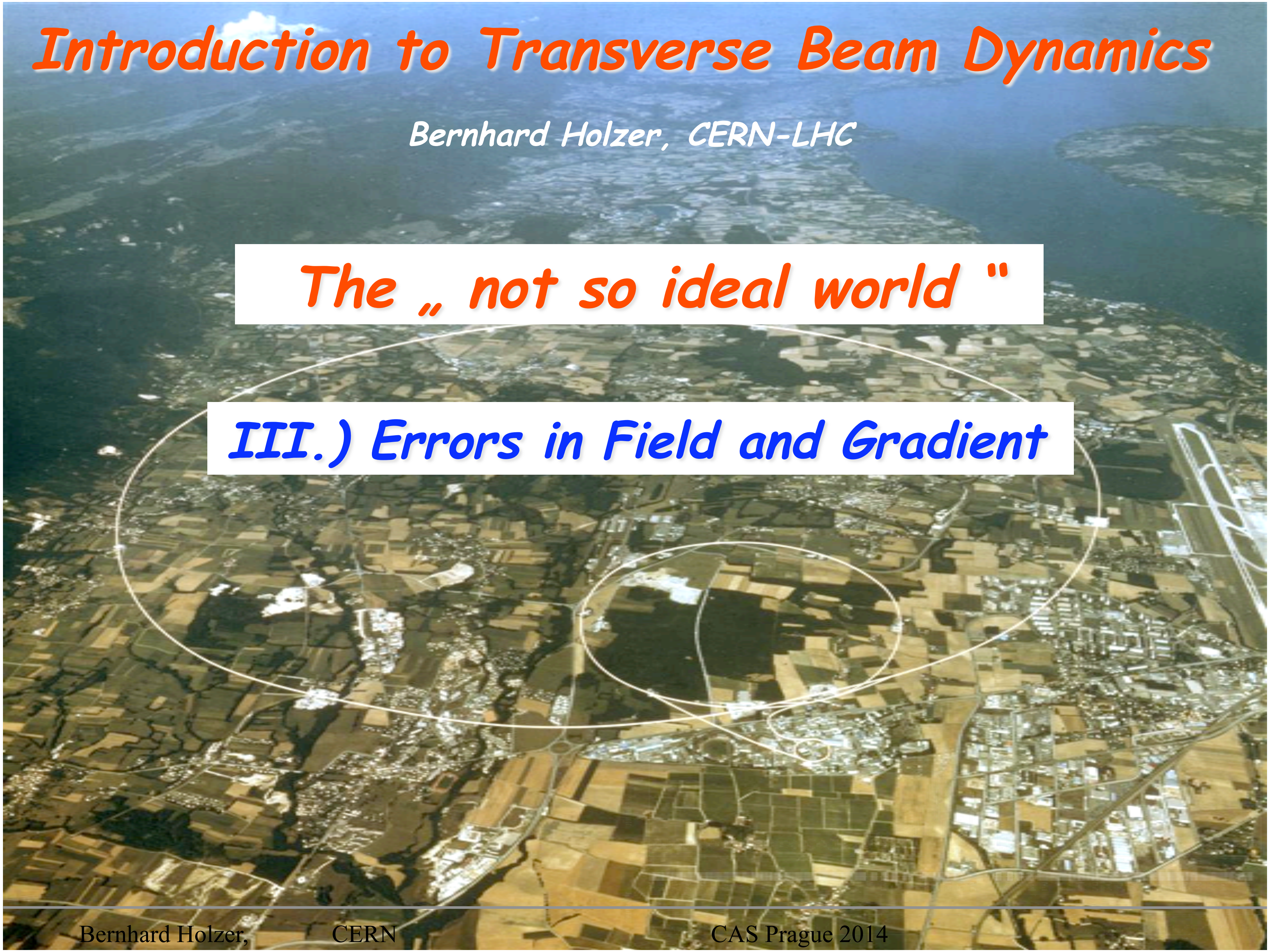
Introduction to Transverse Beam Dynamics

Bernhard Holzer, CERN-LHC

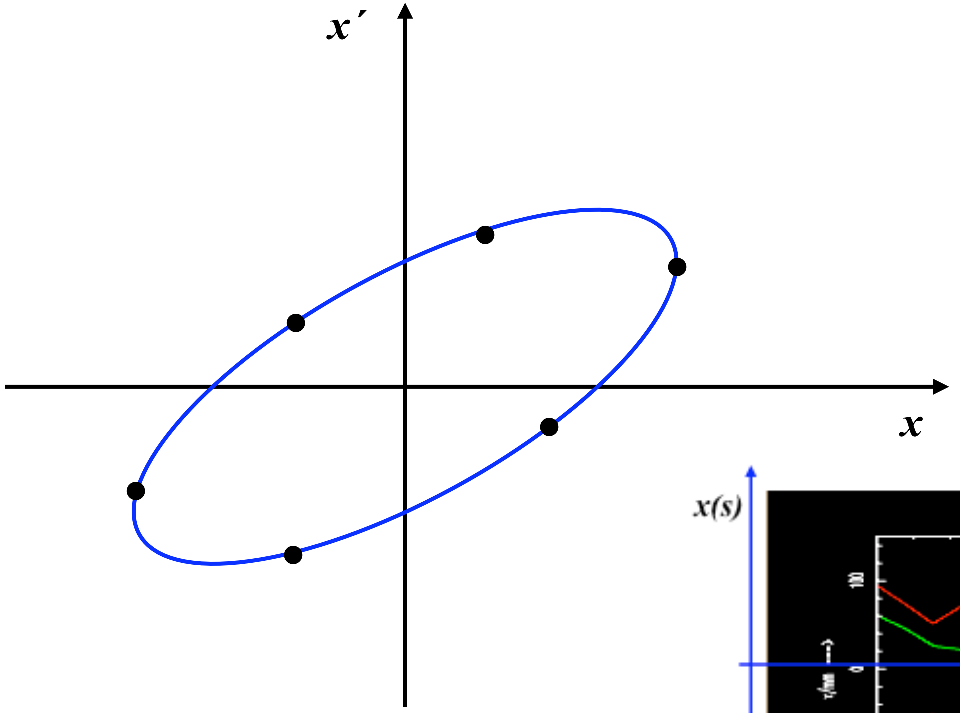


The „ not so ideal world "

III.) Errors in Field and Gradient

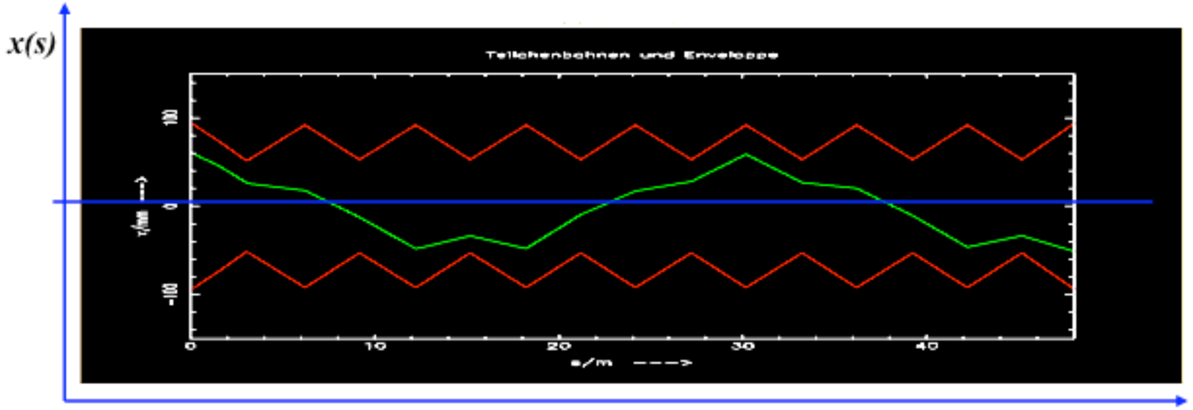
Remember: Beam Emittance and Phase Space Ellipse

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$



Liouville: in reasonable storage rings area in phase space is constant.

$A = \pi * \varepsilon = const$

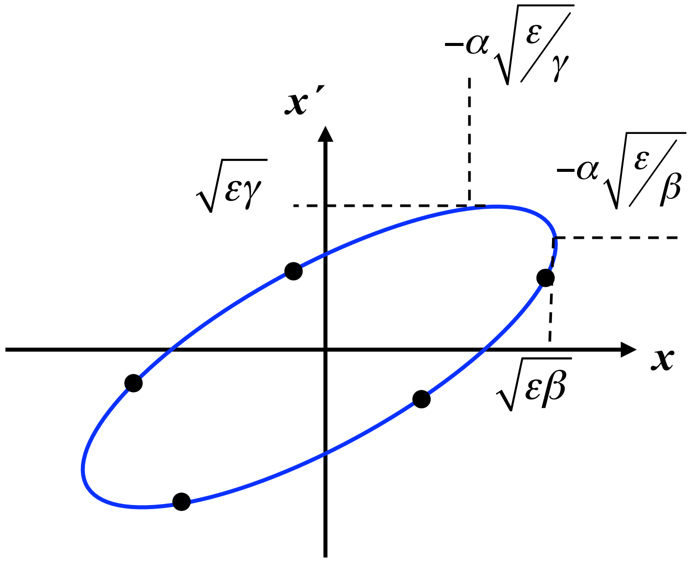


ε beam emittance = woozilycity of the particle ensemble, intrinsic beam parameter, cannot be changed by the foc. properties.

