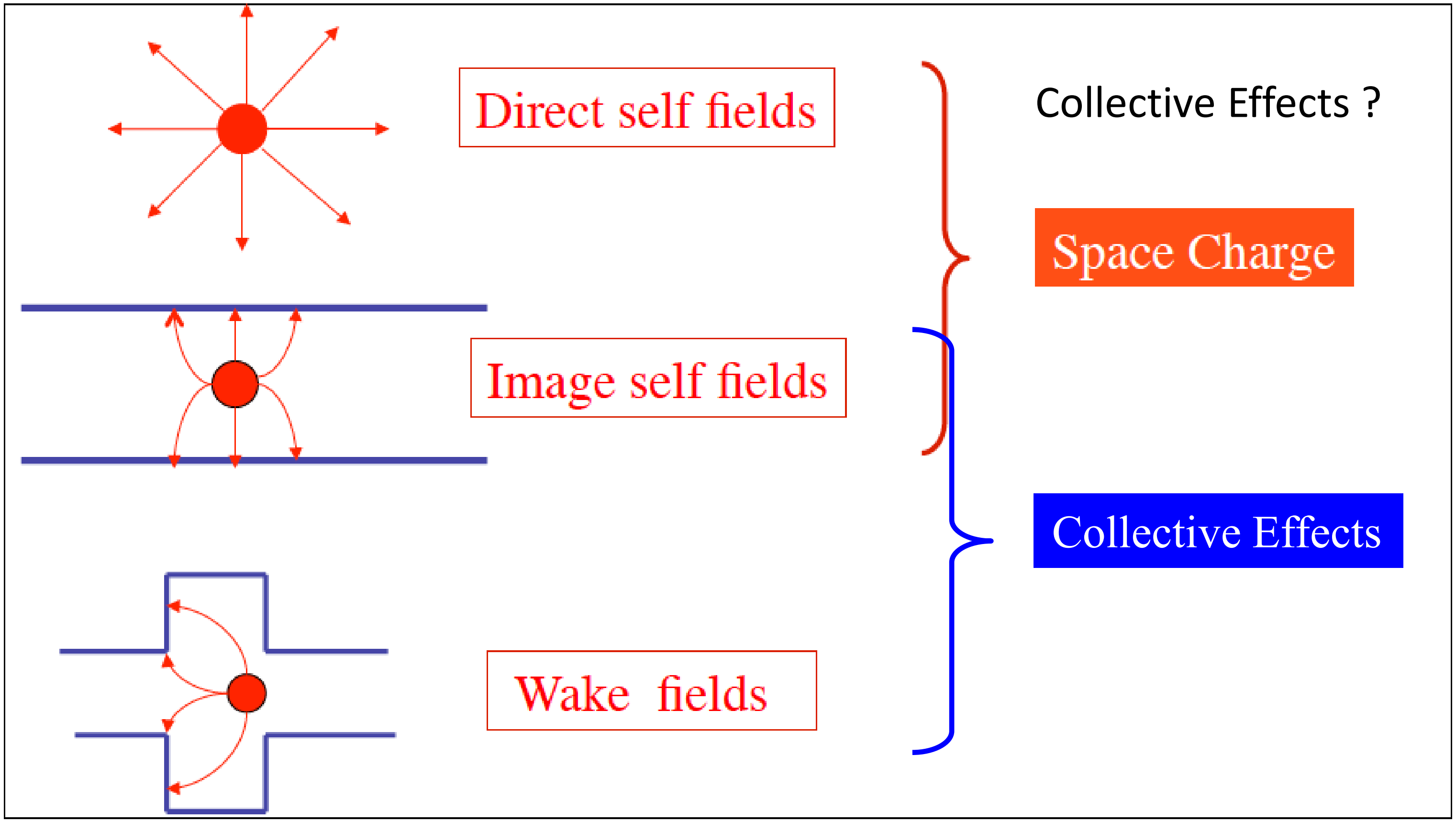
Collective Effect II

Giuliano Franchetti, GSI

CERN Accelerator – School

Prague

Type of fields



Robinson Instability

Robinson Instability

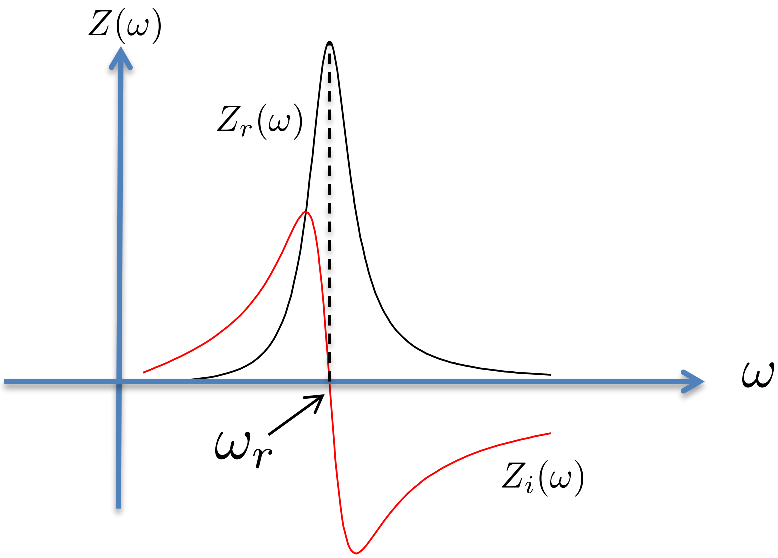
It is an instability arising from the coupling of the impendence and longitudinal motion

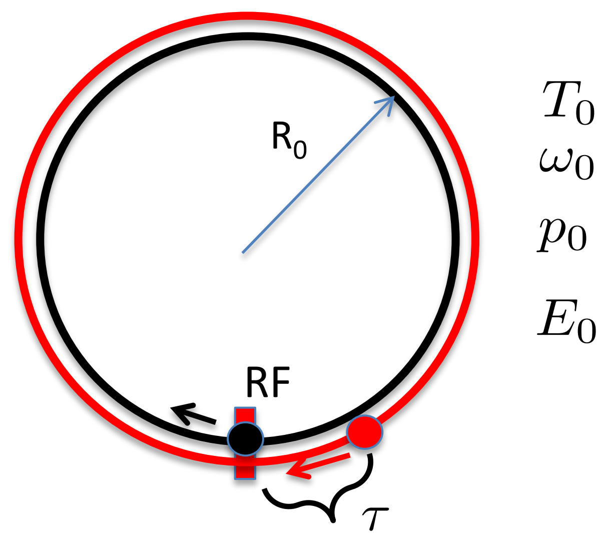
the revolution frequency controls the impendence

real part control the energy loss in a bunch

change particle energy

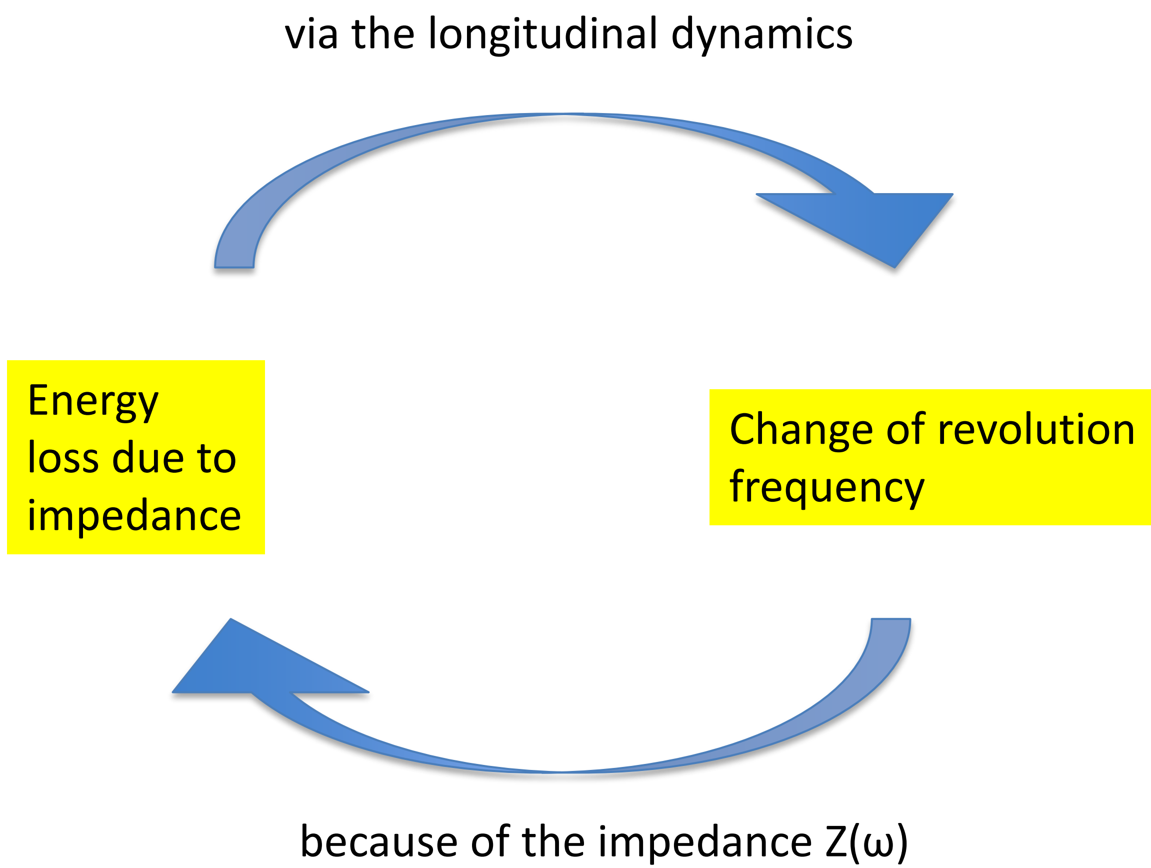
change revolution frequency



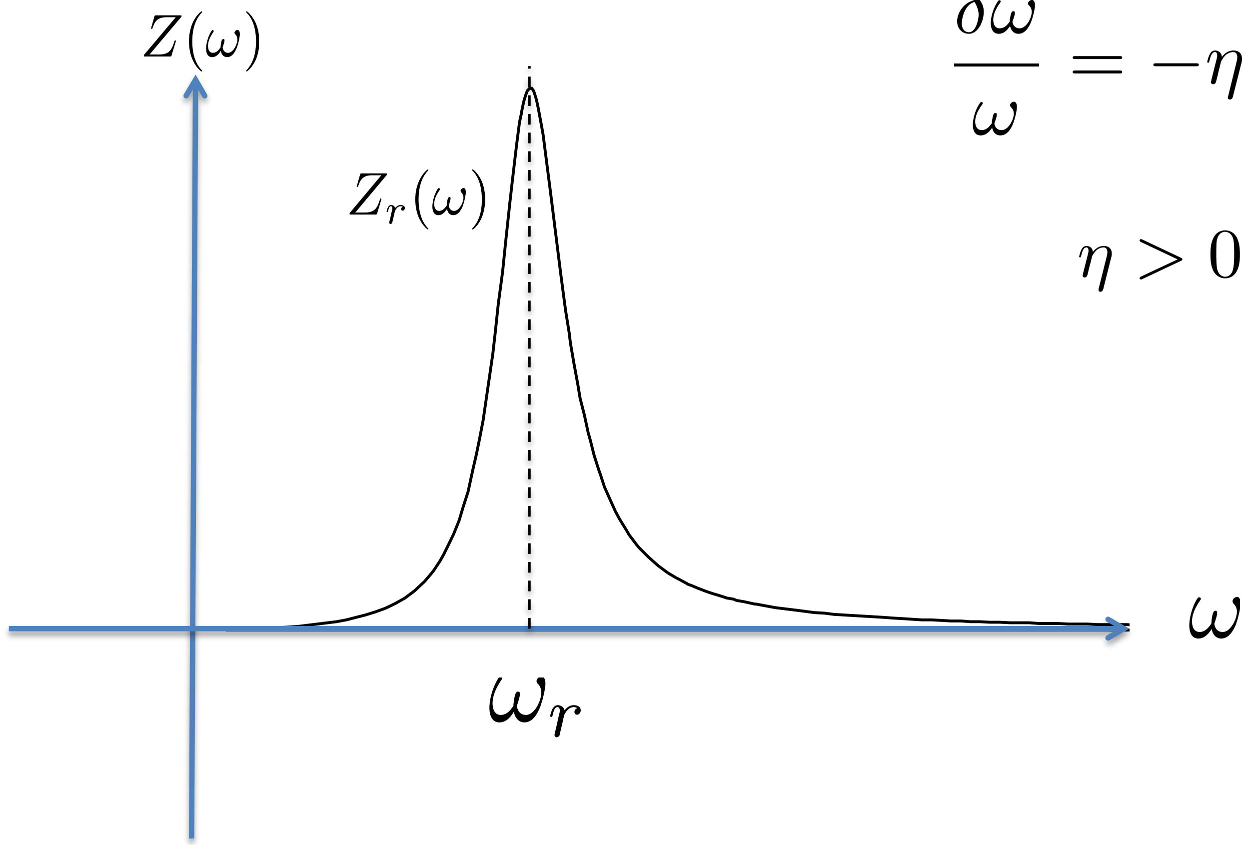


slower particle
(if below transition)

The coupling of two effects



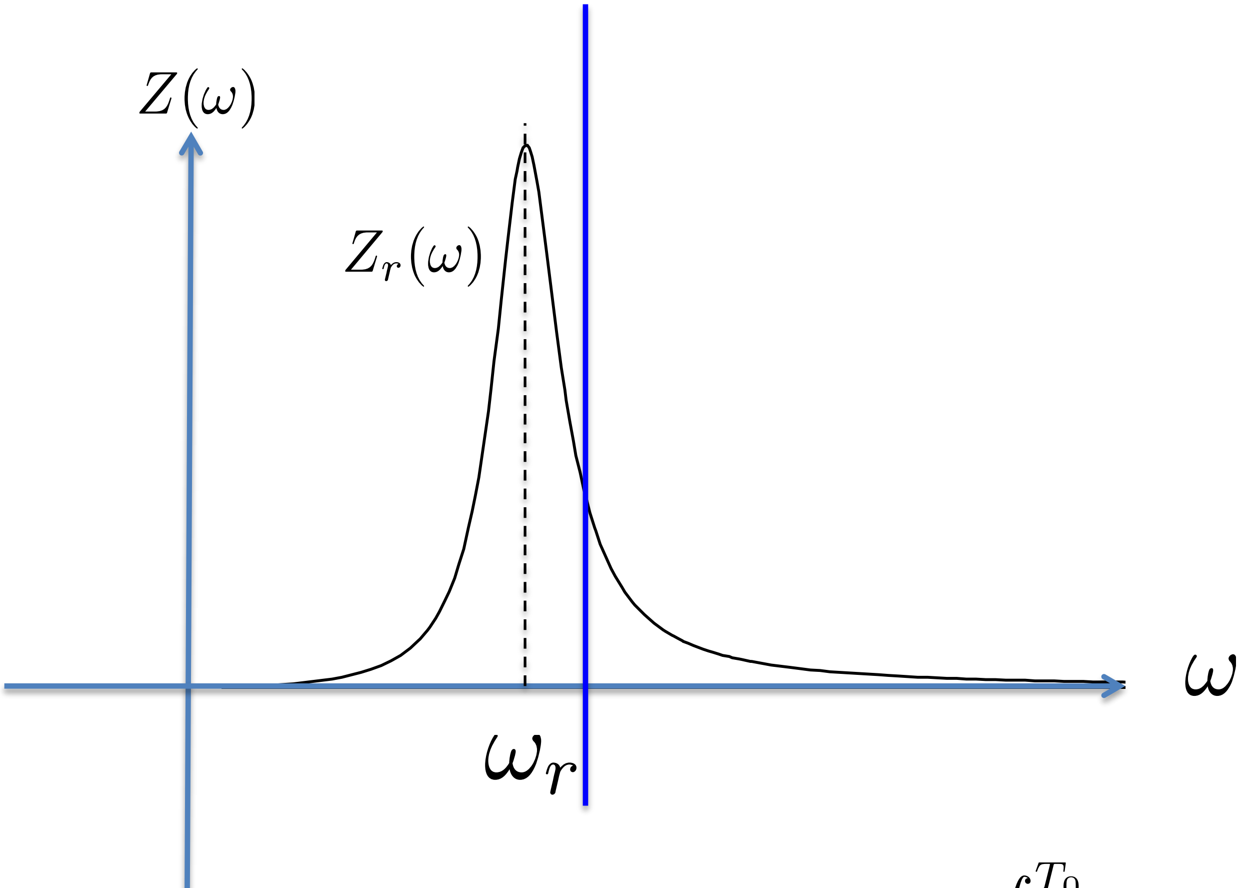
Below transition



$$\frac{\delta\omega}{\omega} = -\eta\frac{\delta E}{E}$$

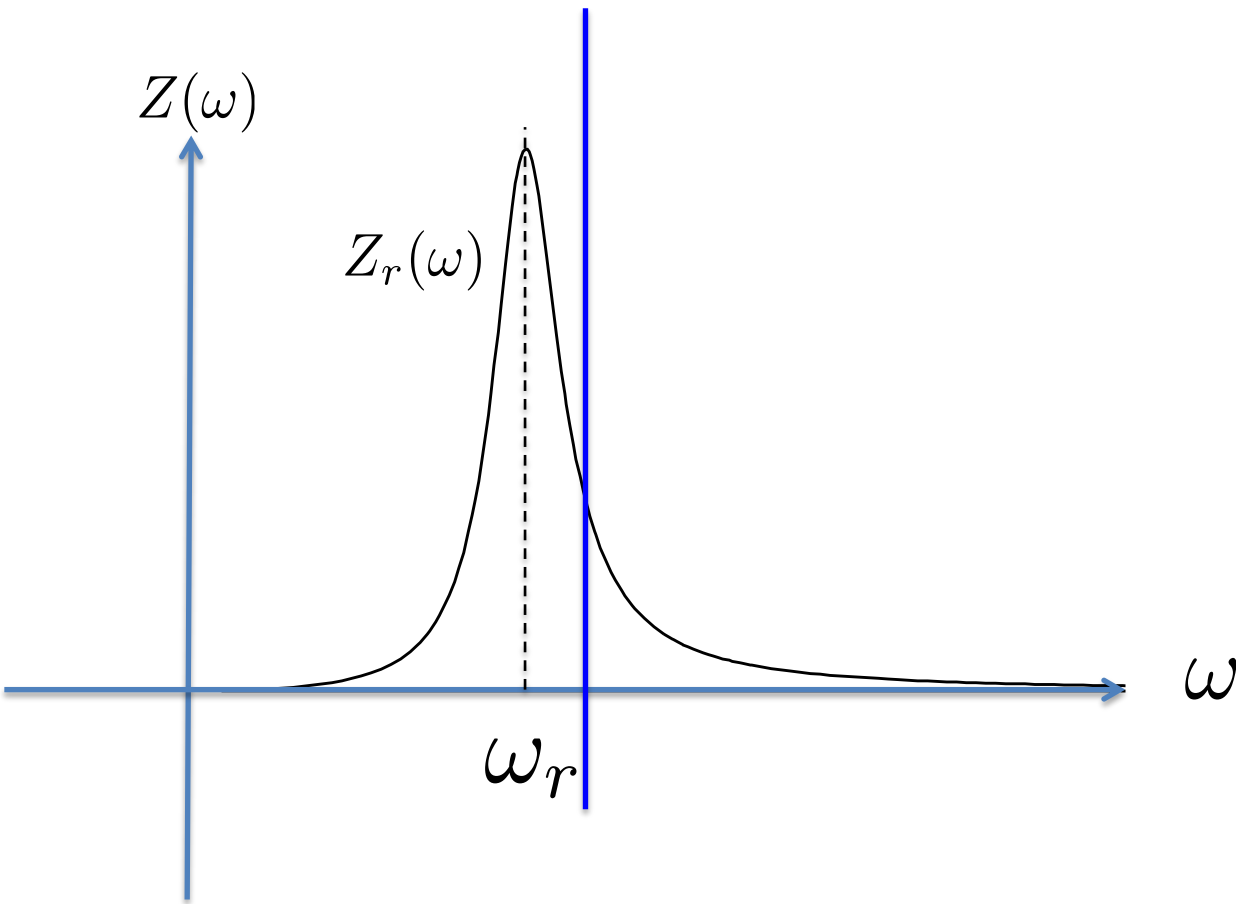
$$\eta > 0$$

Below transition



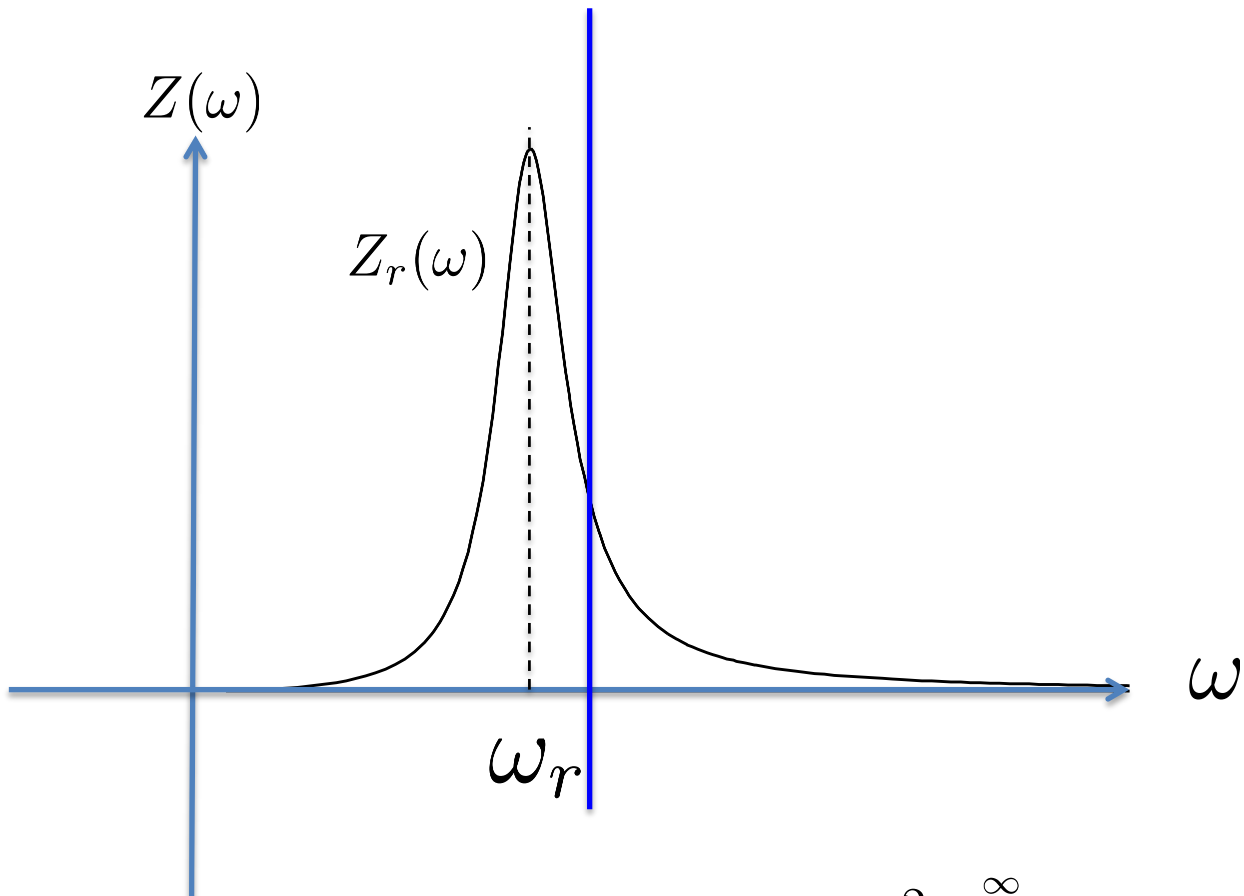
Energy lost in one turn $$W_b = \int_0^{T_0} I(t)V(t)dt$$

