# Space Charge

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This lecture abundantly uses previous material available in former CERN Accelerator Schools. In particular from A. Hofmann, M. Ferrario, G. Rumolo, K. Schindl.

What is the difference ?

A personal view and understanding of the subjects

# The dynamics of particles follow the Lorenz law

$$\frac{d\vec{p}}{dt} = e\vec{E} + e\vec{v} \times \vec{B}$$

$$\vec{p} = m\gamma\vec{v}$$

E,B can be external field. From magnets and RF systems

But E,B can be field also generated by the beam itself

# The beam generate the fields B, E through Maxwell laws

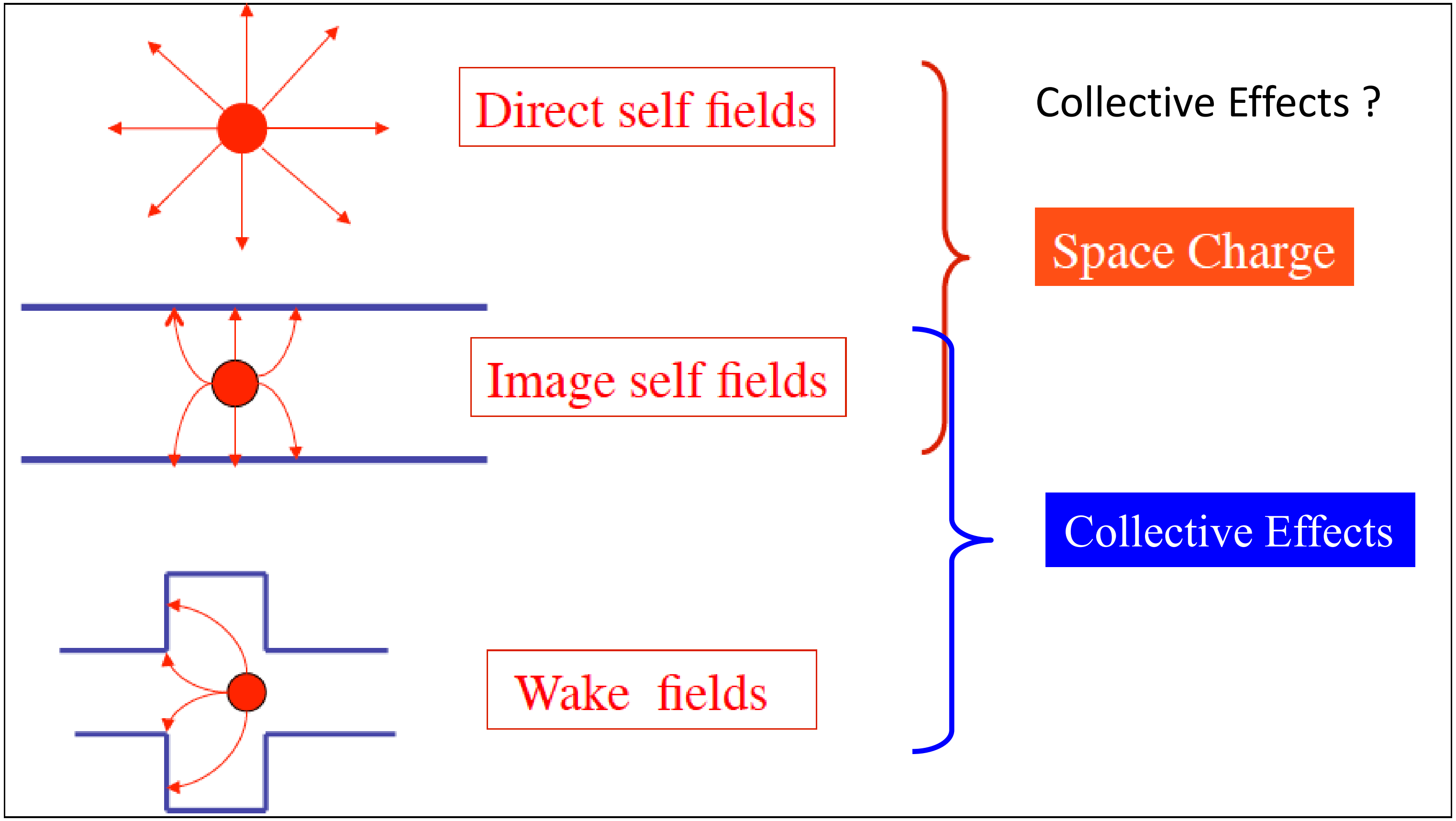
$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

# Type of fields



# How does it looks?

The dynamics of each particle follows the equation

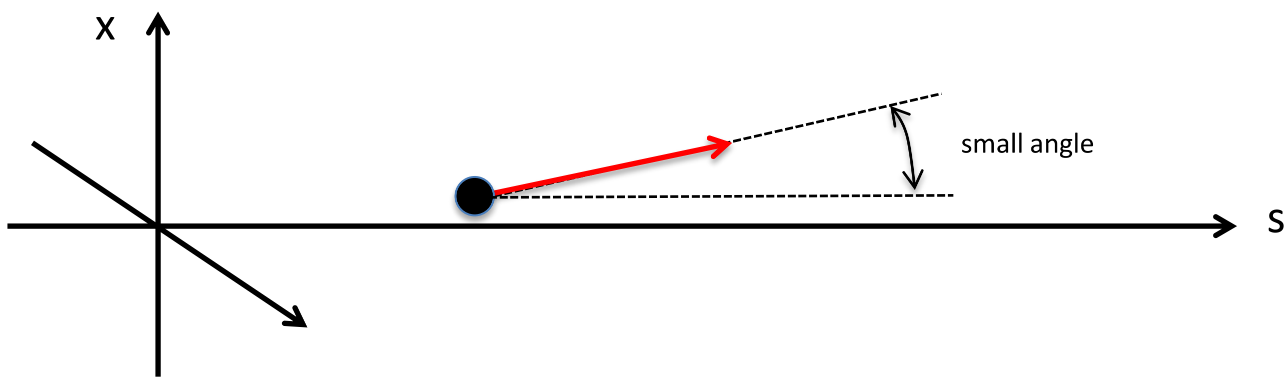
$$\frac{d\vec{p}}{dt} = \underbrace{e\vec{E}_{RF} + e\vec{v} \times \vec{B}_M}_{\text{the origin of the fields is independent on the beam. External fields}} + \underbrace{e\vec{E}_b + e\vec{v} \times \vec{B}_b}_{\text{The origin of the fields is dependent on the beam itself}}$$

the origin of the fields is
independent on the beam.
External fields

The origin of the fields is
dependent on the beam
itself

# Parallaxial approximation

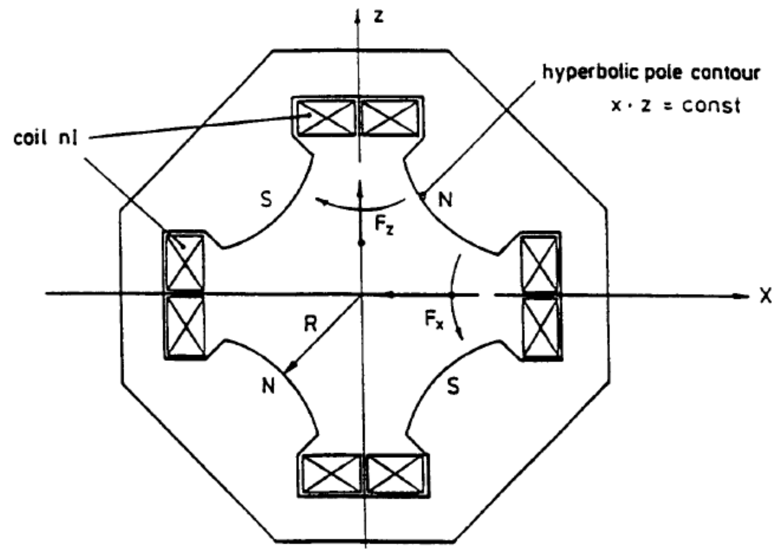
in parallaxial approximation $\quad |v_z| \simeq v_0 = |\vec{v}|$




$$\frac{d^2\vec{r}}{ds^2} = \frac{1}{m\gamma v_0^2} e\vec{E}_{RF} + \frac{1}{m\gamma v_0^2} e\vec{v} \times \vec{B}_M + \frac{1}{m\gamma v_0^2} e\vec{E}_b + \frac{1}{m\gamma v_0^2} e\vec{v} \times \vec{B}_b$$

# Transverse equations of motion

Focusing




$$B_y = \alpha x \qquad B_x = \alpha y \qquad B_z = 0$$

$$(\vec{v} \times \vec{B}_M)_x = v_y B_z - v_z B_y = -v_z \alpha x$$
$$(\vec{v} \times \vec{B}_M)_y = v_z B_x - v_x B_z = v_z \alpha y$$

# Final form of the transverse equation of motion with space charge

$$\frac{d^2x}{ds^2} + \boxed{k_x x} = \left( \frac{e}{m\gamma v_0^2}\vec{E}_b + \frac{e}{m\gamma v_0^2}\vec{v} \times \vec{B}_b \right)_x$$

$$\frac{d^2y}{ds^2} + \boxed{k_y y} = \left( \frac{e}{m\gamma v_0^2}\vec{E}_b + \frac{e}{m\gamma v_0^2}\vec{v} \times \vec{B}_b \right)_y$$

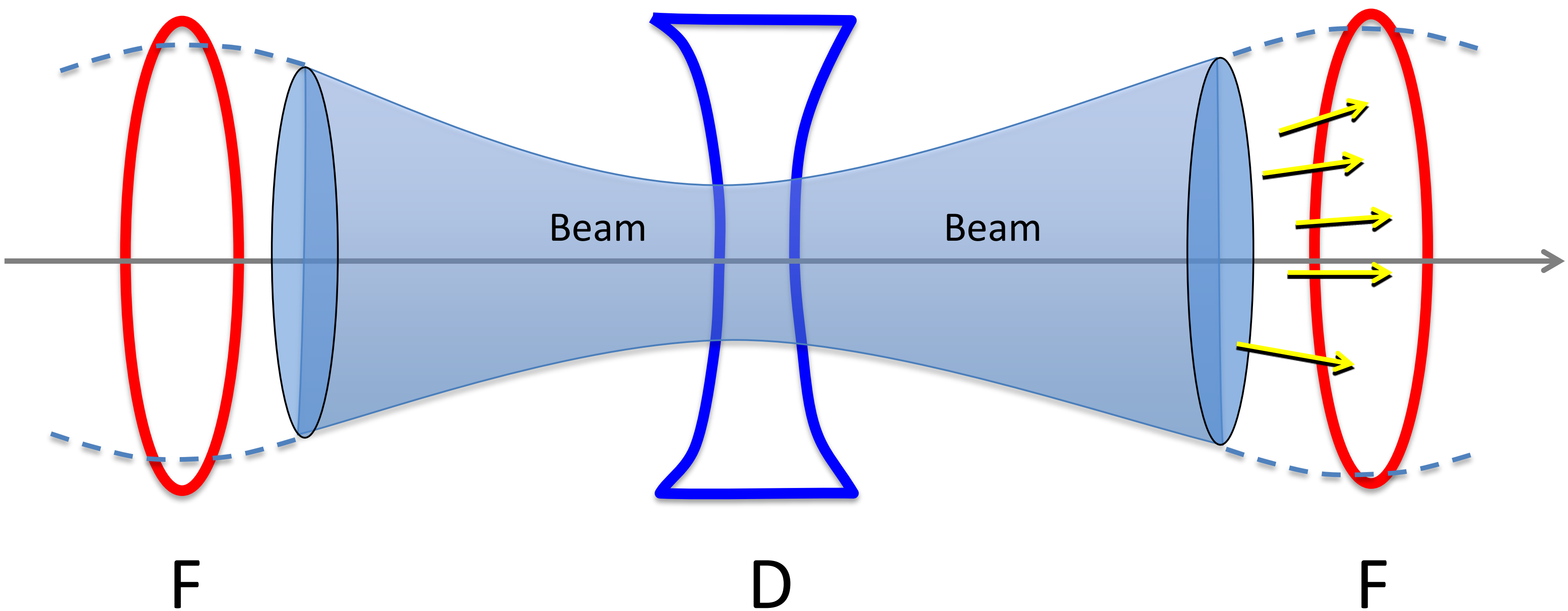
$K_x$, $K_y$ govern the linear optics

$K_x$ = $-\frac{e}{m\gamma v_0^2}\alpha$ $K_x$ = $\frac{e}{m\gamma v_0^2}\alpha$

# Model of beam

We neglect the longitudinal forces.

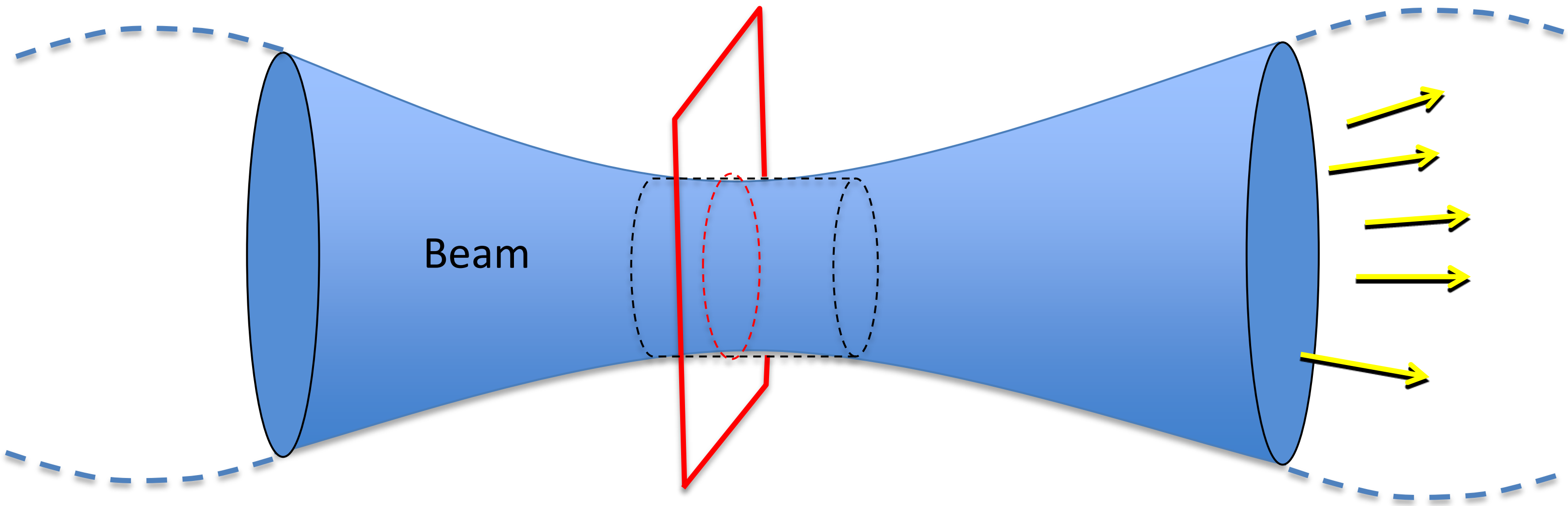
Locally the beam can be seen as a “piece” of a coasting beam



# Model of beam

We neglect the longitudinal forces.