Scientifiquely spoken: area covered in transverse x, x´ phase space … and it is constant !!!

13.) Liouville during Acceleration

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$



***Beam Emittance** corresponds to the area covered in the x, x´ Phase Space Ellipse*

***Liouville:** Area in phase space is constant.*

But so sorry ... $\varepsilon \neq$ const !

Classical Mechanics:

phase space = diagram of the two canonical variables
position & momentum
x p_x

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad ; \quad L = T - V = kin.\,Energy - pot.\,Energy$$

According to Hamiltonian mechanics:
phase space diagram relates the variables q and p

$$q = position = x$$
$$p = momentum = \gamma m v = mc\gamma\beta_x$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad ; \quad \beta_x = \frac{\dot{x}}{c}$$

Liouvilles Theorem: $\int p\,dq = const$

for convenience (i.e. because we are lazy bones) we use in accelerator theory:

$$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{\beta_x}{\beta}$$

where $\beta_x = v_x / c$

$$\int p\,dq = mc\int \gamma\beta_x dx$$

$$\int p\,dq = mc\gamma\beta \underbrace{\int x' dx}_{\varepsilon}$$

$$\boxed{\Rightarrow \quad \varepsilon = \int x' dx \propto \frac{1}{\beta\gamma}}$$

the beam emittance shrinks during acceleration $\varepsilon \sim 1/\gamma$

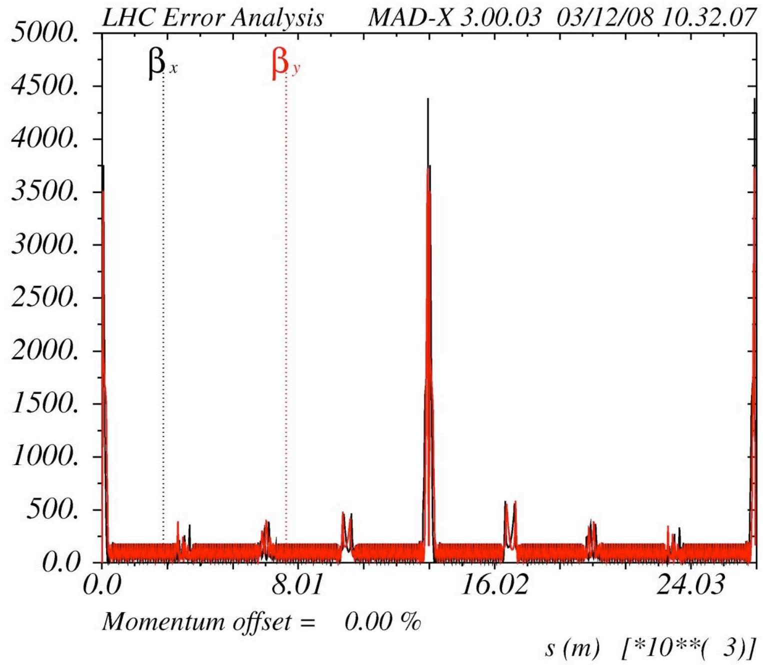
Nota bene:

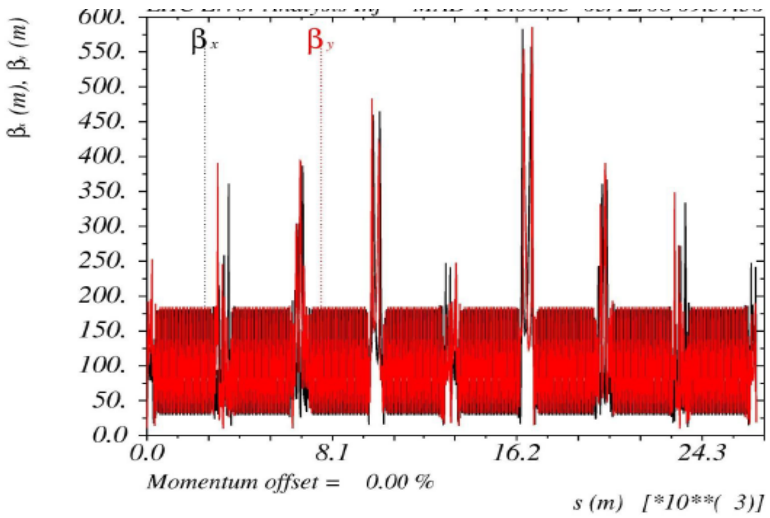
1.) A proton machine … or an electron linac … needs the highest aperture at injection energy !!!
as soon as we start to accelerate the beam size shrinks as $\gamma^{-1/2}$ in both planes.

$$\sigma = \sqrt{\varepsilon\beta}$$

2.) At lowest energy the machine will have the major aperture problems,
→ here we have to minimise $\hat{\beta}$

3.) we need different beam optics adopted to the energy:
A Mini Beta concept will only be adequate at flat top.





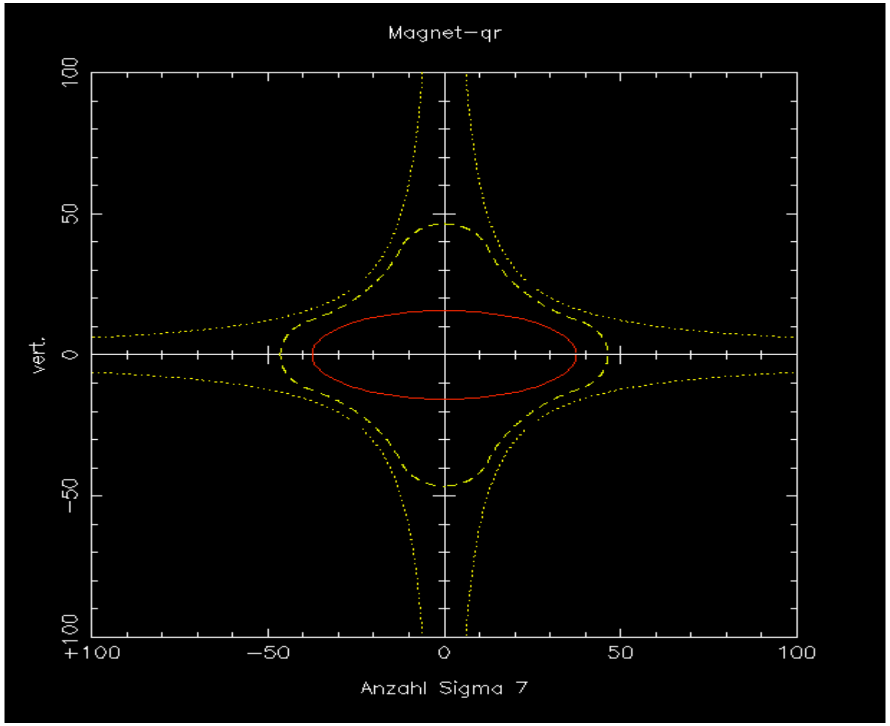
LHC injection optics at 450 GeV

LHC mini beta optics at 7000 GeV

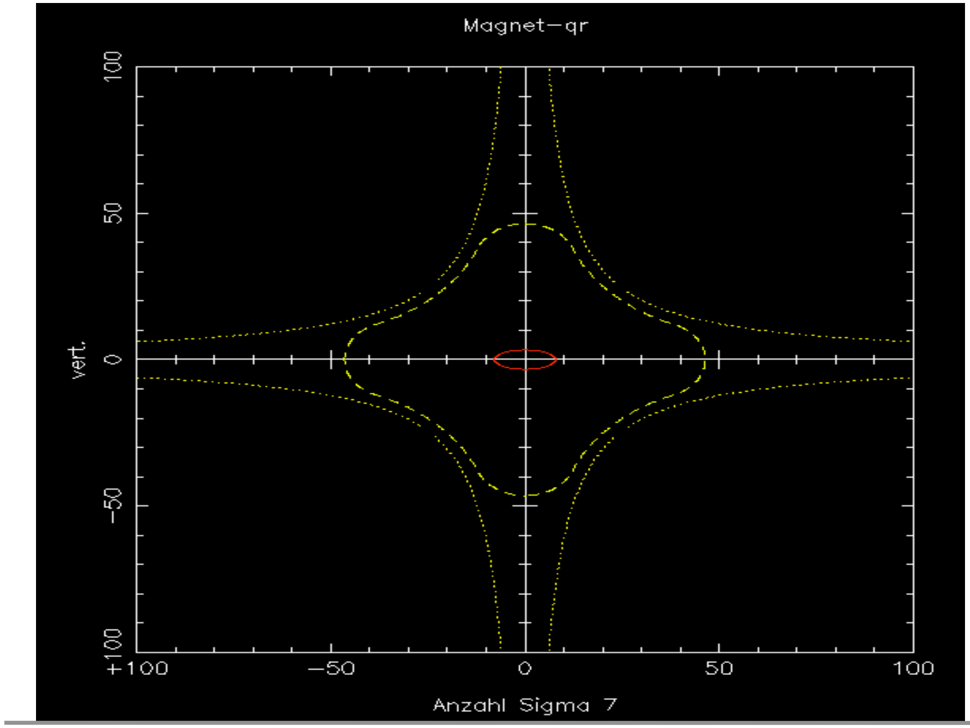
Example: HERA proton ring

injection energy: 40 GeV $\gamma = 43$
flat top energy: 920 GeV $\gamma = 980$

emittance ε *(40GeV)* $= 1.2 * 10^{-7}$
ε *(920GeV)* $= 5.1 * 10^{-9}$



7 σ beam envelope at E = 40 GeV



… and at E = 920 GeV

The „ not so ideal world "