Below transition



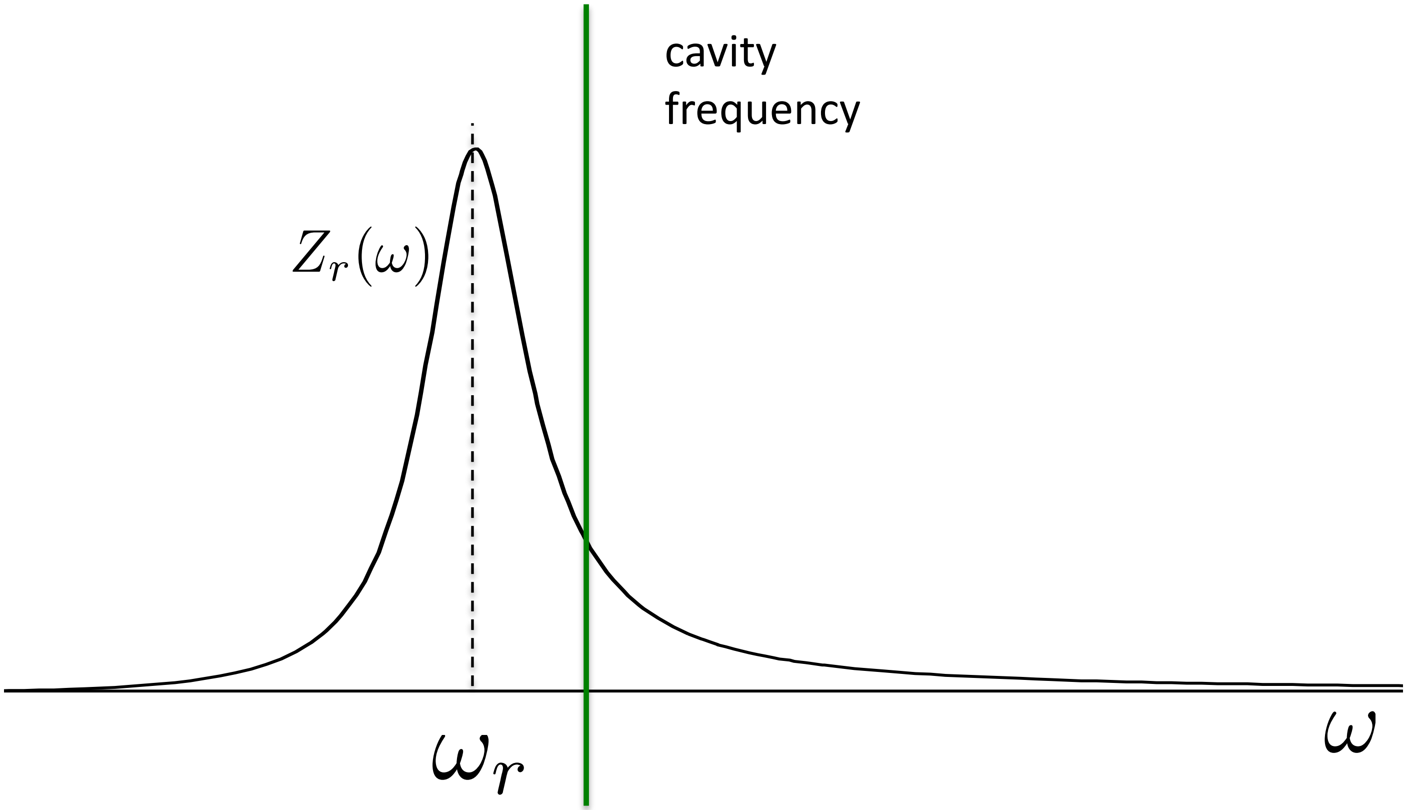
Where $V(t)$ is given by the impedance $Z_r(\omega)$

Below transition



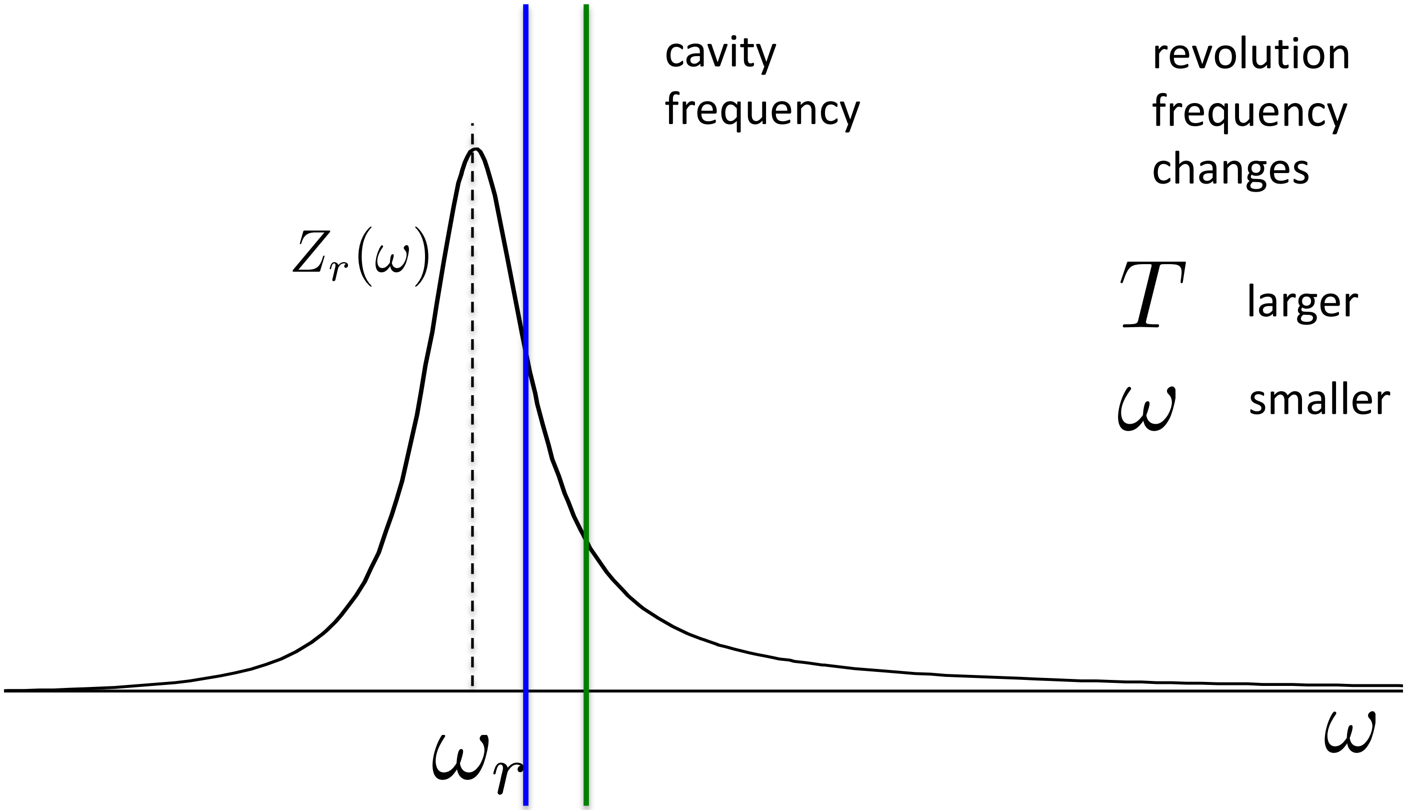
energy lost per particle for non oscillating bunch $U = \frac{2e}{I_0} \sum_1^{\infty} I_p^2 Z_r(p\omega_0)$

Below transition



In one turn energy is lost but compensated by the RF

Below transition



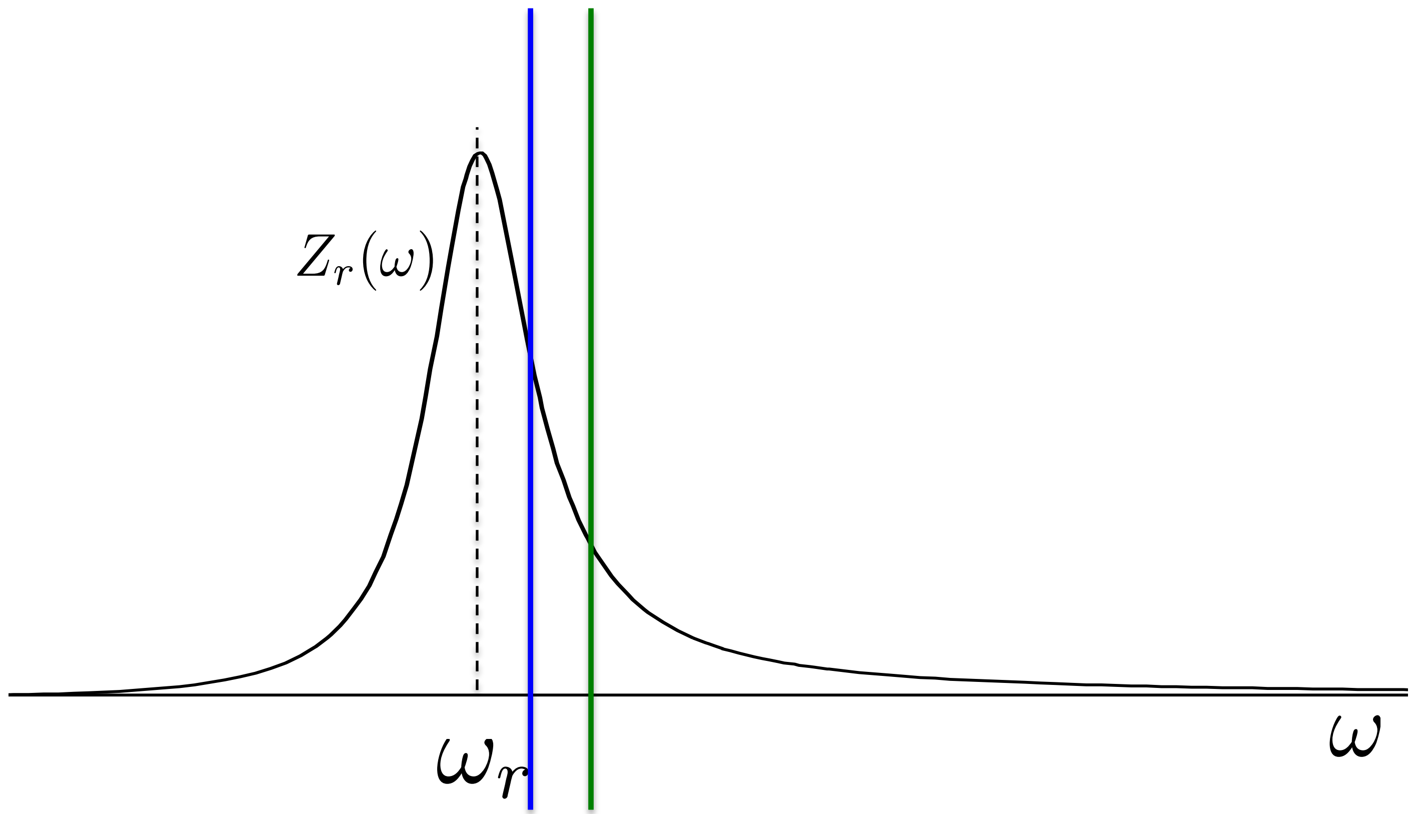
cavity frequency

revolution frequency changes

T larger

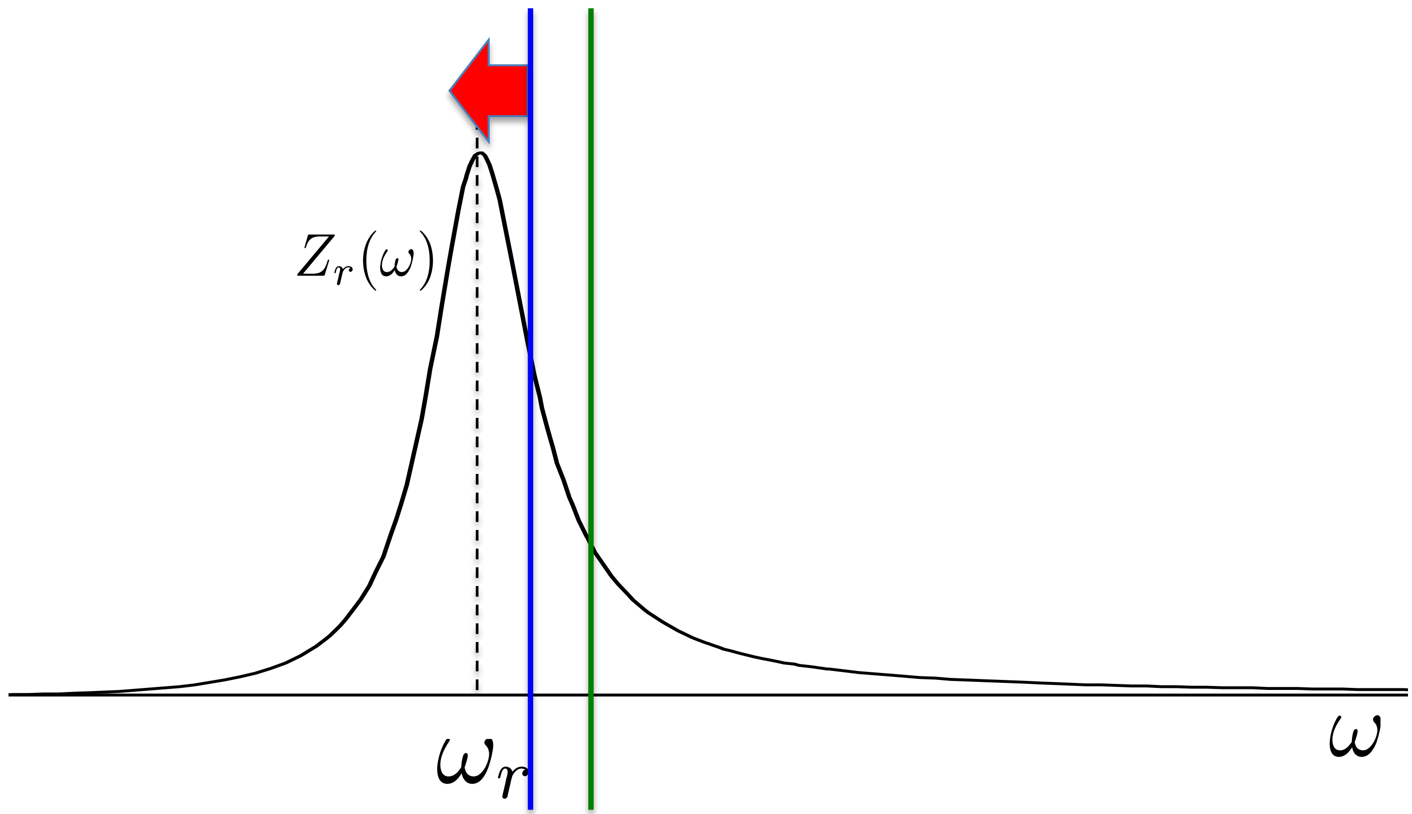
ω smaller

Below transition

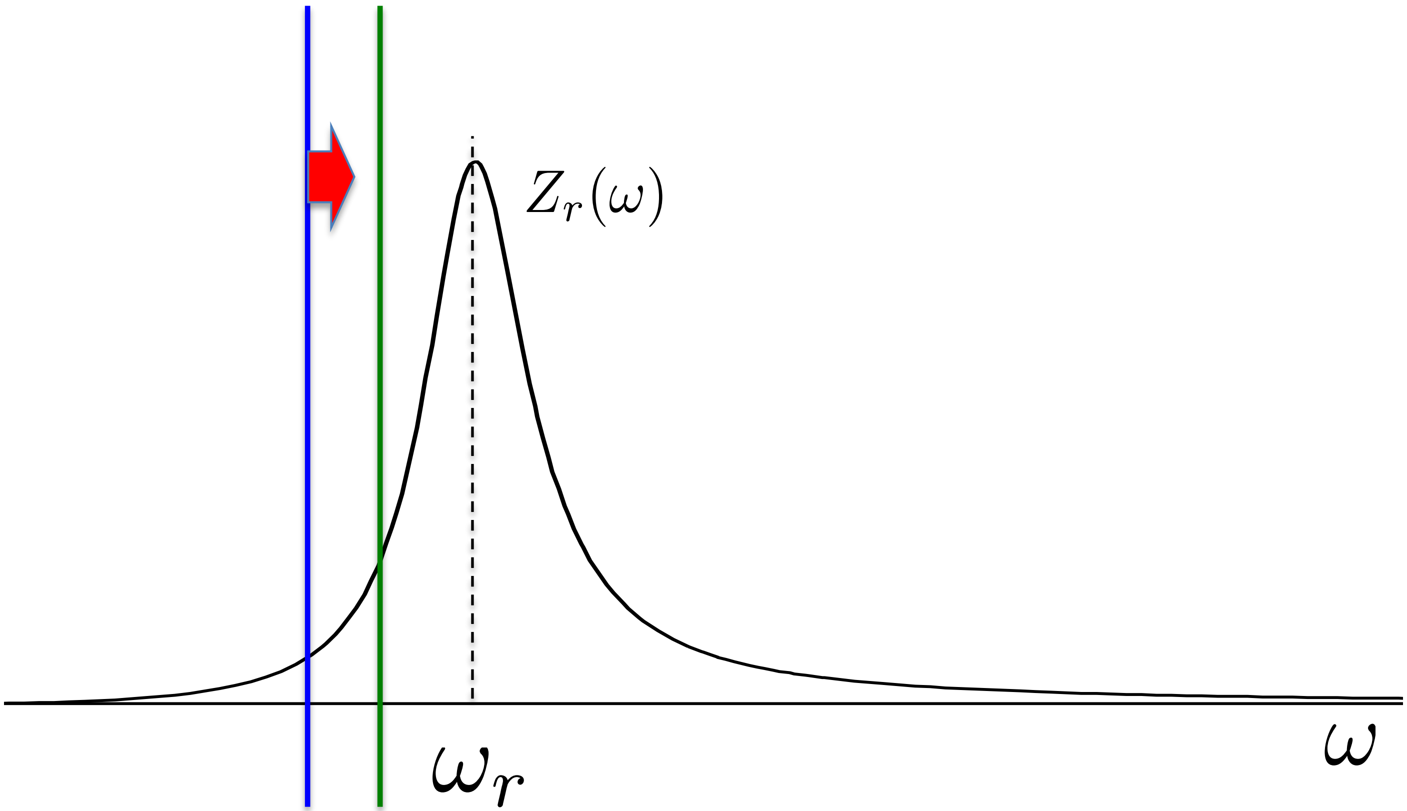


Energy lost → increase ω → increase Z_r → increase energy loss !!!