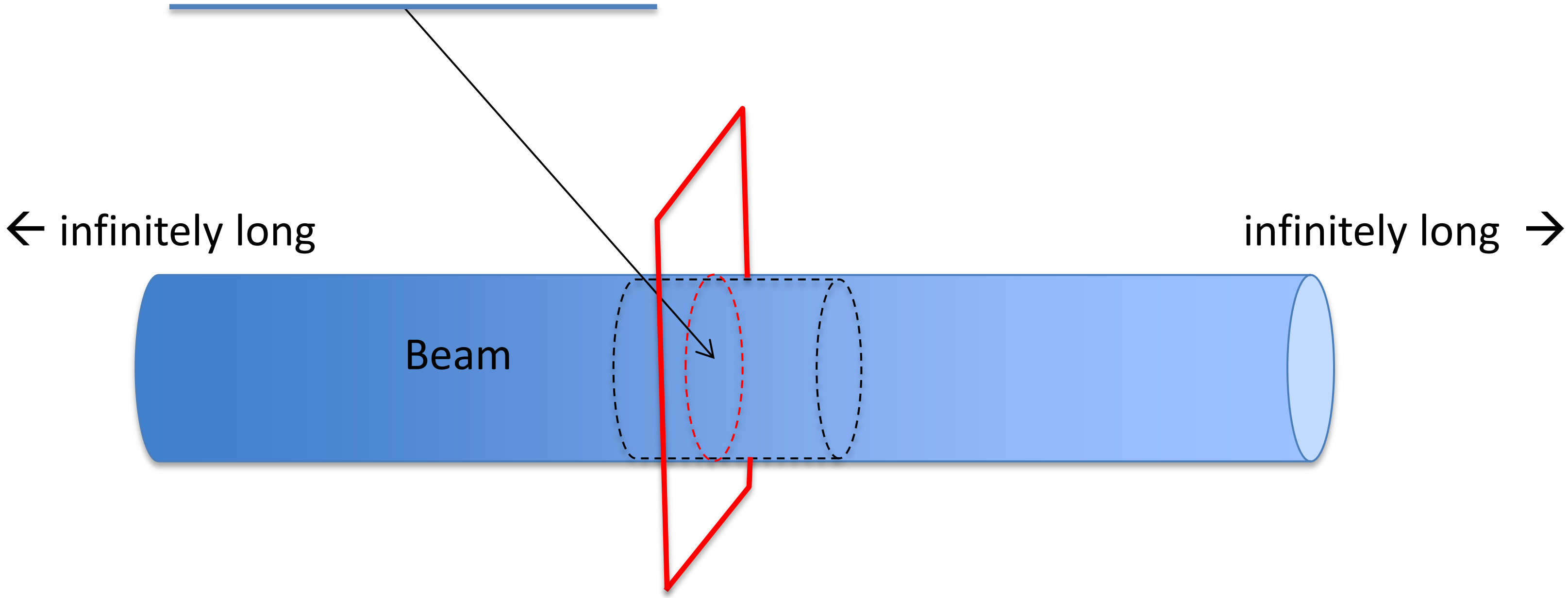
Locally the beam can be seen as a “piece” of a coasting beam




transverse section

# From the point of view of space charge

Space charge forces here are like those created by a coasting beam




transverse section

The lattice strength is adjusted to have the prescribed optics in absence of space charge. That is the functional shape of $k_x(s)$, $k_y(s)$ is independent on the beam energy

However the space charge forces are **not under our control** !

Analysis in the case the beam energy is small

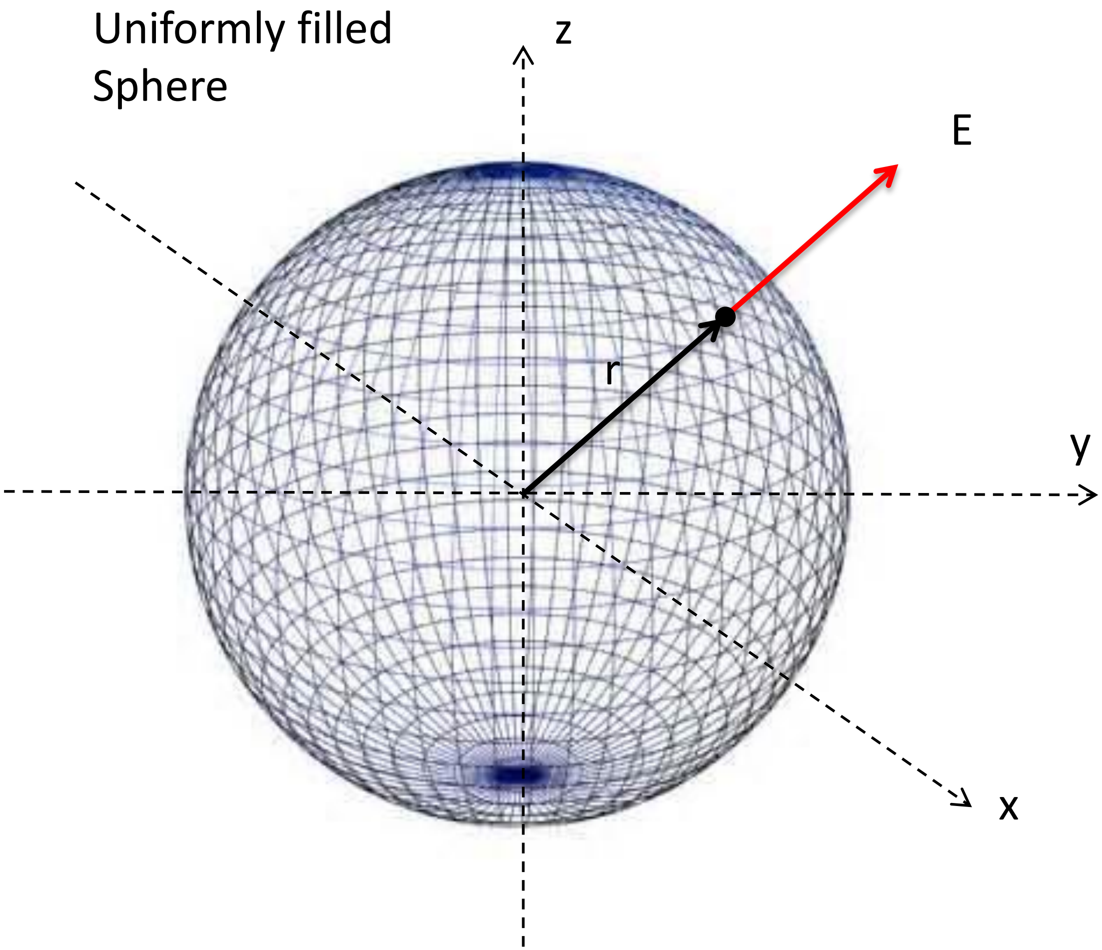
# For non moving particles

Coulomb electric field

$$\vec{E}(r) = \frac{e}{4\pi\epsilon_0}\sum_i \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3}$$

Much easier

# Coulomb Forces




Inside the sphere

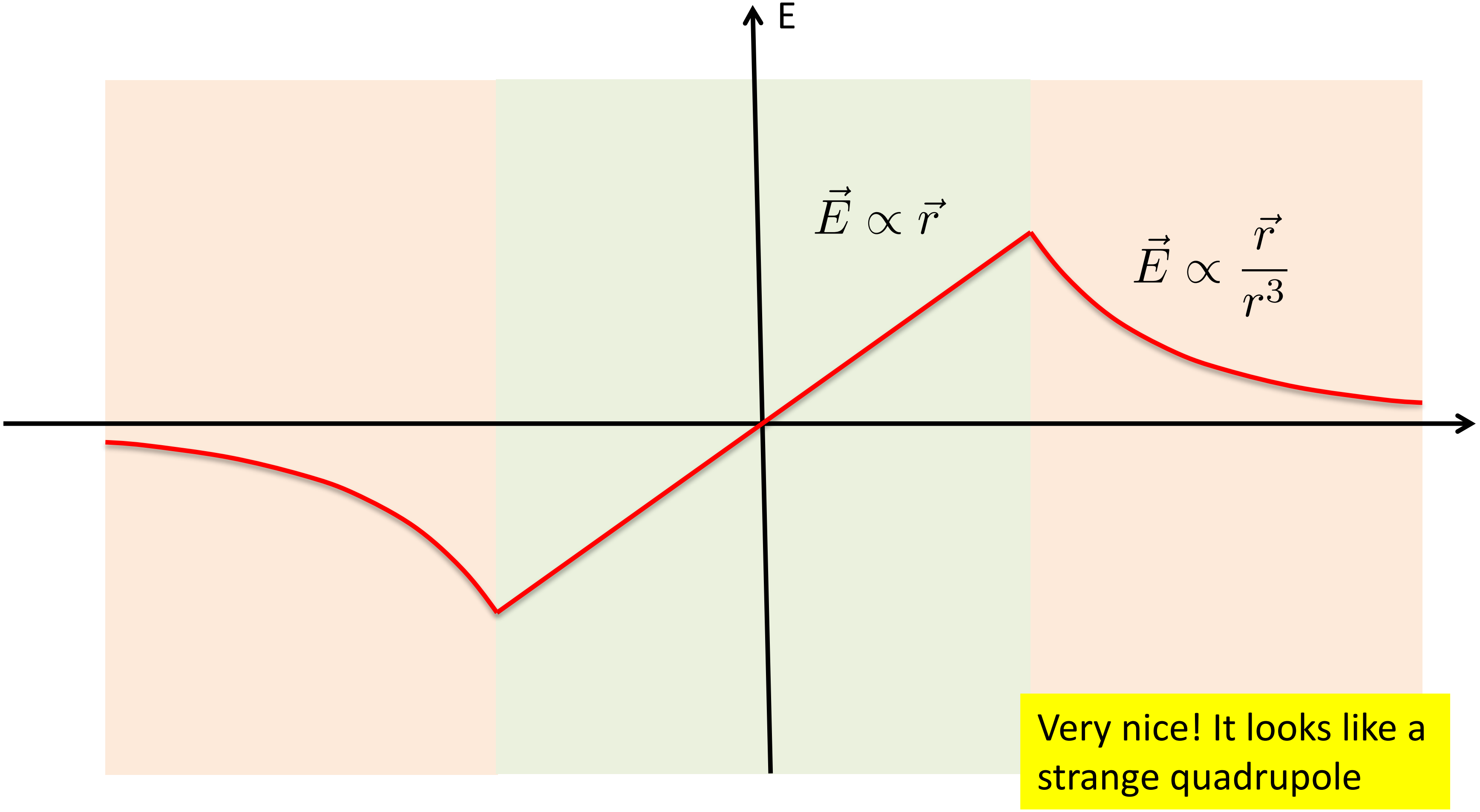
$$E = \frac{\rho}{3\epsilon_0} r$$

$\rho$ = charge density

Outside the sphere

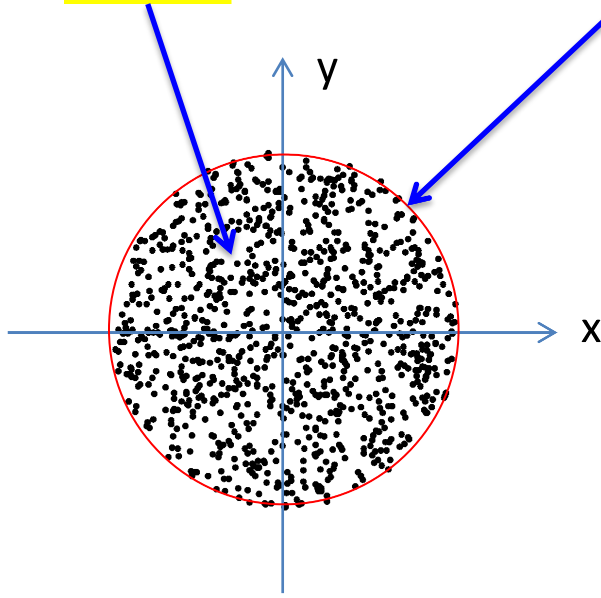
$$E = \frac{R^3}{3\epsilon_0} \frac{\rho}{r^2}$$

# Radial Electric field (along x)

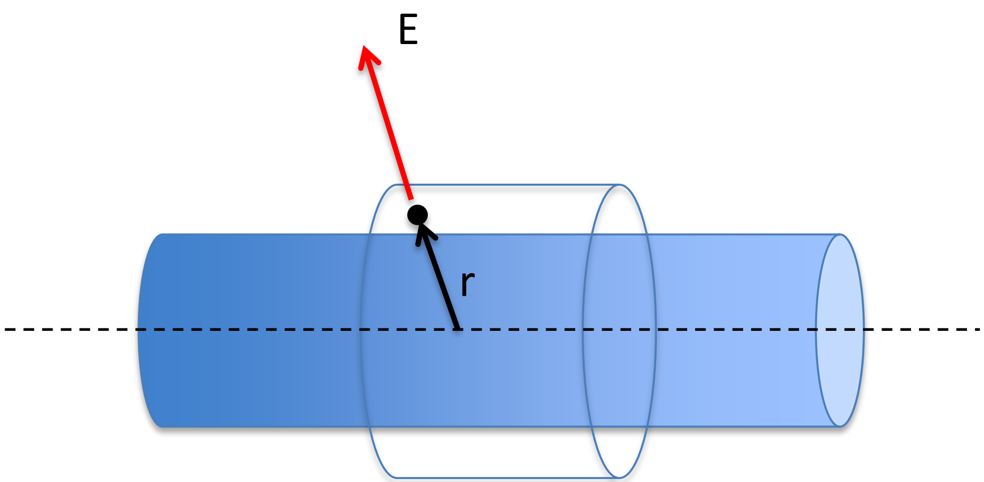


# Beam distribution ansatz

We assume in this first discussion that the beam distribution in (x,y) is always uniform and the beam is round