14.) The „ $\Delta p / p \neq 0$" Problem

ideal accelerator: *all particles will see the same accelerating voltage.*

→ $\Delta p / p = 0$

„nearly ideal" accelerator: *Cockroft Walton or van de Graaf*

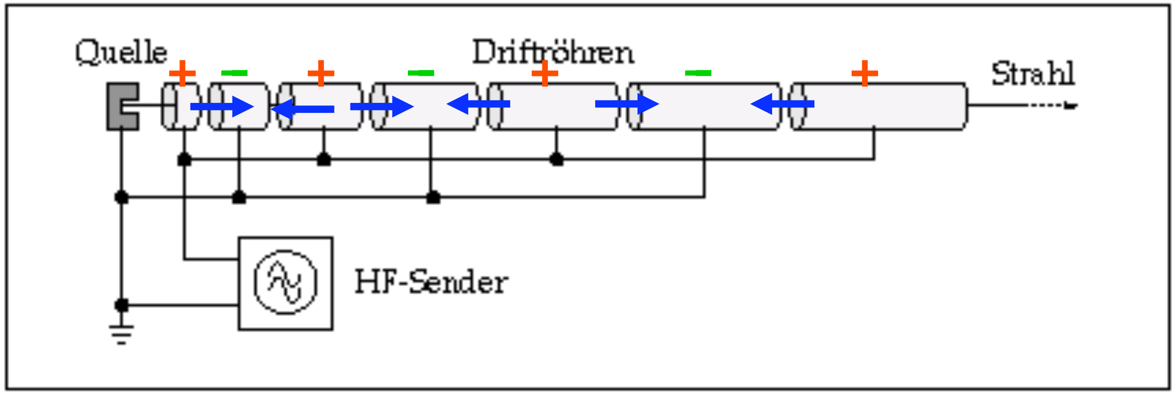
$\Delta p / p \approx 10^{-5}$

Vivitron, Straßbourg, inner structure of the acc. section

MP Tandem van de Graaf Accelerator at MPI for Nucl. Phys. Heidelberg

RF Acceleration

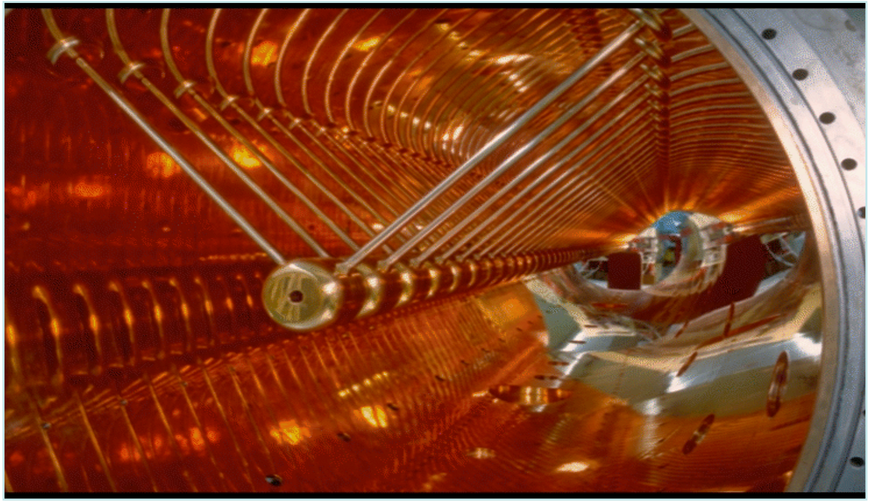
1928, Wideroe



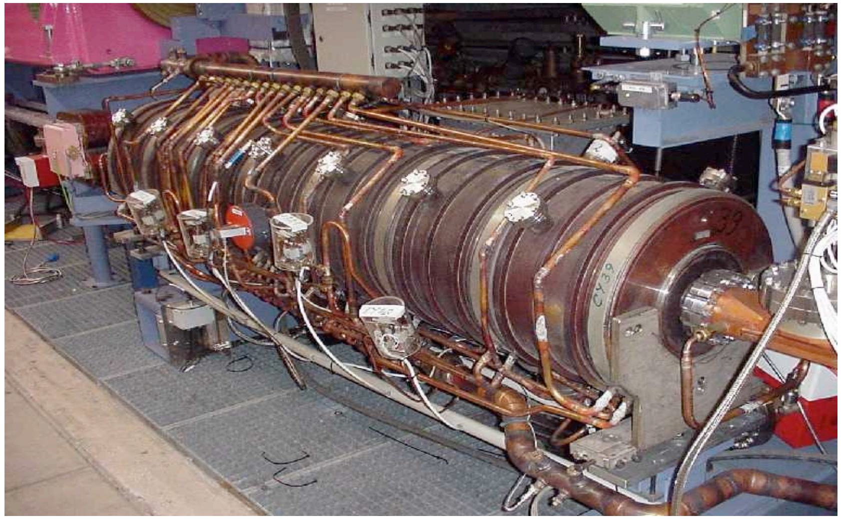
Energy Gain per „Gap":

$$W = q\, U_0 \sin \omega_{RF} t$$

drift tube structure at a proton linac (GSI Unilac)



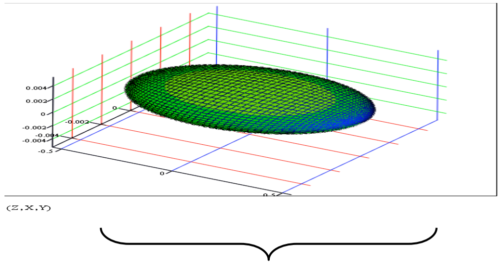
500 MHz cavities in an electron storage ring



* RF Acceleration: **multiple application of the same acceleration voltage; brillant idea to gain higher energies**

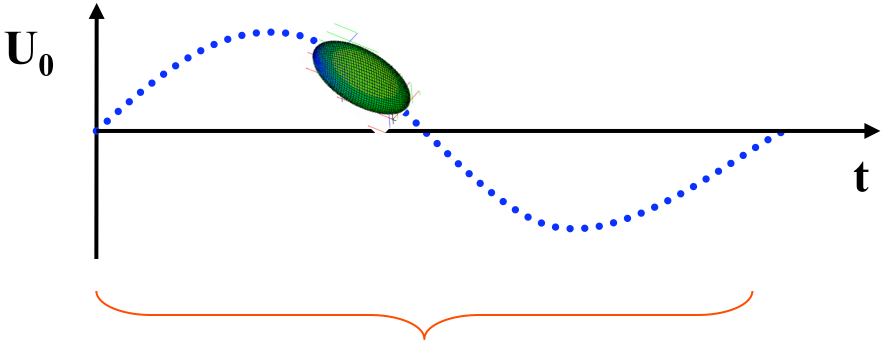
Problem: panta rhei !!!

(Heraklit: 540-480 v. Chr.)



Bunch length of Electrons ≈ 1cm

Example: HERA RF:



$\lambda = 60\ cm$

$$\left.\begin{aligned}\nu &= 500\,MHz\\ c &= \lambda\,\nu\end{aligned}\right\}\quad \lambda = 60\ cm$$

$$\sin(90^o) = 1$$

$$\sin(84^o) = 0.994$$

$$\frac{\Delta U}{U} = 6.0\ \ 10^{-3}$$

typical momentum spread of an electron bunch: $\frac{\Delta p}{p} \approx 1.0\ \ 10^{-3}$

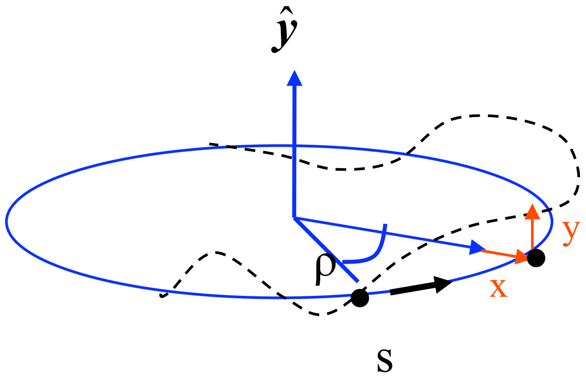
Dispersive and Chromatic Effects: $\Delta p/p \neq 0$

Are there any Problems ???

Sure there are !!!

15.) Dispersion: trajectories for $\Delta p / p \neq 0$

Question: do you remember last session, page 12 ? … sure you do



Force acting on the particle

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v$$

remember: $x \approx mm$, $\rho \approx m$ … → develop for small x

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1-\frac{x}{\rho}) = eB_yv$$

consider only linear fields, and change independent variable: $t \rightarrow s$

$$B_y = B_0 + x\frac{\partial B_y}{\partial x}$$

$$x'' - \frac{1}{\rho}(1-\frac{x}{\rho}) = \frac{e\,B_0}{\boxed{mv}} + \frac{e\,x\,g}{\boxed{mv}}$$

$p = p_0 + \Delta p$

… but now take a small momentum error into account !!!

