]

\[ f_r = \frac{\beta c}{2\pi R} \Rightarrow \frac{df_r}{f_r} = \frac{d\beta}{\beta} - \frac{dR}{R} \]

\[ \frac{dR}{R} = \alpha \frac{dp}{p} \]

\[ p = mv = \beta \gamma \frac{E_0}{c} \Rightarrow \frac{dp}{p} = \frac{d\beta}{\beta} + \frac{d(1 - \beta^2)^{-\frac{1}{2}}}{(1 - \beta^2)^{-\frac{1}{2}}} = (1 - \beta^2)^{-1} \frac{d\beta}{\beta} \]

\[ \frac{d\beta}{\beta} = \frac{1}{\gamma^2} \frac{dp}{p} \]

\[ \frac{df_r}{f_r} = \left( \frac{1}{\gamma^2} - \alpha \right) \frac{dp}{p} \]

\[ \eta = \frac{1}{\gamma^2} - \alpha \]

The change of revolution frequency depends on the particle energy \( \gamma \) and changes sign during acceleration.

Particles get faster in the beginning – and arrive earlier at the cavity: **classic regime**

Particles travel at \( v = c \) and get more massive – and arrive later at the cavity: **relativistic regime**

boundary between the two regimes: no frequency dependence on \( dp/p \), \( \eta = 0 \) “transition energy”

\[ \gamma_{tr} = \frac{1}{\sqrt{\alpha}} \]
14.) The Acceleration for $\Delta p/p \neq 0$

"Phase Focusing" below transition

ideal particle •

particle with $\Delta p/p > 0$ • faster

particle with $\Delta p/p < 0$ • slower

Focussing effect in the longitudinal direction keeping the particles close together ...
... forming a “bunch”
... so sorry, here we need help from Albert:

\[ \gamma = \frac{E_{\text{total}}}{mc^2} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \rightarrow \quad \frac{v}{c} = \sqrt{1 - \frac{mc^2}{E^2}} \]

\( v/c \)

... some when the particles do not get faster anymore

.... but heavier!
15.) The Acceleration for $\Delta p/p \neq 0$

"Phase Focusing" above transition

ideal particle

particle with $\Delta p/p > 0$ • heavier

particle with $\Delta p/p < 0$ • lighter

oscillation frequency: $f_s = f_{rev} \sqrt{\frac{-h\alpha_s}{2\pi} \cdot \frac{qU_0 \cos \phi_s}{E_s}} \approx$ some Hz
LHC Commissioning: RF

A proton bunch: focused longitudinal by the RF field

RF off

RF on, wrong phase condition
Energy ramping is simply obtained by varying the B field:

\[ p = eB \rho \Rightarrow \frac{dp}{dt} = e \rho \dot{B} \Rightarrow (\Delta p)_{\text{turn}} = e \rho \dot{B} T_r = \frac{2\pi e \rho R \dot{B}}{v} \]

Energy Gain per turn:

\[ E^2 = E_0^2 + p \frac{2}{c^2} \quad \Rightarrow \quad \Delta E = v \Delta p \]

* The energy gain depends on the rate of change of the dipole field

* The number of stable synchronous particles is equal to the harmonic number \( h \). They are equally spaced along the circumference.

* Each synchronous particle satisfies the relation \( p = eB\rho \). They have the nominal energy and follow the nominal trajectory.
The Synchrotron: Frequency Change

During the energy ramping, the RF frequency increases to follow the increase of the revolution frequency:

\[ \omega_r = \frac{\omega_{RF}}{h} = \omega(B, R_s) \]

hence:

\[ \frac{f_{RF}(t)}{h} = \frac{v(t)}{2\pi R_s} = \frac{1}{2\pi} \frac{e}{m} < B(t) > \quad \Rightarrow \quad \frac{f_{RF}(t)}{h} = \frac{1}{2\pi} \frac{e c^2}{E_S(t) R_s} \frac{r}{B(t)} \]

Since

\[ E^2 = (m_0 c^2)^2 + p^2 c^2 \]

The RF frequency must follow the variation of the B field with the law:

\[ \frac{f_{RF}(t)}{h} = \frac{c}{2\pi R_s} \left\{ \frac{B(t)^2}{(m_0 c^2 / e c r)^2 + B(t)^2} \right\}^{1/2} \]

and as soon as

\[ B > \frac{m_0 c^2}{ecr} \]

\[ \frac{f_{RF}(t)}{h} \approx \frac{c}{2\pi R_s} = \text{const} \]

which is true for LHC at high energy and for electrons from the start.
**Longitudinal Dynamics: synchrotron motion**

We have to follow two coupled variables:

* the energy gained by the particle

* and the RF phase experienced by the same particle.

Since there is a well defined synchronous particle which has always the same phase $\phi_s$ and the nominal energy $E_s$, it is sufficient and elegant to follow other particles with respect to that particle.