# Infinitely long uniform axi-symmetric cylinder




Longitudinal electric field is zero

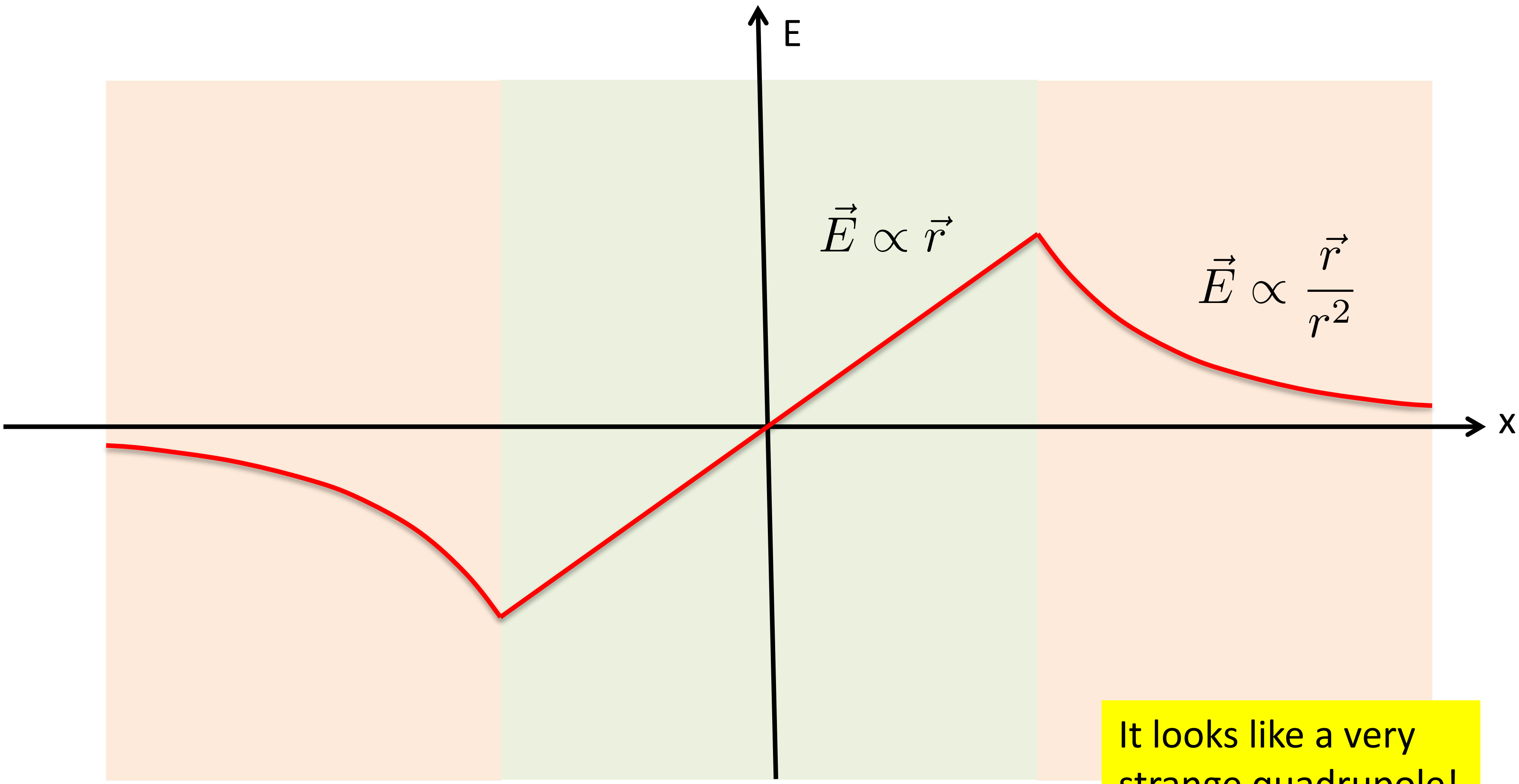
From Gauss law inside

$$E = \frac{\rho}{2\epsilon_0} r$$

Outside the cylinder

$$E = \frac{\rho R^2}{2\epsilon_0} \frac{1}{r}$$

# Transverse Electric field




It looks like a very strange quadrupole!

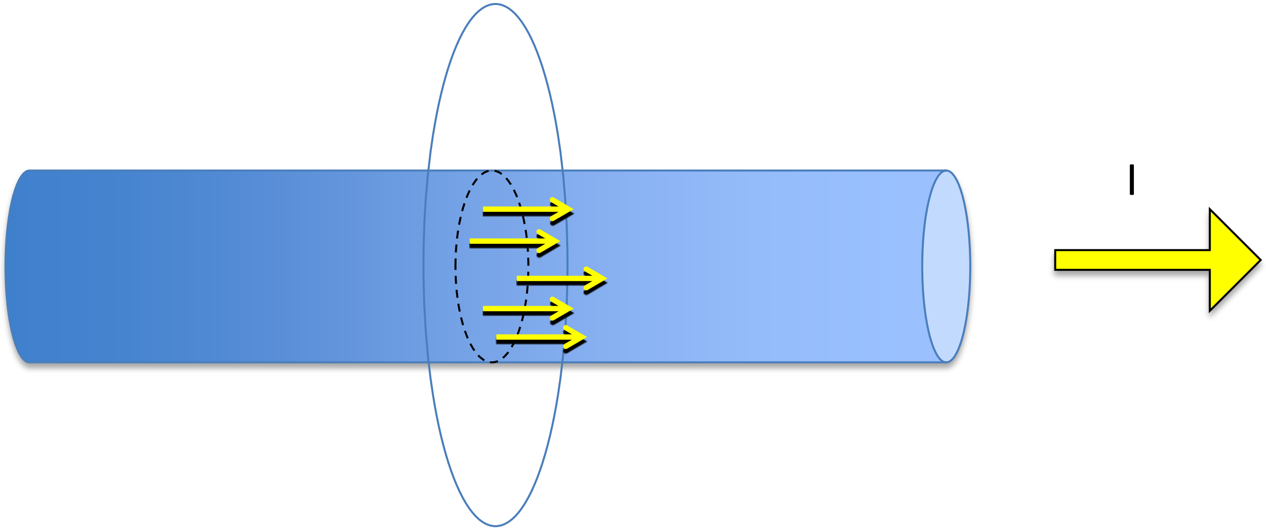
# This is an approximation ... real beam infinitely long does not exists

Such a beam would require infinite energy…
in fact the energy a particle gain is infinite

$$\int_R^\infty E(r)dr = \int_R^\infty \frac{\rho R^2}{2\epsilon_0}\frac{1}{r}dr = \frac{\rho R^2}{2\epsilon_0}[\log(\infty) - \log(R)] \to \infty$$

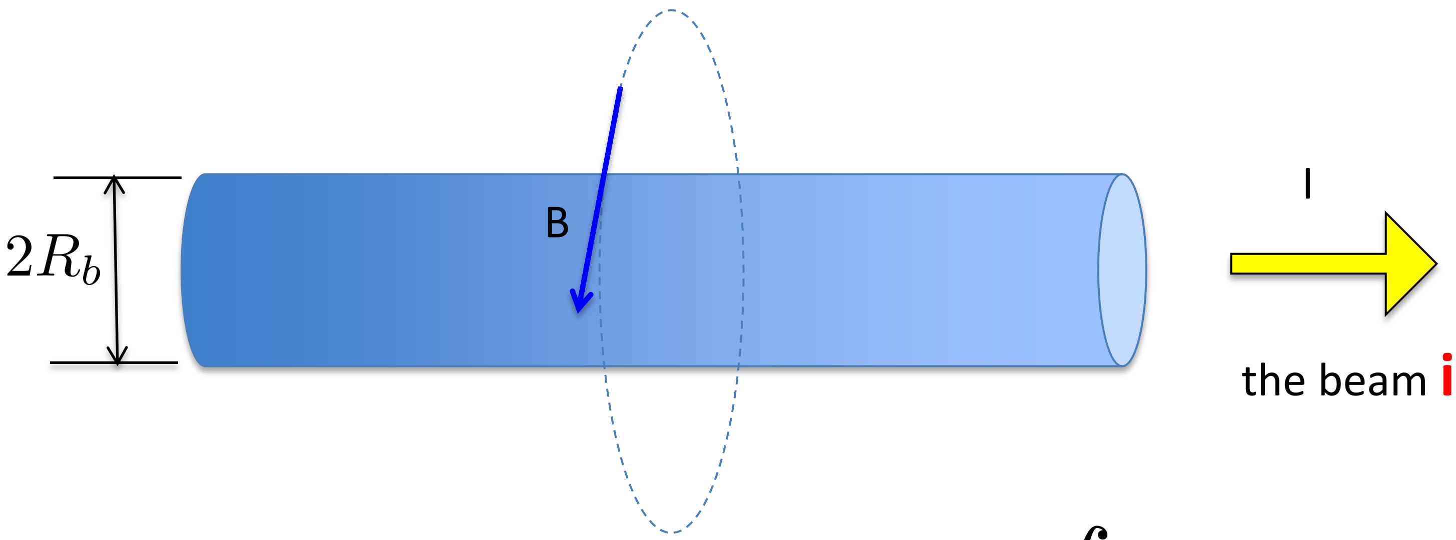
Also $$\int_0^\infty E_r^2(r)dr \to \infty$$ the energy of the beam is infinite !

# Magnetic field generated by an infinitely long beam




We apply BIOT-SAVART law

# Axi-symmetric beam




the beam **is** the current

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I$$

# Example for uniform, round beam

Outside the beam

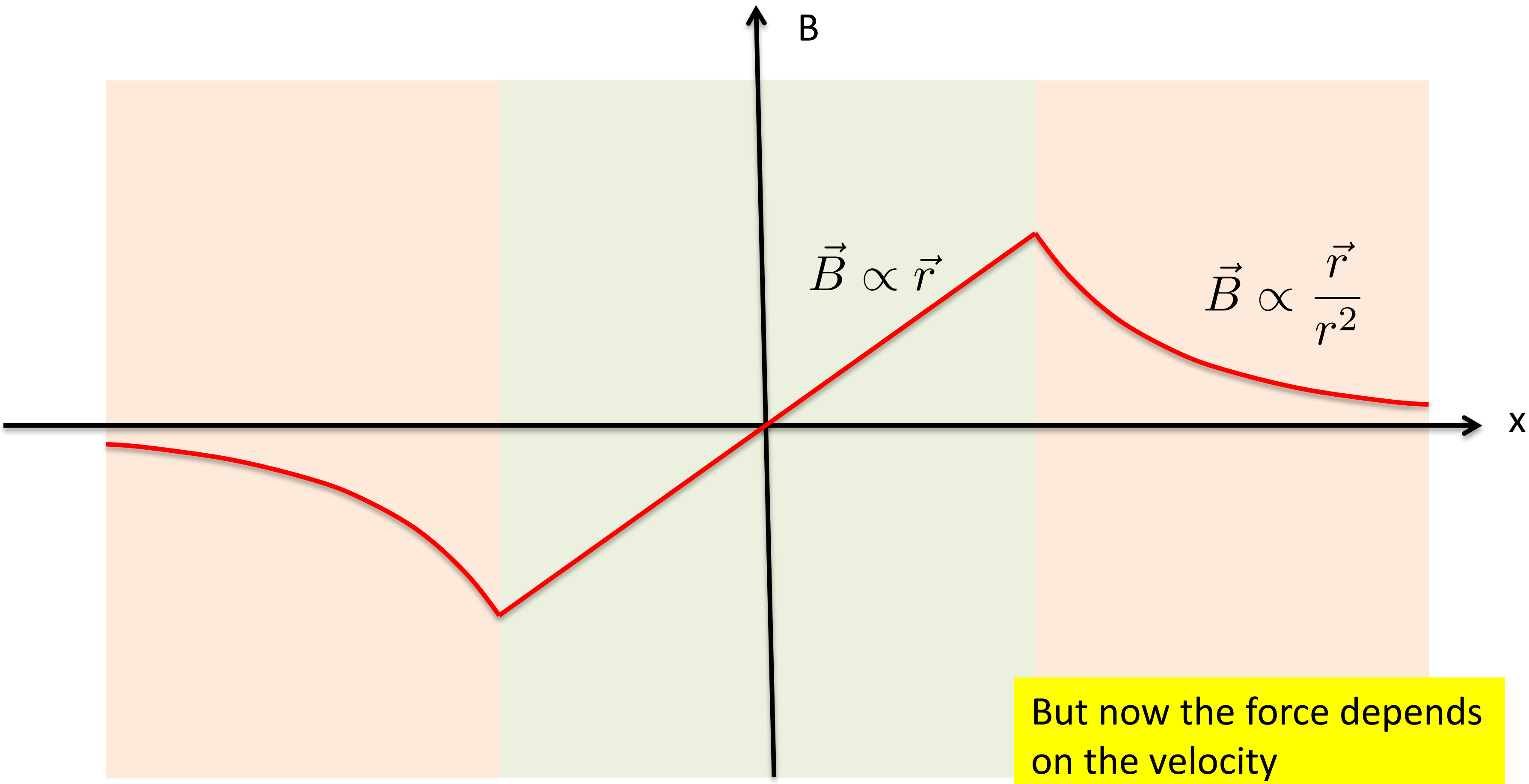
$$B = \frac{\mu_0}{2\pi}\frac{I}{r}$$

Inside the beam

$$B = \frac{\mu_0}{2\pi}\frac{I}{r}\frac{r^2}{R_b^2} = \frac{\mu_0}{2\pi}\frac{I}{R_b^2} r$$

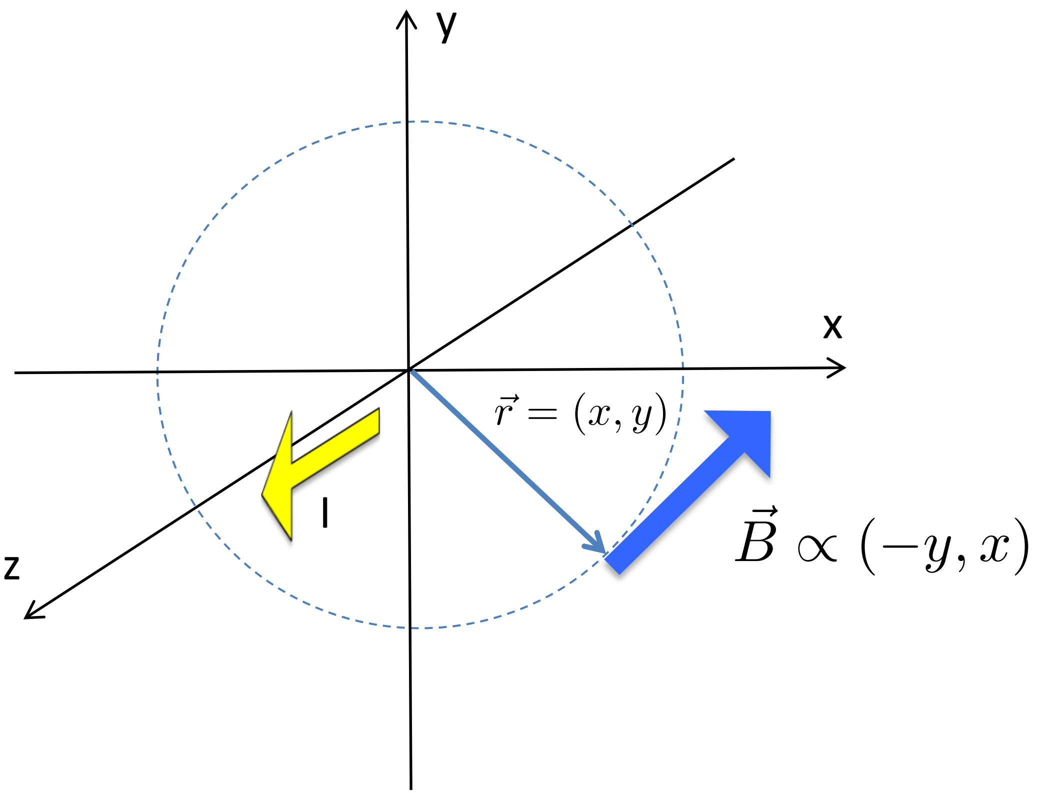
Exactly the same dependence as for the electric field of a uniform coasting beam