Dispersion:

develop for small momentum error

$$\Delta p << p_0 \Rightarrow \frac{1}{p_0 + \Delta p} \approx \frac{1}{p_0} - \frac{\Delta p}{p_0^2}$$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} \approx \underbrace{\frac{e\ B_0}{p_0}}_{-\frac{1}{\rho}} - \frac{\Delta p}{p_0^2} eB_0 + \underbrace{\frac{xeg}{p_0}}_{k*x} - \underbrace{xeg\frac{\Delta p}{p_0^2}}_{\approx 0}$$

$$x'' + \frac{x}{\rho^2} \approx \frac{\Delta p}{p_0} * \underbrace{\frac{(-eB_0)}{p_0}}_{\frac{1}{\rho}} + k * x = \frac{\Delta p}{p_0} * \frac{1}{\rho} + k * x$$

$$x'' + \frac{x}{\rho^2} - kx = \frac{\Delta p}{p_0}\frac{1}{\rho} \quad \longrightarrow \quad \boxed{x'' + x(\frac{1}{\rho^2} - k) = \frac{\Delta p}{p_0}\frac{1}{\rho}}$$

***Momentum spread** of the beam adds a term on the r.h.s. of the equation of motion.*
*→ **inhomogeneous differential equation.***

Dispersion:

$$x'' + x(\frac{1}{\rho^2} - k) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) \boxed{+\, x_i(s)}$$

$$\left\{ \begin{array}{l} x_h''(s) + K(s) \cdot x_h(s) = 0 \\ x_i''(s) + K(s) \cdot x_i(s) = \frac{1}{\rho} \cdot \frac{\Delta p}{p} \end{array} \right.$$

Normalise with respect to Δp/p:

$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

Dispersion function D(s)

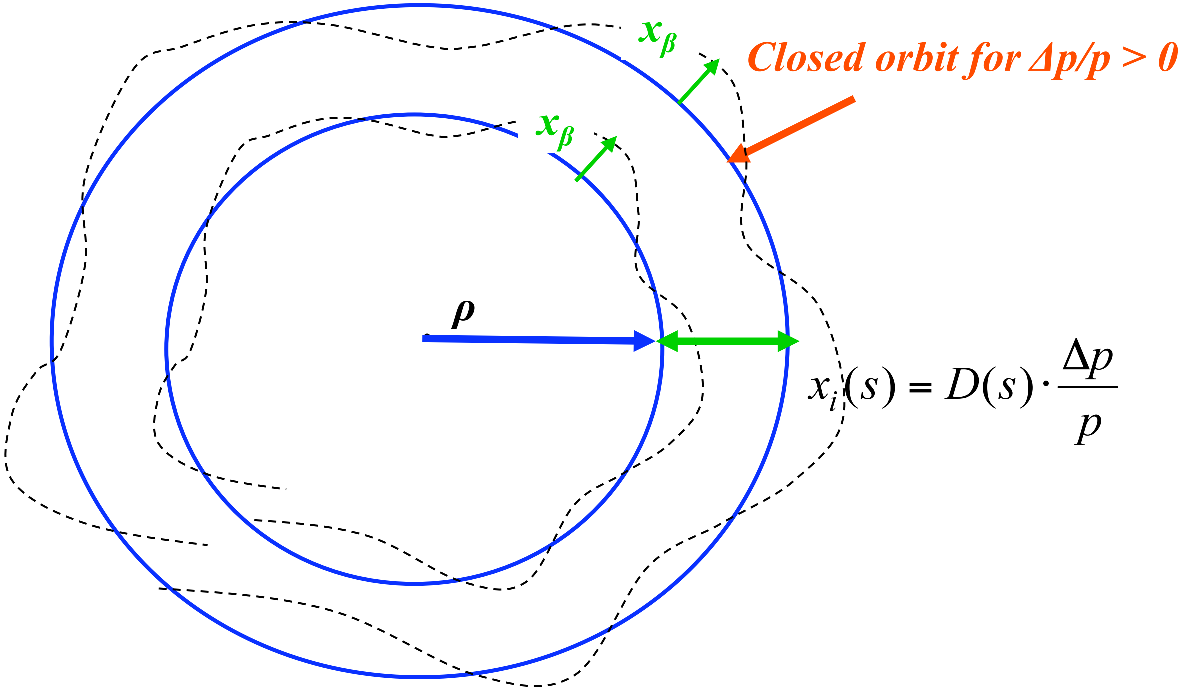
**** is that special orbit, an ideal particle would have for Δp/p = 1***

**** the orbit of any particle is the sum of the well known x_β and the dispersion***

**** as D(s) is just another orbit it will be subject to the focusing properties of the lattice***

Dispersion

Example: homogeneous dipole field



$$x_i(s) = D(s) \cdot \frac{\Delta p}{p}$$

Matrix formalism:

$$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$

$$x(s) = C(s) \cdot x_0 + S(s) \cdot x_0' + D(s) \cdot \frac{\Delta p}{p}$$

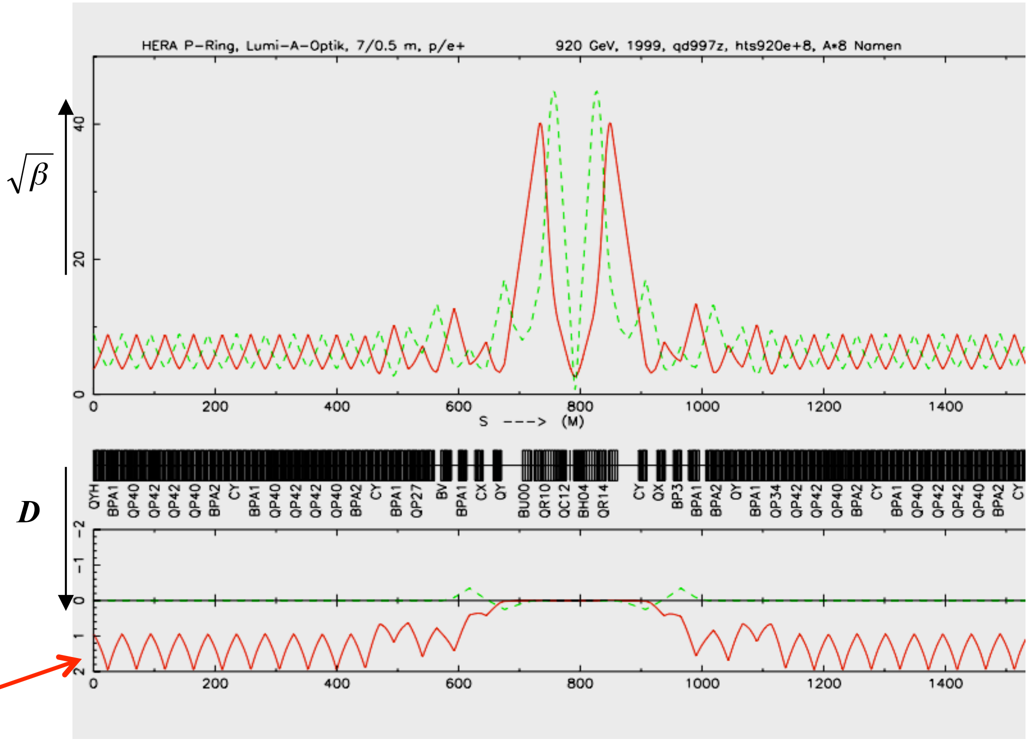
$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}$$

$$C = \cos(\sqrt{|k|}s) \qquad S = \frac{1}{\sqrt{|k|}} \sin(\sqrt{|k|}s)$$

$$C' = \frac{dC}{ds} \qquad S' = \frac{dS}{ds}$$

or expressed as 3x3 matrix

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0$$



Example

$$x_\beta = 1 \dots 2\,mm$$

$$D(s) \approx 1 \dots 2\,m$$

$$\Delta p/p \approx 1 \cdot 10^{-3}$$

Amplitude of Orbit oscillation

contribution due to Dispersion ≈ beam size

→ Dispersion must vanish at the collision point

!