We will introduce the following relative variables:

- revolution frequency: $\Delta f_r = f_r - f_{rs}$
- particle RF phase: $\Delta \phi = \phi - \phi_s$
- particle momentum: $\Delta p = p - p_s$
- particle energy: $\Delta E = E - E_s$
- azimuth angle: $\Delta \theta = \theta - \theta_s$
The Equation of Motion:

Energy-Phase Relations in a Synchrotron

energy offset $\leftrightarrow$ phase change
**First Energy-Phase Equation:**

*energy offset $\leftrightarrow$ phase change*

\[ f_{RF} = hf_r \quad \Rightarrow \quad \Delta \phi = -h\Delta \theta \quad \text{with} \quad \theta = \int \omega_r dt \]

For a given particle with respect to the reference one:

\[ d\omega_r = \frac{d}{dt} (\Delta \theta) = -\frac{1}{h} \frac{d}{dt} (\Delta \phi) = -\frac{1}{h} \frac{d\phi}{dt} \]

Since:

\[ \eta = \frac{p_s}{\omega_{rs}} \left( \frac{d\omega_r}{dp} \right)_s \quad \Rightarrow \quad dp = \frac{p}{\eta} \left( \frac{d\omega_r}{dp} \right)_s \]

and from relativity we know:

\[ \Delta E = v_s \Delta p = \omega_{rs} R_s \Delta p \]

one gets:

\[ \frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta \omega_{rs}} \frac{d(\Delta \phi)}{dt} = -\frac{p_s R_s}{h\eta \omega_{rs}} \dot{\phi} \]

The energy deviation from the synchronous particle depends on the rate of change of the phase.
**Second Energy-Phase Equation**

energy offset $\leftrightarrow$ RF voltage

energy gain per turn:

$$\Delta E_{\text{turn}} = e\hat{V}\sin\phi$$

$$\Delta p_{\text{turn}} = \frac{e\hat{V}}{\omega R}\sin\phi$$

momentum rate of change:

$$\dot{p} = \frac{\Delta p_{\text{turn}}}{T} = \frac{\Delta p_{\text{turn}}}{2\pi}\omega = \frac{e\hat{V}}{2\pi R}\sin\phi$$

$$2\pi R\dot{p} = e\hat{V}\sin\phi$$

difference to the synchr. particle:

$$2\pi \Delta(R\dot{p}) = e\hat{V}(\sin\phi - \sin\phi_s)$$
Second Energy-Phase Equation

momentum offset <-> geometry

\[ \Delta (R\dot{p}) = R\dot{p} - R_s\dot{p}_s = (R_s + \Delta R)(\dot{p}_s + \Delta \dot{p}) - R_s\dot{p}_s \]

\[ = R_s\dot{p}_s + R_s\Delta \dot{p} + \Delta R\dot{p}_s + \Delta R\Delta \dot{p} - R_s\dot{p}_s \approx 0 \]

\[ = R_s\Delta \dot{p} + \Delta R\dot{p}_s = R_s\Delta \dot{p} + \dot{p}_s\left(\frac{dR}{dp}\right)_s \Delta p = R_s\Delta \dot{p} + \frac{dp_s}{dt}\left(\frac{dR}{dp}\right)_s \Delta p \]

\[ = R_s\Delta \dot{p} + \dot{R}_s\Delta p = \frac{d}{dt}(R_s\Delta p) = \frac{d}{dt}\left(\frac{\Delta E}{\omega_s}\right) \]

... put into the green equation ... to get

\[ 2\pi \Delta (R\dot{p}) = e\hat{V} (\sin \phi - \sin \phi_s) \]

\[ 2\pi \frac{d}{dt}\left(\frac{\Delta E}{\omega_s}\right) = e\hat{V} (\sin \phi - \sin \phi_s) \]

the rate of energy change is determined by the distance in phase in the sinusoidal rf voltage function
Equations of Longitudinal Motion

\[ \frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta \omega_{rs}} \frac{d(\Delta \phi)}{dt} = -\frac{p_s R_s}{h\eta \omega_{rs}} \dot{\phi} \]

\[ 2\pi \frac{d}{dt} \left( \frac{\Delta E}{\omega_{rs}} \right) = e \hat{V} (\sin \phi - \sin \phi_s) \]

This rather formidable looking differential equation simplifies a lot if we consider ...

\( R_s, p_s, \omega_s, \eta \) as constant (or slowly varying with time).
Small Amplitude Oscillations

Let’s assume constant parameters \( R_s, p_s, \omega_s \) and \( \eta \):

\[
\ddot{\phi} + \frac{\Omega_s^2}{\cos \phi_s}(\sin \phi - \sin \phi_s) = 0
\]

with

\[
\Omega_s^2 = \frac{h \eta \omega_r e \hat{V} \cos \phi_s}{2\pi R_s p_s}
\]

Consider now small phase deviations from the reference particle:

\[
\sin \phi - \sin \phi_s = \sin(\phi_s + \Delta \phi) - \sin \phi_s \equiv \cos \phi_s \Delta \phi
\]

and the corresponding linearized motion reduces to a harmonic oscillation:

\[
\ddot{\phi} + \Omega_s^2 \Delta \phi = 0
\]

stable for \( \Omega_s^2 > 0 \) and \( \Omega_s \) real
**Small Amplitude Oscillations:** *phase stability*