# Transverse Magnetic Field




But now the force depends on the velocity

# Orientation




Inside the beam

$$B_x = -\frac{\mu_0}{2\pi}\frac{I}{R_b^2}y, \quad B_y = \frac{\mu_0}{2\pi}\frac{I}{R_b^2}x$$

# Magnetic force in the equation of motion

$$\frac{d^2x}{ds^2} + k_x x = \left( \frac{e}{m\gamma v_0^2}\vec{E}_b + \frac{e}{m\gamma v_0^2}\vec{v} \times \vec{B}_b \right)_x$$

$$(\vec{v} \times \vec{B}_b)_x = v_y B_z - v_z B_y = -v_z B_y = -v_z \frac{\mu_0}{2\pi}\frac{I}{R_b^2} x$$

$B_z$ absent

# therefore

$$\frac{d^2x}{ds^2} + k_x x = \frac{e}{m\gamma v_0^2} E_{b,x} - \frac{e}{m\gamma v_0^2} v_z \frac{\mu_0}{2\pi} \frac{I}{R_b^2} x$$

$$I = v_z \pi R_b^2 \rho$$

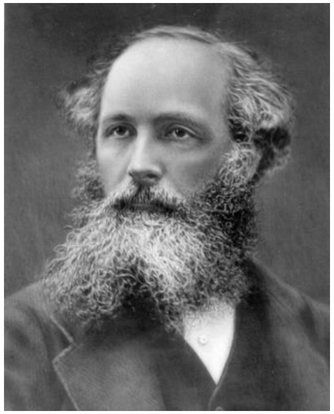
$$E_{b,x} = \frac{\rho}{2\epsilon_0} x$$

Therefore the electric + magnetic field are written as a “modified” electric field

$$\frac{d^2x}{ds^2} + k_x x = \frac{e}{m\gamma v_0^2} E_{b,x}(1 - v_z^2 \mu_0 \epsilon_0)$$

But the fundamental constants combines as follow

$$\boxed{\mu_0 \epsilon_0 = \frac{1}{c^2}}$$



therefore

$$\frac{d^2x}{ds^2} + k_x x = \frac{e}{m\gamma v_0^2} E_{b,x} \left(1 - \frac{v_z^2}{c^2}\right)$$

As $|v_z| \simeq v_0 = |\vec{v}|$ therefore we reach the result

$$\boxed{\frac{d^2x}{ds^2} + k_x x = \frac{e}{m\gamma^3 v_0^2} E_{b,x}}$$

# Equation of motion for coasting beams axi-symmetric

$$\frac{d^2x}{ds^2} + k_x x = \frac{e}{m\gamma^3 v_0^2} E_{b,x}$$

$$\frac{d^2y}{ds^2} + k_y y = \frac{e}{m\gamma^3 v_0^2} E_{b,y}$$

result valid for any axi-symmetric distribution

Space charge is suppressed as $1/\gamma^2$

# Uniform distribution

Suppose that the beam **"remains"** always uniform in x-y circle, then

$$I = v_z \pi R_b^2 \rho$$

only $\mathrm{I}$ is constant ! (not $\rho$, not $R_b$)

and the electric field becomes

$$E_x = \frac{\rho}{2\epsilon_0} x = \frac{1}{2\epsilon_0} \frac{I}{v_z \pi R_b^2} x$$

# then ….

$$\frac{d^2x}{ds^2} + k_x x = \frac{e}{m\gamma^3 v_0^2} E_{b,x}$$

$$\frac{d^2x}{ds^2} + k_x x = \frac{eI}{2\pi\epsilon_0 m\gamma^3 v_0^2 v_z} \frac{x}{R_b^2}$$

but $eI/v_z > 0$ $\Rightarrow$ $eI/v_z \simeq |eI|/v_0$ (positive)

# Perveance

It is convenient to define the quantity

$$K = \frac{eI}{2\pi\epsilon_0 m\gamma^3\beta^3c^3}$$

(positive)

General form of the transverse equation of motion for a uniform axi-symmetric coasting beam

$$\frac{d^2x}{ds^2} + k_x x = K\frac{x}{R_b^2}$$

# Everything is linear !

$$\frac{d^2x}{ds^2} + \underbrace{\left(k_x - \frac{K}{R_b^2}\right)}\, x = 0$$

This is like a quadrupole with changed strength:
too beautiful to be true !!

# Consequences for the motion of one particle

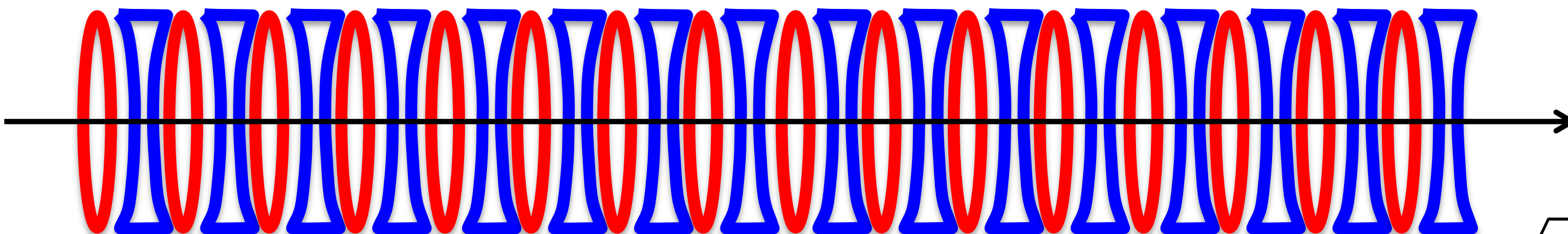
A particle experiences an modified optics

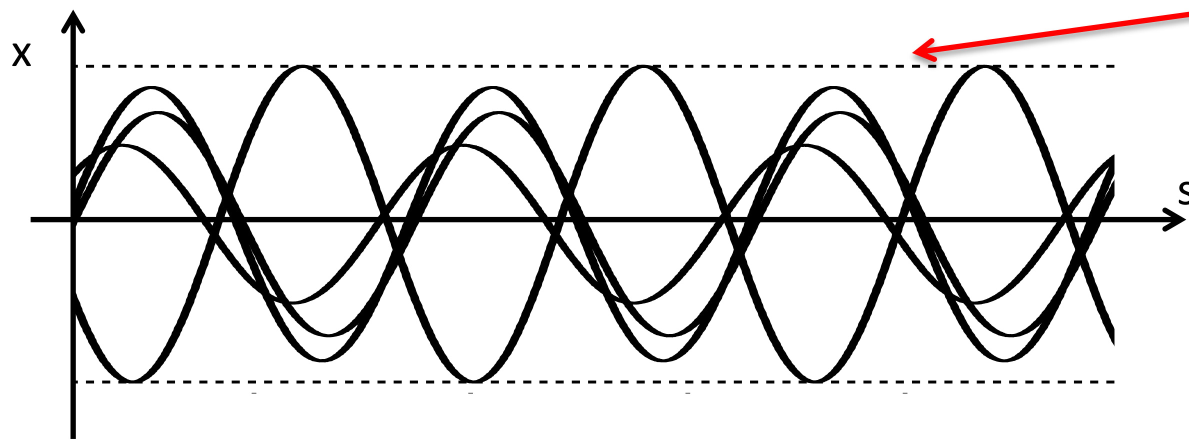
$$k_{x,eff}(s) = k_x(s) - \frac{K}{R_b^2}$$

$$k_{y,eff}(s) = k_y(s) - \frac{K}{R_b^2}$$

# Is it $R_b$ constant? Example with constant focusing lattice

We have to remember that the radius of the beam depends on the optics






$\sqrt{\beta_x \epsilon_x}$

Beam Envelope

But if there is a linear space charge we have a beta function that **depends also on the radius of the envelope**

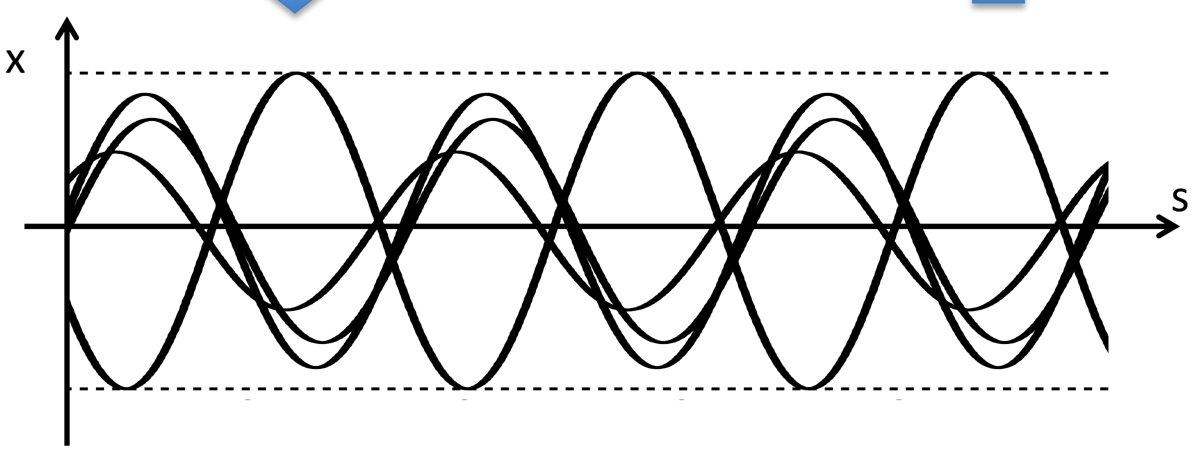
# Strange situation

$$\boxed{k_{x,eff}(s) = k_x(s) - \frac{K}{R_b^2}}$$

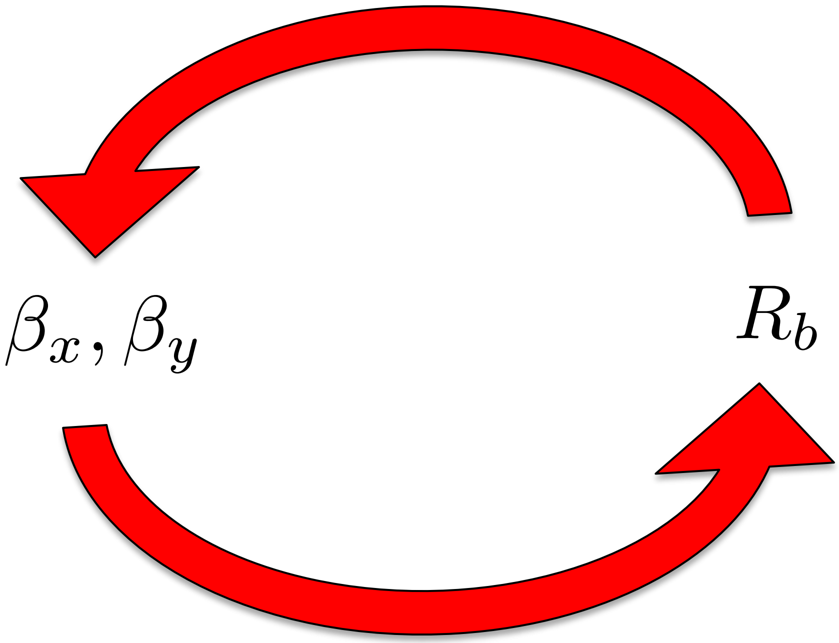
$\beta_x, \beta_y$

set optics:
this is taken
constant

$R_b$



# Optics sets the beam → beam sets space charge → space charge sets the optics !



# Is there a stationary solution ?

$$k_{x,eff}(s) = k_x(s) - \frac{K}{R_b^2}$$

For a constant focusing channel

$$k_{x,eff} = \frac{1}{\beta_x^2}$$

and the beam radius is

$$R_b^2 = \beta_x \epsilon_x$$

Therefore given $k_x$, K, $\epsilon_x$

$$\frac{1}{(\beta_x^*)^2} = k_x - \frac{K}{\beta_x^* \epsilon_x}$$

there is one $\beta_x^*$ which creates a beam such that **space charge + linear optics** creates exactly $\beta_x^*$

# What does it mean ?

This means that we have to create a beam of radius

$$R_b^* = X^* = \sqrt{\beta_x^* \epsilon_x}$$

which is the only beam that, for an emittance of $\epsilon_x$, lattice strength of $k_x$, perveance K, can create an effective optics of $\beta_x^*$

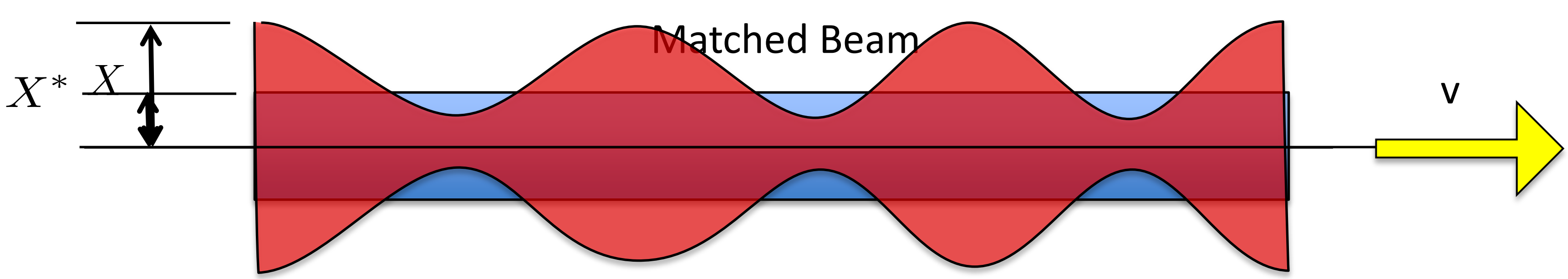
This beam is called **MATCHED** with the effective optics deriving from **linear optics + linear space charge forces**

# When we inject a non matched beam

The optics created by the **lattice + space charge forces** makes the beam mismatched

Mismatch oscillations

Mis-matched Beam



# Summary of finding for a uniform coasting beam

1) the lattice focusing strength is affected by space charge

2) there exists a beam that is matched

# Important consequences of the modified optics (constant focusing)

Equation of motion

tune

without space charge

$$\frac{d^2x}{ds^2} + k_x x = 0 \quad \Rightarrow \quad Q_{x0} = \sqrt{k_x}\frac{L}{2\pi}$$

with space charge

$$\frac{d^2x}{ds^2} + \left(k_x - \frac{K}{R^2}\right)x = 0 \quad \Rightarrow \quad Q_x = \sqrt{k_x - \frac{K}{R_b^2}}\frac{L}{2\pi}$$

# Space charge tune-shift

$$\Delta Q_x = Q_x - Q_{x0}$$ is the space charge tune-shift

$$\Delta Q_x = \sqrt{k_x - \frac{K}{R_b^2}}\frac{L}{2\pi} - \sqrt{k_x}\frac{L}{2\pi}$$

for K/($k_x$ R^2) small

$$\Delta Q_x = -Q_{x0}\frac{K}{2k_x R_b^2} = -Q_{x0}\frac{K}{2R_b^2}\frac{L^2}{4\pi^2 Q_{x0}^2}$$

Detuning created by an axi-symmetric coasting beam, with weak intensity

$$\boxed{\Delta Q_x = -\frac{R_m^2}{2R_b^2}\frac{K}{Q_{x0}}}$$

$R_m$ is the accelerator radius
$R_b$ is the radius of the beam
$Q_{x0}$ is the bare tune
$K$ is the perveance

# Envelope

For a uniform beam

$$\frac{d^2x}{ds^2} + \left(k_x - \frac{K}{R_b^2}\right)x = 0$$

$$\beta_x\,,\beta_y$$

$$R_b = \sqrt{\beta_x \epsilon_x}$$

We can compute the evolution of $R_b$ !