Calculate D, D´: ... takes a couple of sunny Sunday evenings !

$$D(s) = S(s) \int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s})\, d\tilde{s} - C(s) \int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s})\, d\tilde{s}$$

Example: Drift

$$M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

$$D(s) = S(s)\underbrace{\int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s})\,d\tilde{s}}_{=0} - C(s)\underbrace{\int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s})\,d\tilde{s}}_{=0}$$

Example: Dipole

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}s \\ -\sqrt{|K|}\sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0$$

$$\boldsymbol{K} = \frac{1}{\rho^2} - \cancel{k}$$

$$\boldsymbol{s} = \boldsymbol{l_B}$$

$$M_{Dipole} = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho} \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} \end{pmatrix} \quad \rightarrow$$

$$D(s) = \rho\cdot(1-\cos\frac{l}{\rho})$$

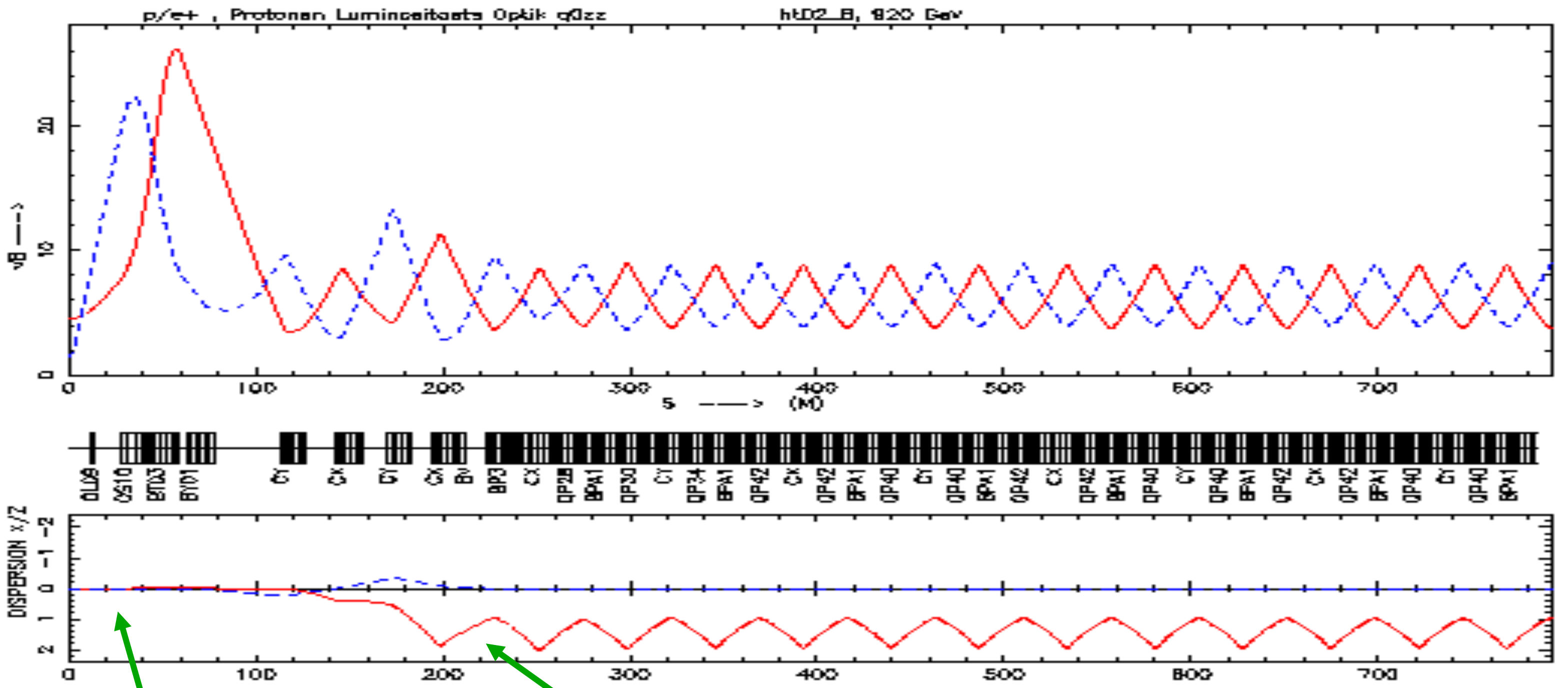
$$D'(s) = \sin\frac{l}{\rho}$$

Example: Dispersion, calculated by an optics code for a real machine

$$x_D = D(s)\,\frac{\Delta p}{p}$$

* D(s) is created by the dipole magnets

… and afterwards focused by the quadrupole fields



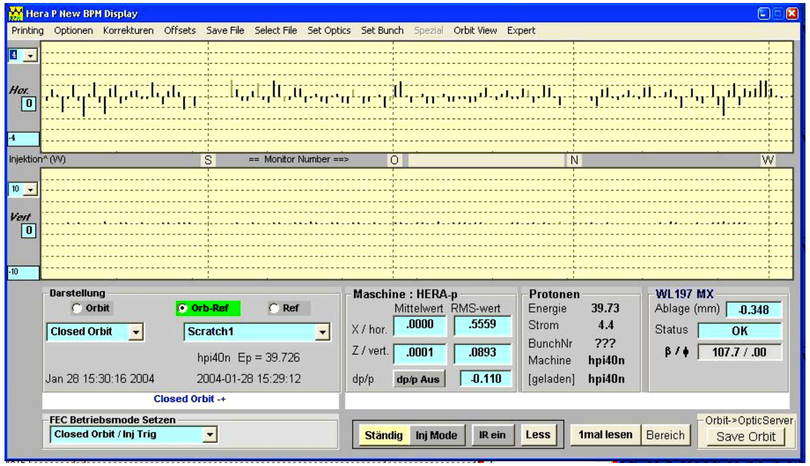
→ s

Mini Beta Section,
→ no dipoles !!!

D(s) ≈ 1 … 2 m

Dispersion is visible

HERA Standard Orbit



Darstellung: Closed Orbit, Scratch1

hpi40n Ep = 39.726

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Closed Orbit -+

Maschine : HERA-p

	Mittelwert	RMS-wert
X / hor.	.0000	.5559
Z / vert.	.0001	.0893
dp/p	dp/p Aus	-0.110

Protonen
Energie 39.73
Strom 4.4
BunchNr ???
Machine hpi40n
[geladen] hpi40n

WL197 MX
Ablage (mm) -0.348
Status OK
β / ϕ 107.7 / .00

FEC Betriebsmode Setzen: Closed Orbit / Inj Trig

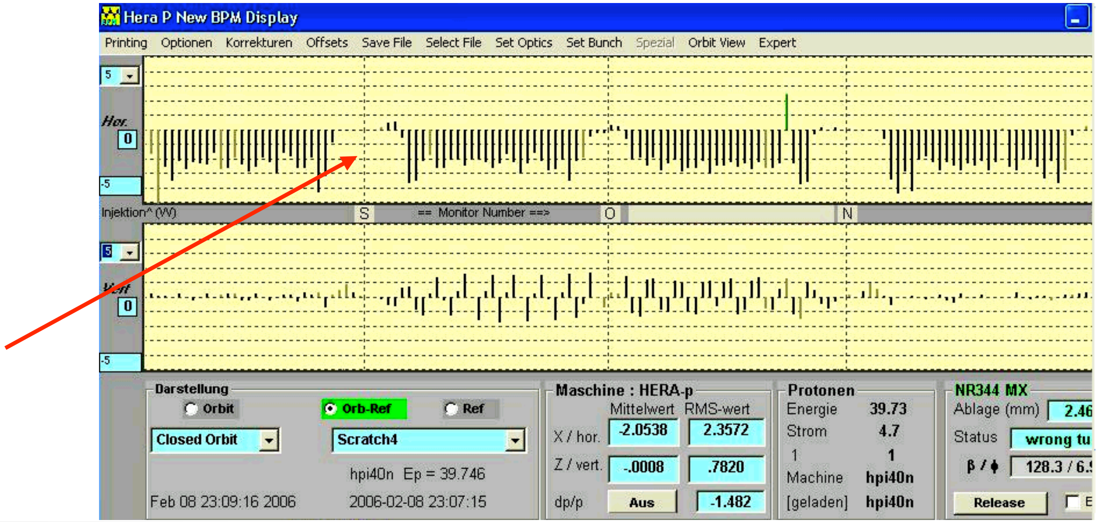
Orbit->OpticServer

HERA Dispersion Orbit

dedicated energy change of the stored beam

→ closed orbit is moved to a dispersions trajectory

$$x_D = D(s) * \frac{\Delta p}{p}$$



Darstellung: Closed Orbit, Scratch4

hpi40n Ep = 39.746

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Maschine : HERA-p

	Mittelwert	RMS-wert
X / hor.	-2.0538	2.3572
Z / vert.	-.0008	.7820
dp/p	Aus	-1.482

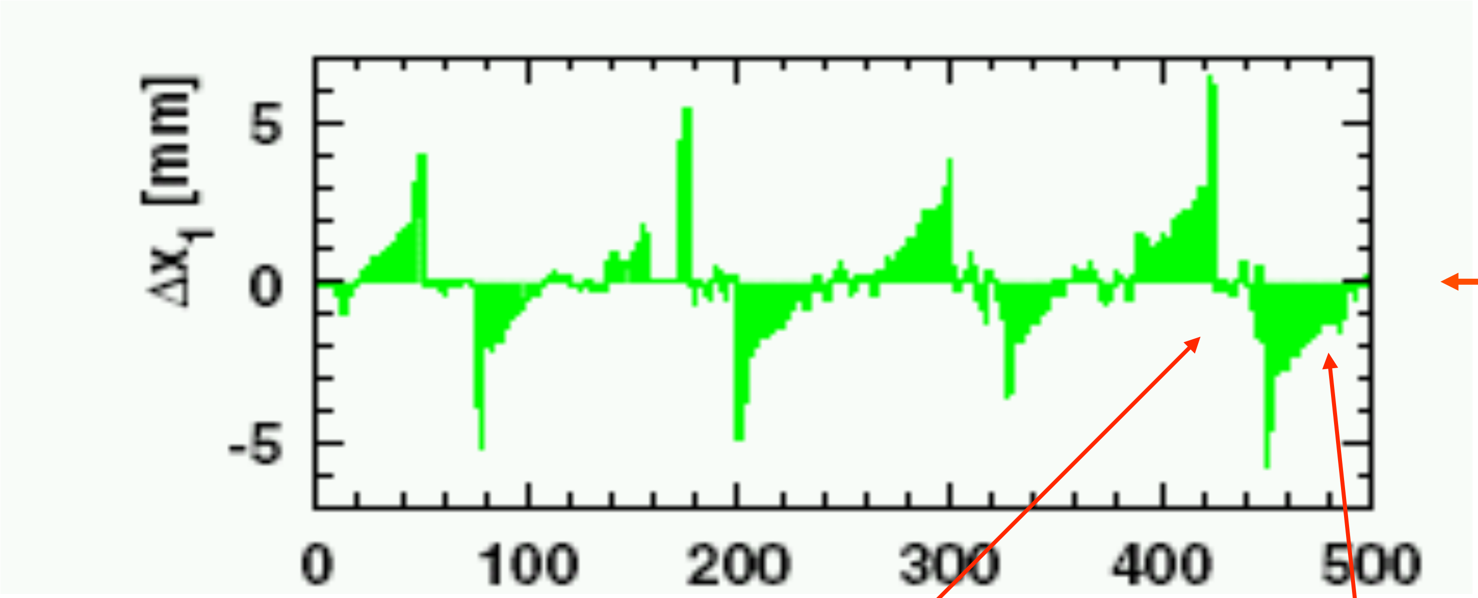
Protonen
Energie 39.73
Strom 4.7
1 1
Machine hpi40n
[geladen] hpi40n