We get a harmonic oscillation of the particle phase with the oscillation frequency

\[
\ddot{\phi} + \Omega_s^2 \Delta \phi = 0
\]

\[
\Omega_s = \sqrt{\frac{\hbar \eta \omega_v e \hat{V} \cos \phi_s}{2 \pi R_s p_s}}
\]

Stability condition: \( \Omega_s \) real \( \Omega_s^2 > 0 \)

\[
\begin{align*}
\gamma &< \gamma_{tr} \quad \eta > 0 \quad 0 < \phi_s < \pi/2 \\
\gamma &> \gamma_{tr} \quad \eta < 0 \quad \pi/2 < \phi_s < \pi
\end{align*}
\]

And we will find this situation “\( h \)”-times in the machine

**LHC:**
35640 Possible Bunch Positions (“buckets”)
2808 Bunches

*oscillation frequency depends on*
* the square root
* of an electrical potential
* weak force <-> small frequency*
The RF system: IR4

4xFour-cavity cryo module 400 MHz, 16 MV/beam
Nb on Cu cavities @4.5 K (=LEP2)
Beam pipe diam.=300mm
(small) ... Synchrotron Oscillations in Energy and Phase

\[ \ddot{\phi} + \Omega_s^2 \Delta \phi = 0 \]

Ansatz: \[ \Delta \phi = \Delta \phi_{\text{max}} \cos(\Omega_s t) \]

take the first derivative and put it into the first energy-phase relation

\[ \frac{d(\Delta \phi)}{dt} = -\Delta \phi_{\text{max}} \sin(\Omega_s t) \Omega_s \]

to get the energy oscillations

\[ \Delta E = \Delta E_{\text{max}} \sin(\Omega_s t) \]

which defines an ellipse in phase space \( \Delta \Phi, \Delta E \):

\[ \left( \frac{\Delta \Phi}{\Delta \Phi_{\text{max}}} \right)^2 + \left( \frac{\Delta E}{\Delta E_{\text{max}}} \right)^2 = 1 \]
Large Amplitude Oscillations

Equation of motion: \[ \ddot{\phi} + \frac{\Omega_s^2}{\cos \phi_s} (\sin \phi - \sin \phi_s) = 0 \]

There are two positions (in fact three) where a particle does not get any phase focusing force, \( \Phi = \Phi_s \) (i.e. the ideal position)

and at \( \Phi = \pi - \Phi_s \)

When \( \phi \) reaches \( \pi - \phi_s \) the force goes to zero and beyond it becomes non restoring. Hence \( \pi - \phi_s \) is an extreme amplitude for a stable motion which in the phase space (\( \dot{\phi}/\Omega_s, \Delta \phi \)) is shown as closed trajectories.

The phase curve, that belongs to \( \Phi = \Phi_s \) separates the stable from the unstable regime

-> “Separatrix”

Equation of the separatrix:

\[ \frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos \phi_s} (\cos \phi + \phi \sin \phi_s) = - \frac{\Omega_s^2}{\cos \phi_s} (\cos(\pi - \phi_s) + (\pi - \phi_s) \sin \phi_s) \]
LHC Commissioning: RF

We have to match these conditions:
- phase (i.e. timing between rf and injected bunch) has to correspond to $\phi_s$.
- long. acceptance of injected beam has to be smaller than bucket area of the synchrotron.

Max stable energy: set $\phi = \phi_s$ and calculate $\Delta E$

$$\left(\Delta E_{\text{max}}\right)_{\text{sep}} = \sqrt{\frac{p_s v_s e V_0}{2\pi\hbar\eta_s}} \sqrt{|4\cos\phi_s - (2\pi - 4\phi_s)\sin\phi_s|}$$

LHC injection:
- acceptance: 1.4eVs
- long emittance: 1.0 eVs
Than‘x

I am still confused, but now I am confused at a much higher level!
A long bunch coming from the gun enters an RF cavity; the reference particle is the one which has no velocity change. The others get accelerated or decelerated. After a distance L bunch gets shorter while energies are spread: bunching effect. This short bunch can now be captured in the following rf structures.
Improved Capture With Pre-buncher

The bunching effect is a space modulation that results from a velocity modulation and is similar to the phase stability phenomenon. Let’s look at particles in the vicinity of the reference one and use a classical approach.

Energy gain as a function of cavity crossing time:

$$\Delta W = \Delta \left( \frac{1}{2} m_0 v^2 \right) = m_0 v_0 \Delta v = eV_0 \sin \phi \approx eV_0 \phi$$

Perfect linear bunching will occur after a time delay $\tau$, corresponding to a distance $L$, when the path difference is compensated between a particle and the reference one:

$$\Delta v \cdot \tau = \Delta z = v_0 t = v_0 \frac{\phi}{\omega_{RF}}$$

(assuming the reference particle enters the cavity at time $t=0$)

Since $L = v\tau$ one gets:

$$L = \frac{2v_0 W}{eV_0 \omega_{RF}}$$