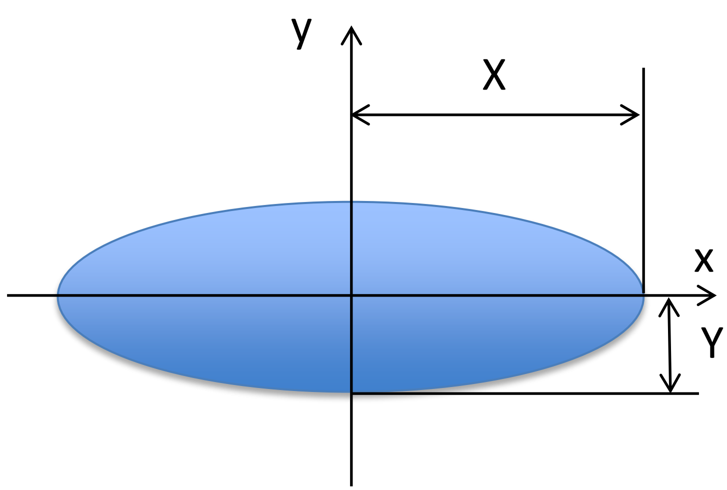
# Envelope equation for an axi-symmetric beam

$$\boxed{R_b'' + k_x R_b - \frac{K}{R_b} - \frac{\epsilon^2}{R_b^3} = 0}$$

Extra term
due to space charge

# Non axi-symmetric uniform beams

For uniform beams the electric field becomes




Inside the beam

$$E_x = \frac{I}{\pi\epsilon_0 v}\frac{x}{X(X+Y)}$$

$$E_y = \frac{I}{\pi\epsilon_0 v}\frac{y}{Y(X+Y)}$$

# Equation of motion

$$\frac{d^2x}{ds^2} + \left[k_x - \frac{2K}{X(X+Y)}\right] x = 0$$

$$\frac{d^2y}{ds^2} + \left[k_y - \frac{2K}{Y(X+Y)}\right] y = 0$$

# Modified beta function

The lattice optics is modified in x, and y

$$k_{x,eff} = k_x - \frac{2K}{X(X+Y)} \Rightarrow \beta_x^*$$

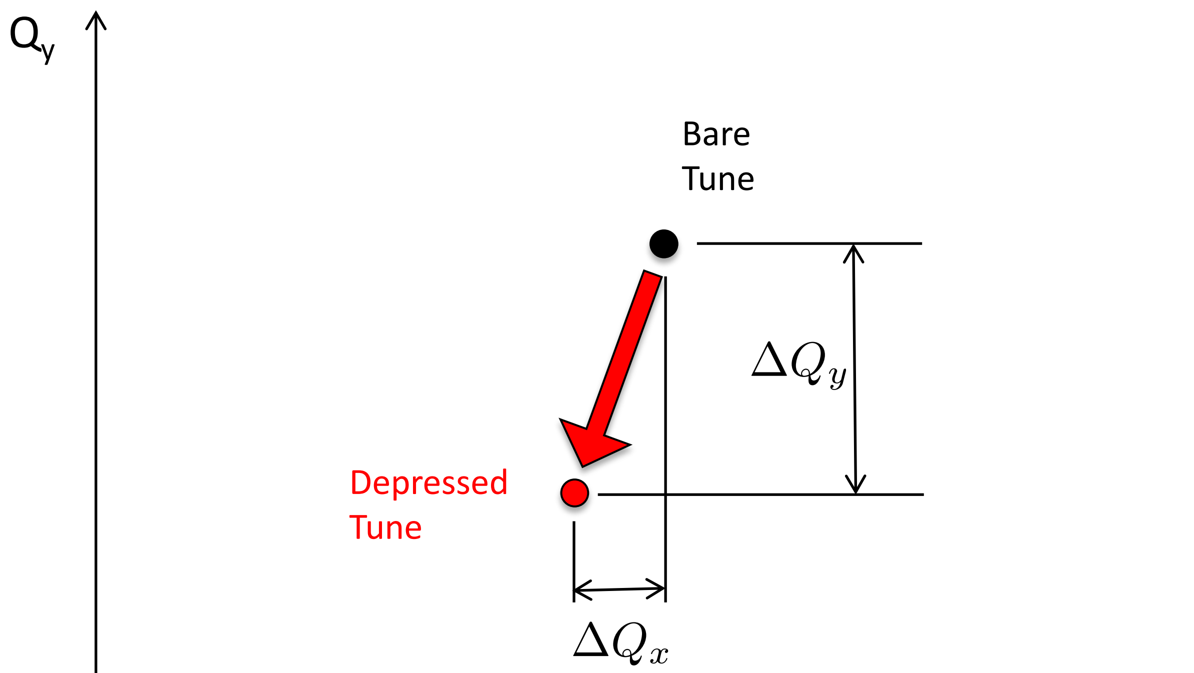
$$k_{y,eff} = k_y - \frac{2K}{Y(X+Y)} \Rightarrow \beta_y^*$$

# Space charge tune-shift

$$\Delta Q_x = -\frac{K}{X(X+Y)}\frac{R_m^2}{Q_{x0}}$$

$$\Delta Q_y = -\frac{K}{Y(X+Y)}\frac{R_m^2}{Q_{y0}}$$

# Situation in a tune diagram



# Envelope equations

$$X'' + k_x X - \frac{2K}{X+Y} - \frac{\epsilon_x^2}{X^3} = 0$$

$$Y'' + k_y Y - \frac{2K}{X+Y} - \frac{\epsilon_y^2}{Y^3} = 0$$

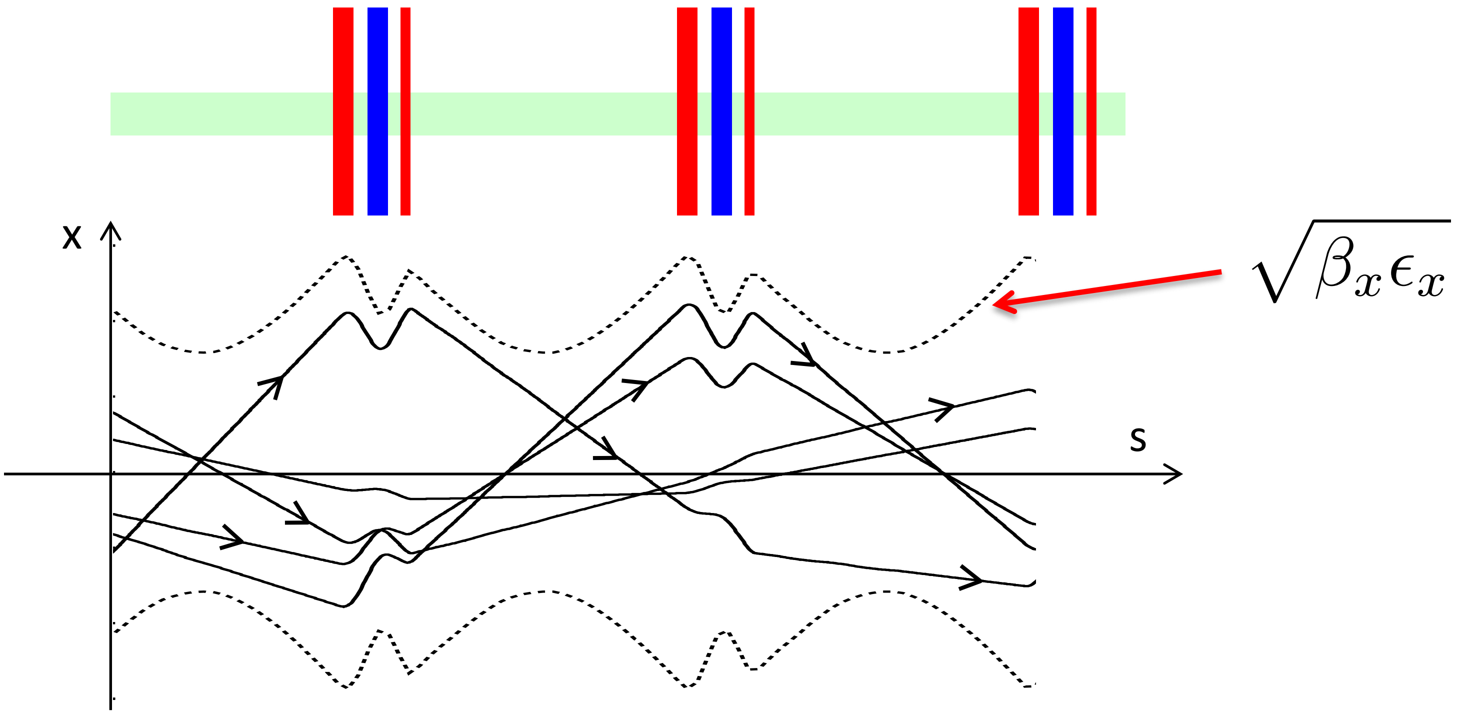
# Conclusion for the constant focusing

Space charge changes the particle tune, in both planes according to the beam sizes, and the optics

Again we can describe the beam via envelope equations which are coupled through the space charge

# For varying focusing

All formulation remains the same, but the difference is in what happens to the beta functions and the detuning



# New optics

We continue to keep the ansatz that the beam remains uniform, and with the same transverse emittances

$$\beta_{x,0}(s), \beta_{y,0}(s)$$

$$\frac{d^2x}{ds^2} + \left[k_x - \frac{2K}{X(X+Y)}\right] x = 0$$

$$\frac{d^2y}{ds^2} + \left[k_y - \frac{2K}{Y(X+Y)}\right] y = 0$$

$$\beta_{x,1}(s), \beta_{y,1}(s)$$

Go on until $\beta_{x,n}(s), \beta_{y,n}(s)$ converges

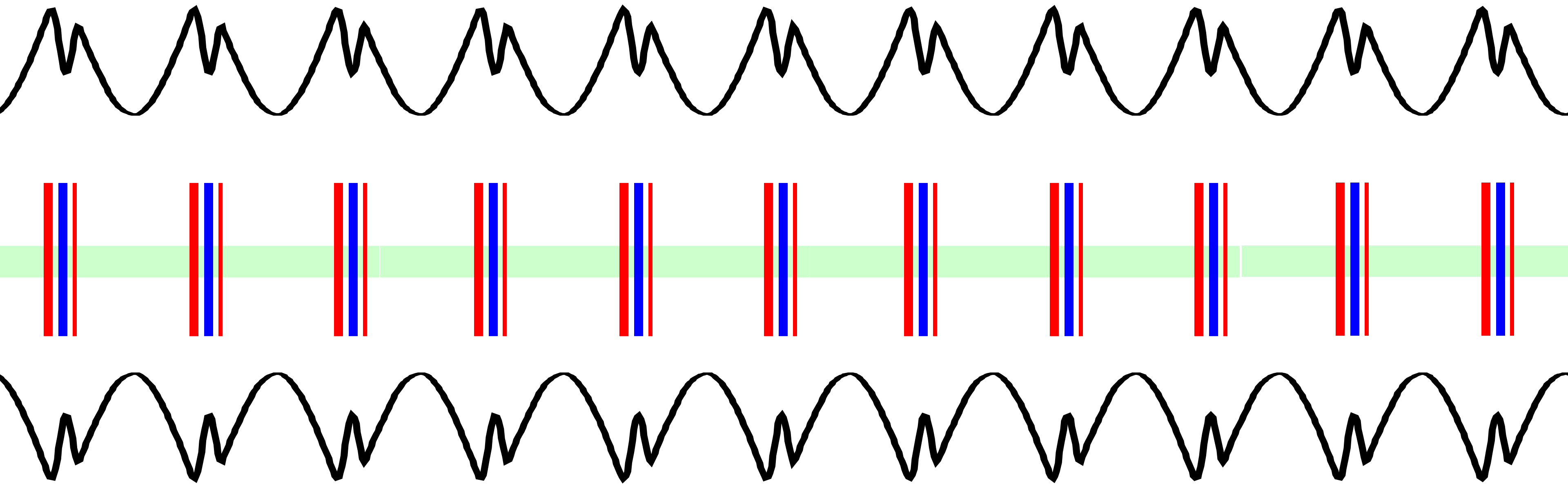
# Space charge tune-shift

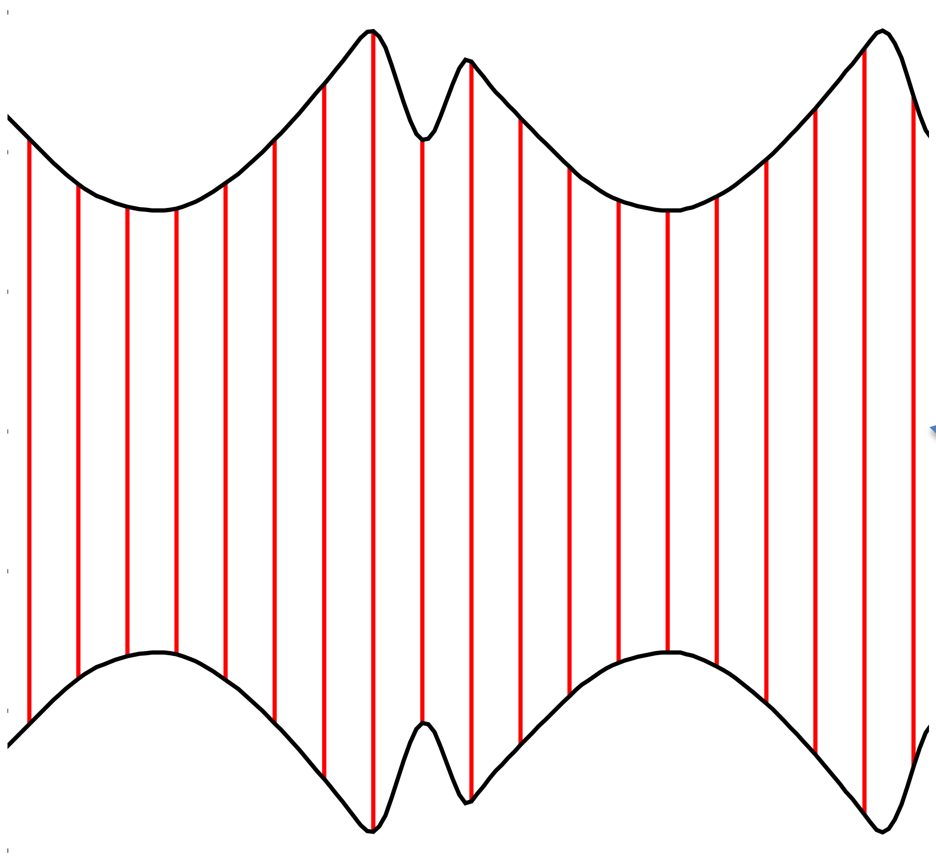
Now we have a matched optics for a beam with perveance K, and transverse emittances Ex, E,y. Therefore injecting a beam matched with

$$\beta_x^*(s), \alpha_x^*(s), \beta_y^*(s), \alpha_y^*(s)$$

will create a matched optical function.