NR344 MX
Ablage (mm) 2.46
Status wrong tu
β / ϕ 128.3 / 6.

Attention: at the Interaction Points
we require D=D´= 0

Periodic Dispersion:

„Sawtooth Effect“ at LEP (CERN)



Electron course

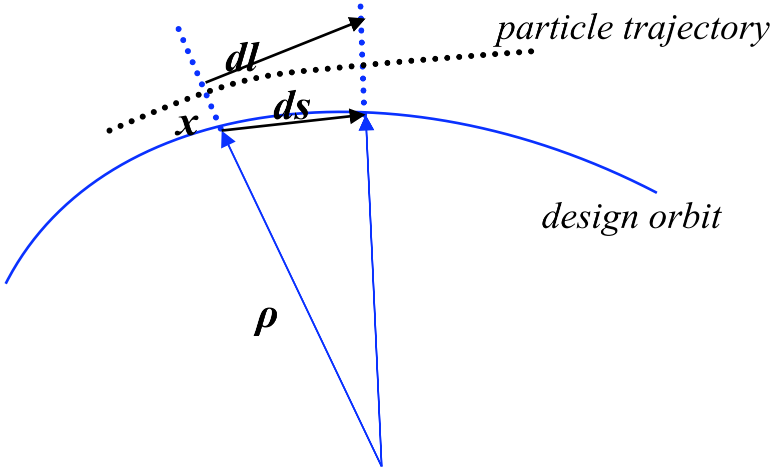
BPM Number

In the straight sections they are accelerated by the rf cavities so much that they „overshoot“ and reach nearly the outer side of the vacuum chamber.

In the arc the electron beam loses so much energy in each octant that the particle are running more and more on a dispersion trajectory.

16.) Momentum Compaction Factor: α_p

particle with a displacement x to the design orbit
→ path length dl ...



$$\frac{dl}{ds} = \frac{\rho + x}{\rho}$$

$$\rightarrow \quad dl = \left(1 + \frac{x}{\rho(s)}\right) ds$$

circumference of an off-energy closed orbit

$$l_{\Delta E} = \oint dl = \oint \left(1 + \frac{x_{\Delta E}}{\rho(s)}\right) ds$$

remember:

$$x_{\Delta E}(s) = D(s)\frac{\Delta p}{p}$$

$$\delta l_{\Delta E} = \frac{\Delta p}{p}\oint\left(\frac{D(s)}{\rho(s)}\right) ds$$

**** The lengthening of the orbit for off-momentum particles is given by the dispersion function and the bending radius.***

Definition:

$$\frac{\delta l_\varepsilon}{L} = \alpha_p \frac{\Delta p}{p}$$

$$\rightarrow \alpha_p = \frac{1}{L}\oint\left(\frac{D(s)}{\rho(s)}\right)ds$$

For first estimates assume:

$$\frac{1}{\rho} = const.$$

$$\int_{dipoles} D(s)\,ds \approx l_{\Sigma(dipoles)} \cdot \langle D \rangle_{dipole}$$

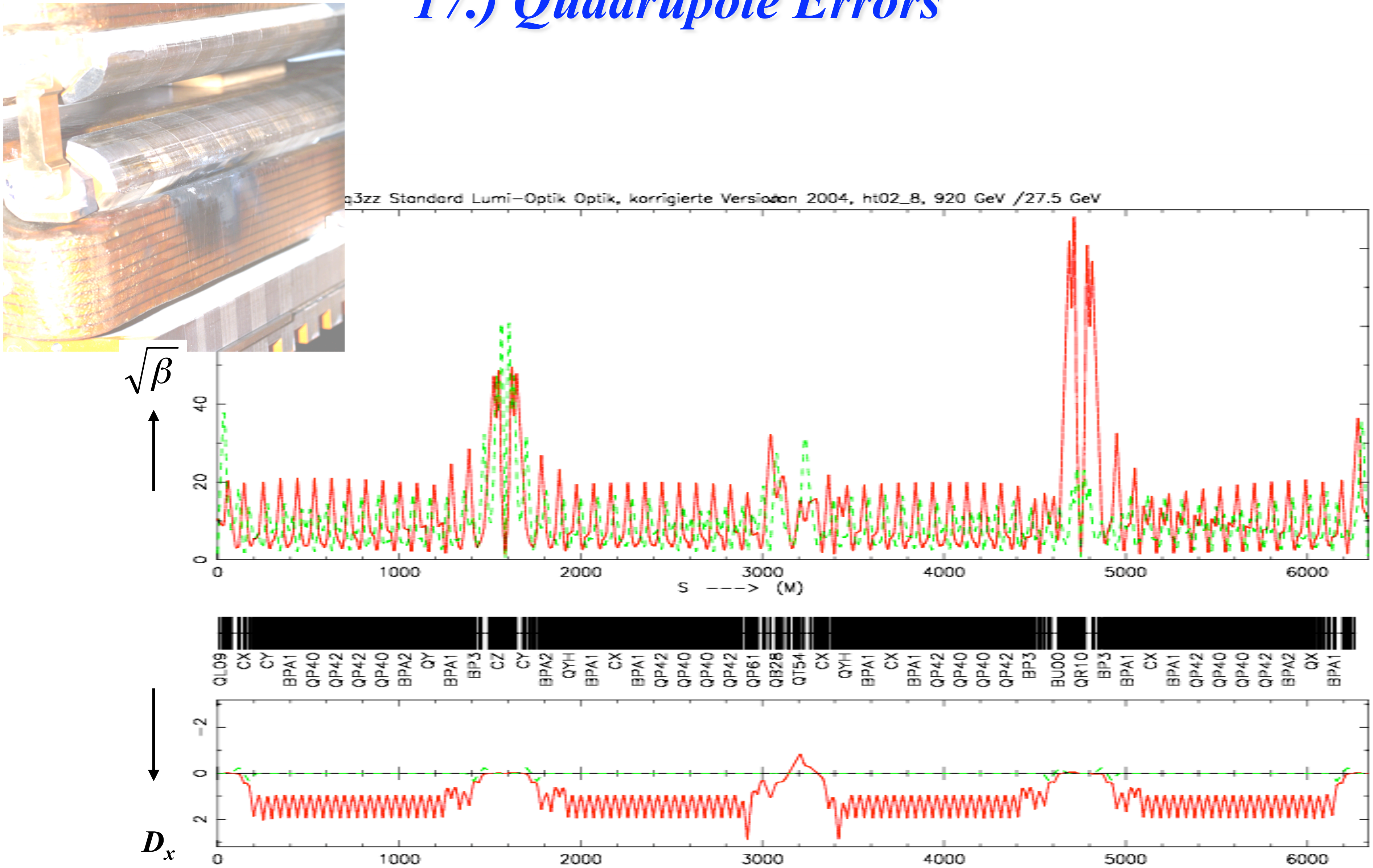
$$\alpha_p = \frac{1}{L}\, l_{\Sigma(dipoles)} \cdot \langle D \rangle \frac{1}{\rho} = \frac{1}{L}\, 2\pi\rho \cdot \langle D \rangle \frac{1}{\rho} \quad \rightarrow \quad \alpha_p \approx \frac{2\pi}{L}\langle D \rangle \approx \frac{\langle D \rangle}{R}$$

Assume: $v \approx c$

$$\rightarrow \quad \frac{\delta T}{T} = \frac{\delta l_\varepsilon}{L} = \alpha_p \frac{\Delta p}{p}$$

α_p combines via the dispersion function the momentum spread with the longitudinal motion of the particle.