Below transition

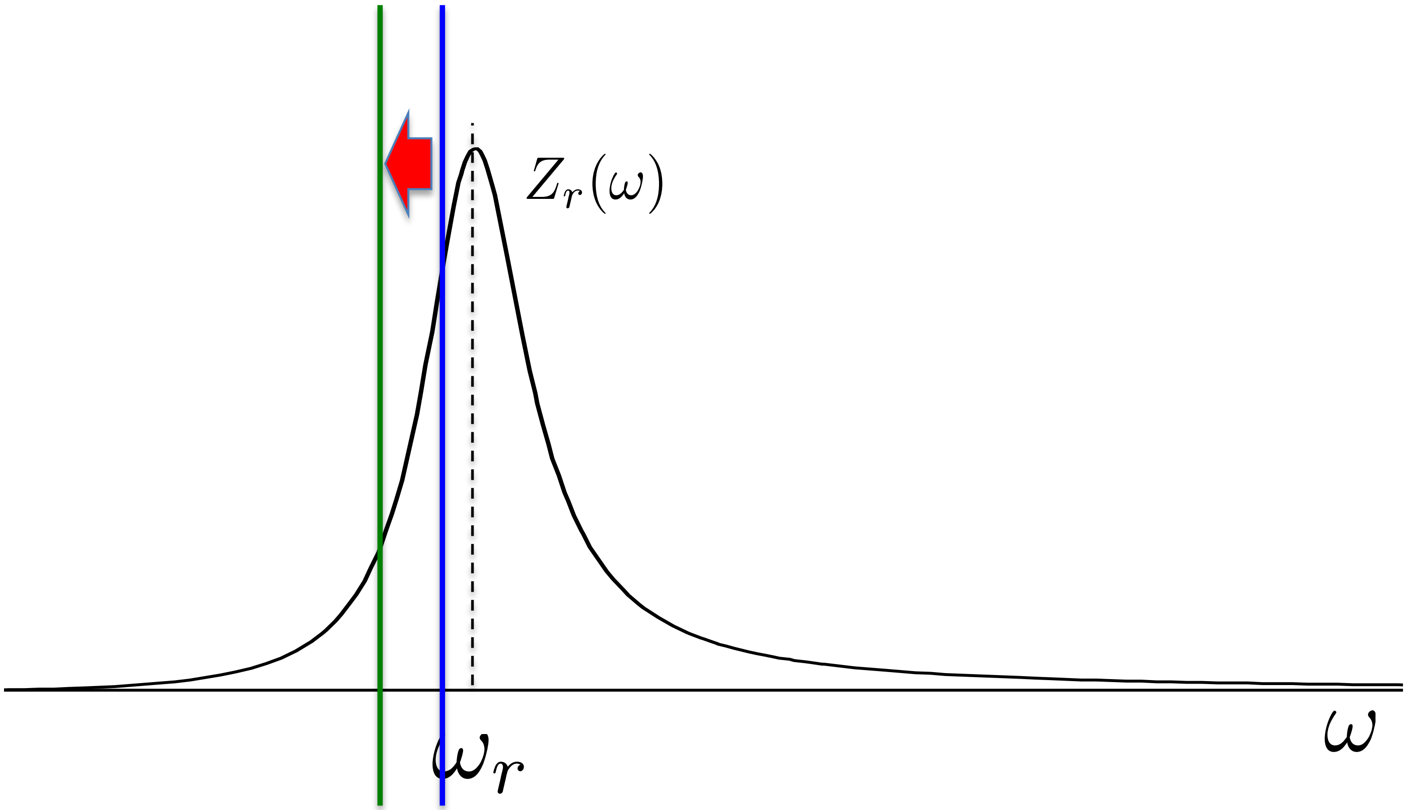


Below transition



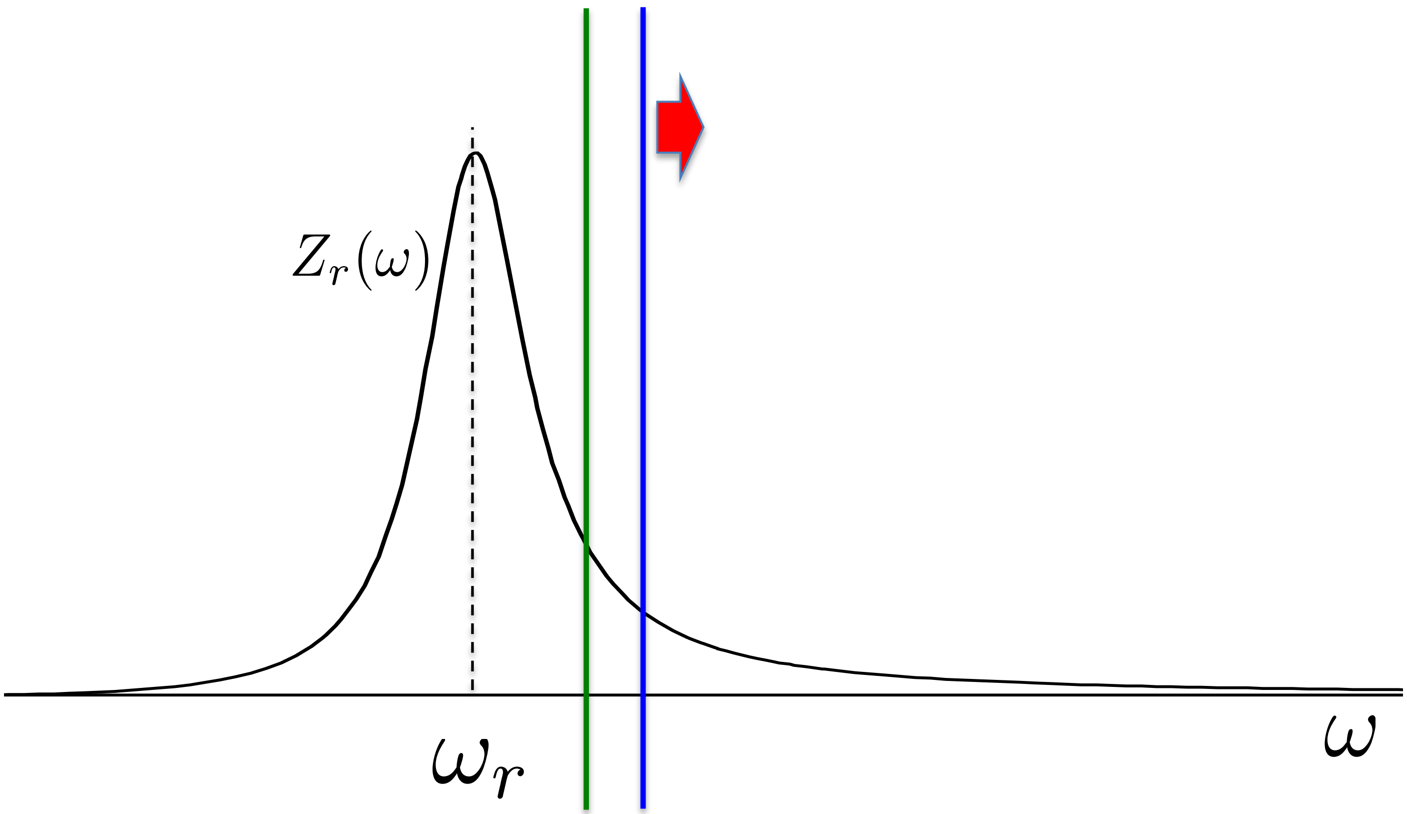
Stable

Below transition



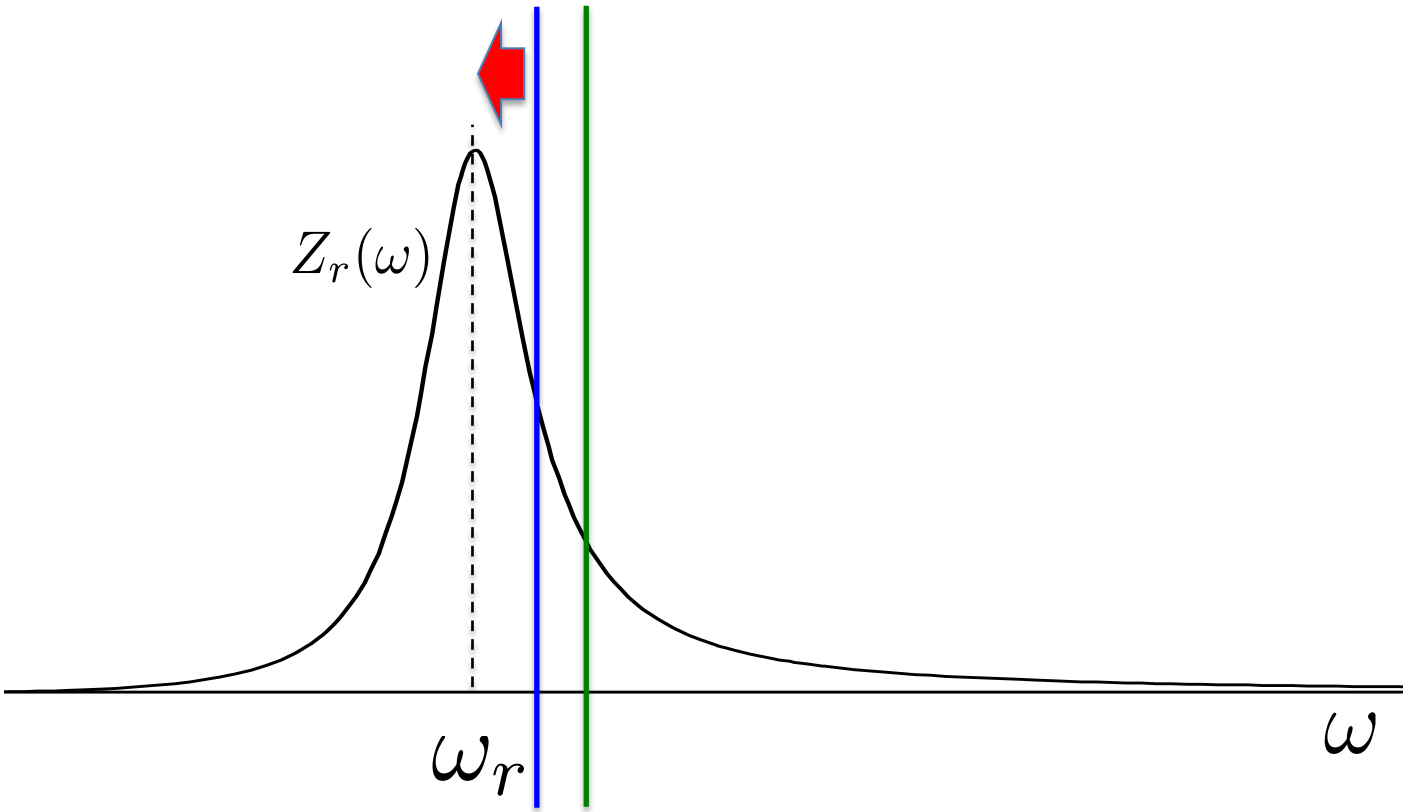
Stable

Below transition



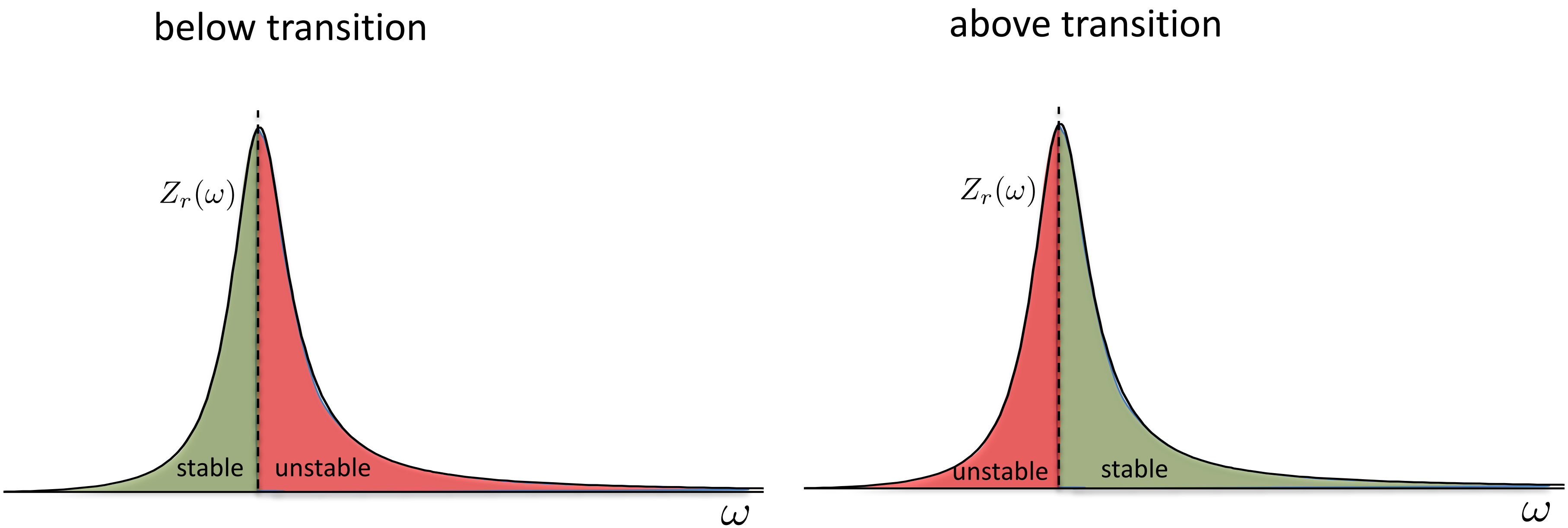
Unstable

Below transition

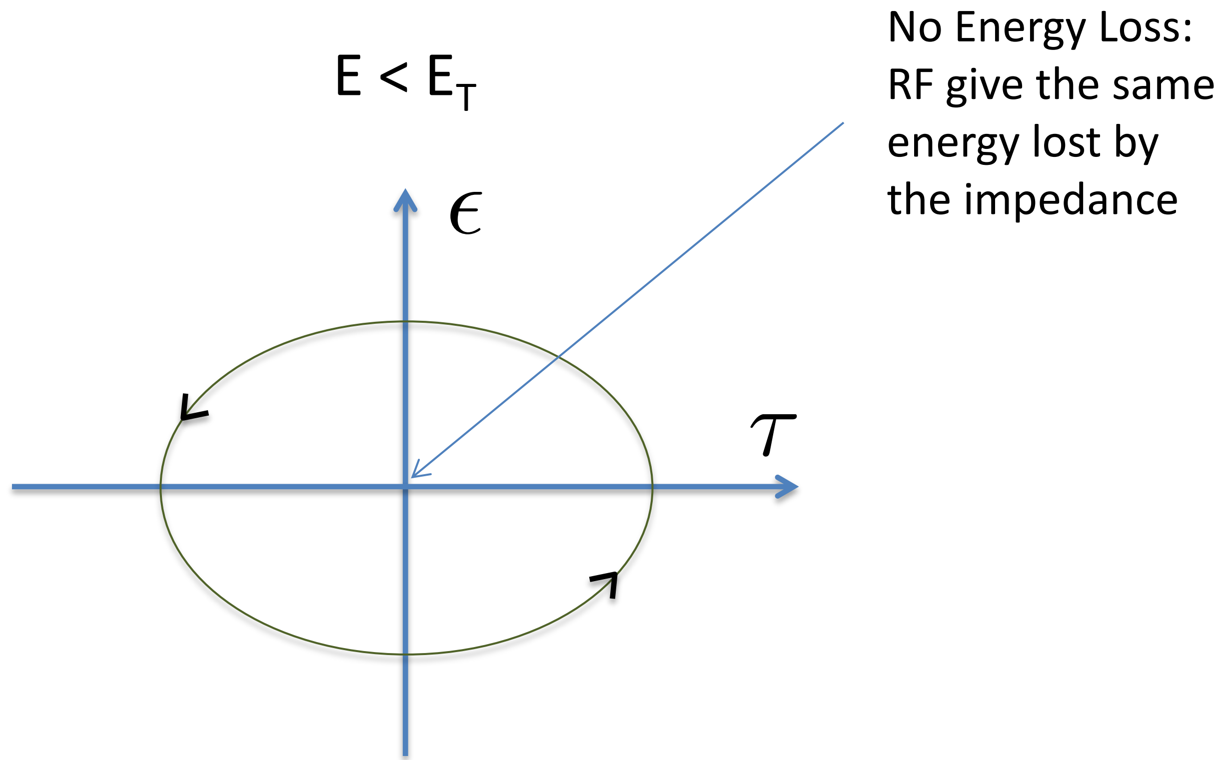


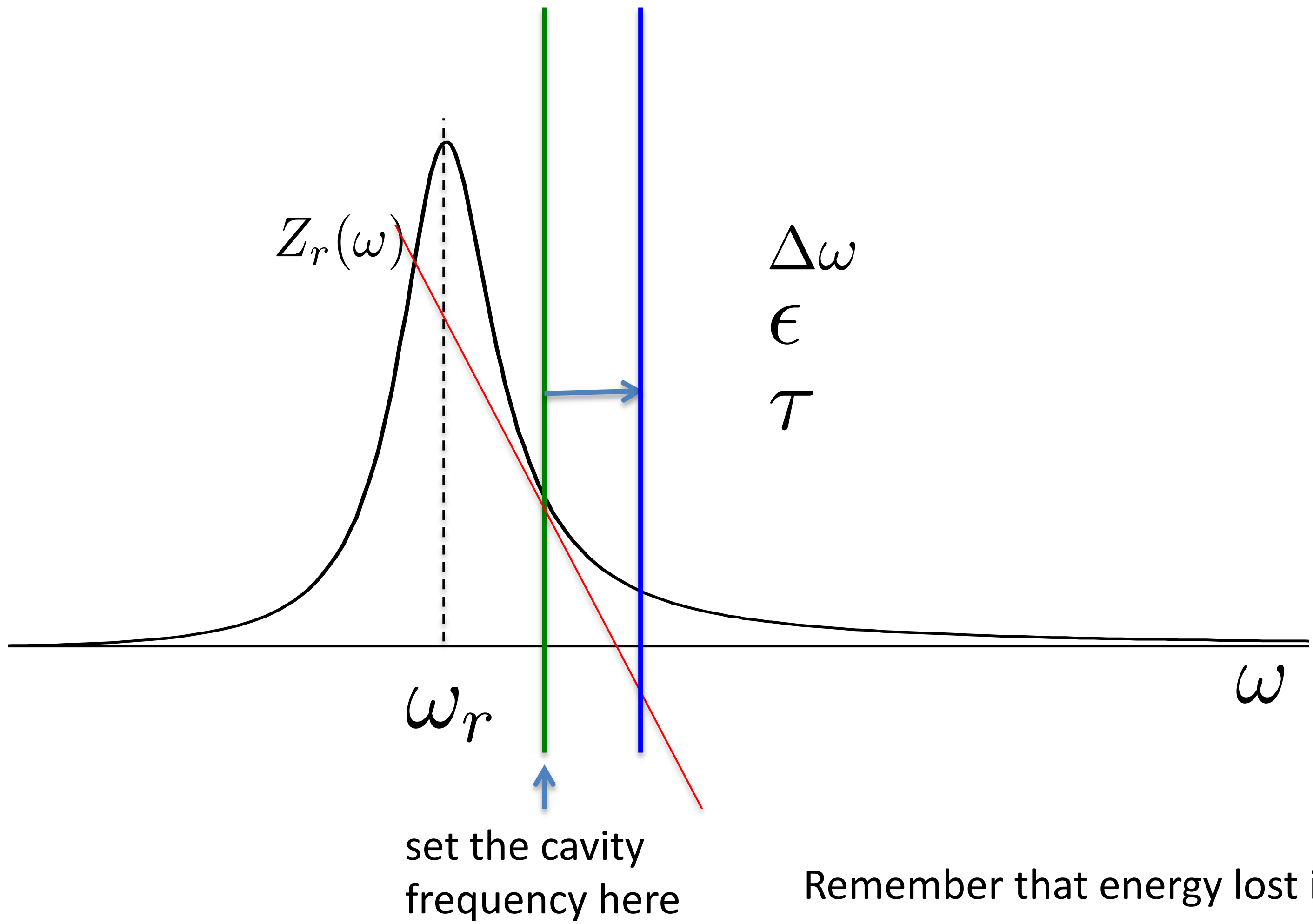
Unstable

Summary of the reasoning



More complicated



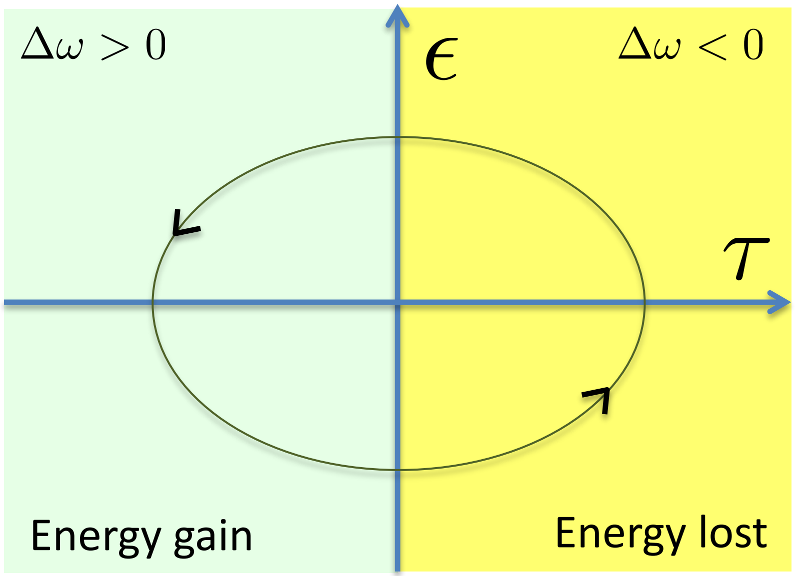


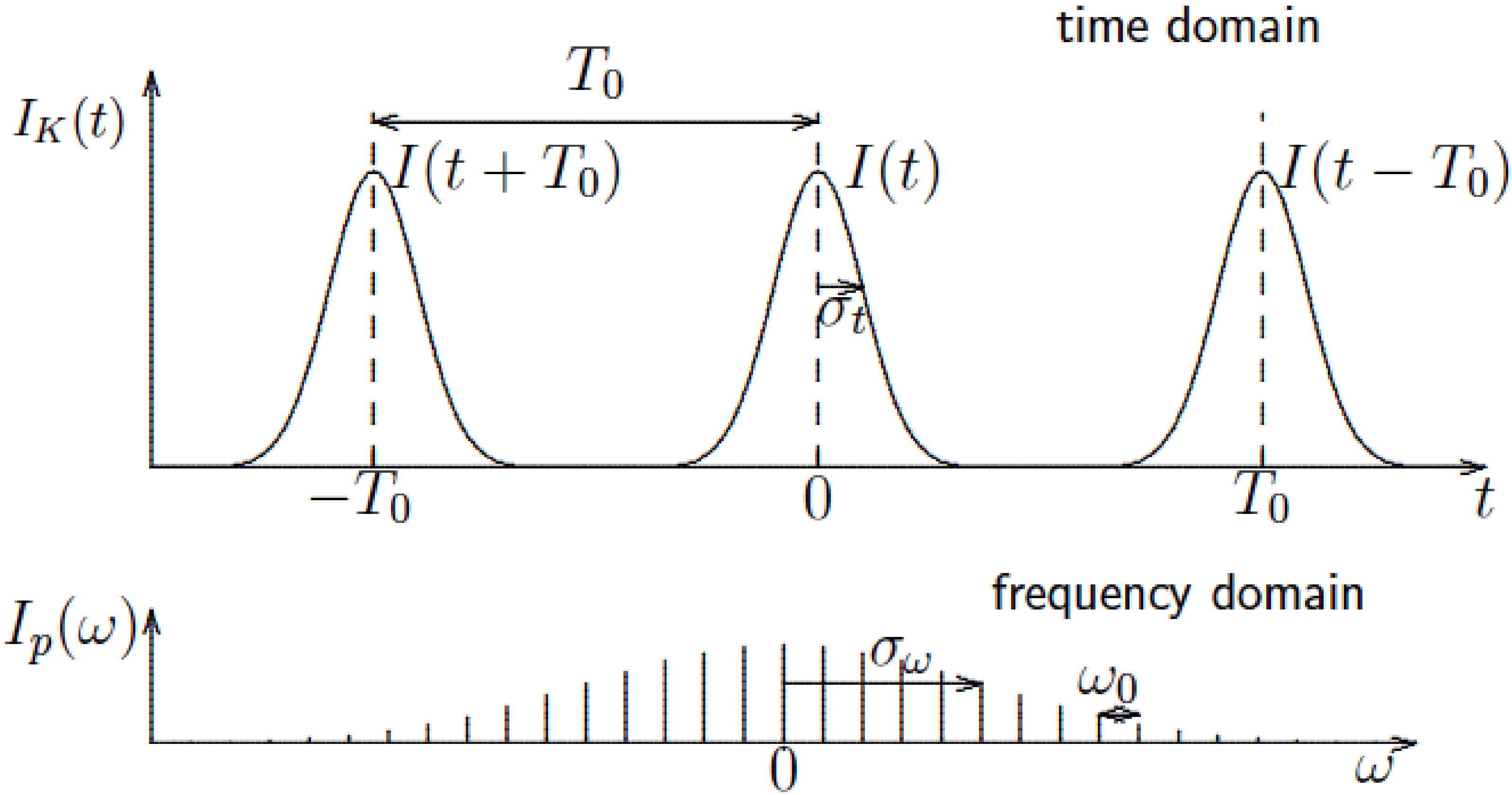
Remember that energy lost is V*I

Source of difficulty

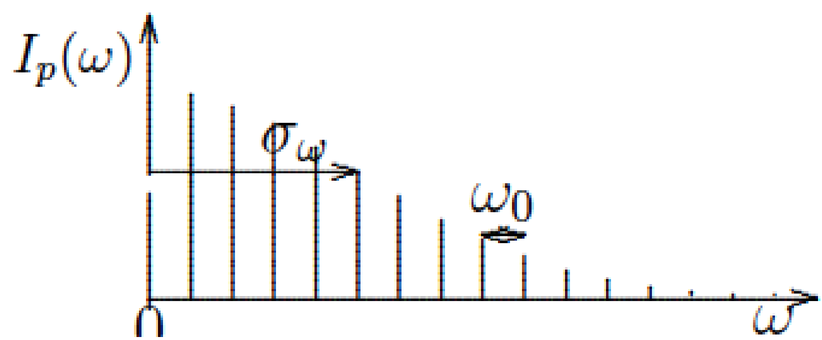
E < E_T

Impedance effect →

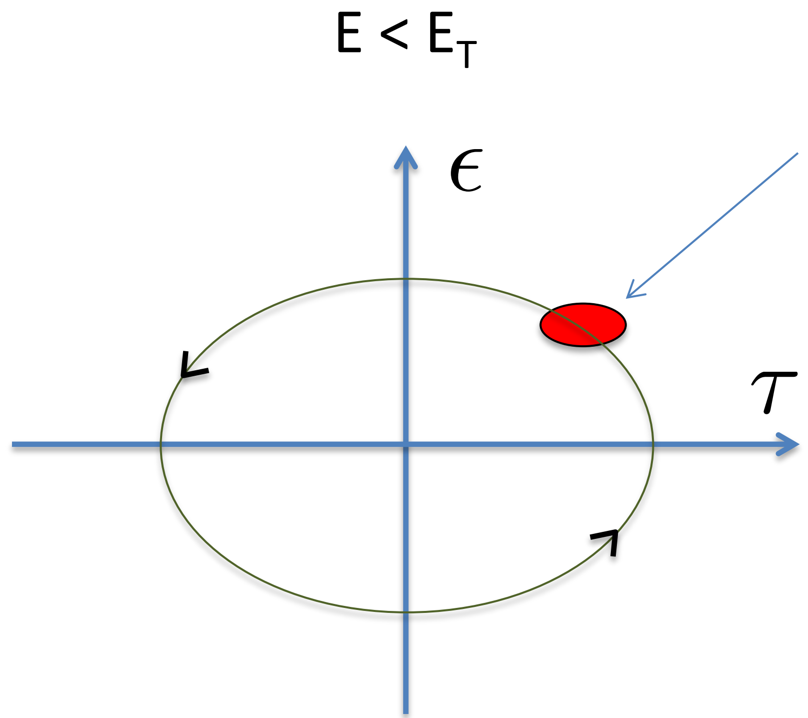




$$I_k(t) = \sum_{p=-\infty}^{\infty} I_p e^{ip\omega_0 t}$$



Still we neglect something



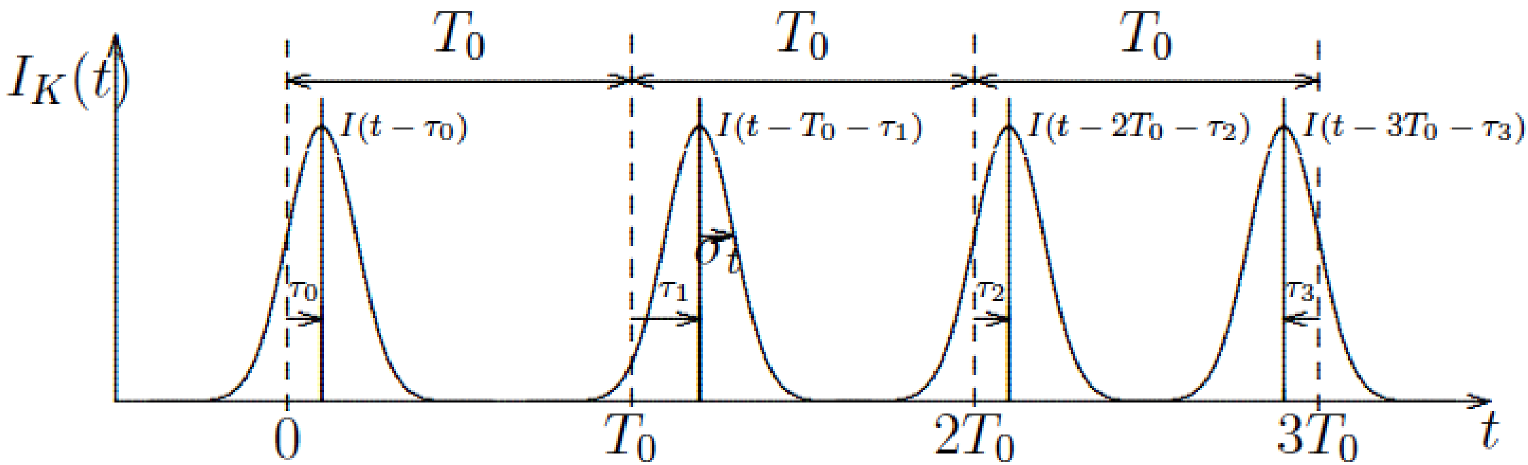
position of the bunch at turn "k"

$$\tau_k = \hat{\tau}\cos(2\pi Q_s k)$$

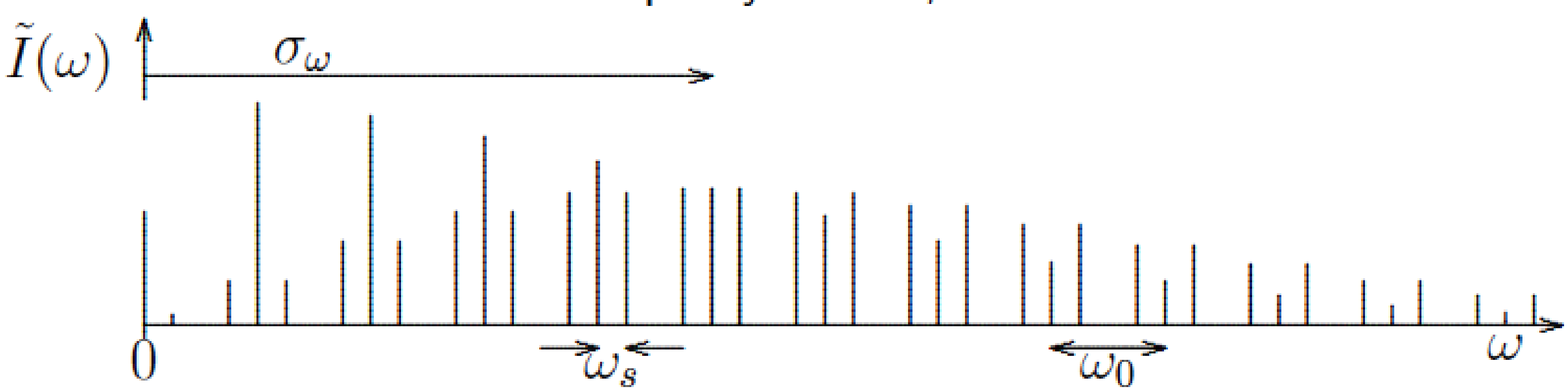
Q_s is the synchrotron tune

$$\tau_k = \hat{\tau}\cos(\omega_s t)$$

time domain



frequency domain, $\omega > 0$

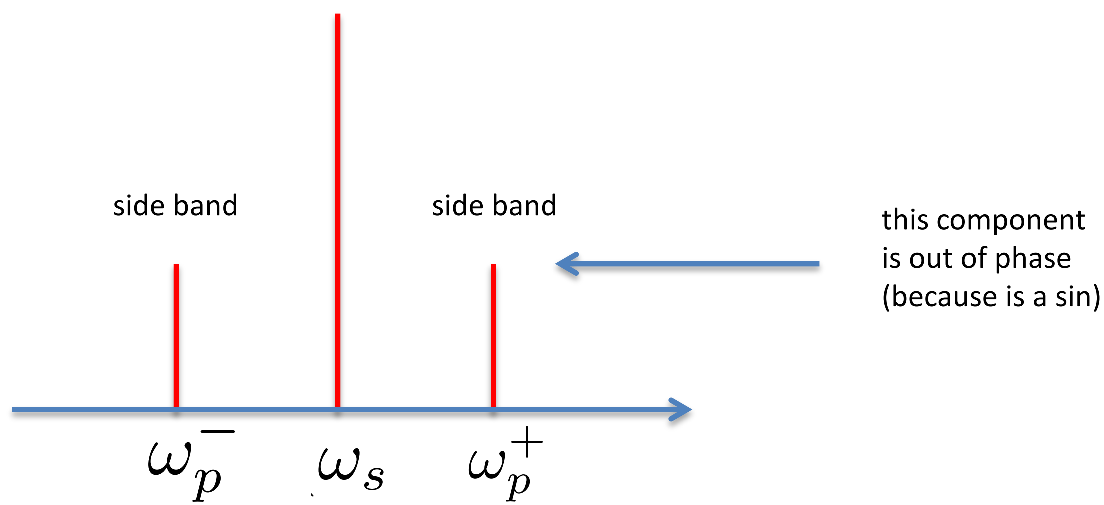


$$t \to t + \hat{\tau}\cos(\omega_s t) \quad \Rightarrow \quad I_k(t) = \sum_{p=-\infty}^{\infty} I_p e^{ip\omega_0[t+\hat{\tau}\cos(\omega_s t)]}$$