Now you can look at the space charge as a distribution of many space charge "kicks"



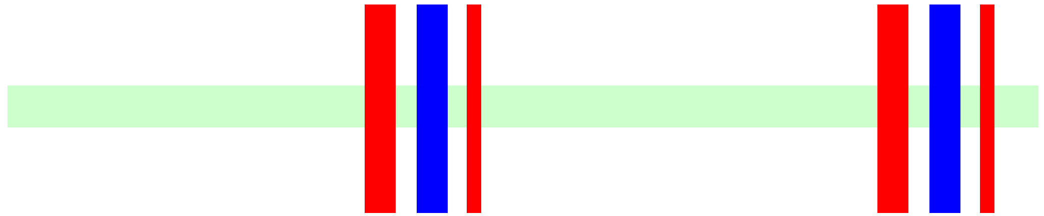


$$\frac{d^2x}{ds^2} + \left[k_x - \frac{2K}{X(X+Y)}\right] x = 0$$

Space charge kick

$$\tilde{E}_x = \frac{2K}{X(X+Y)} ds\ x$$

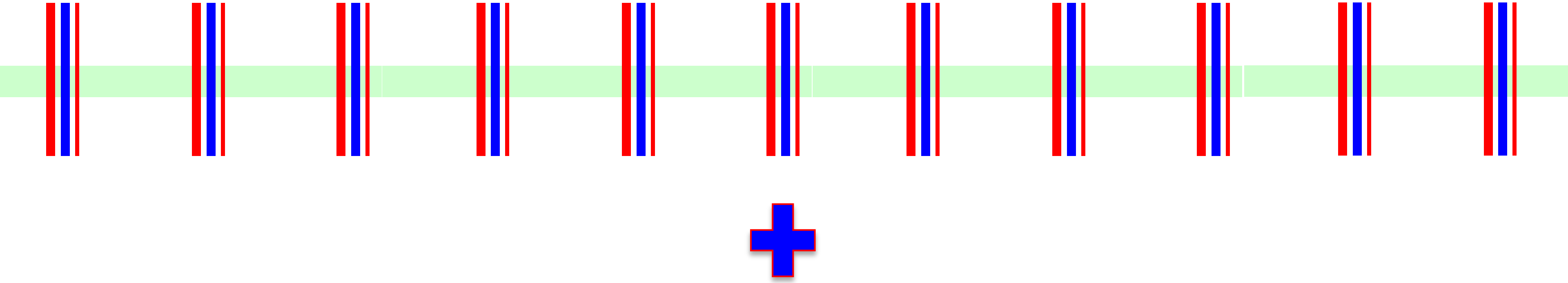
$$\tilde{E}_y = \frac{2K}{Y(X+Y)} ds\ y$$



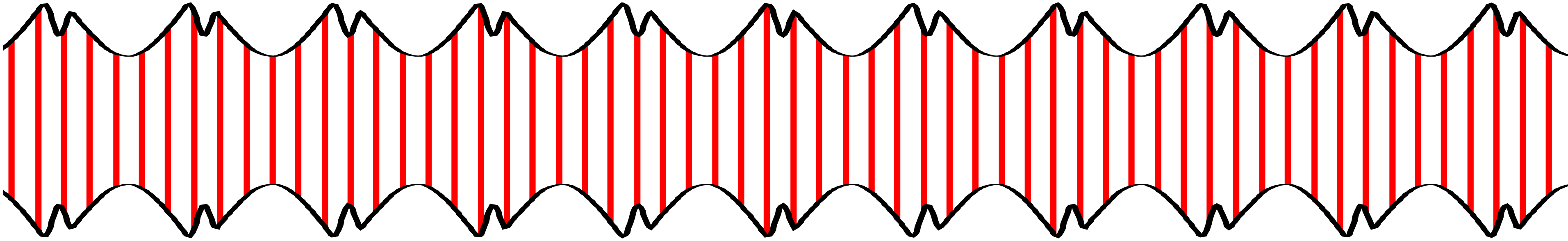
in units of the equation of motion

# Situation

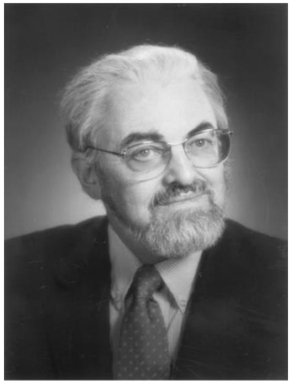
Linear optics $$\frac{d^2x}{ds^2} + k_x x = 0 \quad \frac{d^2y}{ds^2} + k_y y = 0$$



Space charge kicks $$\tilde{E}_x = \frac{2K}{X(X+Y)} ds\ x \quad \tilde{E}_y = \frac{2K}{Y(X+Y)} ds\ y$$



*E. Courant*



$$\Delta\nu = \frac{\Delta\mu}{2\pi} = -\frac{\Delta(\cos\mu)}{2\pi\sin\mu_0} = \frac{1}{4\pi}\int_0^C \beta(s)k(s)\,ds.$$

$$\Delta Q_x = \frac{1}{4\pi}\int_0^C \beta_x(s)\tilde{E}_x(s)ds = -\frac{1}{4\pi}\int_0^C \beta_x(s)\frac{2K}{X(s)(X(s)+Y(s))}ds$$

$$\Delta Q_y = \frac{1}{4\pi}\int_0^C \beta_y(s)\tilde{E}_y(s)ds = -\frac{1}{4\pi}\int_0^C \beta_y(s)\frac{2K}{Y(s)(X(s)+Y(s))}ds$$

$$\Delta Q_x = -\frac{KR_m}{\epsilon_x}\left\langle \frac{1}{1+\sqrt{\frac{\epsilon_y \beta_y(s)}{\epsilon_x \beta_x(s)}}} \right\rangle$$

It is a usual approximation that

$$\left\langle \frac{1}{1+\sqrt{\frac{\epsilon_y \beta_y(s)}{\epsilon_x \beta_x(s)}}} \right\rangle \simeq \frac{1}{1+\sqrt{\frac{\epsilon_y \langle\beta_y\rangle}{\epsilon_x \langle\beta_x\rangle}}}$$

(not really
obvious…)

Therefore

$$\Delta Q_x \simeq -\frac{K R_m}{\epsilon_x} \frac{1}{1+\sqrt{\frac{\epsilon_y \langle \beta_y \rangle}{\epsilon_x \langle \beta_x \rangle}}} = -K R_m \frac{\langle \beta_x \rangle}{\sqrt{\epsilon_x \langle \beta_x \rangle}(\sqrt{\epsilon_x \langle \beta_x \rangle} + \sqrt{\epsilon_y \langle \beta_y \rangle})}$$

Taking $\langle \beta_x \rangle \simeq \frac{R_m}{Q_{x0}}$

$$\Delta Q_x \simeq -K \frac{R_m^2}{Q_{x0}} \frac{1}{\sqrt{\epsilon_x \langle \beta_x \rangle}(\sqrt{\epsilon_x \langle \beta_x \rangle} + \sqrt{\epsilon_y \langle \beta_y \rangle})}$$

# Exactly the same formula of the constant focusing channel

Ring with constant focusing

$$\Delta Q_x = -\frac{K}{X(X+Y)}\frac{R_m^2}{Q_{x0}}$$

Ring with AG focusing

$$\Delta Q_x \simeq -K\frac{R_m^2}{Q_{x0}}\frac{1}{\sqrt{\epsilon_x\langle\beta_x\rangle}(\sqrt{\epsilon_x\langle\beta_x\rangle}+\sqrt{\epsilon_y\langle\beta_y\rangle})}$$

# What is the meaning?

It seems that the space charge detuning is governed by the same type of law, provided we use some kind of "effective" beam size.

This **seems** to suggest that when two beams have the same "effective" size, and they are in a machine with the same radius, and the same tune, they have the same space charge detuning !!

(nice, but not obvious)

# About the ansatz of the uniformity

Is it true that if we start with a beam distribution uniform, that is remains uniform ?

Beam distribution evolves according to the Vlasov equation

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left( \frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) = 0$$

with $f(q,p,t) = \dfrac{\Delta N}{\Delta V}$ particle density in phase space

A very complex and difficult equation !!

# Self-consistency

Is there a distribution that does not change “functional shape” ?

That is, that it is not time dependent ?

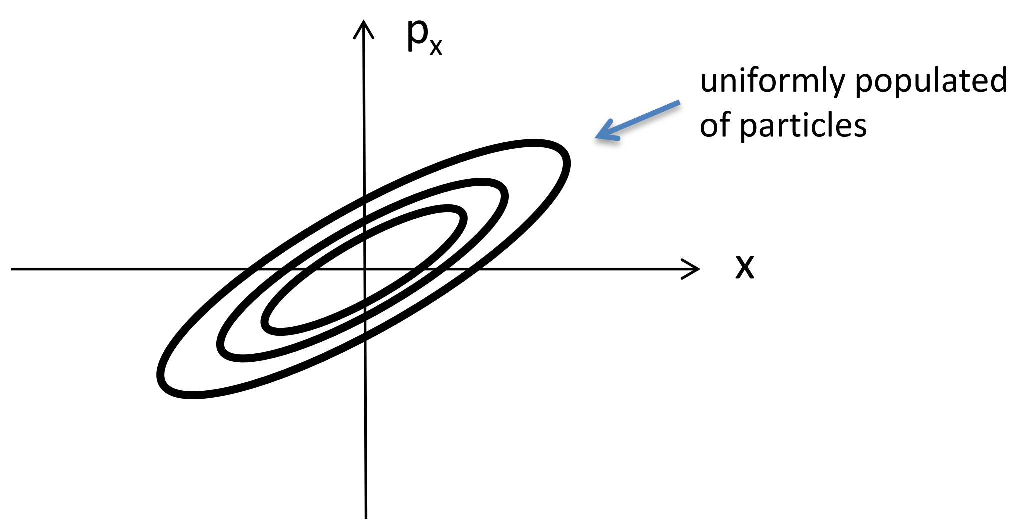
**Without space charge**

for a linear uncoupled lattice → Answer: YES

take $$f(x, x', y, y', t) = g(\epsilon_x, \epsilon_y)$$

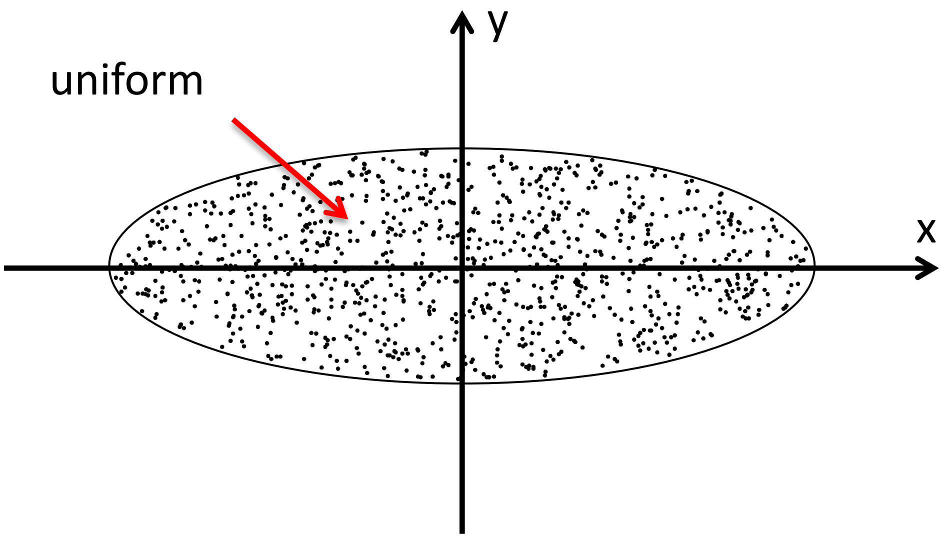
This type of distributions are all self-consistent → MATCHED with the lattice in fact $$\frac{\partial}{\partial t} f(x, x', y, y', t) = \frac{\partial}{\partial t} g(\epsilon_x, \epsilon_y) = 0$$

Practically it means that the Courant-Snyder ellipses are populated with constant particle density



# Self-consistent distribution

If a distribution is x-y uniformly populated of particles




Forces are linear

$$\tilde{E}_x = \frac{2K}{X(X+Y)}x$$

$$\tilde{E}_y = \frac{2K}{Y(X+Y)}y$$

But we are not sure that the x-y distribution remains uniform during beam propagation

# KAPCHINSKY-VLADIMIRSKY (KV)

But any distribution $f(x, x', y, y', t) = g(\epsilon_x, \epsilon_y)$

remains of the same type if forces are **linear**

But then, if we choose a distribution that creates linear space charge forces, then that distribution also will remain of the same type !

$$f = \delta\left(\frac{\epsilon_x}{E_x} + \frac{\epsilon_y}{E_y} - 1\right)$$

This distribution creates a uniform x-y distribution

it will remain of the same type !!

This allows to make a complete use of the envelope equations !

# NON uniform distributions

Non-uniform beam distributions exhibits a more complex behaviour.

1) These distribution can be generated to be matched with a linear lattice without space charge

2) When the beam has space charge effects, these distributions are not self-consistent, hence they change with time, BUT for short time they keep their form.