17.) Quadrupole Errors



Quadrupole Errors

go back to Lecture I, page 1

single particle trajectory

$$\begin{pmatrix} x \\ x' \end{pmatrix}_2 = M_{QF} * \begin{pmatrix} x \\ x' \end{pmatrix}_1$$

Solution of equation of motion

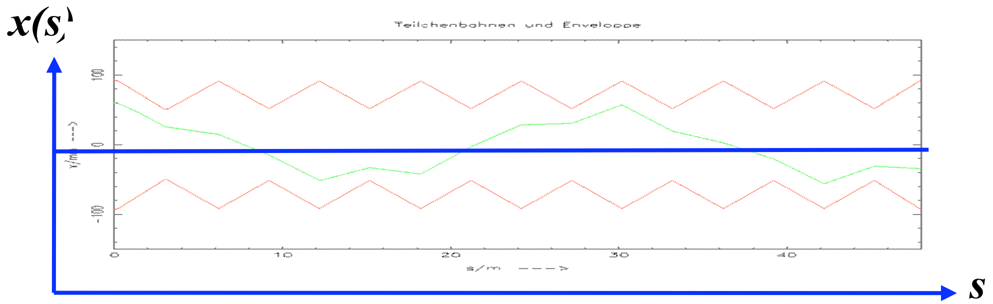
$$x = x_0 \cos(\sqrt{k}\, l_q) + x'_0 \frac{1}{\sqrt{k}} \sin(\sqrt{k}\, l_q)$$

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{k}\, l_q) & \frac{1}{\sqrt{k}} \sin(\sqrt{k}\, l_q) \\ -\sqrt{k} \sin(\sqrt{k}\, l_q) & \cos(\sqrt{k}\, l_q) \end{pmatrix} \quad , \quad M_{thinlens} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

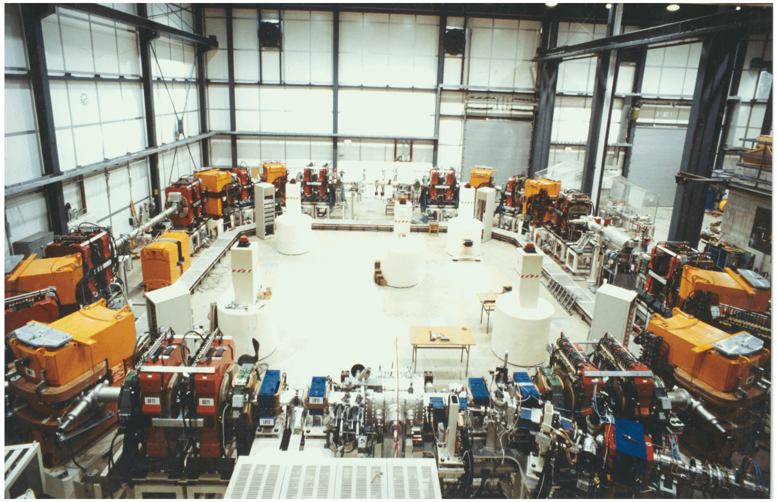
$$M_{turn} = M_{QF} * M_{D1} * M_{QD} * M_{D2} * M_{QF} \ \dots$$

Definition:** phase advance of the particle oscillation per revolution in units of 2π is called **tune

$$Q = \frac{\psi_{turn}}{2\pi}$$



Matrix in Twiss Form



Transfer Matrix from point „0“ in the lattice to point „s“:

$$\boldsymbol{M(s)} = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}(\cos\psi_s + \alpha_0 \sin\psi_s) & \sqrt{\beta_s \beta_0}\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos(\psi_s - (1 + \alpha_0 \alpha_s)\sin\psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}(\cos(\psi_s - \alpha_0 \sin\psi_s) \end{pmatrix}$$

For one complete turn the Twiss parameters have to obey periodic bundary conditions:

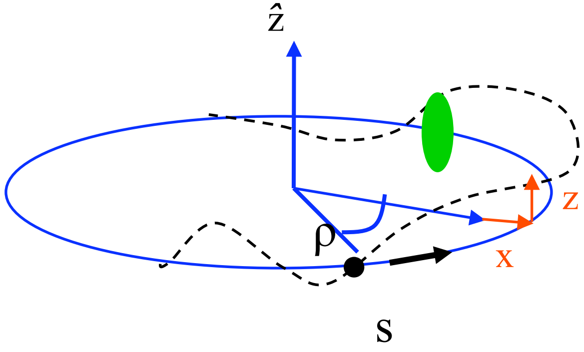
$$\beta(\boldsymbol{s}+\boldsymbol{L}) = \beta(\boldsymbol{s})$$
$$\alpha(\boldsymbol{s}+\boldsymbol{L}) = \alpha(\boldsymbol{s})$$
$$\gamma(\boldsymbol{s}+\boldsymbol{L}) = \gamma(\boldsymbol{s})$$

$$\boldsymbol{M(s)} = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_s & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$

Quadrupole Error in the Lattice

optic perturbation described by thin lens quadrupole

$$M_{dist} = M_{\Delta k} \cdot M_0 = \underbrace{\begin{pmatrix} 1 & 0 \\ \Delta k ds & 1 \end{pmatrix}}_{quad\ error} \cdot \underbrace{\begin{pmatrix} \cos\psi_{turn} + \alpha \sin\psi_{turn} & \beta \sin\psi_{turn} \\ -\gamma \sin\psi_{turn} & \cos\psi_{turn} - \alpha \sin\psi_{turn} \end{pmatrix}}_{ideal\ storage\ ring}$$



$$M_{dist} = \begin{pmatrix} \cos\psi_0 + \alpha \sin\psi_0 & \beta \sin\psi_0 \\ \Delta k ds (\cos\psi_0 + \alpha \sin\psi_0) - \gamma \sin\psi_0 & \Delta k ds \, \beta \sin\psi_0 + \cos\psi_0 - \alpha \sin\psi_0 \end{pmatrix}$$

rule for getting the tune

$$Trace(M) = 2\cos\psi = 2\cos\psi_0 + \Delta k ds \beta \sin\psi_0$$

Quadrupole error → Tune Shift

$$\boxed{\psi = \psi_0 + \Delta\psi} \longrightarrow \cos(\psi_0 + \Delta\psi) = \cos\psi_0 + \frac{\Delta \boldsymbol{kds}\,\beta \sin\psi_0}{2}$$

remember the old fashioned trigonometric stuff and assume that the error is small !!!

$$\cos\psi_0 \underbrace{\cos\Delta\psi}_{\approx 1} - \sin\psi_0 \underbrace{\sin\Delta\psi}_{\approx \Delta\psi} = \cos\psi_0 + \frac{\boldsymbol{kds}\,\beta \sin\psi_0}{2}$$

$$\Delta\psi = \frac{\boldsymbol{kds}\,\beta}{2}$$

and referring to Q instead of ψ:

$$\psi = 2\pi \boldsymbol{Q}$$

$$\boxed{\Delta \boldsymbol{Q} = \int_{s0}^{s0+l} \frac{\Delta \boldsymbol{k}(\boldsymbol{s})\beta(\boldsymbol{s})\boldsymbol{ds}}{4\pi}}$$

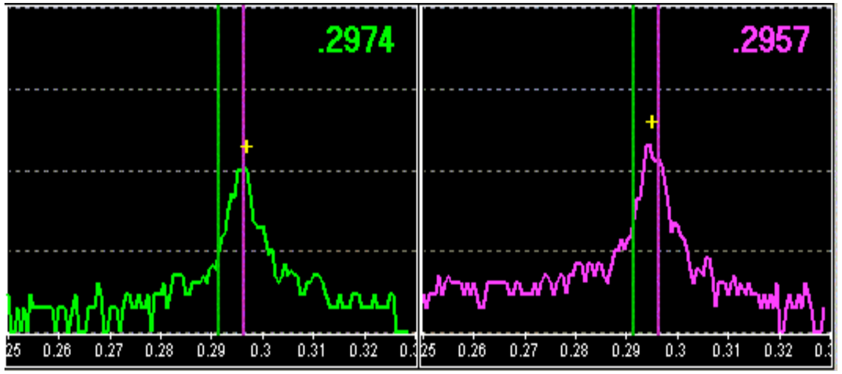
! the tune shift is proportional to the β-function at the quadrupole

!! field quality, power supply tolerances etc are much tighter at places where β is large

!!! mini beta quads: β ≈ 1900 m
arc quads: β ≈ 80 m

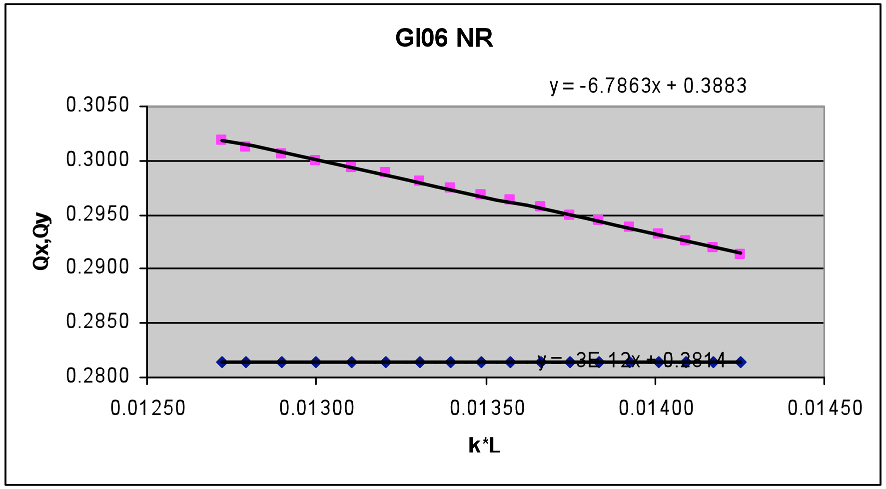
!!!! β is a measure for the sensitivity of the beam

a quadrupol error leads to a shift of the tune:



$$\Delta Q = \int_{s0}^{s0+l} \frac{\Delta k\,\beta(s)}{4\pi}\,ds \approx \frac{\Delta k l_{quad}\,\overline{\beta}}{4\pi}$$