Current

$$I_k(t) \simeq \sum_{\omega>0} I_p \left[\cos(p\omega_0 t) + \frac{p\omega_0\tau}{2}\sin(\underbrace{(p+Q_s)\omega_0}_{\omega_p^+} t) + \frac{p\omega_0\tau}{2}\sin(\underbrace{(p-Q_s)\omega_0}_{\omega_p^-} t)\right]$$

The bunch current can be described by 3 components with frequency very close



That means that the energy loss due to the impedance has to be computed on the 3 currents…

Voltage created by the resistive impedance

Main component

$$V = 2\sum_{\omega>0}^{\infty} I_p Z_r(p\omega_0)\cos(p\omega_0 t)$$

1st sideband

$$V = \sum_{\omega>0}^{\infty} I_p p\omega_0 \hat{\tau} Z_r(\omega_p^+)\sin(\omega_p^+ t)$$

2nd sideband

$$V = \sum_{\omega>0}^{\infty} I_p p\omega_0 \hat{\tau} Z_r(\omega_p^-)\sin(\omega_p^- t)$$

Prosthaphaeresis formulae

$$\boxed{\begin{aligned}\sin(\omega_p^- t) &= \sin(p\omega_0 t)\cos(\omega_s t) - \cos(p\omega_0 t)\sin(\omega_s t)\\ \sin(\omega_p^+ t) &= \sin(p\omega_0 t)\cos(\omega_s t) + \cos(p\omega_0 t)\sin(\omega_s t)\end{aligned}}$$

But $\tau = \hat{\tau}\cos(\omega_s t)$ $\Rightarrow$

$$\begin{cases}\cos(\omega_s t) = \dfrac{\tau}{\hat{\tau}}\\[2ex] \sin(\omega_s t) = -\dfrac{\dot{\tau}}{\hat{\tau}\omega_s}\end{cases}$$

$$\boxed{\begin{aligned}\sin(\omega_p^+ t) &= \sin(p\omega_0 t)\frac{\tau}{\hat{\tau}} - \cos(p\omega_0 t)\frac{\dot{\tau}}{\hat{\tau}\omega_s}\\ \sin(\omega_p^- t) &= \sin(p\omega_0 t)\frac{\tau}{\hat{\tau}} + \cos(p\omega_0 t)\frac{\dot{\tau}}{\hat{\tau}\omega_s}\end{aligned}}$$

Voltage created by the resistive impedance

Main component
$$V = 2\sum_{\omega>0}^{\infty} I_p Z_r(p\omega_0)\cos(p\omega_0 t)$$

1st sideband
$$V = \sum_{\omega>0}^{\infty} I_p p\omega_0 Z_r(\omega_p^+)[\sin(p\omega_0 t)\tau - \cos(p\omega_0 t)\frac{\dot\tau}{\omega_s}]$$

2nd sideband
$$V = \sum_{\omega>0}^{\infty} I_p p\omega_0 Z_r(\omega_p^-)[\sin(p\omega_0 t)\tau + \cos(p\omega_0 t)\frac{\dot\tau}{\omega_s}]$$

Therefore the induced Voltage depends on $\tau, \dot\tau$

Energy lost in one turn

$$E_l = \int_0^{T_0} V(t) I(t) dt$$

energy lost
per particle
per turn

$$U = \frac{2e}{I_0}\left[I_p^2 Z_r(p\omega_0) - \frac{I_p^2 p\omega_0}{2}\left(Z_r(\omega_p^+) - Z_r(\omega_p^-)\right)\frac{\dot{\tau}}{\omega_s}\right]$$

this term can give rise to
a constant loss, or a constant
gain of energy

In terms of the energy of a particle

$$U = \frac{2e}{I_0}\left[I_p^2 Z_r(p\omega_0) - \frac{I_p^2 p\omega_0}{2}(Z_r(\omega_p^+) - Z_r(\omega_p^-))\frac{\eta\epsilon}{\omega_s}\right]$$

$$\frac{\partial U}{\partial \epsilon} = -\frac{e}{I_0}\sum_{\omega>0}\frac{I_p^2 p\omega_0}{2}(Z_r(\omega_p^+) - Z_r(\omega_p^-))\frac{\eta}{\omega_s}$$

This is a slope in the energy, and the sign of the slope depends on

$Z_r(\omega_p^+) - Z_r(\omega_p^-)$ and η

The longitudinal motion now!

$$\ddot{\tau} + 2\alpha_s\dot{\tau} + \omega_{s0}^2\tau = 0$$

$$\alpha_S = \frac{1}{2}\frac{\omega_0}{2\pi}\frac{\partial U}{\partial E} = \frac{\omega_0}{4\pi E}\frac{\partial U}{\partial \epsilon} = \frac{\omega_s \sum p I_p^2 (Z_r(\omega_p^+) - Z_r(\omega_p^-))}{2 I_0 h \hat{V} \cos\phi_s}$$

Robinson Instability

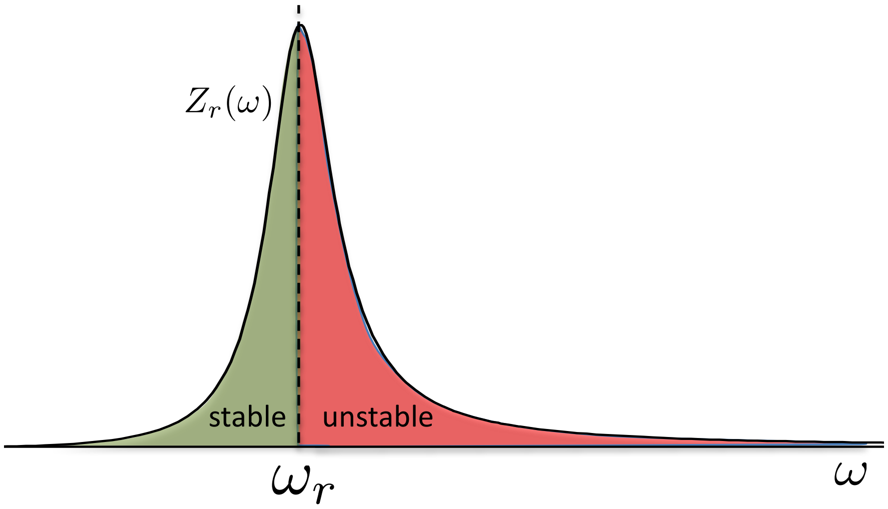
If $\alpha_S > 0$ there is a damping

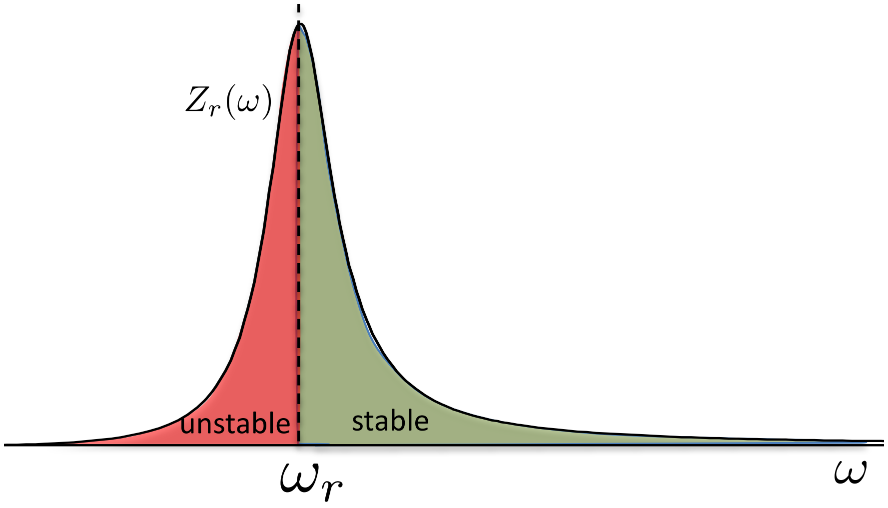
If $\alpha_s < 0$ there is an instability

Robinson Instability

below transition

above transition



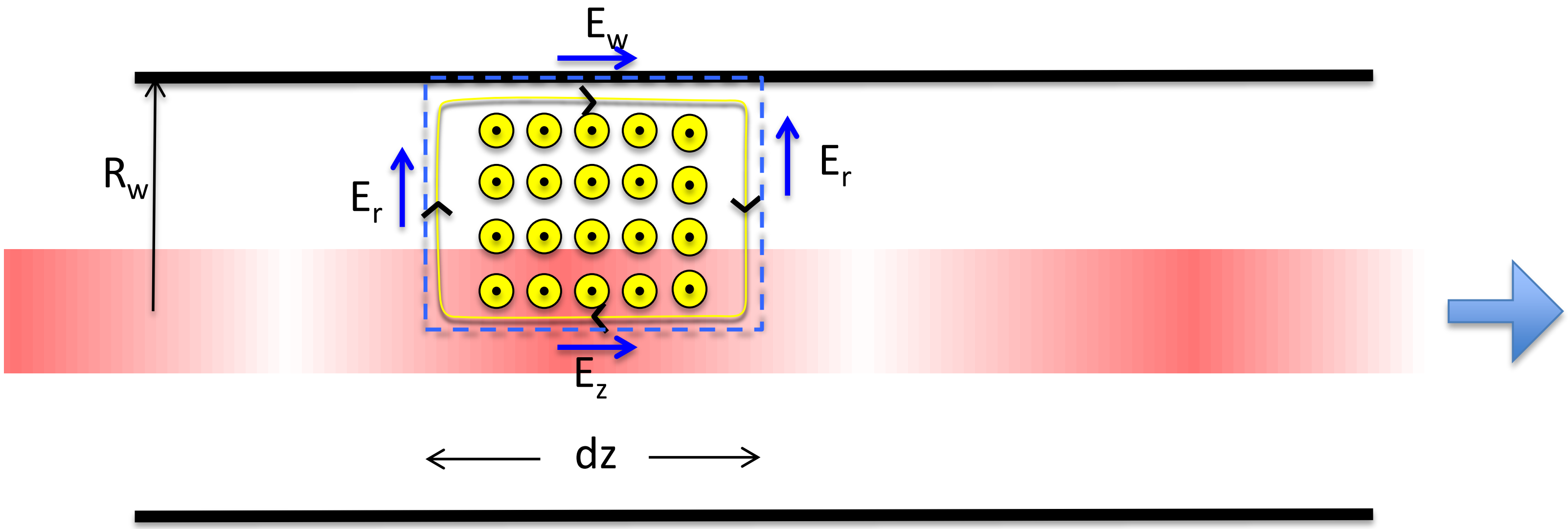


Longitudinal space charge and resistive wall impedance

Space charge longitudinal field

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0}$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int \vec{B} d\vec{a}$$



$$\oint \vec{E} \cdot d\vec{l} = \int E_r(z) dr + E_w \Delta z - \int E_r(z + \Delta z) dr - E_z \Delta z$$

For a KV beam

Electric Field

$$E_r = \begin{cases} \dfrac{\lambda(z)}{2\epsilon_0} r & \text{if} \quad r < r_0 \\ \dfrac{\lambda(z) r_0^2}{2\epsilon_0} \dfrac{1}{r} & \text{if} \quad r > r_0 \end{cases}$$

$$\int_0^{r_w} E_r(z) dr = \int_0^{r_0} \frac{\lambda(z)}{2\epsilon_0} r dr + \int_{r_0}^{r_w} \frac{\lambda(z) r_0^2}{2\epsilon_0} \frac{1}{r} dr$$

$$\Downarrow$$

$$\int_0^{r_w} E_r(z) dr = \frac{\lambda(z) r_0^2}{4\epsilon_0} \left[1 + 2 \ln\left(\frac{r_w}{r_0}\right) \right]$$

Therefore

$$\int E_r(z)dr - \int E_r(z+\Delta z)dr = -\frac{r_0^2}{4\epsilon_0}\left[1 + 2\ln\left(\frac{r_w}{r_0}\right)\right]\frac{\partial\lambda(z)}{\partial z}\Delta z$$

$$\Downarrow$$

$$\oint \vec{E}\cdot d\vec{l} = (E_w - E_z)\Delta z - \frac{r_0^2}{4\epsilon_0}\left[1 + 2\ln\left(\frac{r_w}{r_0}\right)\right]\frac{\partial\lambda(z)}{\partial z}\Delta z$$

Magnetic Field

$$B_\perp = \begin{cases} \dfrac{\mu_0 v_z \lambda(z)}{2} r & \text{if} \quad r < r_0 \\ \dfrac{\mu_0 v_z \lambda(z) r_0^2}{2} \dfrac{1}{r} & \text{if} \quad r > r_0 \end{cases}$$

$$\int B_\perp da = \int_0^{r_0} dr \int_z^{z+\Delta z} \frac{\mu_0 v_z \lambda(z)}{2} r + \int_{r_0}^{r_w} dr \int_z^{z+\Delta z} \frac{\mu_0 v_z \lambda(z) r_0^2}{2} \frac{1}{r}$$

$$\int B_\perp da = \frac{\mu_0 v_z r_0^2 \lambda \Delta z}{4} \left[1 + 2 \ln \left(\frac{r_w}{r_0} \right) \right]$$

Maxwell-Faraday Law

$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int \vec{B} d\vec{a}$$

$$(E_w - E_z)\Delta z - \frac{r_0^2}{4\epsilon_0}\left[1 + 2\ln\left(\frac{r_w}{r_0}\right)\right]\frac{\partial \lambda(z)}{\partial z}\Delta z = +\frac{\mu_0 v_z r_0^2 \Delta z}{4}\left[1 + 2\ln\left(\frac{r_w}{r_0}\right)\right]\frac{\partial \lambda}{\partial t}$$

from the equation of continuity $\frac{\partial \lambda}{\partial t} + v_z \frac{\partial \lambda}{\partial z} = 0$

$$E_z = E_w - \frac{r_0^2}{4\epsilon_0}\left[1 + 2\ln\left(\frac{r_w}{r_0}\right)\right]\frac{1}{\gamma^2}\frac{\partial \lambda}{\partial z}$$

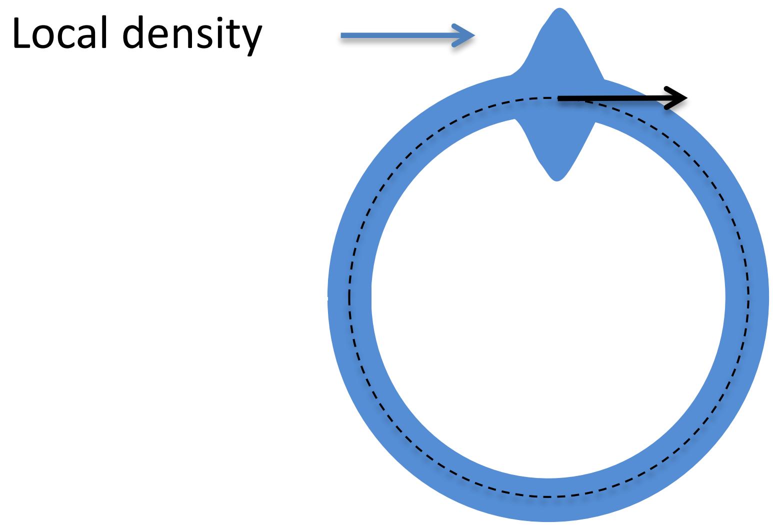
again we find the factor $1/\gamma^2$!

Space charge impedance

$$\lambda(\theta,t) = \sum_n \lambda_n e^{i(n\theta - \omega_n t)}$$

$$\theta = 2\pi \frac{z}{L}$$

$$\omega_n = n\omega_0$$



$$V_{z0} = 2\pi R E_{zw} - i \sum_n \frac{I_n}{4\pi\epsilon_0} \frac{2\pi n}{\beta c \gamma^2} \left[1 + 2 \ln\left(\frac{r_w}{r_0}\right) \right] e^{i(n\theta - \omega_n t)}$$

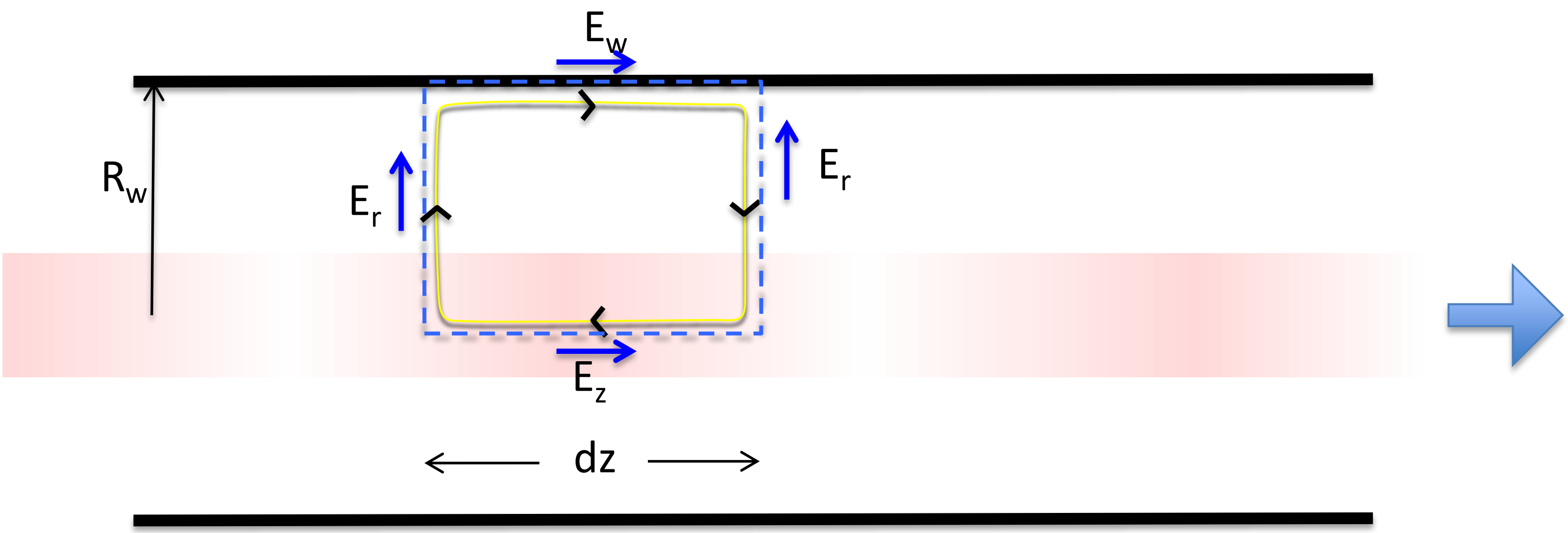
Perfect vacuum chamber $E_{zw} = 0$

$$I = I_n e^{i(n\theta - \omega_n t)} \quad \Rightarrow \quad V = -i \frac{I_n}{4\pi\epsilon_0} \frac{2\pi n}{\beta c \gamma^2} \left[1 + 2 \ln\left(\frac{r_w}{r_0}\right) \right] e^{i(n\theta - \omega_n t)}$$

$$Z_{||sc} = \frac{\hat{V}}{\hat{I}} \quad \Rightarrow \quad \boxed{Z_{||sc} = -i \frac{1}{\epsilon_0 c} \frac{n}{2\beta\gamma^2} \left[1 + 2 \ln\left(\frac{r_w}{r_0}\right) \right]}$$

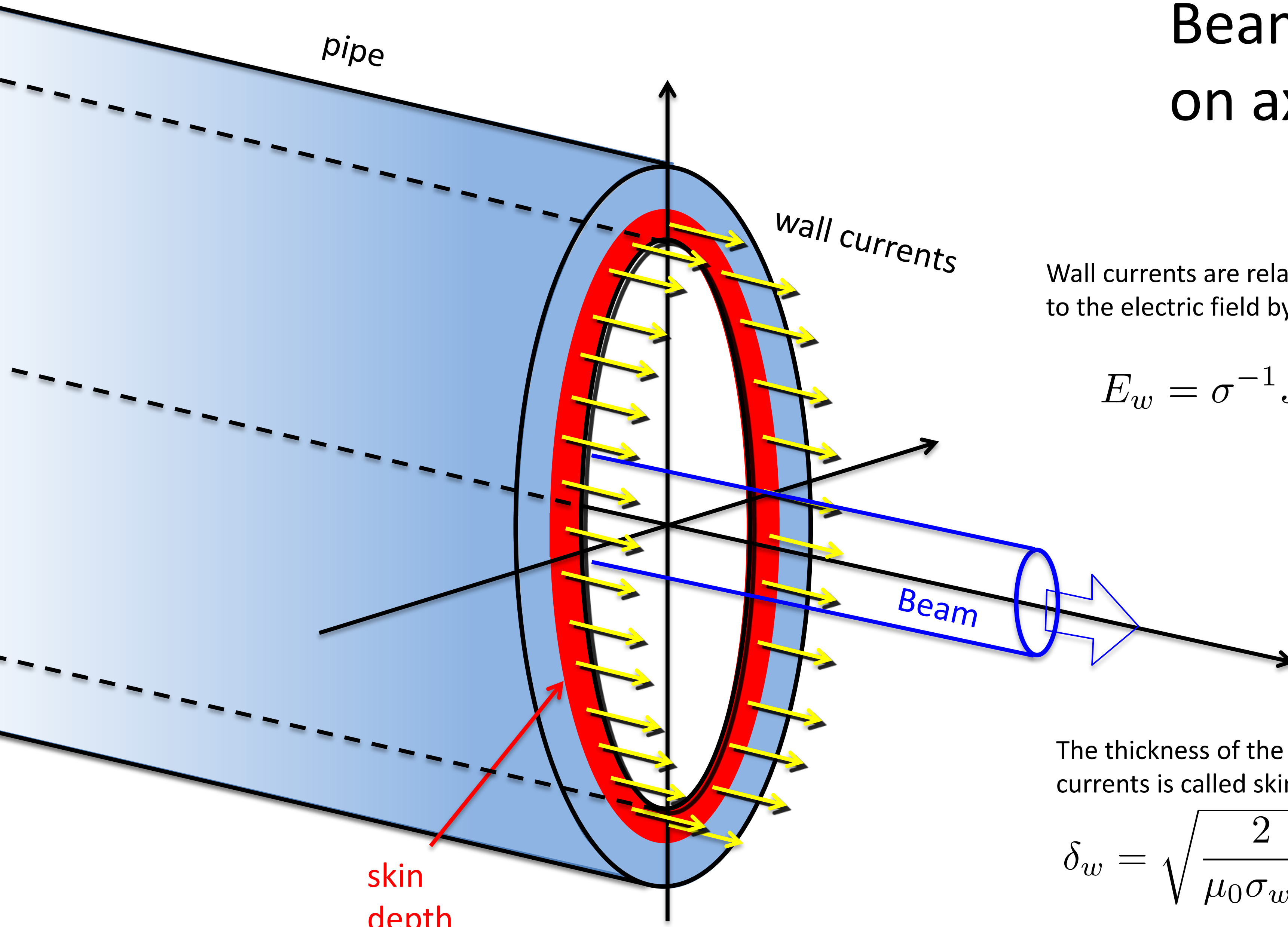
Resistive Wall impedance

Do not take into account B



$E_w = E_z$

Beam on axis



Wall currents are related to the electric field by Ohm' law

$$E_w = \sigma^{-1} J_w$$

The thickness of the wall currents is called skin depth

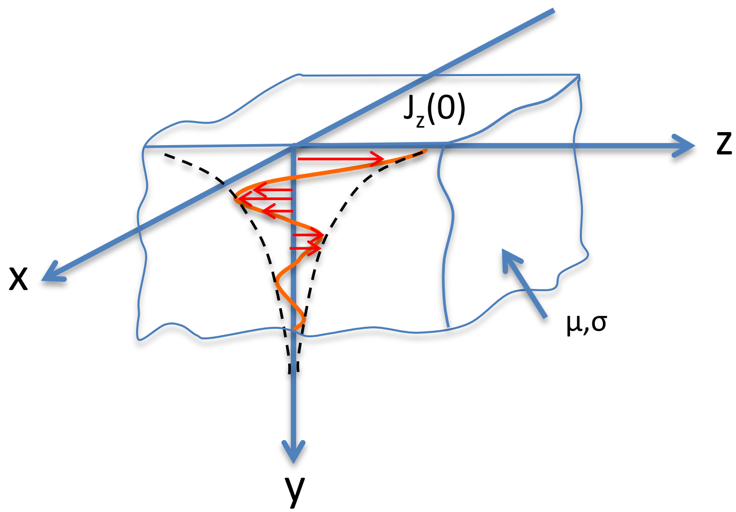
$$\delta_w = \sqrt{\frac{2}{\mu_0 \sigma_w \omega}}$$