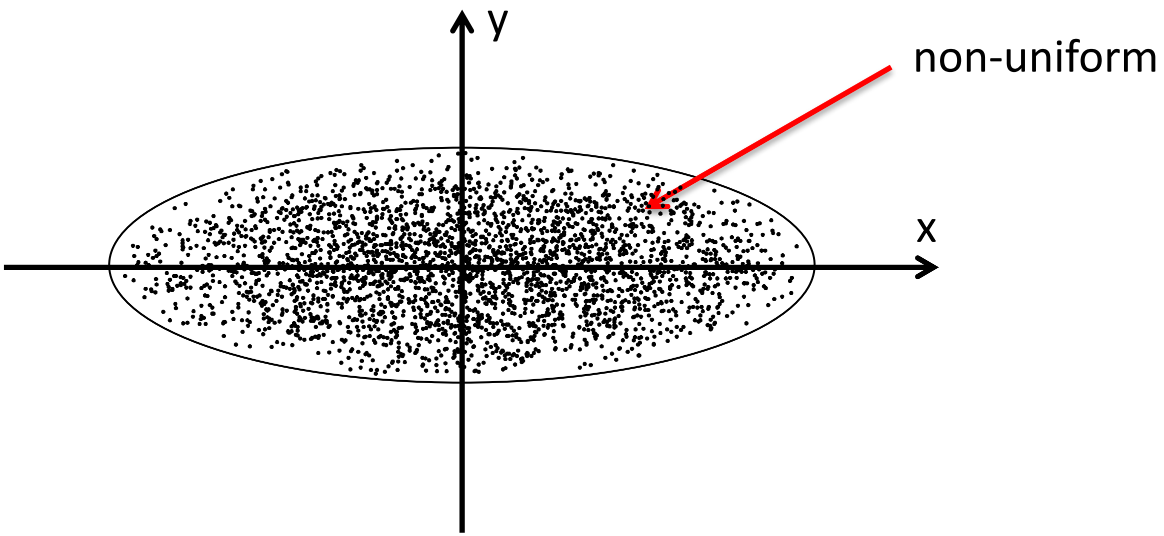
# WATERBAG

$$f = \Theta\left(\frac{\epsilon_x}{E_x} + \frac{\epsilon_y}{E_y} - 1\right)$$

with $\Theta\left(x\right)$

the Heaviside function

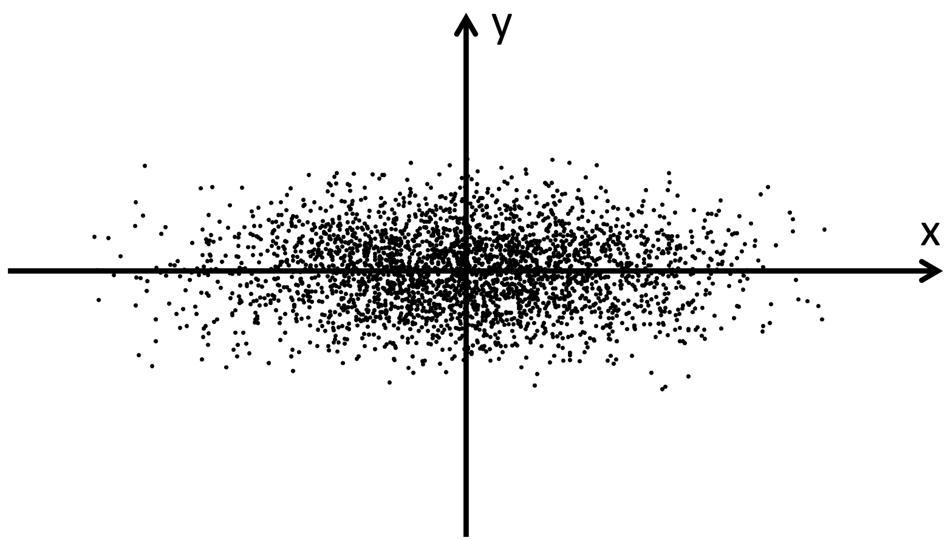
It is a 4D sphere completely filled



# GAUSSIAN

$$f = \propto e^{-\frac{1}{2}\left(\frac{\epsilon_x}{E_x}+\frac{\epsilon_y}{E_y}\right)}$$

The distribution is not bounded, and is the product of two 1D Gaussians



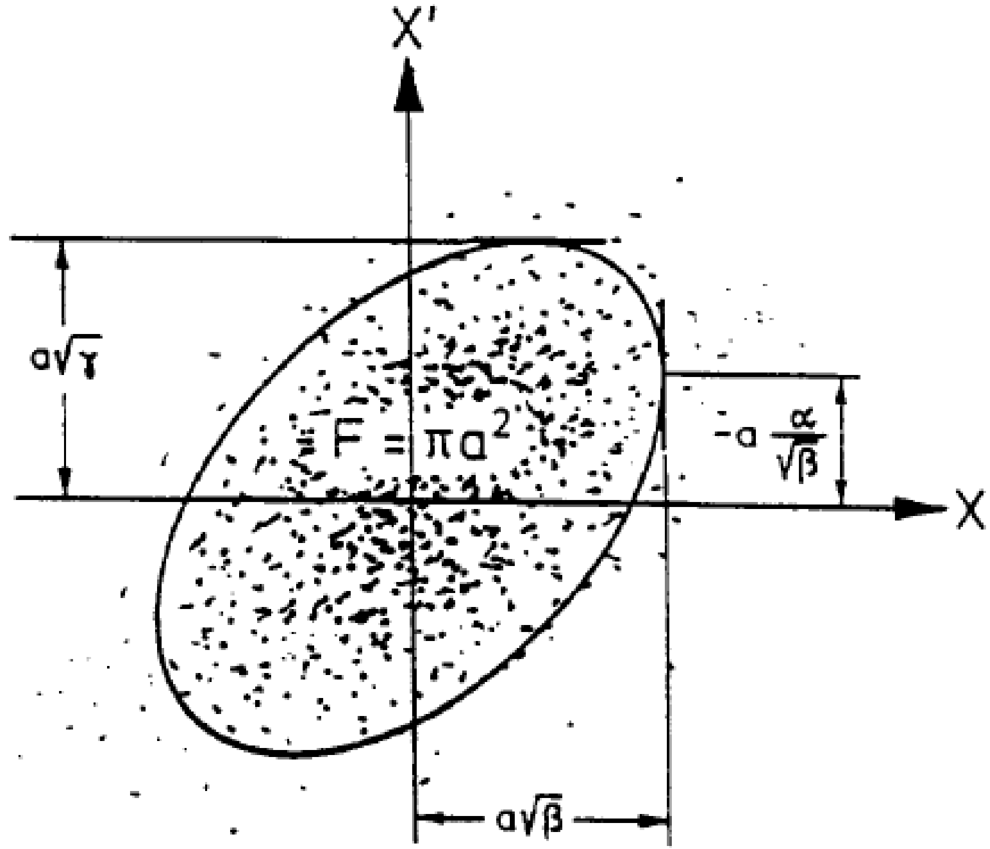
# Moments

$$\langle x^2 \rangle = \frac{1}{N}\sum_{i=1}^{N} x_i^2$$

$$\langle p_x^2 \rangle = \frac{1}{N}\sum_{i=1}^{N} p_{x,i}^2$$

$$\langle x p_x \rangle = \frac{1}{N}\sum_{i=1}^{N} x_i\, p_{x,i}$$

$$E_x^2 = \langle x^2\rangle\langle p_x^2\rangle - \langle x p_x\rangle^2$$




RMS emittance depends
on the beam distribution

Defining $$X = \sqrt{\langle x^2 \rangle}$$

$$X'' = \frac{\langle x x'' \rangle}{X} + \frac{E_{x,rms}^2}{X^3}$$

# Without space charge

$$x'' + k(s)x = 0 \quad \Rightarrow \quad \langle x x'' \rangle = -k(s)\langle x^2 \rangle$$

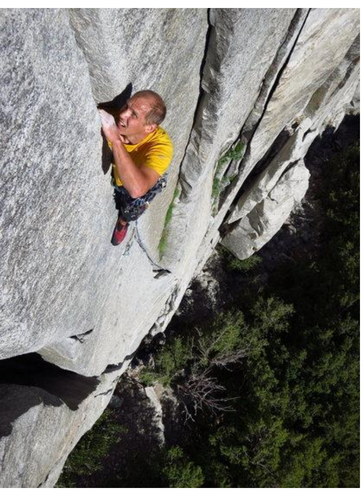
$$X'' = \frac{-k(s)\langle x^2 \rangle}{X} + \frac{E_{x,rms}^2}{X^3}$$

$$\Downarrow$$

$$\boxed{X'' + k(s)X - \frac{E_{x,rms}^2}{X^3} = 0}$$

# Including space charge

Frank Sacherer
1940 - 1978



Sacherer Cracker,
Yosemite (and 33 peaks climbed)

Equation of motion

space
charge
force

$$x'' = -k(s)x + \mathcal{E}_x$$

Therefore

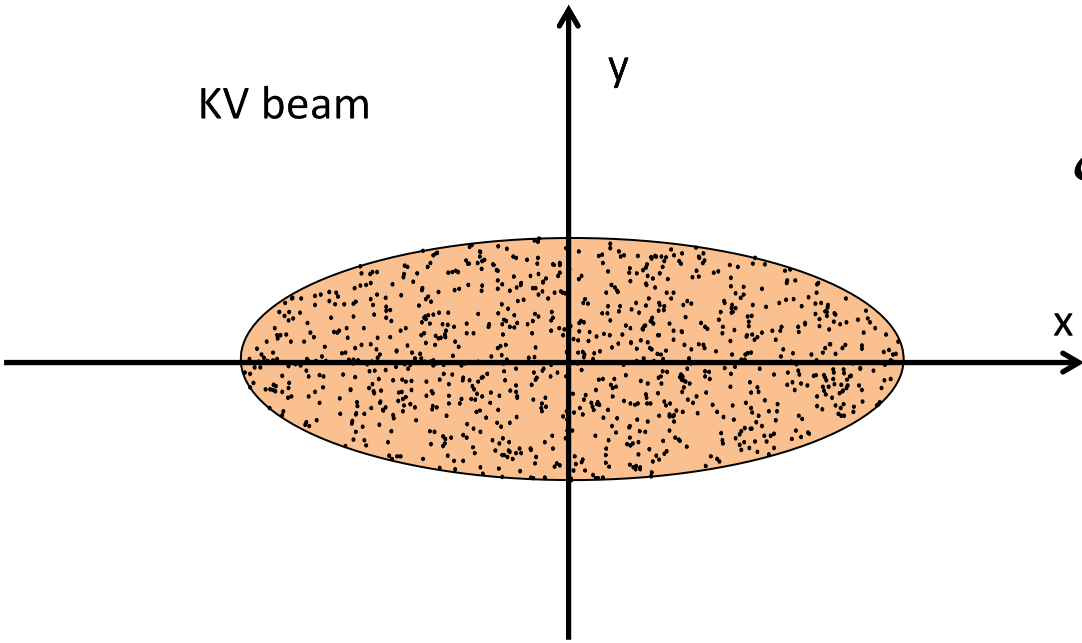
$$\langle xx'' \rangle = -k(s)\langle x^2 \rangle + \langle x\mathcal{E}_x \rangle$$

$$X'' + k(s)X - \frac{\langle x\mathcal{E}_x\rangle}{X} - \frac{E_{x,rms}^2}{X^3} = 0$$

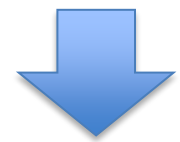
What is it $\langle x\mathcal{E}_x\rangle$ ?

Well: If $\mathcal{E}_x = \lambda x$ $\Rightarrow$ $\langle x\mathcal{E}_x\rangle = \lambda X^2$

# For a KV beam




$$\mathcal{E}_x = 2K \frac{x}{X(X+Y)}$$



$$\langle x\mathcal{E}_x \rangle = 2K \frac{\langle x^2 \rangle}{X(X+Y)} = 2K \frac{X}{(X+Y)}$$

incredible !

# F. Sacherer: very surprising result

If the beam has transverse distribution

$$\rho \propto n\left(\frac{x^2}{X^2}+\frac{y^2}{Y^2}\right)$$

True for any distribution matched with the naked optics

$$\boxed{\langle x\mathcal{E}_x\rangle = 2K\frac{X}{(X+Y)}}$$

# RMS envelope equation

Therefore the rms envelope follows the equation

$$X'' + k(s)X - \frac{2K}{X+Y} - \frac{E_{x,rms}^2}{X^3} = 0$$

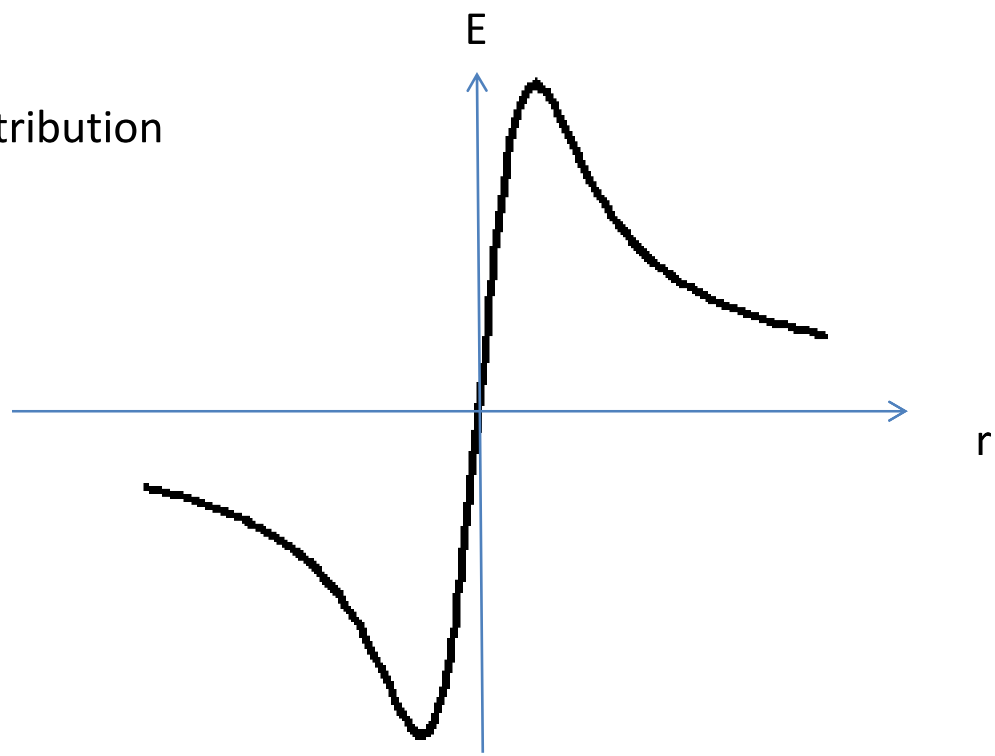
If different beams have the same rms sizes,
the same rms emittance, the same perveance