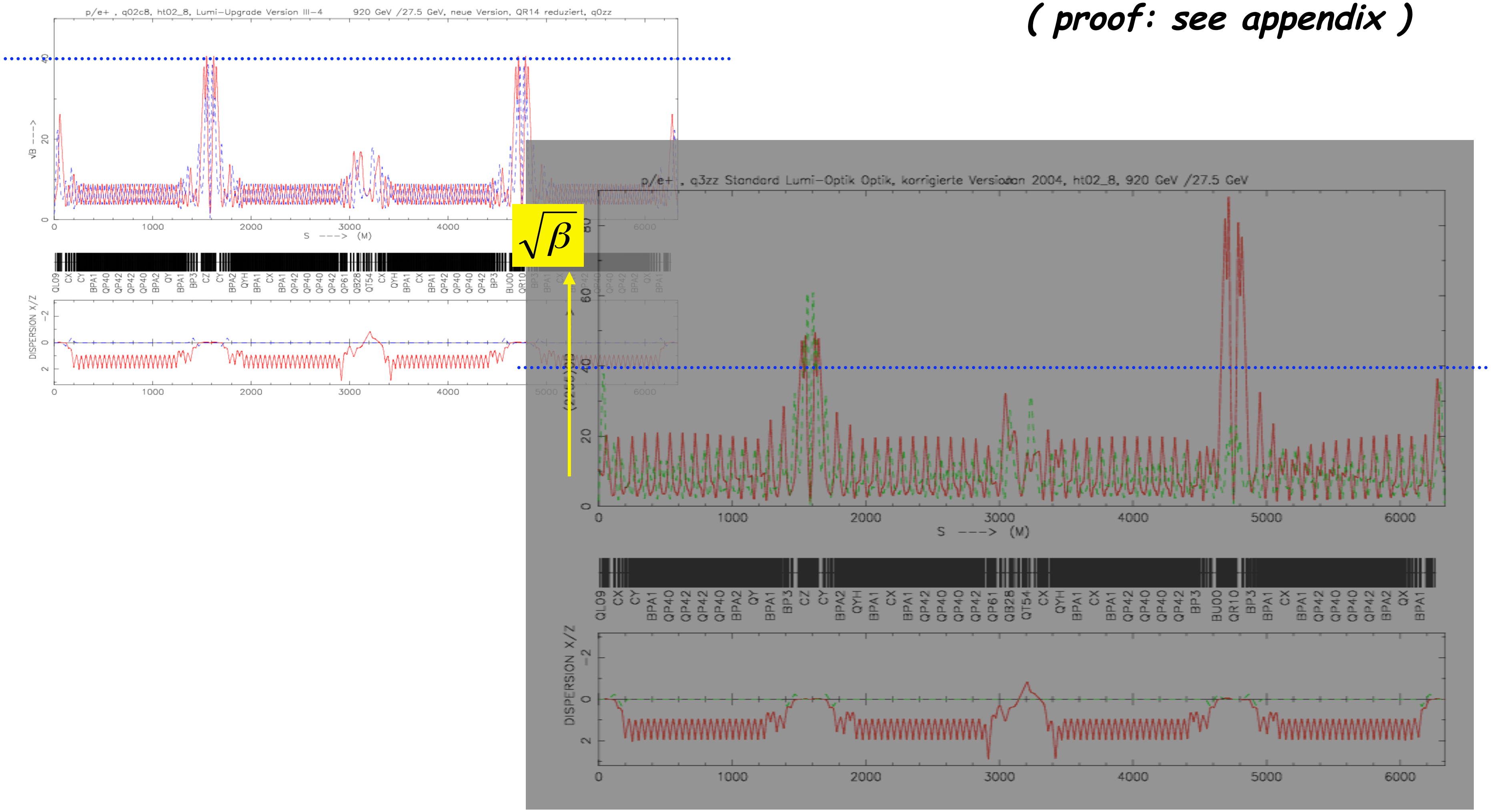
Example: measurement of β in a storage ring:
tune spectrum



Quadrupole error: Beta Beat

$$\Delta\beta(s_0) = \frac{\beta_0}{2\sin 2\pi Q}\int_{s1}^{s1+l}\beta(s_1)\Delta K\cos\left(2|\psi_{s1}-\psi_{s0}|-2\pi Q\right)ds$$

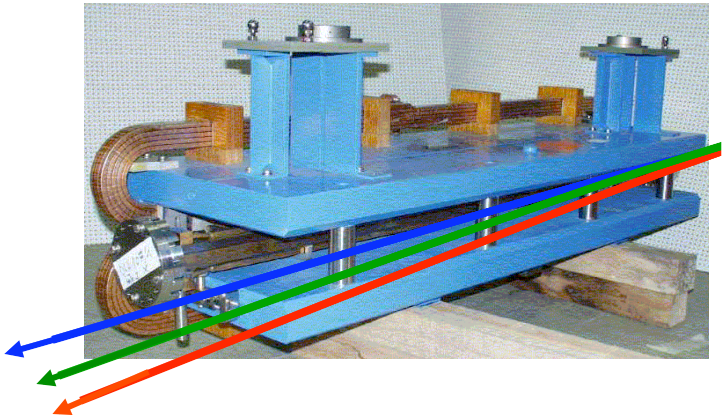
(proof: see appendix)



18.) Chromaticity:
A Quadrupole Error for $\Delta p/p \neq 0$

Influence of external fields on the beam: *prop. to magn. field & prop. zu 1/p*

dipole magnet $$\alpha = \frac{\int B\,dl}{p/e}$$



$$x_D(s) = D(s)\,\frac{\Delta p}{p}$$

focusing lens $$k = \frac{g}{p/e}$$

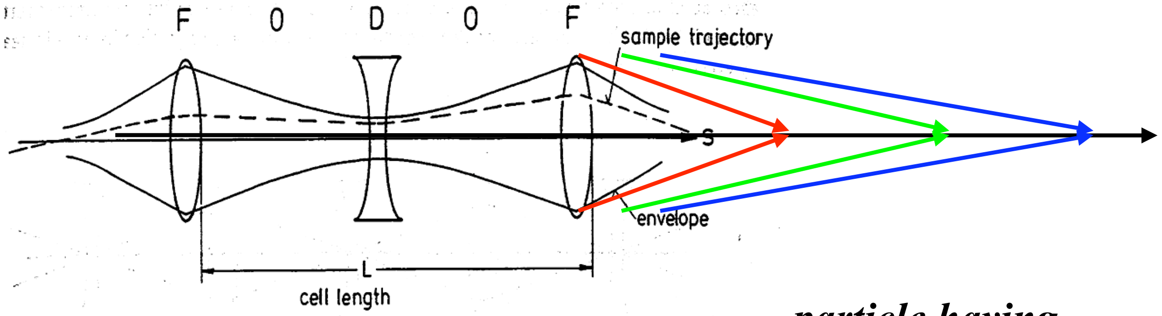


Figure 29: FODO cell

particle having ...

to high energy

to low energy

ideal energy

Chromaticity: Q'

$$k = \frac{g}{p/e} \qquad\qquad p = p_0 + \Delta p$$

in case of a momentum spread:

$$\boldsymbol{k} = \frac{eg}{p_0 + \Delta p} \approx \frac{e}{p_0}\left(1 - \frac{\Delta p}{p_0}\right) g = k_0 + \Delta k$$

$$\Delta k = -\frac{\Delta p}{p_0} k_0$$

… which acts like a quadrupole error in the machine and leads to a tune spread:

$$\Delta Q = -\frac{1}{4\pi} \frac{\Delta p}{p_0} k_0 \beta(s) ds$$

definition of chromaticity:

$$\Delta Q = Q' \frac{\Delta p}{p} \quad ; \qquad Q' = -\frac{1}{4\pi} \oint k(s) \beta(s) ds$$

Resume´:

quadrupole error: *tune shift*

$$\Delta Q \approx \int_{s0}^{s0+l} \frac{\Delta k(s)\,\beta(s)}{4\pi}\,ds \approx \frac{\Delta k(s)\; l_{quad}\;\overline{\beta}}{4\pi}$$

beta beat

$$\Delta\beta(s_0) = \frac{\beta_0}{2\sin 2\pi Q}\int_{s1}^{s1+l}\beta(s_1)\Delta k\cos\bigl(2(\psi_{s1}-\psi_{s0})-2\pi Q\bigr)ds$$

chromaticity

$$\Delta Q = Q'\,\frac{\Delta p}{p}$$

$$Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)ds$$

momentum compaction

$$\frac{\delta\, l_{\varepsilon}}{L} = \alpha_p\,\frac{\Delta p}{p}$$

$$\alpha_p \approx \frac{2\pi}{L}\,\langle D\rangle \approx \frac{\langle D\rangle}{R}$$

beta function in a symmateric drift

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$