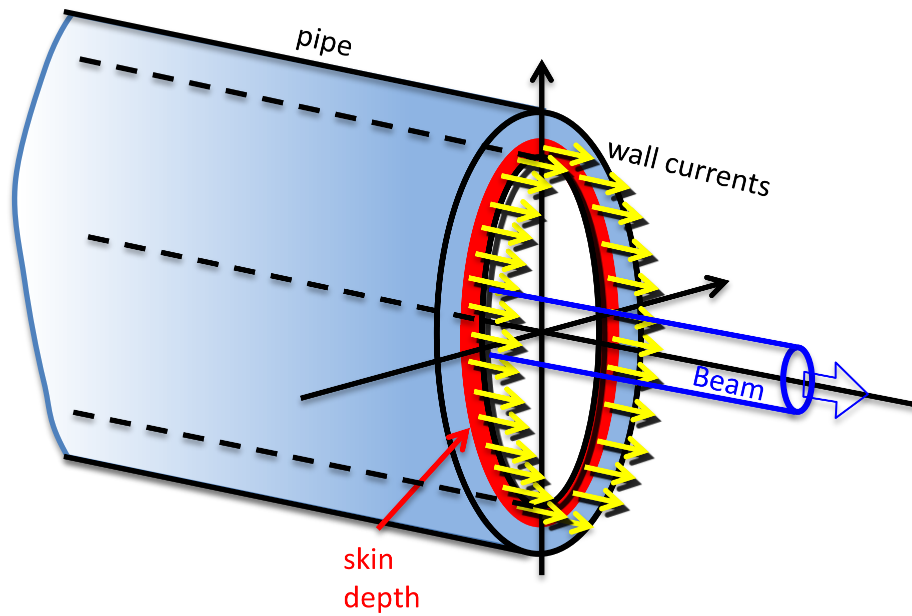
Impedance of the surface (pipe)

$$Z_{surf} = \frac{1+i}{\sigma\delta_w}$$

Longitudinal impedance (beam)

$$\boxed{Z_{||} = \frac{2\pi R}{2\pi r_p}\frac{1+i}{\sigma\delta_w}}$$

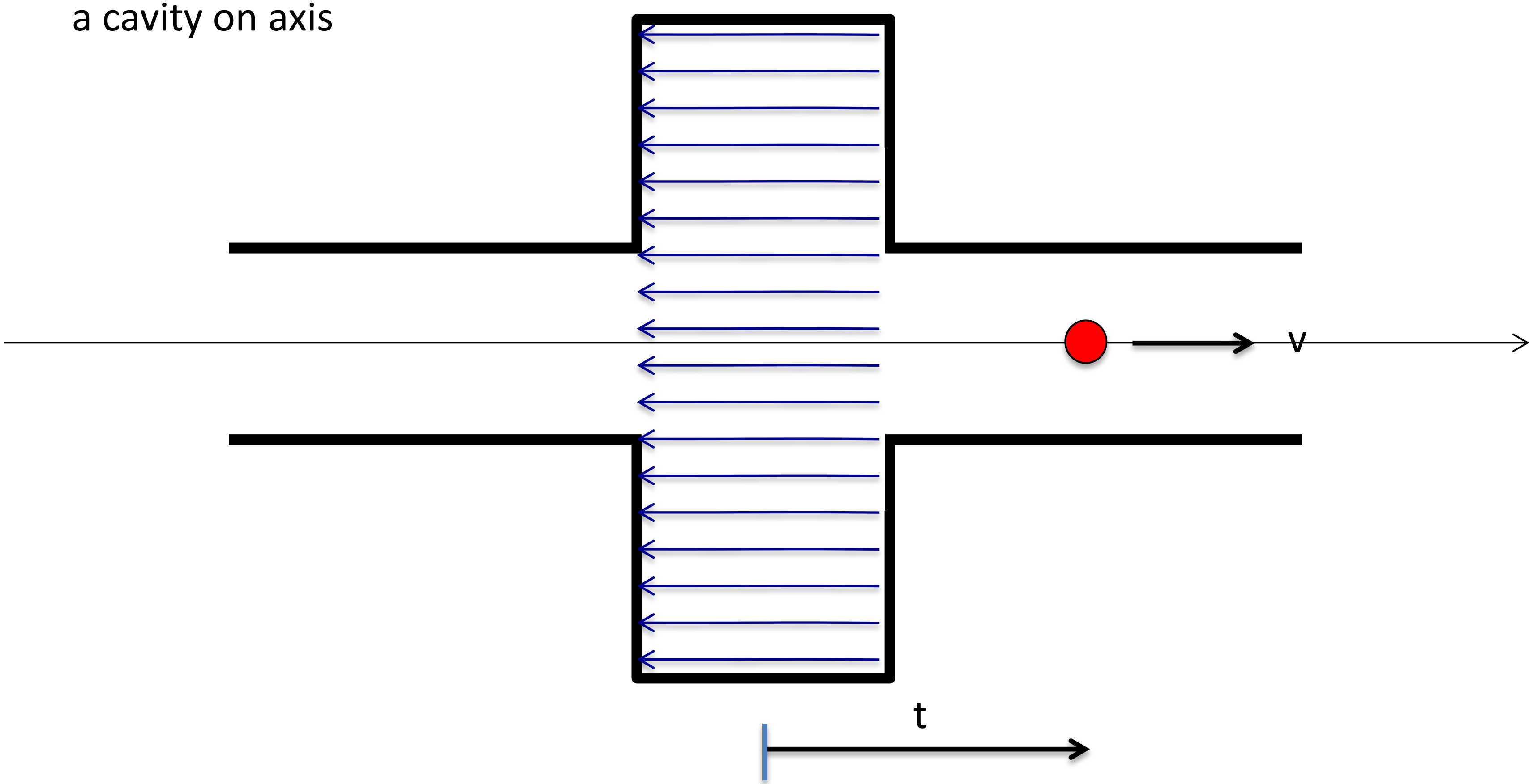




Transverse impedance

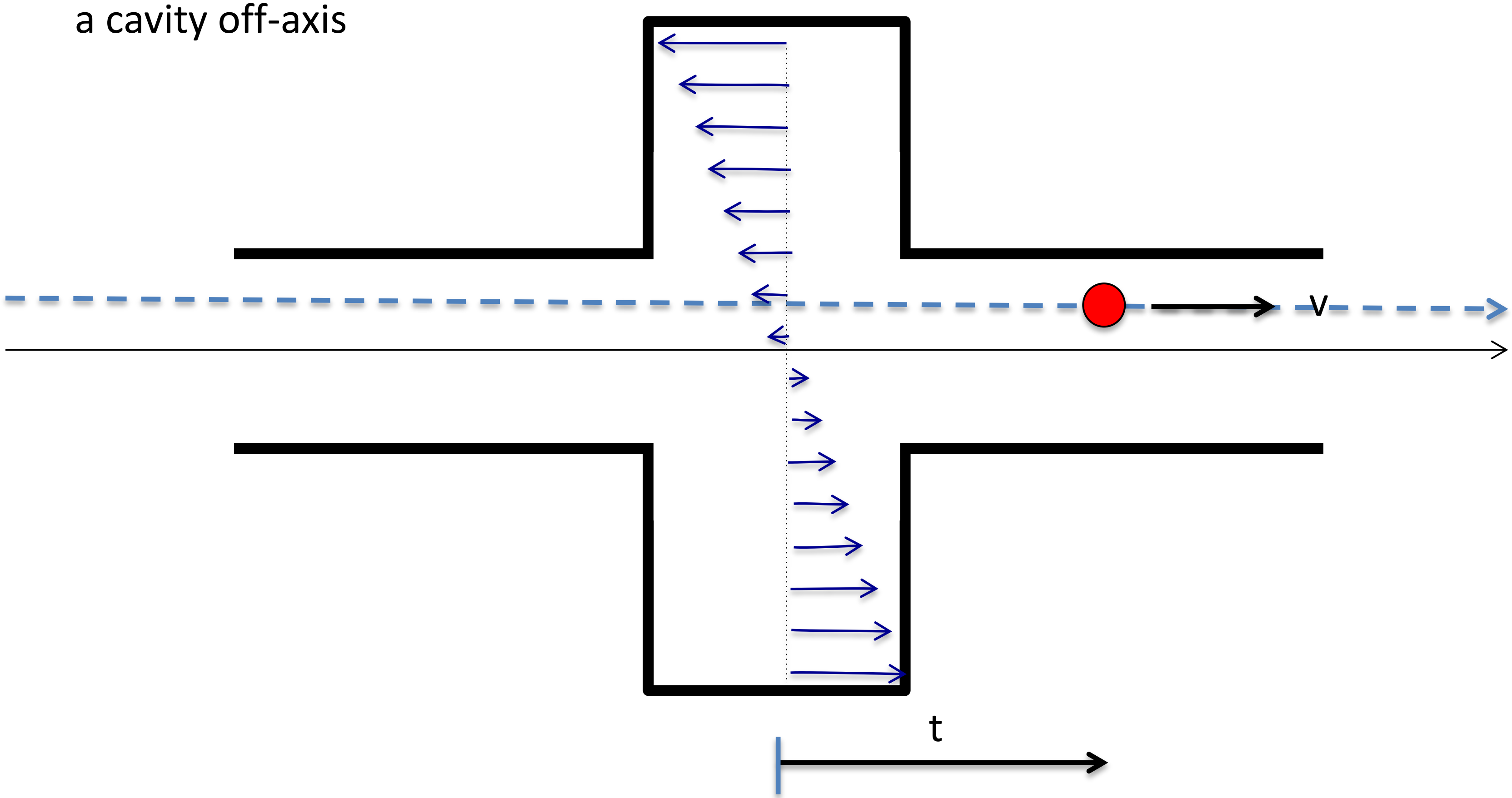
Origin

Beam passing through
a cavity on axis

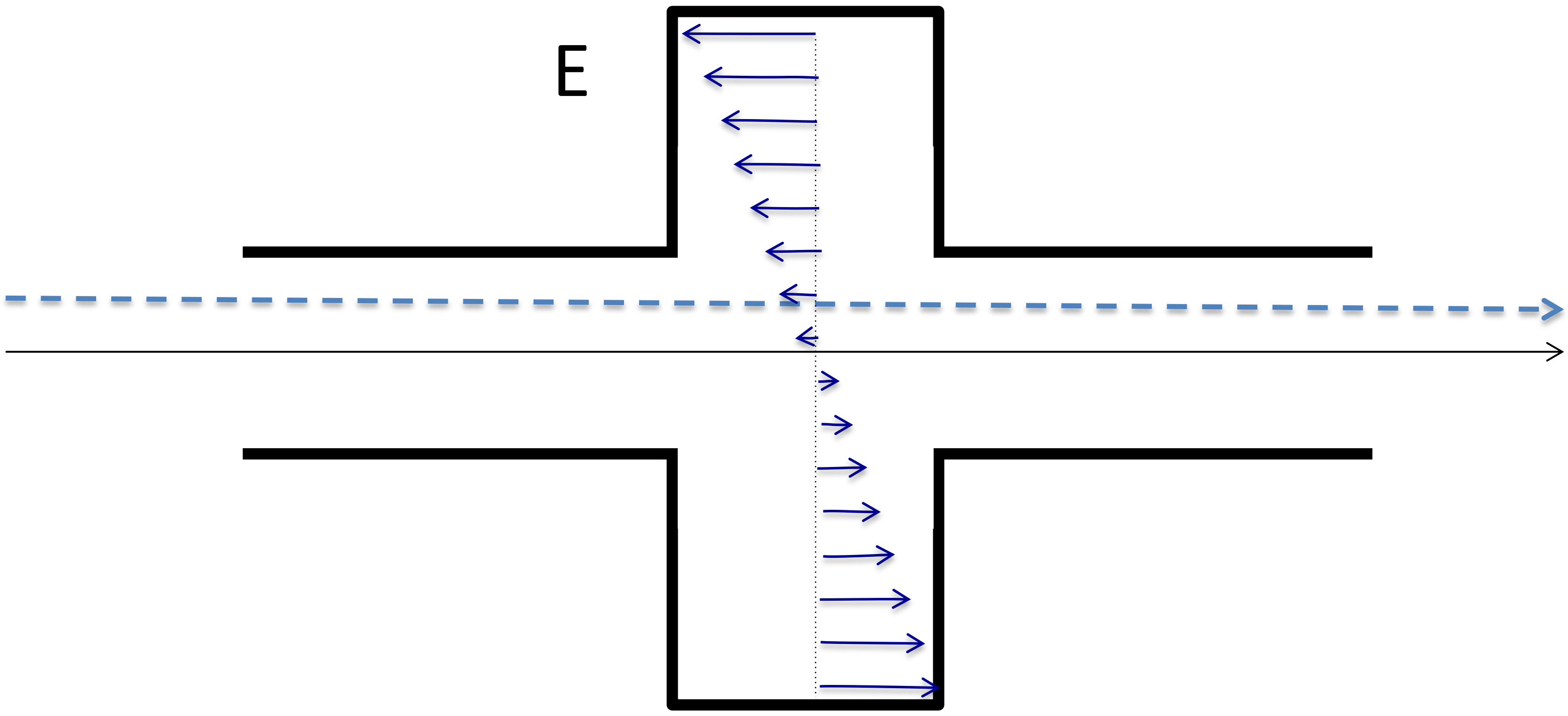


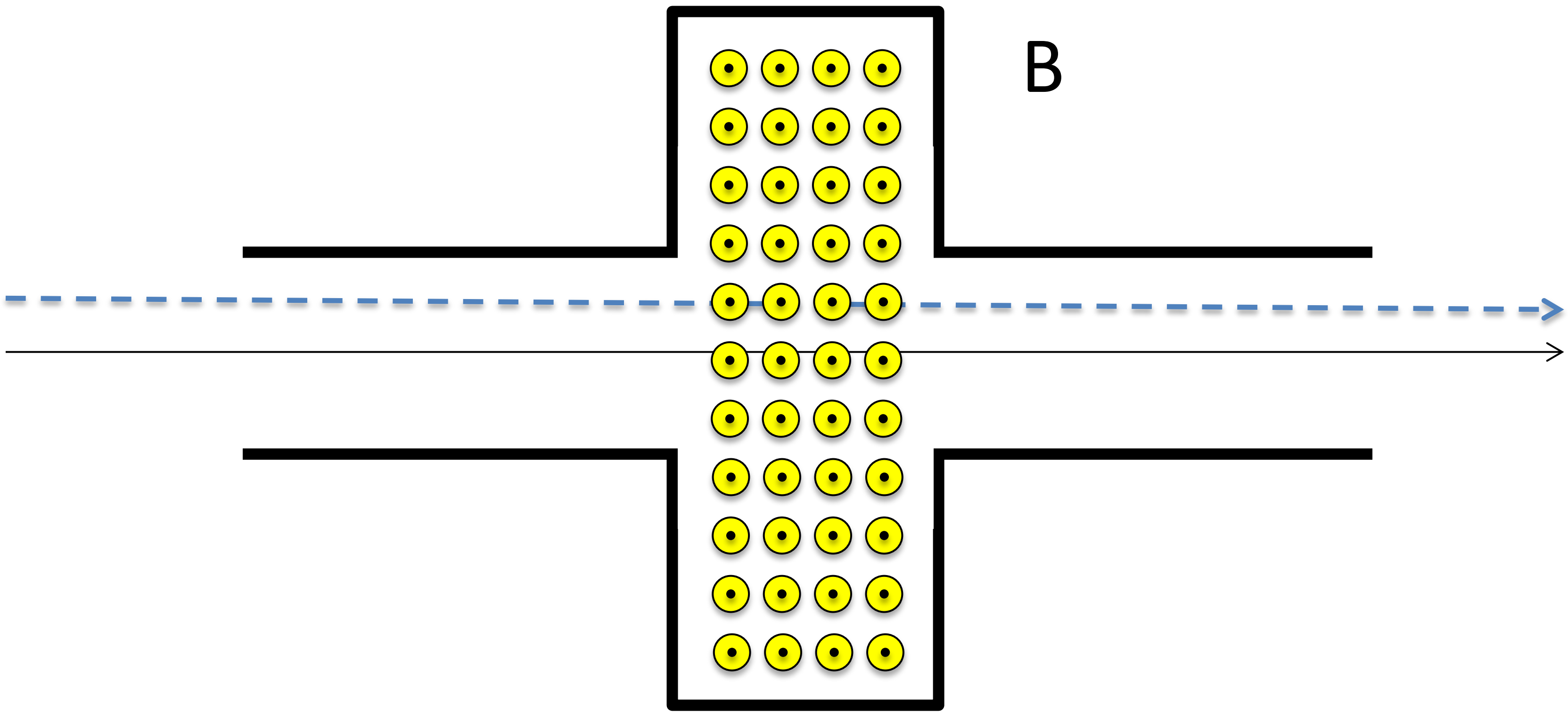
Origin

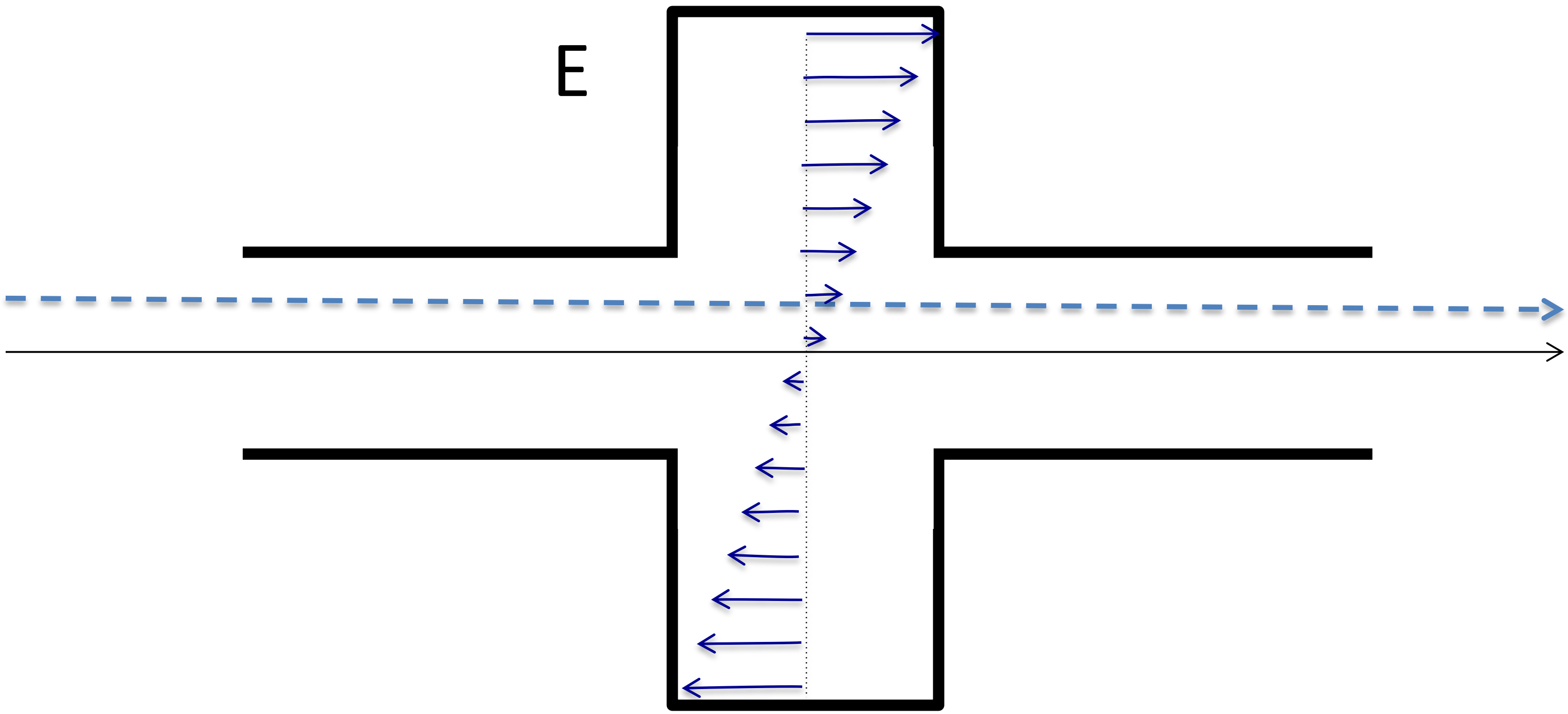
Beam passing through
a cavity off-axis

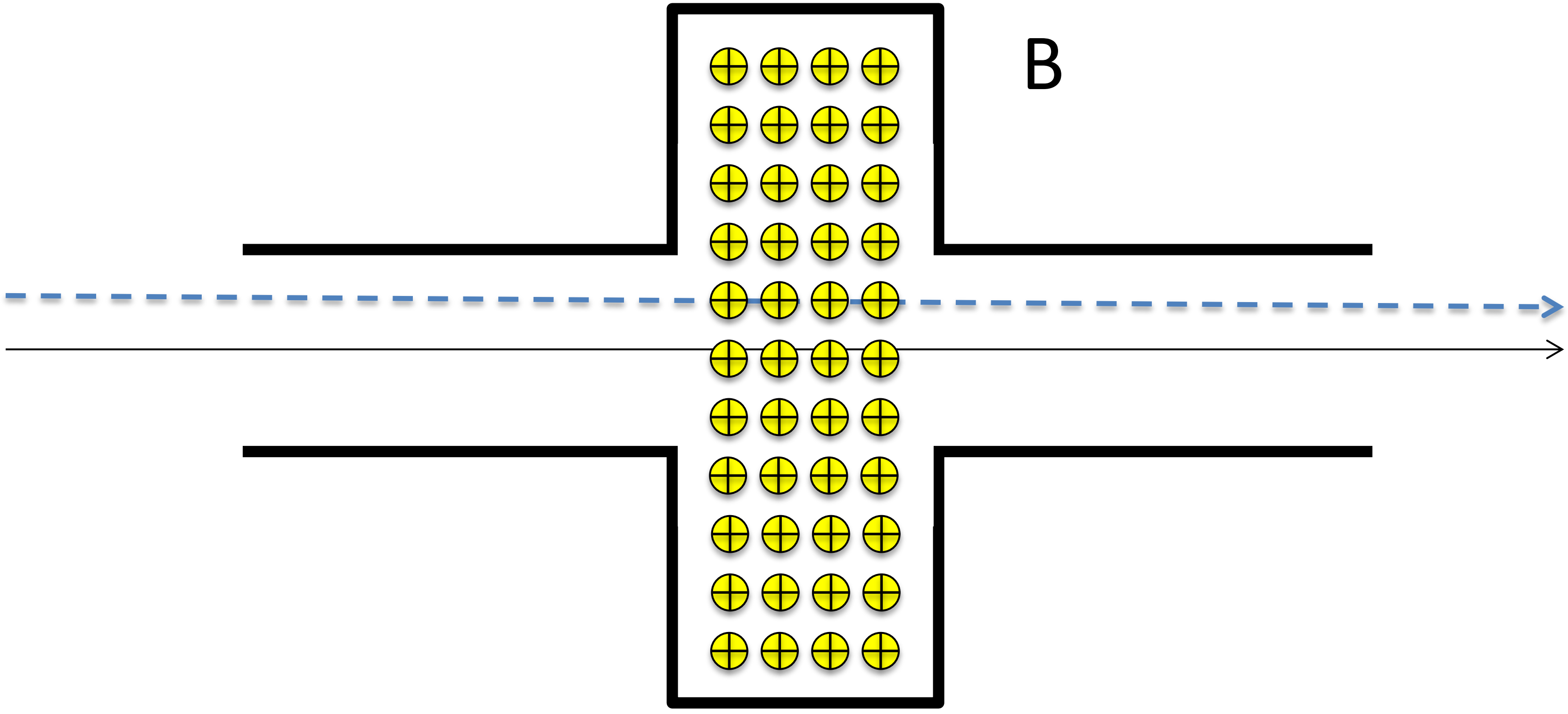


But the field transform it-self !




B


E


B

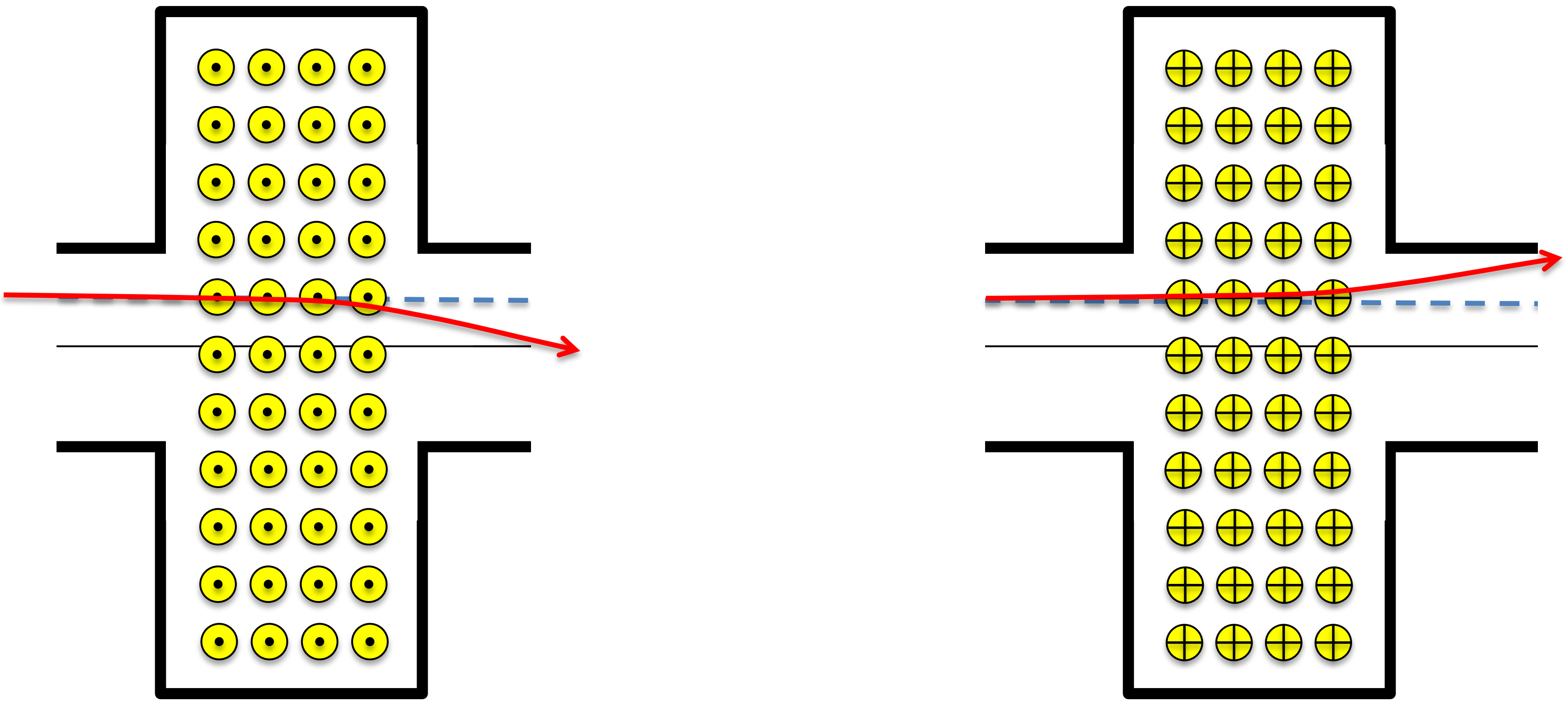
Effect on the dynamics

The dynamics is much more affected by B, than E because

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

this speed is high

The beam creates its own dipolar magnetic field !