**All these beams have the same rms evolution**

# Space Charge Detuning of Non-uniform distribution

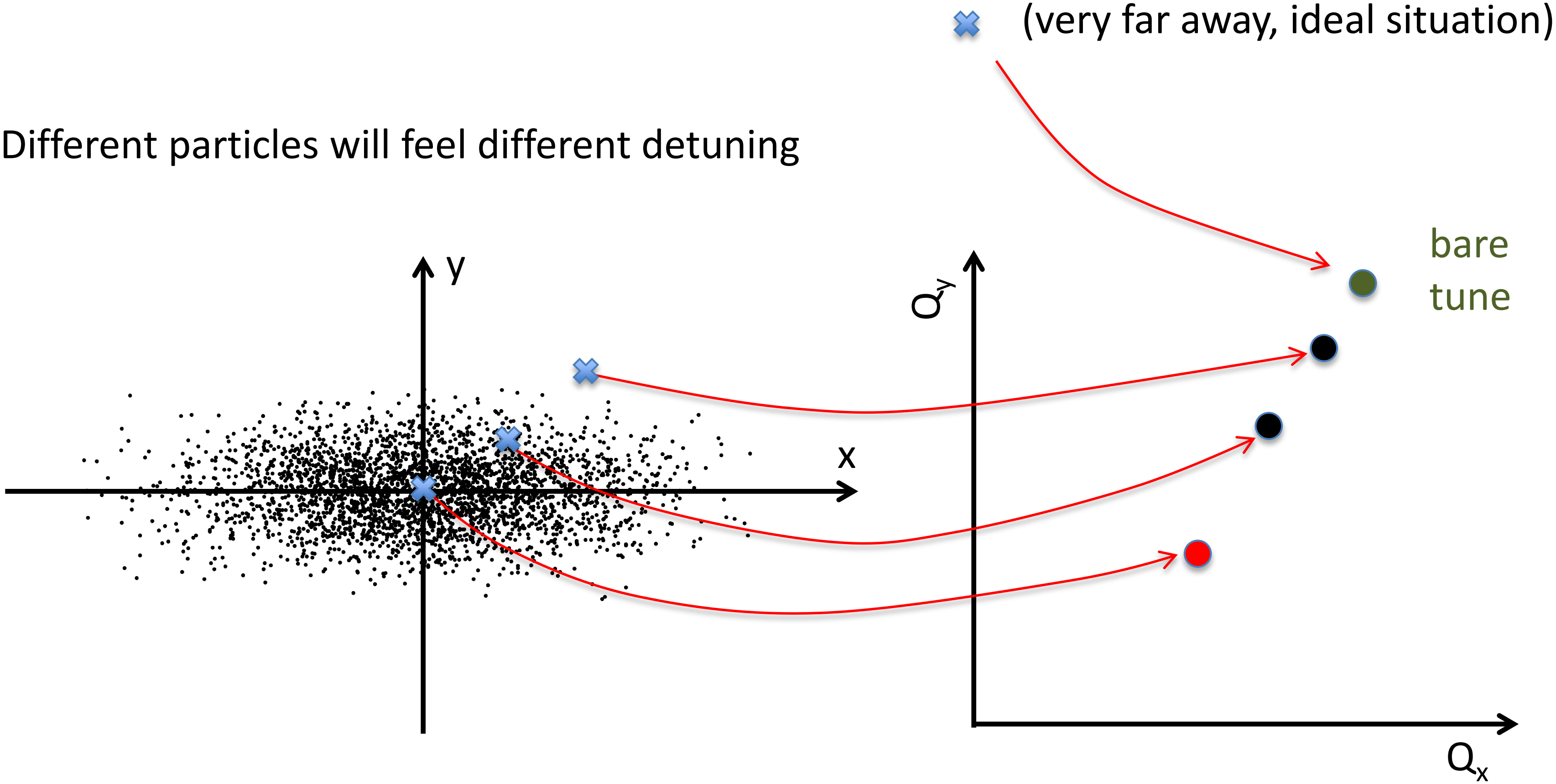
For WB, G distributions the expression of the space charge force is more complex.

Example of a Gaussian distribution



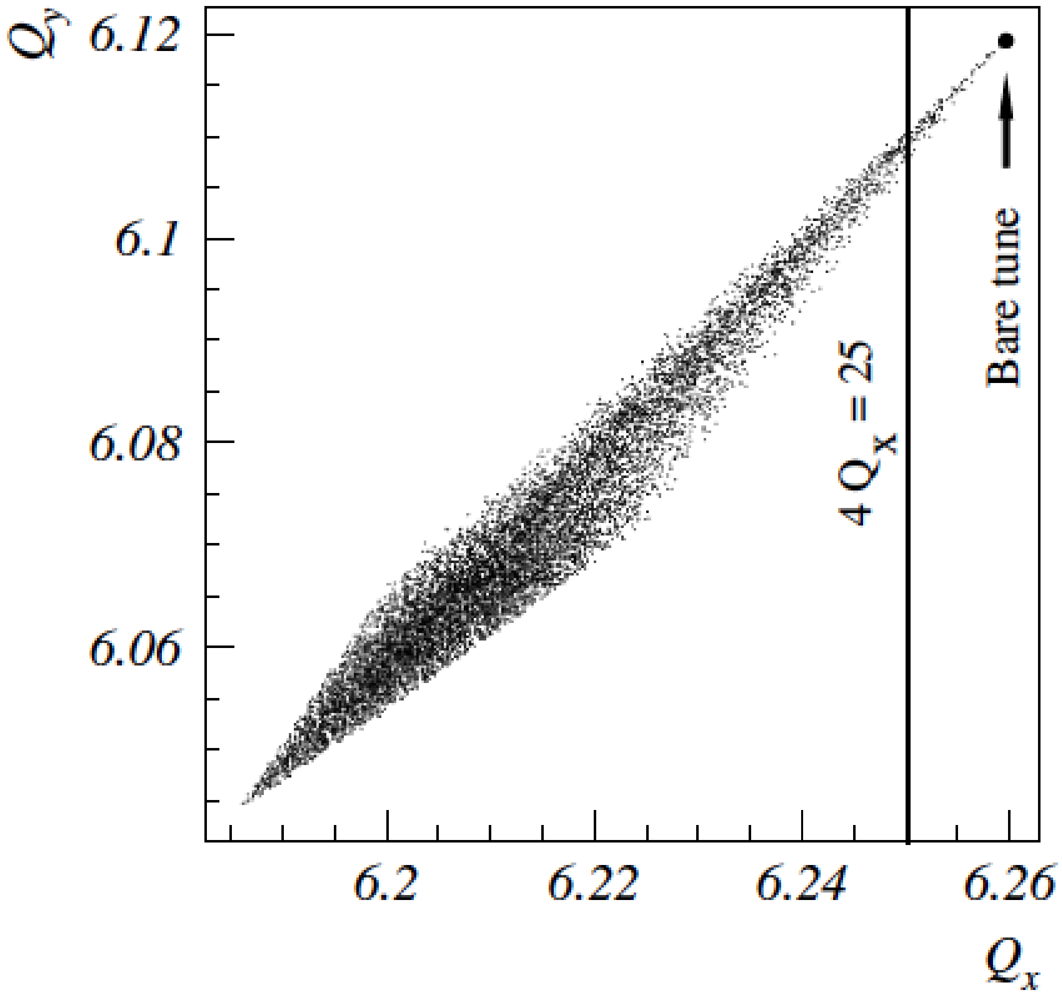
(very far away, ideal situation)

Different particles will feel different detuning



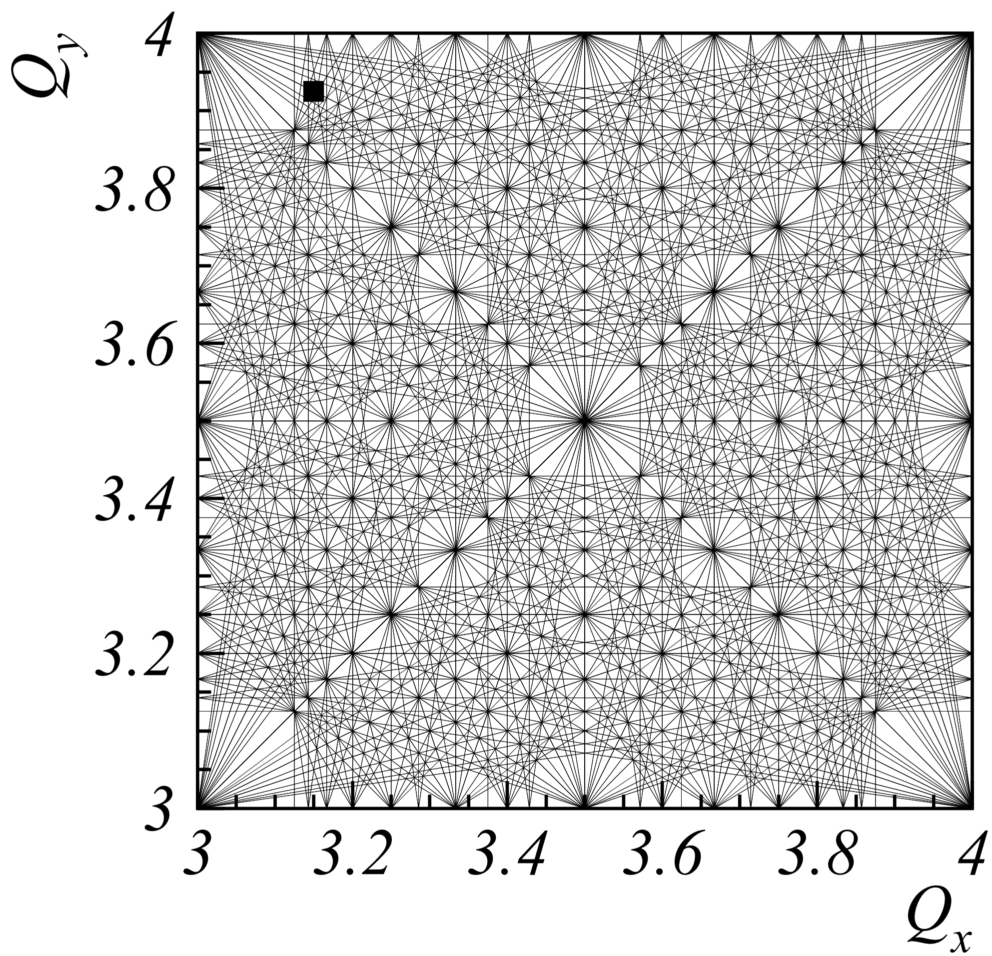
# The space charge tune-spread

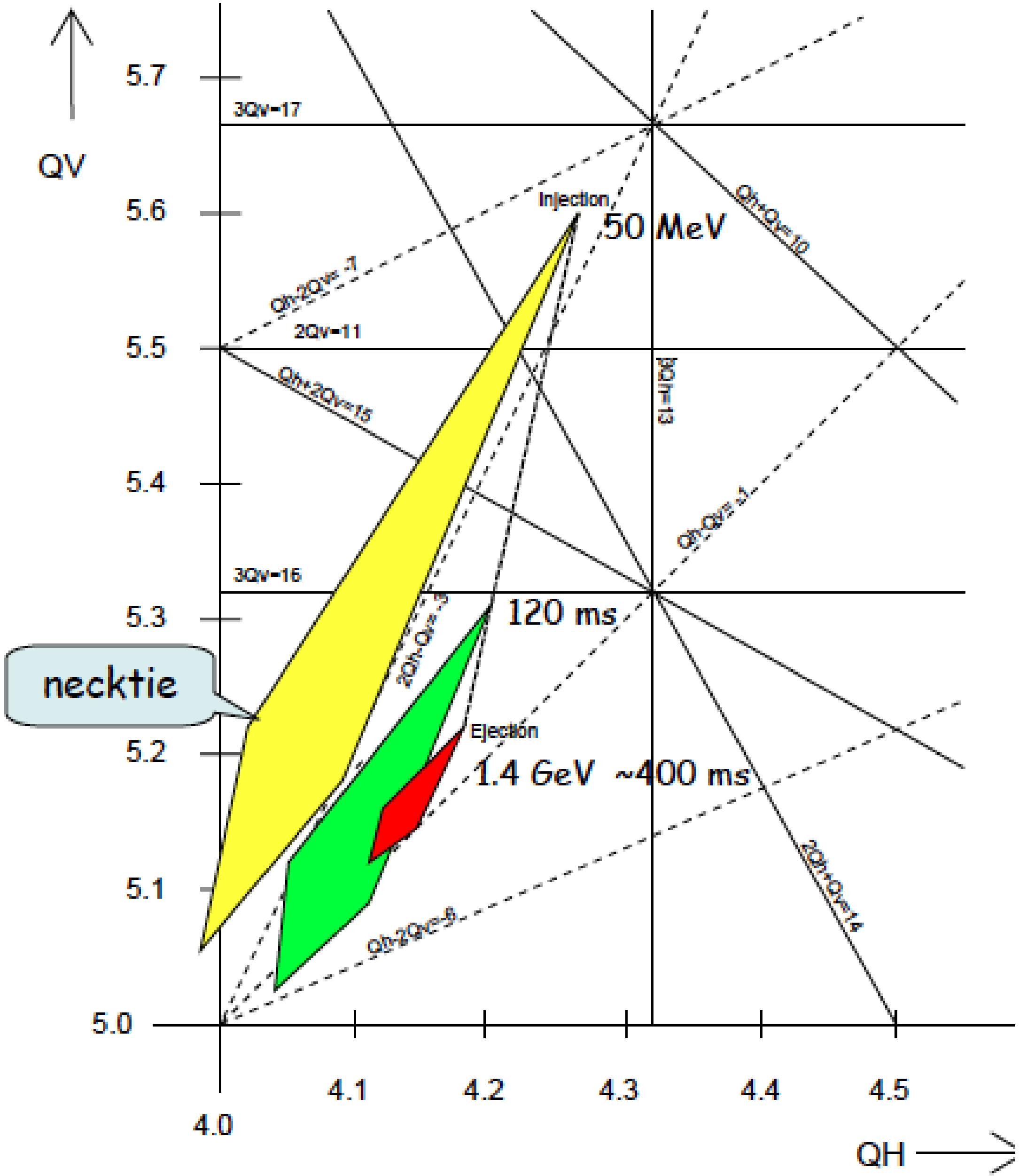
Example



# Consequences

If the space charge induced tune-spread overlaps a machine resonance there is a problem






**Issues**

1) Space charge + resonances in coasting beams
2) Space charge + resonances in bunched beams
3) Collective beam response to direct space charge forces ?

# Summary

1) Space charge is important at low energy
2) Space charge affect the optics
3) It requires a matched beam
4) It creates a tune-spread
5) Beams rms-equivalent behave similarly (in rms sense)
6) Mismatched beams oscillates (mismatch)
7) Self-consistency is important and desired
8) Space charge tune spread creates severe problem in case of resonance overlapping
10) The higher the space charge tune-spread the more difficult is to control the beam

Next lecture → Image charge → Collective effects

# Further readings

**eory and design of charged particle beams** Martin Reiser JOHN WILEY and Son, Inc, New York 19

All previous CAS