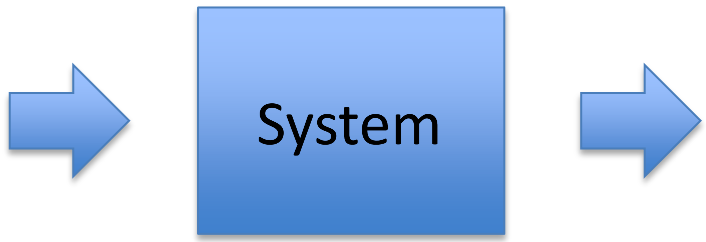


(dipolar errors create integer resonances…. we expect the same…)

Transverse impedance

Definition of longitudinal impedance (classical)

$$I = \hat{I} e^{i\omega t}$$



$$V = \hat{V} e^{i\omega t}$$

Impedance

$$Z(\omega) = \hat{V} / \hat{I}$$

For a displaced beam

source of the effect

$$Ix_0$$

→

Effect

$$\vec{E} + \vec{v} \times \vec{B}$$

↓

this field acts on a single particle

It means that in the equation of motion we have to add this effect

$$\frac{d^2x}{ds^2} + k_x x = \frac{q}{m\gamma v_0^2}[E_x + (\vec{v} \times \vec{E})_x]$$

therefore for a weak effect or distributed we find

$$\frac{d^2x}{ds^2} + \left(\frac{Q_x}{R}\right)^2 x = \frac{q}{m\gamma v_0^2}\frac{1}{2\pi R}\int_0^{2\pi R}[E_x + (\vec{v}\times\vec{E})_x]ds$$

In the time domain

$$\frac{d^2x}{dt^2} + (Q_x\omega_0)^2 x = \frac{q}{m\gamma}\frac{1}{2\pi R}\int_0^{2\pi R}[E_x + (\vec{v}\times\vec{E})_x]ds$$

But $\int_0^{2\pi R} [E_x + (\vec{v} \times \vec{E})_x] ds$ is like a “strange” voltage

$$V = -\int_0^{2\pi R} [\vec{E} + \vec{v} \times \vec{B}]_\perp ds$$

Now the situation is the following:

$I x_0$

System

$$V = -\int_0^{2\pi R} [\vec{E} + \vec{v} \times \vec{B}]_\perp ds$$

it depends on frequency

Transverse beam coupling impedance

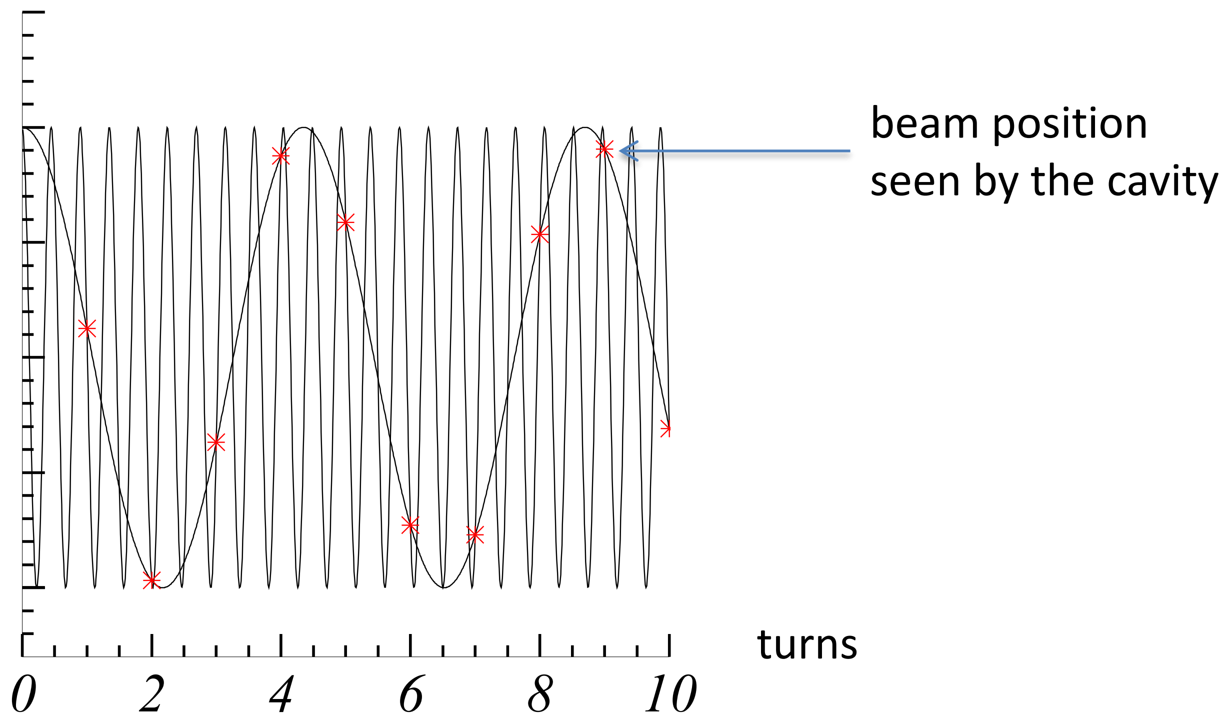
$$Z_\perp(\omega) = i\frac{\int_0^{2\pi R}[\vec{E} + \vec{v} \times \vec{B}]_\perp ds}{\beta I x_0}$$

now the question is
what is ω ?

What is it ω ?

It is given by the fractional tune, as this is the frequency seen in a cavity

Example: $Q = 2.23$ fractional tune $q = 0.23$



B-field induced by beam displacement

From $$\frac{\partial E_z}{\partial x} = kIx_0 \quad \Rightarrow \quad E_z = kIx_0 x$$

electric field at the position of beam x_0 is

$$E_z(x_0) = kIx_0^2$$

Longitudinal impedance

$$Z_{||} = -\frac{E_z(x_0)l}{I} = -kx_0^2 l$$

The magnetic field comes from Maxwell

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0}$$

$$\frac{\partial B_y}{\partial t}\Big|_{x_0} = kIx_0$$

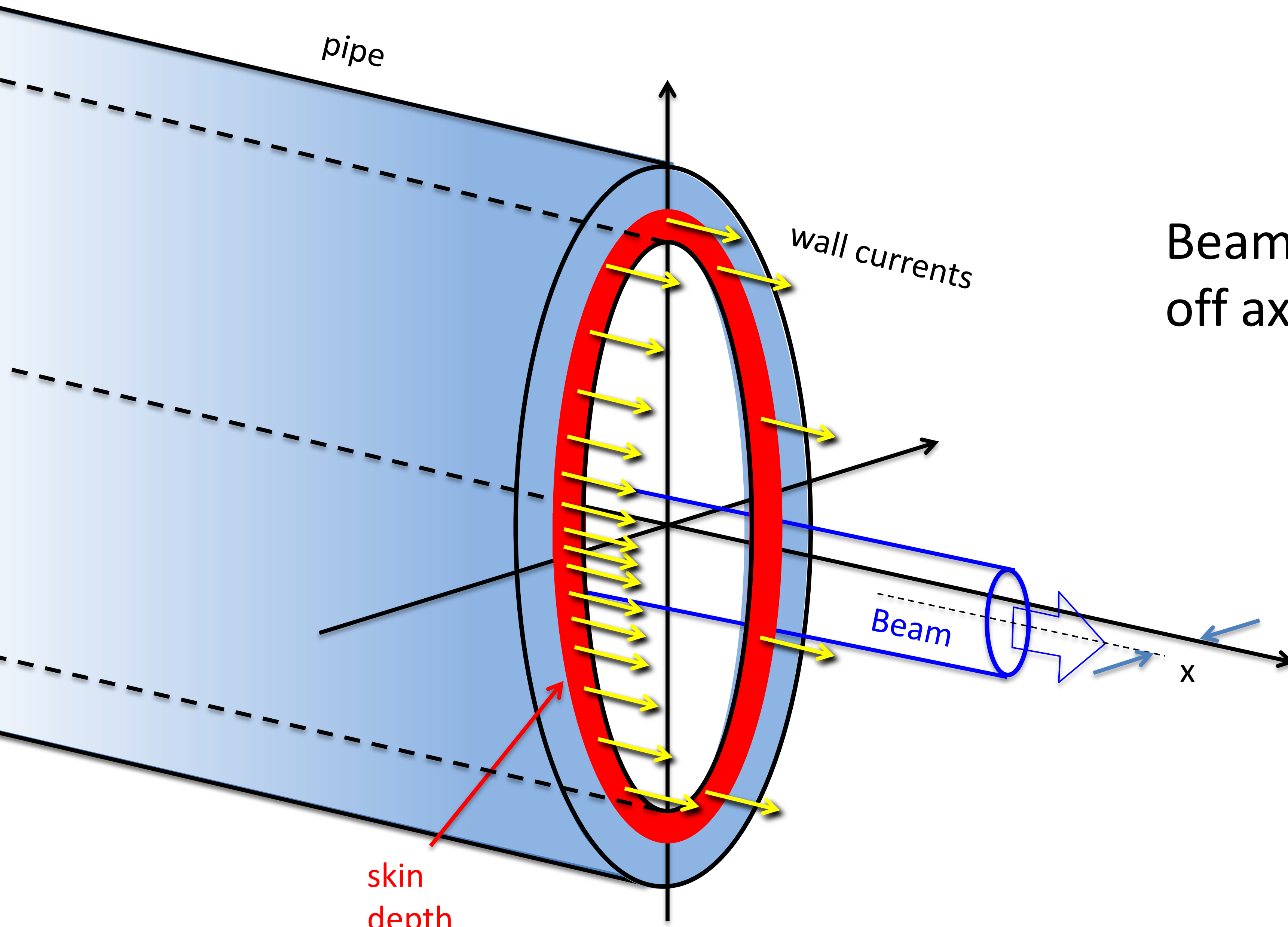
taking $Ix_0 = I\hat{x}e^{i\omega t}$

$$\Rightarrow \quad B_y = \frac{kI\hat{x}}{i\omega}e^{i\omega t} = \frac{kIx_0}{i\omega}$$

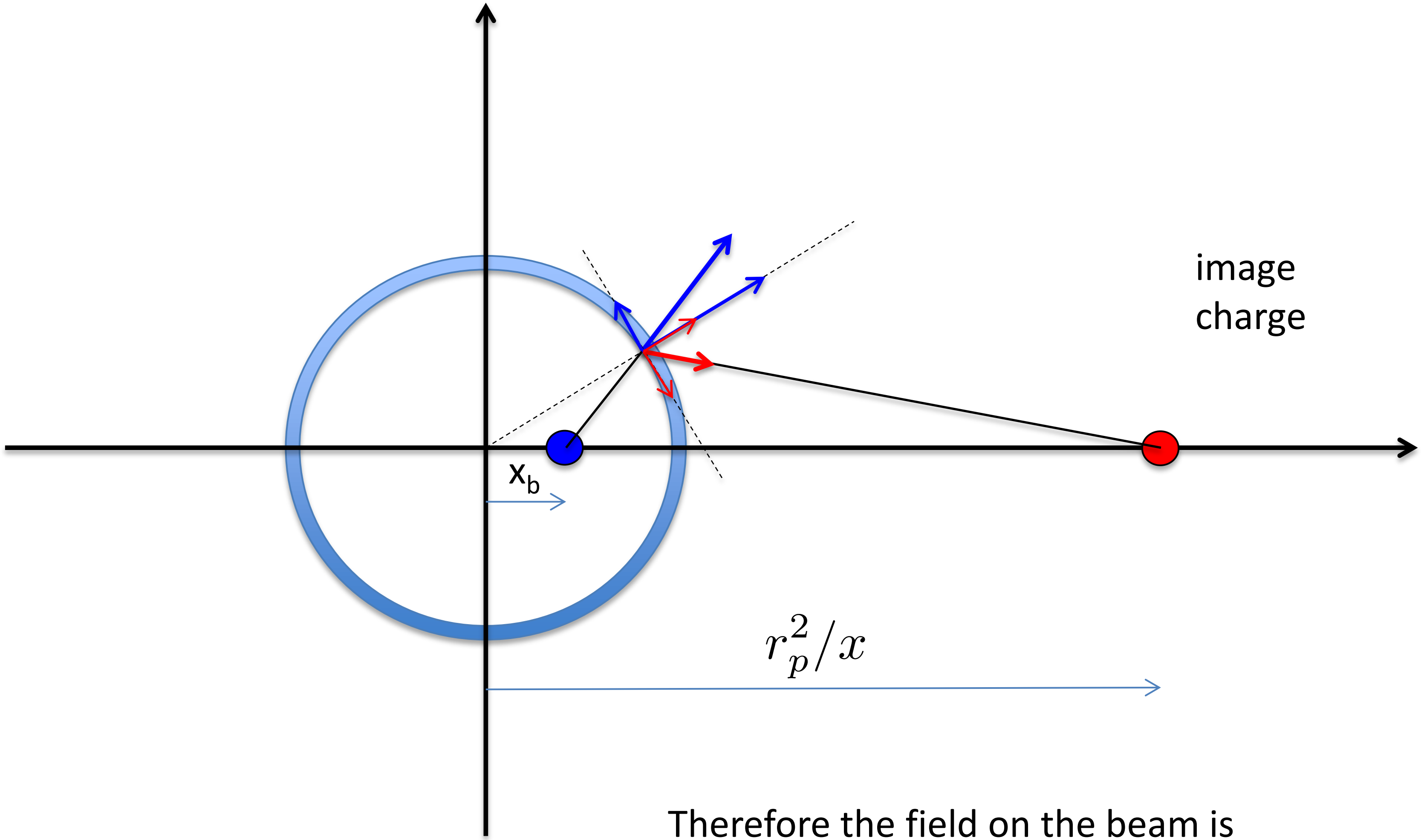
Transverse impedance

$$Z_\perp = i\frac{\int_0^l [\vec{v} \times \vec{B}]_\perp ds}{Ix_0} \quad \Rightarrow \quad Z_\perp = -\frac{v_z kl}{\omega}$$

$$Z_\perp = \frac{v_z}{2\omega}\frac{d^2 Z_{||}(\omega)}{dx}$$



Beam off axis

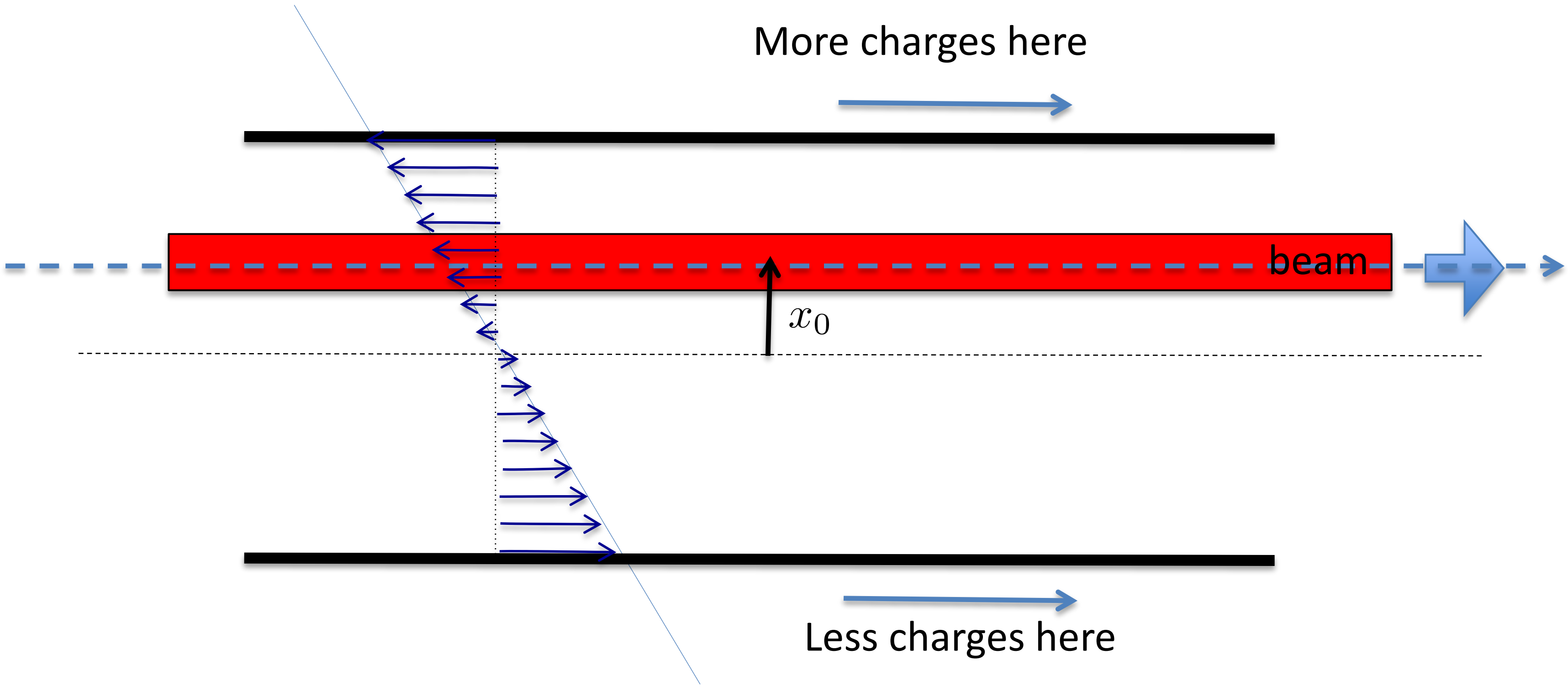


Therefore the field on the beam is

$$E_x = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r_p^2}x_b$$

(for small x_b/r_p)

$$\frac{\partial E}{\partial x} \propto I x_0$$



Transverse resistive Wall impedance

$$Z(\omega_n)_\perp = \frac{2R}{r_p^2}\frac{Z_{||}(\omega_n)}{n}|_{res}$$

Transverse instability

Coasting beam instability

Force due to the impedance (in the complex notation)

$$F_{\perp} = i \frac{q Z_{\perp} I_0}{2\pi R} x_b$$

Equation of motion of one particle for a beam on axis

$$\ddot{x} + Q^2 \omega_0^2 x = 0$$

Equation of motion of a beam particle when the beam is off-axis

$$\ddot{x} + Q^2 \omega_0^2 x = -i \frac{q Z_{\perp} I_0}{2\pi R m \gamma} x_b$$

Collective motion

On the other hand the beam center is $$x_b = \int x\, n(x,y,s)dxdy$$

with $$\int \tilde{n} dV = 1$$

therefore

$$\int \ddot{x}\tilde{n}dV + \int Q^2\omega_0^2 x\tilde{n}dV = -i\frac{qZ_\perp I_0}{2\pi R m\gamma}x_b$$

If all particles have the same frequency, i.e. each particle experience a force

$$Q^2\omega^2 x$$

then $$\ddot{x}_b + Q^2\omega_0^2 x_b = -i\frac{qZ_\perp I_0}{2\pi R m\gamma}x_b$$

$$\ddot{x}_b + Q^2\omega_0^2 x_b = -i\frac{qZ_\perp I_0}{2\pi R m\gamma}x_b$$

We can define a coherent "detuning" because this is a linear equation

$$Q^2\omega_0^2 + i\frac{qZ_\perp I_0}{2\pi R m\gamma} = (Q + \Delta Q^c)^2\omega_0^2$$

$$\Downarrow$$

$$\Delta Q^c = i\frac{1}{2Q\omega_0^2}\frac{qZ_\perp I_0}{2\pi R m\gamma}$$

$$\ddot{x}_b + Q^2\omega_0^2 x_b = -2Q\omega^2 \Delta Q^c x_b$$

that is

$$\ddot{x}_b + (Q^2\omega_0^2 + 2Q\omega_0^2 \Delta Q^c) x_b = 0$$

But now ΔQ^c is a complex number !!

Solution $x_b = A \exp[-\omega_0 I_m(\Delta Q^c)t + i\omega_0[Q + Re(\Delta Q^c)]t]$

$$\tau_I^{-1} = \omega_0 Im(\Delta Q^c)$$

Is the growth rate of the transverse resistive wall instability

$$\frac{1}{\tau} = \frac{qRe\{Z_\perp\}I_0}{4\pi R m\gamma Q\omega_0}$$

This instability always take place

Instability suppression

→ Landau damping

An important assumption

We assumed that all particles have the same frequency so that

$$\int Q^2\omega_0^2 x\tilde{n}dV = Q^2\omega_0^2 \int x\tilde{n}dV = Q^2\omega_0^2 x_b$$

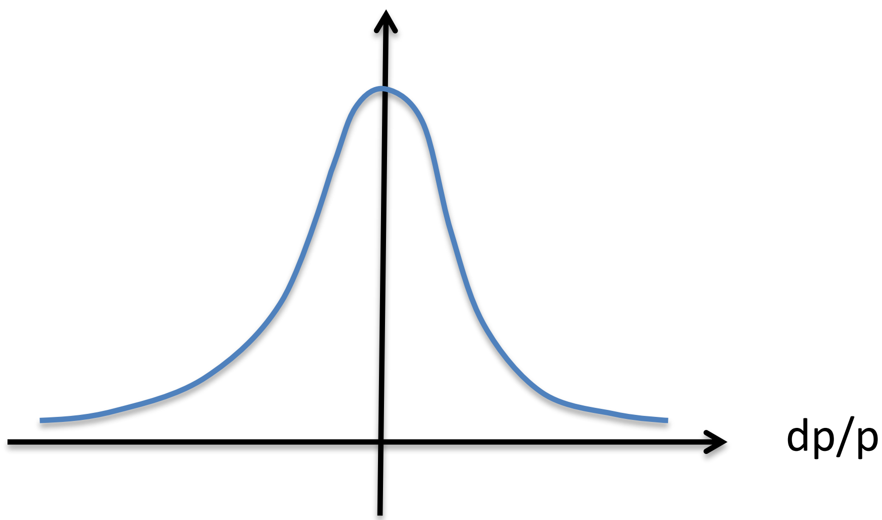
This assumption means that each particle of the beam respond in the same way to a change of particle amplitude

Coherent motion → drive particle motion, which is again coherent

Chromaticity ?

What happened if the incoherent force created by the accelerator do not allow a coherent build up

Momentum spread



one particle with off-momentum dp/p has tune

$$\delta Q = \xi \frac{\delta p}{p}$$

chromaticity

$$Q = Q_0 + \delta Q = Q + \xi \frac{\delta p}{p}$$

If each particle of the beam has different dp/p then the force that the lattice exert on a particle depends on the particle !

$$F_x = \left(Q_0 + \xi \frac{\delta p}{p} \right)^2 \omega^2 x$$

Incoherent motion damps x_b

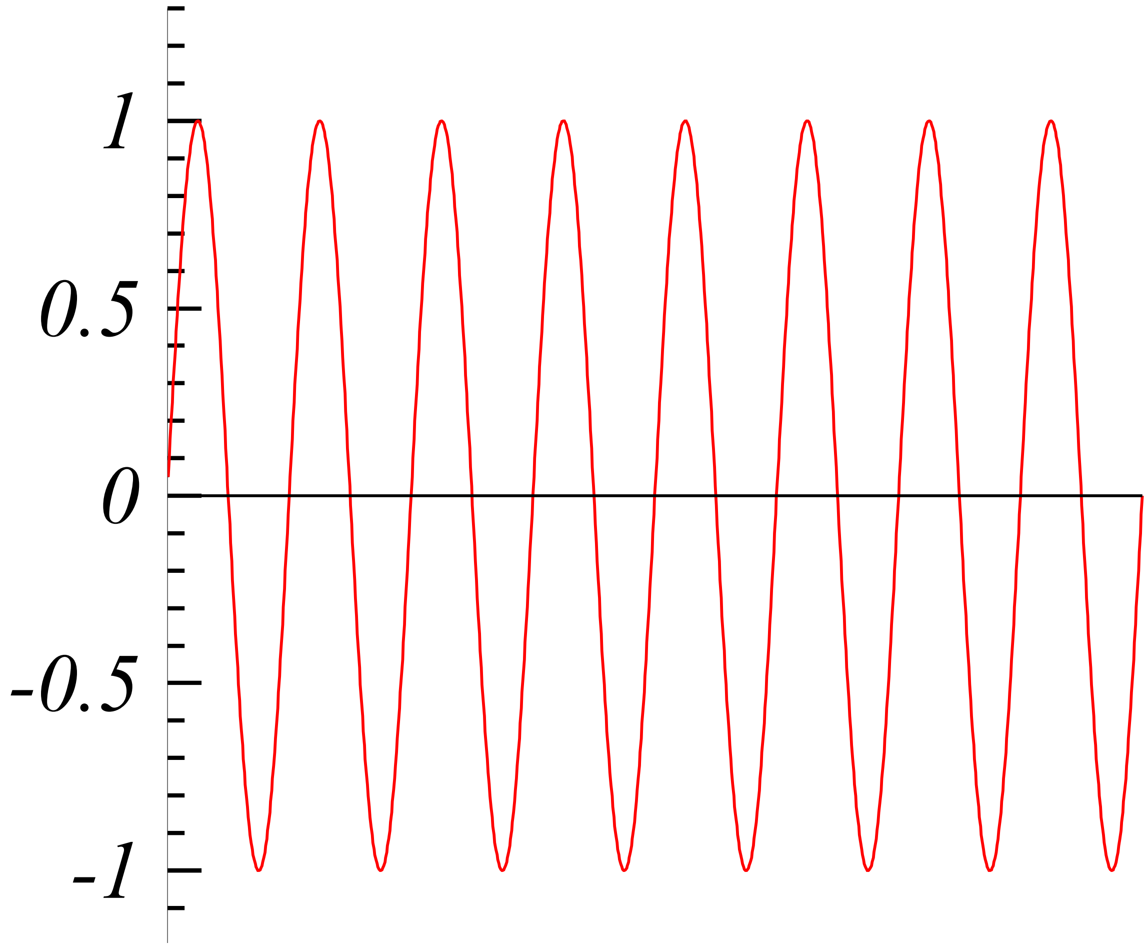
Equation of motion without impedances

$$\ddot{x} + \left(Q_0 + \xi \frac{\delta p}{p}\right)^2 \omega^2 x = 0$$

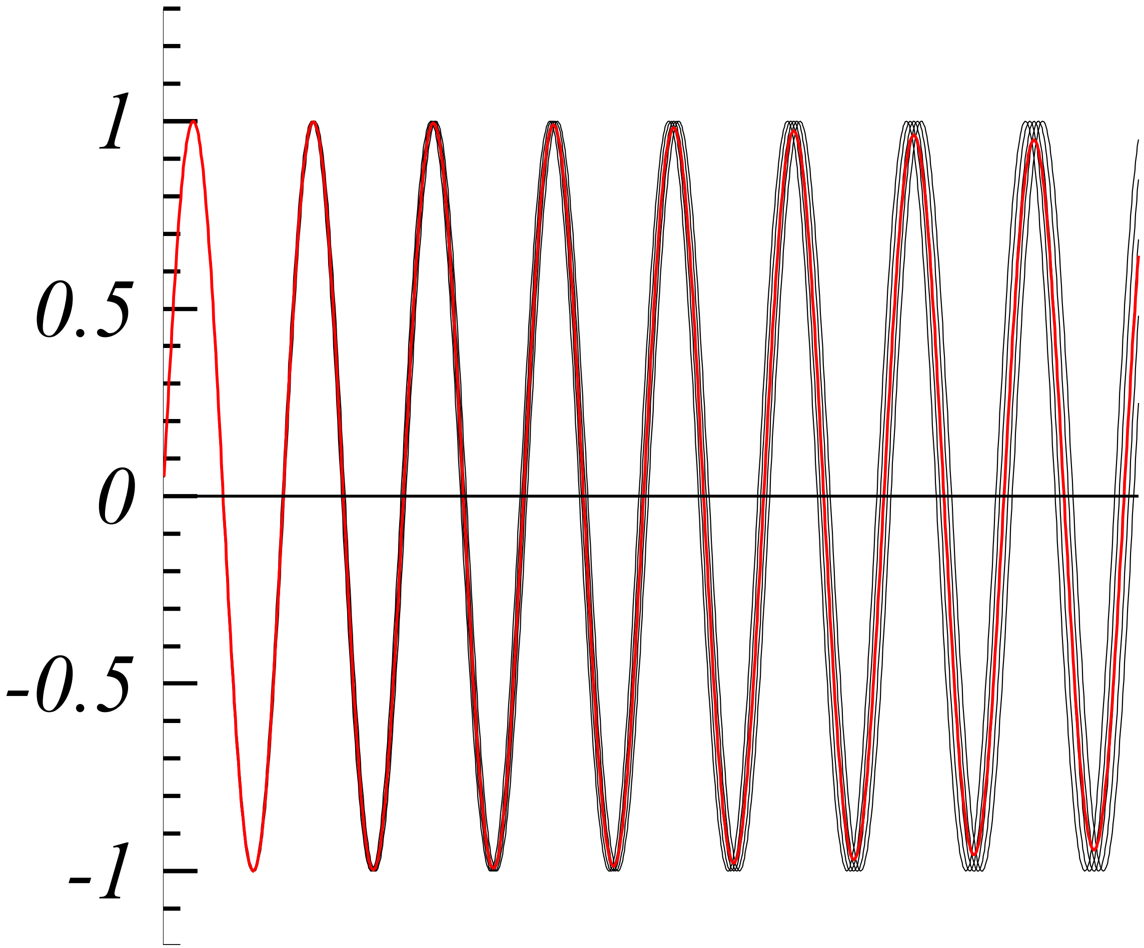
Motion of center of mass as an effect of the spread of the frequencies of oscillation

The momentum compaction also provides a spread of the betatron oscillations

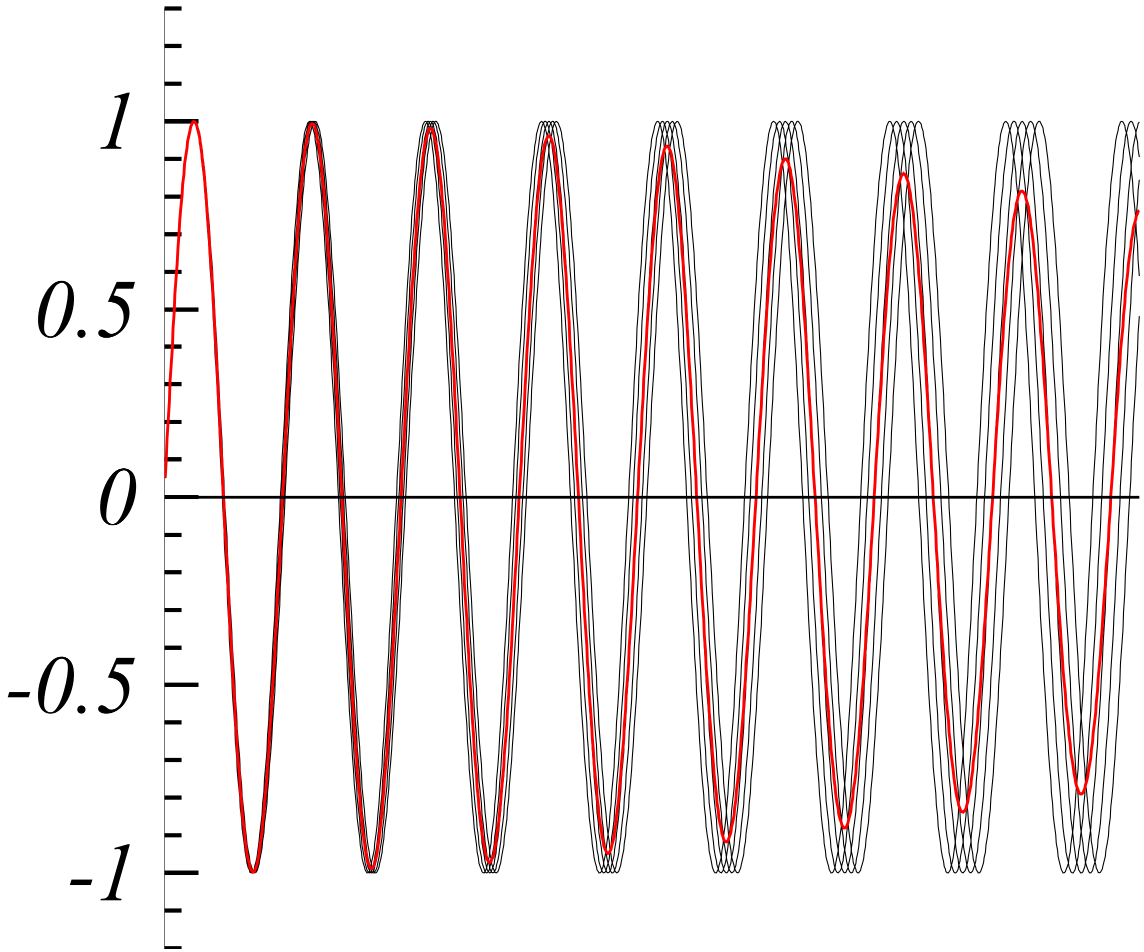
Example:
N. particles = 5
dq/q = 0



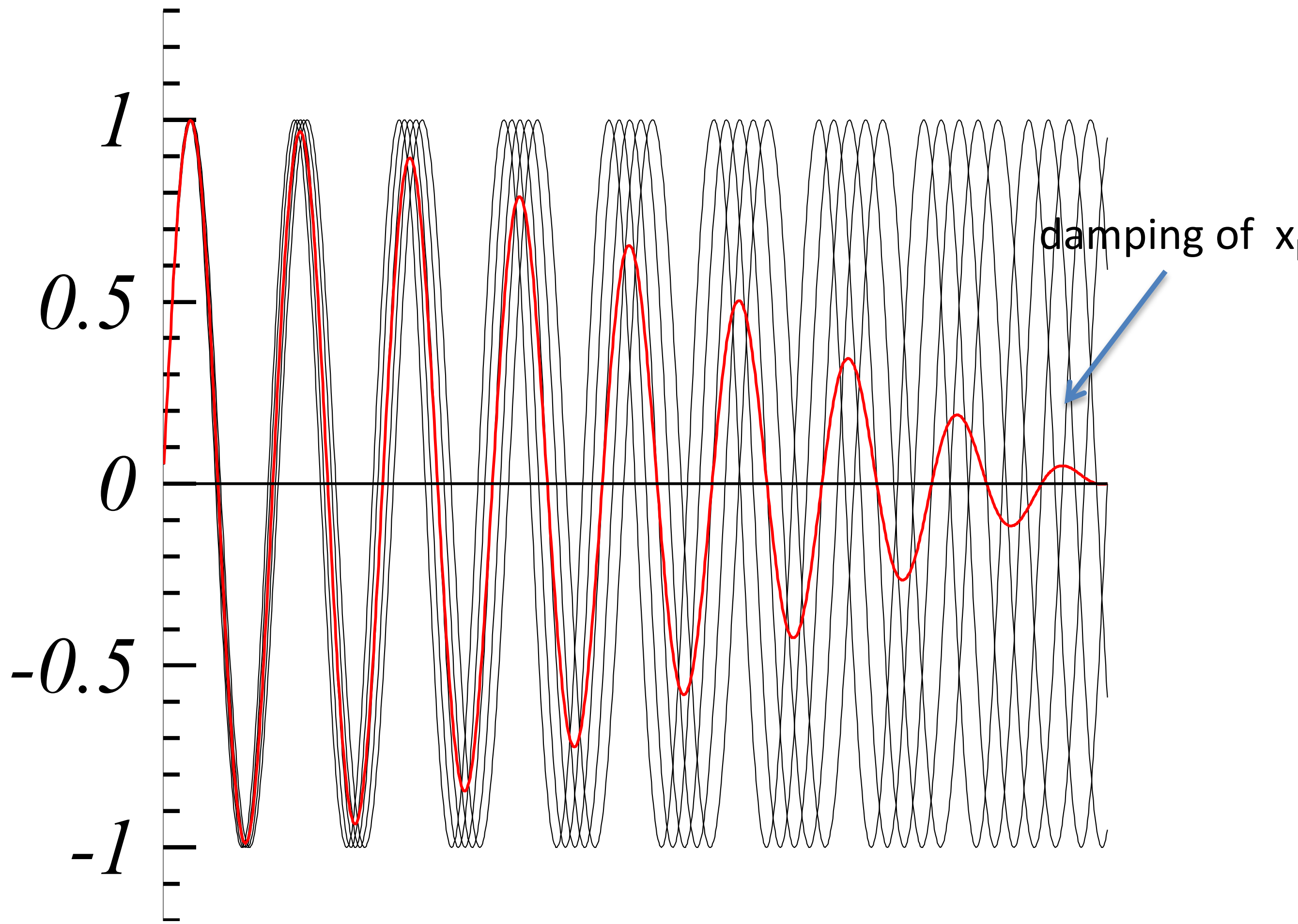
Example:
N. particles = 5
dq/q = 5E-3



Example:
N. particles = 5
dq/q = 1E-2

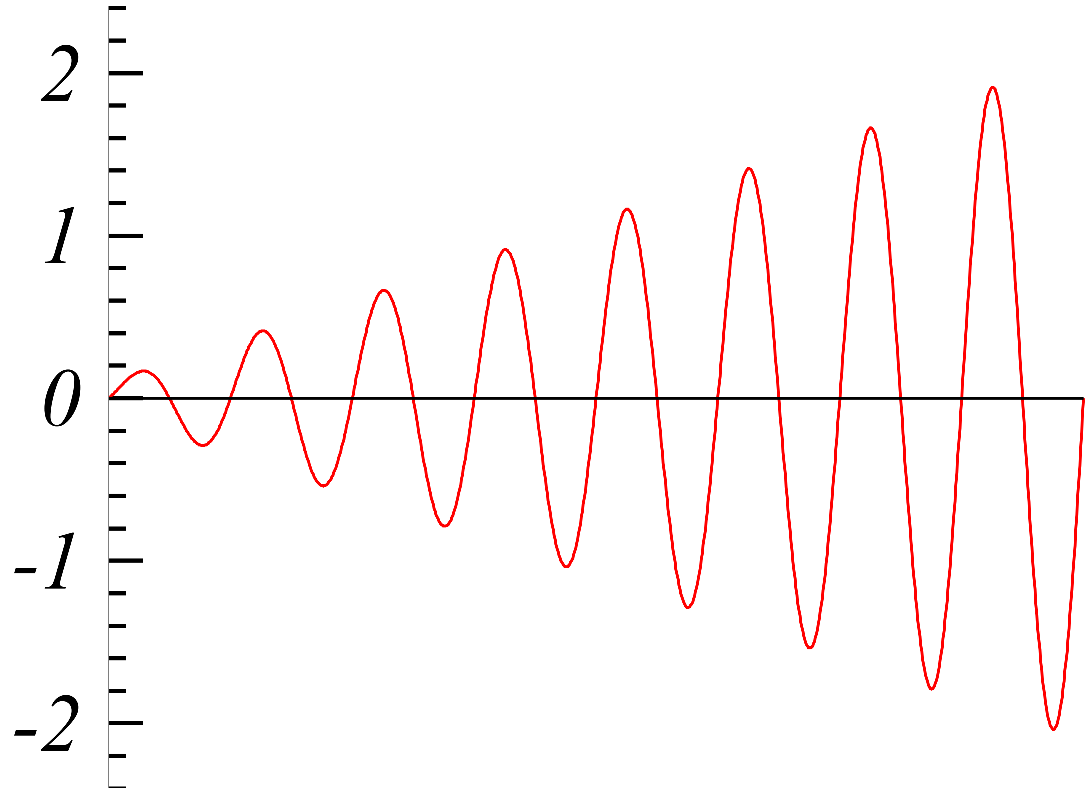


Example:
N. particles = 5
dq/q = 2.5E-2



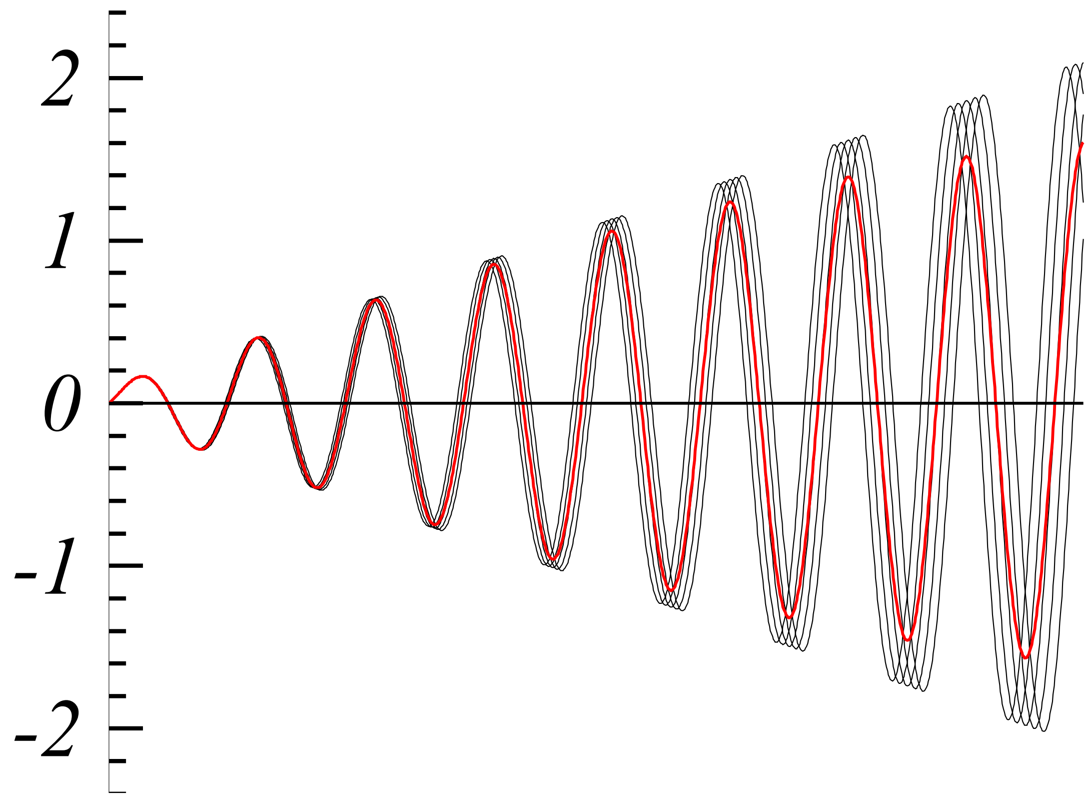
But incoherent motion reduces x_b

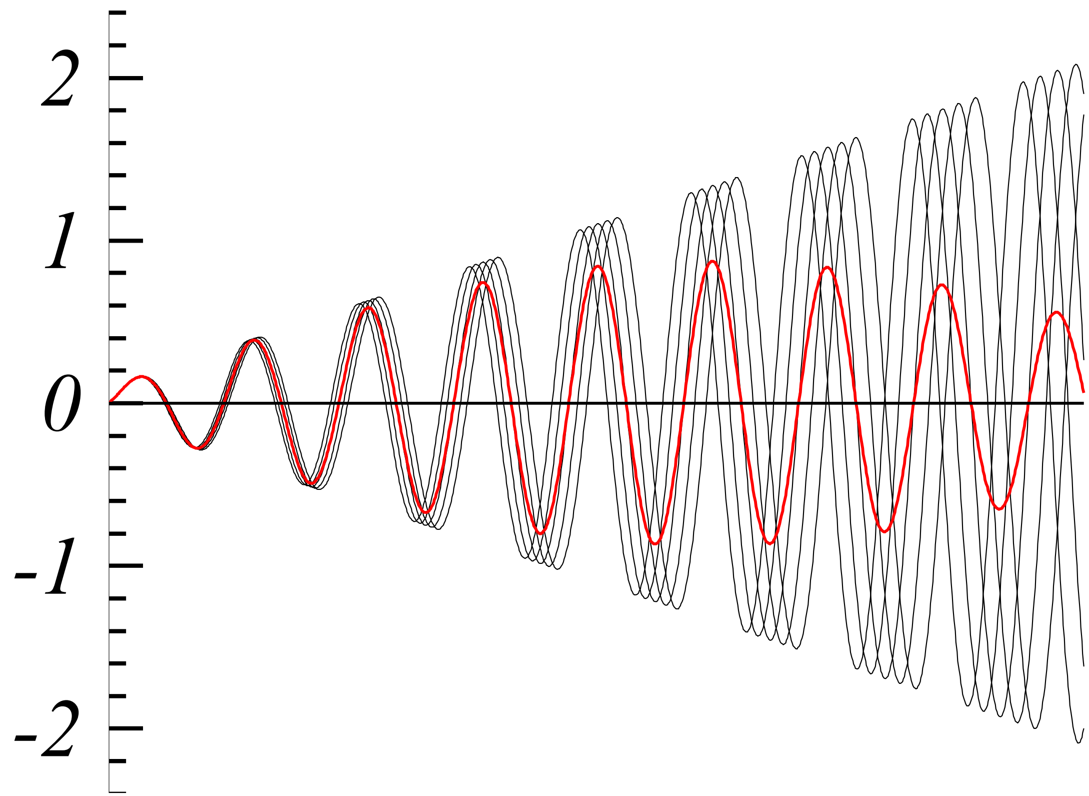
Example:
these are 5 sinusoid with amplitude linearly growth



Example:
now a spread dq/q of 1E-2 is added to the 5 curves

The center of mass growth slower





Example:
now a spread dq/q of 1E-2 is added to the 5 curves

the spread of the particles damps the oscillations of the center of mass → the instability cannot develop

Situation

Coherent effect

Incoherent effect

Growth rate

Damping rate

τ_I

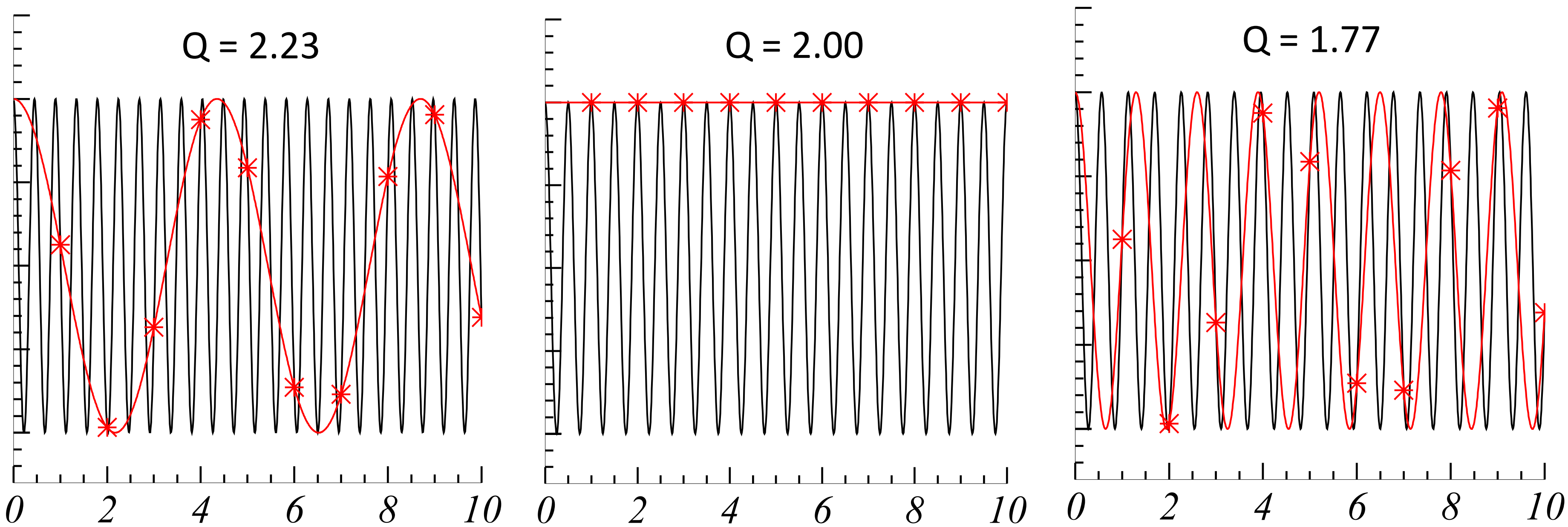
The faster wins

τ_D

instability of a single bunch

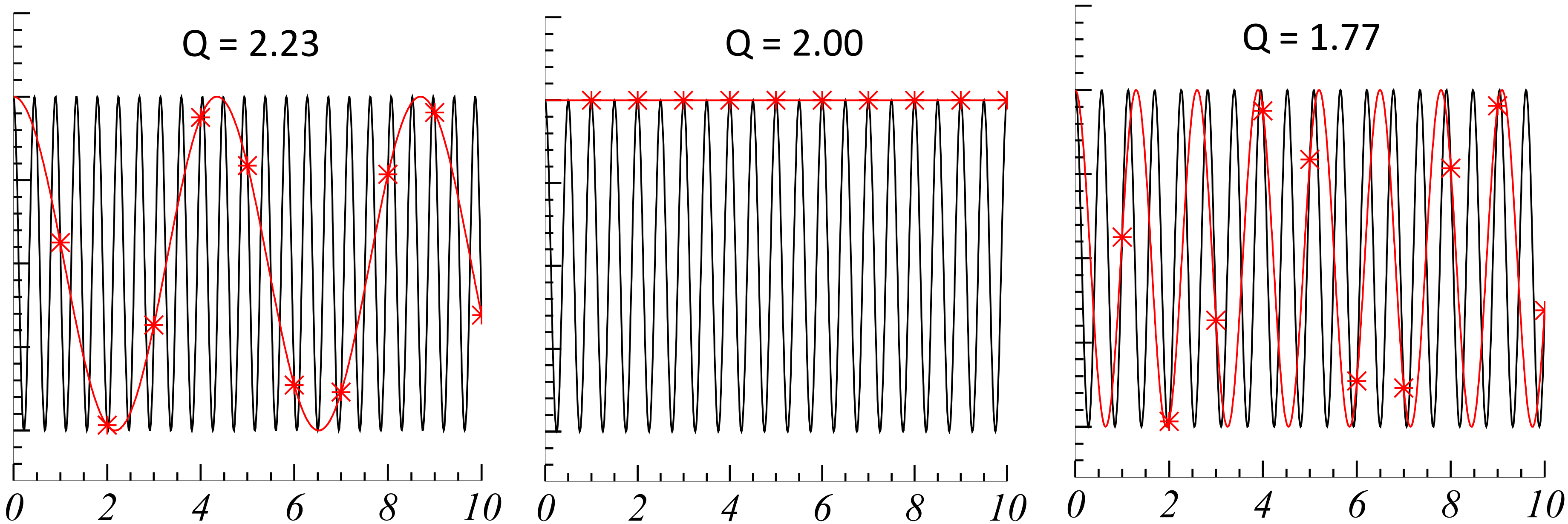
Example

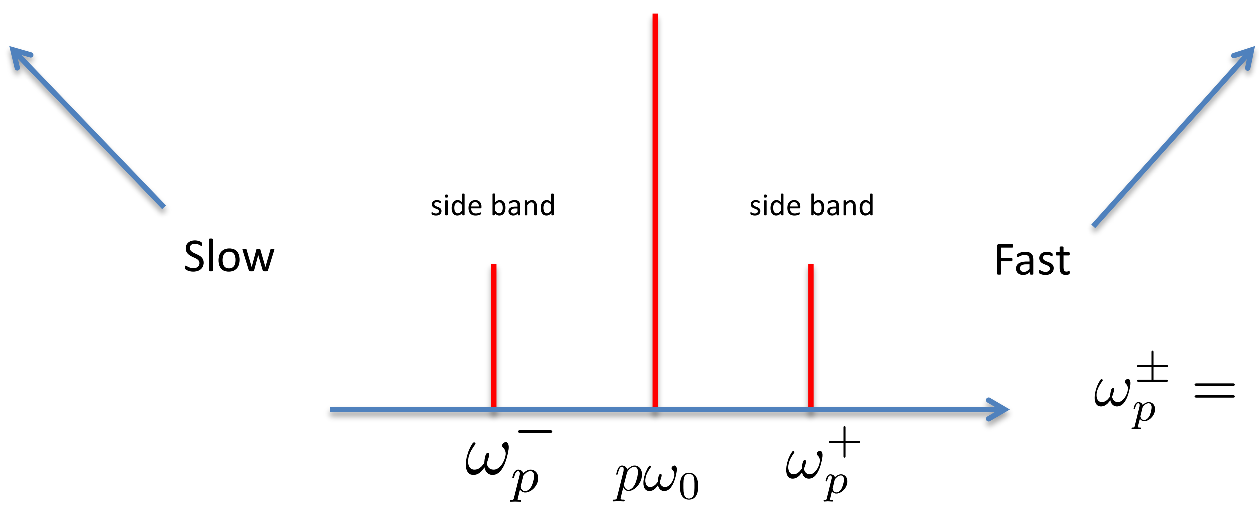
beam position at the cavity



No oscillations →

$$\omega = 0$$

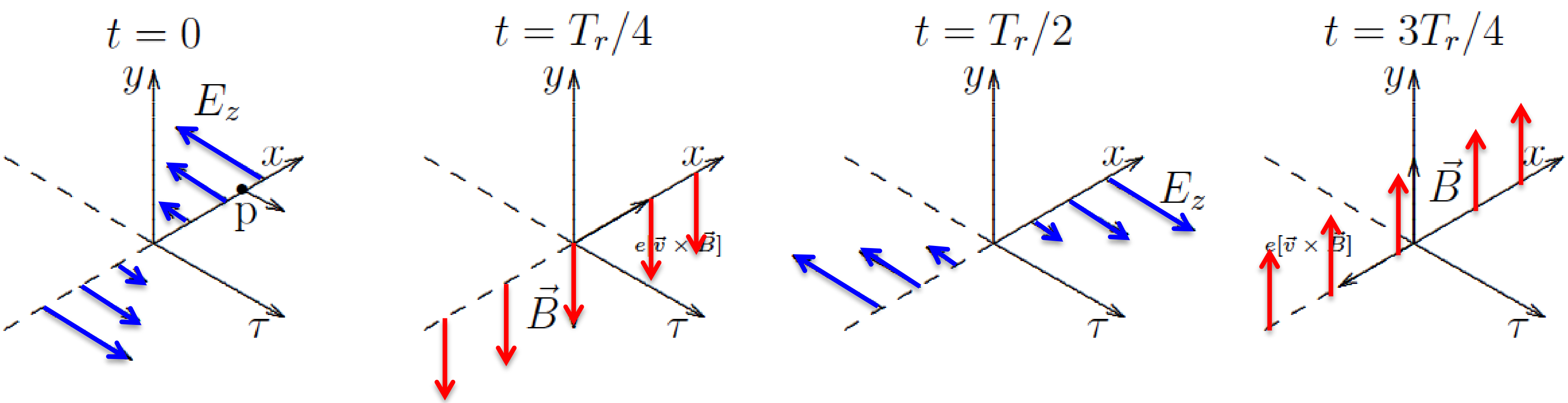




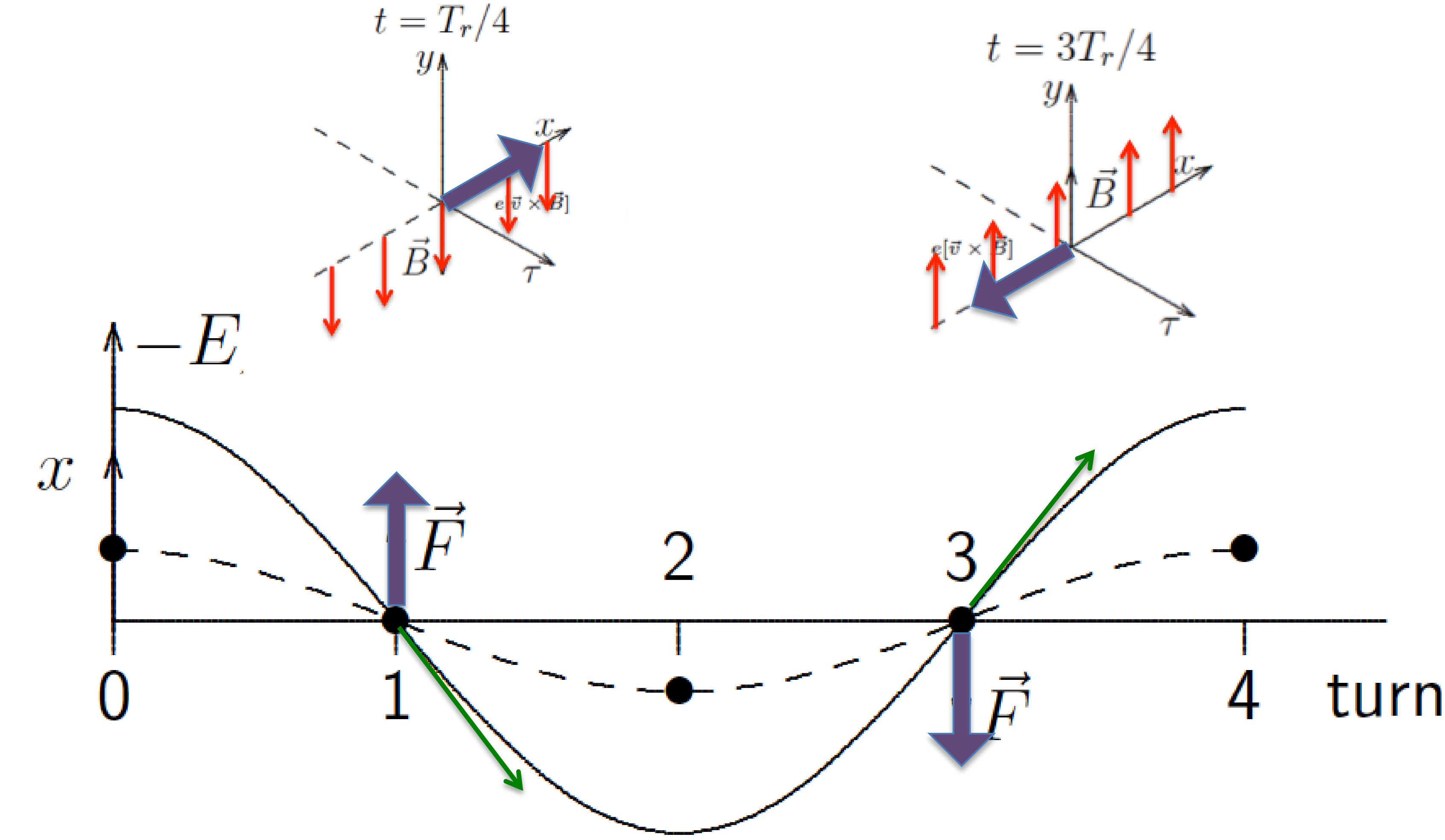
$$\omega_p^{\pm} = (p \pm q)\omega_0$$

behavior of the field in the cavity

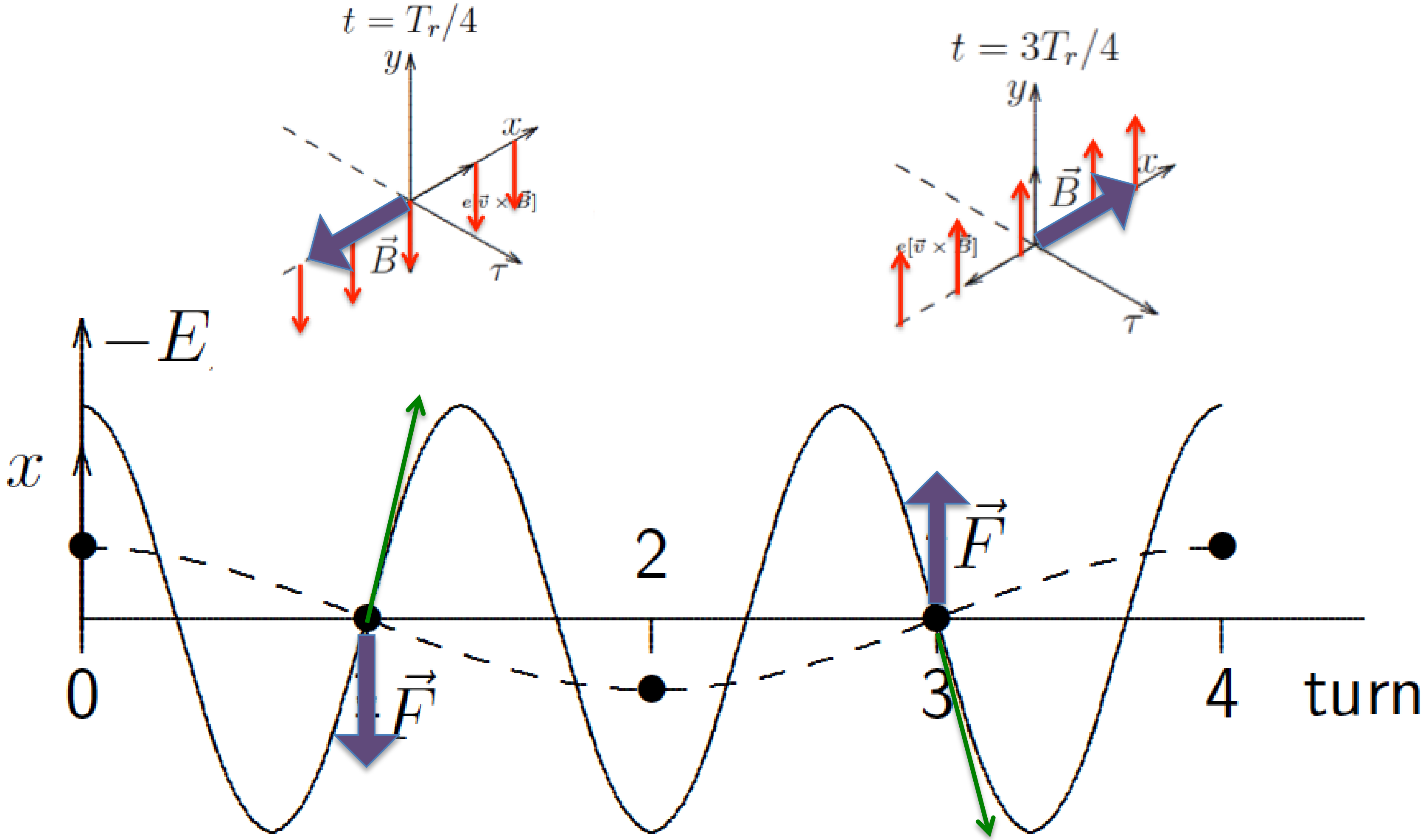
T_r = time of oscillation of the field in the cavity



Cavity tuned upper sideband



Cavity tuned upper sideband



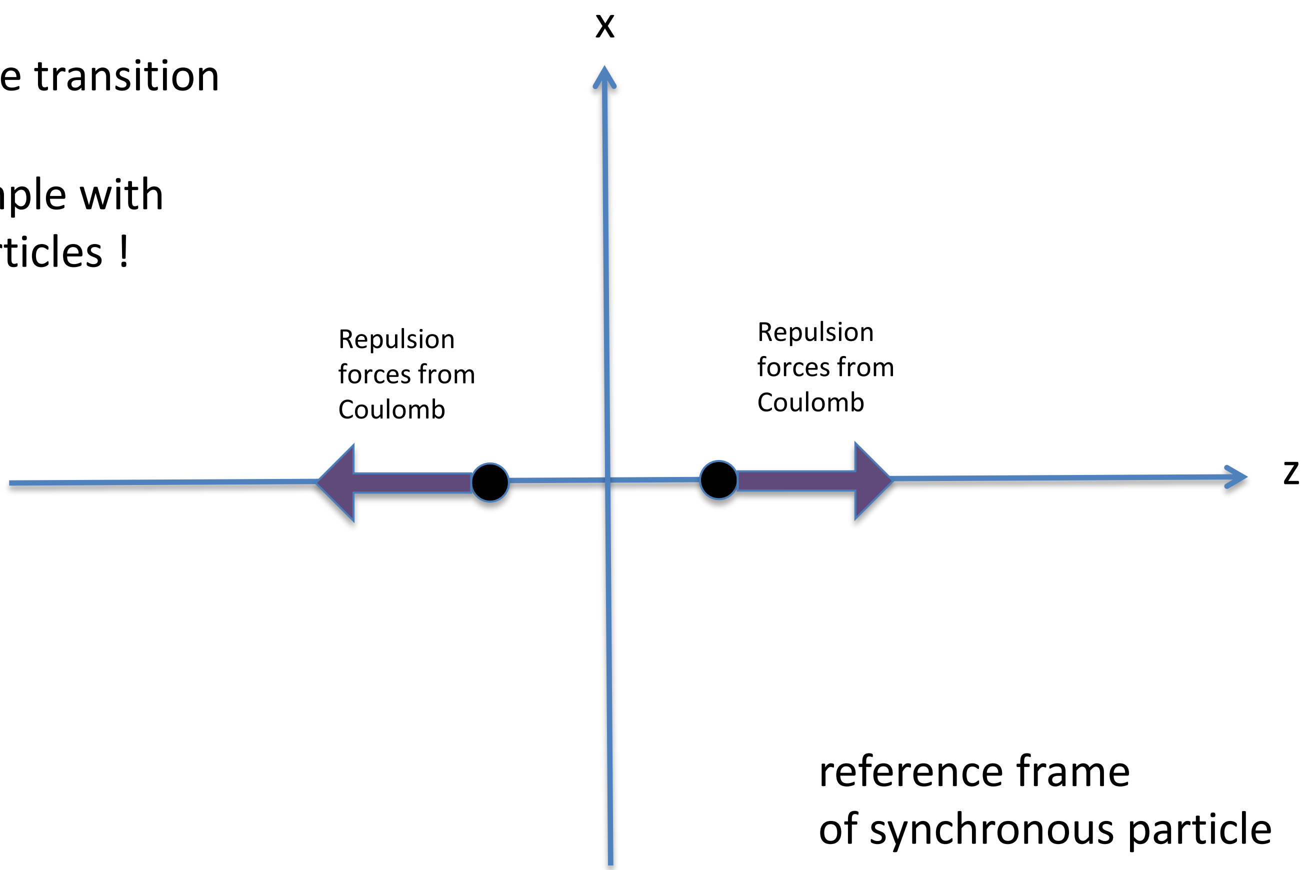
As for the Robinson Instability

$$\alpha_s = \frac{1}{\tau} \propto \sum_p I_p^2 [Z_\perp(\omega_p^+) - Z_\perp(\omega_p^-)]$$

Negative mass instability

Above transition

Example with
2 particles !

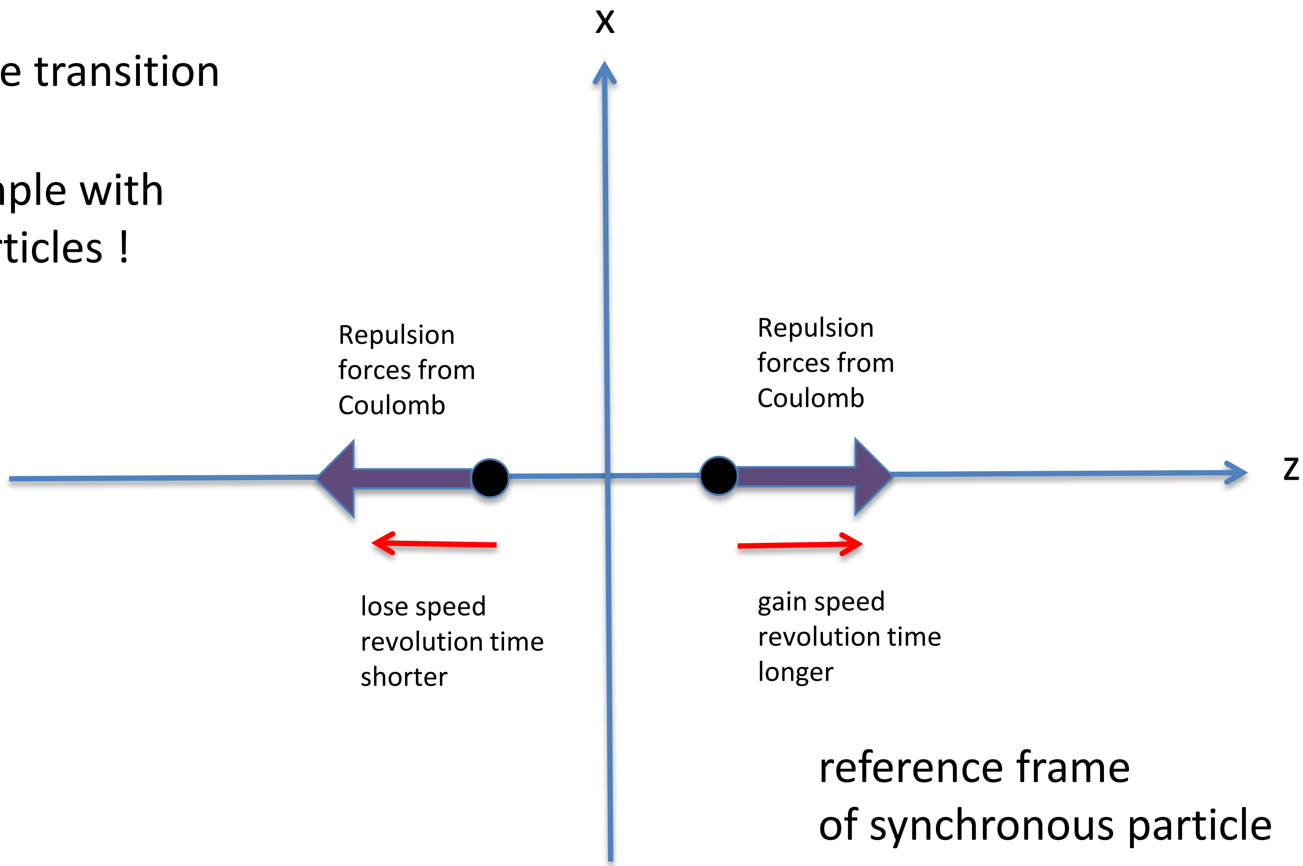


reference frame
of synchronous particle

Negative mass instability

Above transition

Example with
2 particles !

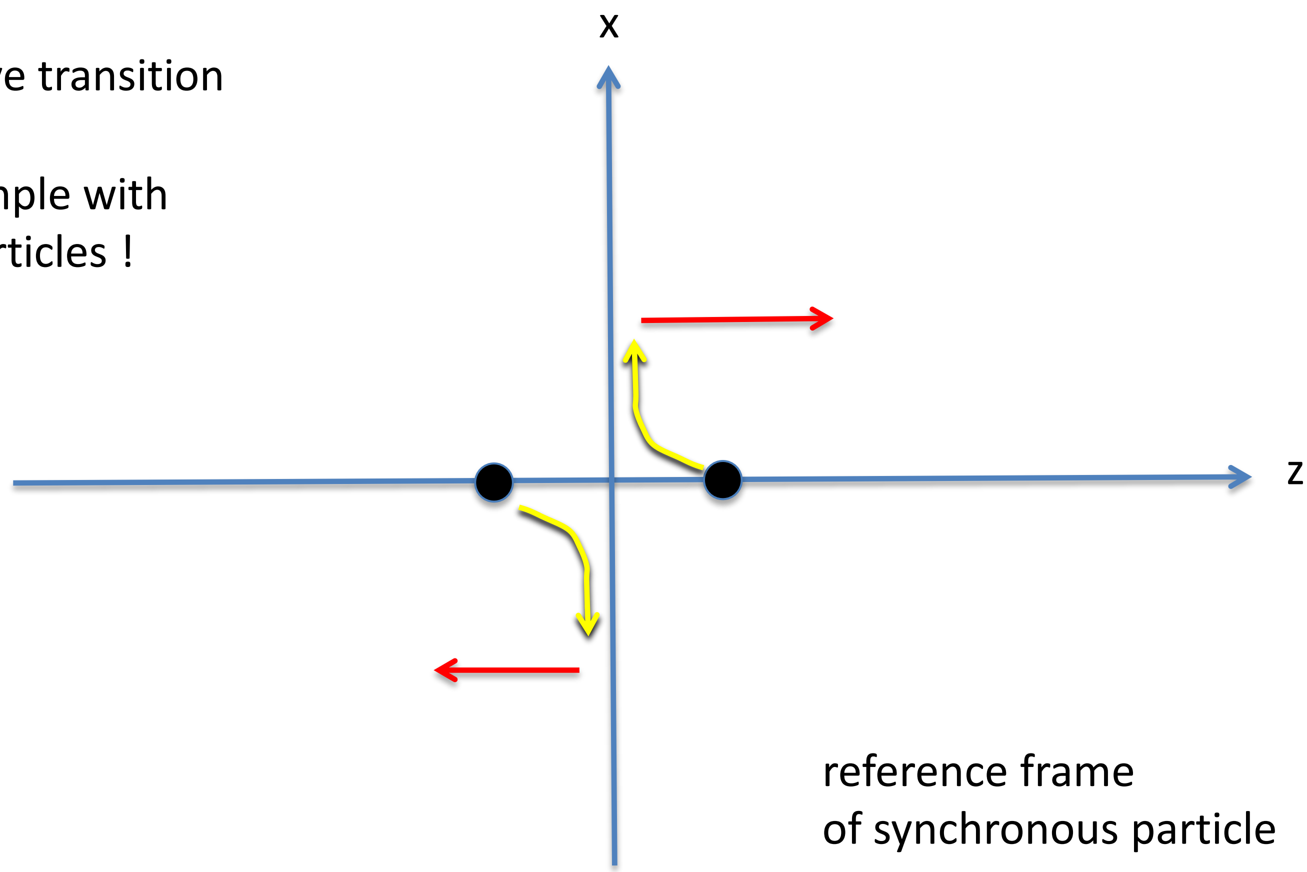


reference frame
of synchronous particle

Negative mass instability

Above transition

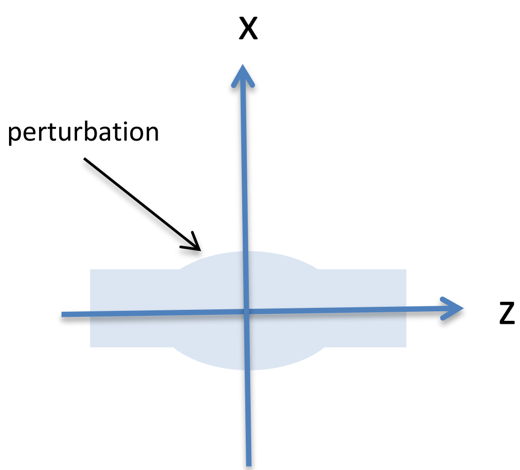
Example with
2 particles !



reference frame
of synchronous particle

Negative mass instability

Above transition



repulsive forces attract particles as if their mass were negative

Summary

Robinson instability
Longitudinal space charge and resistive wall impedance
Transverse impedance
Transverse instability
Landau damping
Single bunch instability
Negative mass instability