Collective Effect I

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Type of fields

- Direct self fields
- Image self fields
- Wake fields

Collective Effects

Space Charge
Image charges
Influence of the chamber wall

the electron in the metal quickly travel on the surface of the metal until the electric field parallel to the surface is zero.

field line at 90°
Image charge

the image charge is a reflection of the particle with exchanged sign
the image charge is a reflection of the particle with exchanged sign
Conducting plates

2D beam \( \rho L \)

charge density

\( h \)

\( y \)
Incoherent motion

Consider a particle of the beam.
Summing image charge contribution in pairs

\[ E_{y,n} = \frac{\rho_L}{2\pi \epsilon_0} (-1)^n \left( \frac{1}{2nh + y} - \frac{1}{2nh - y} \right) \]

\[ h \gg y \]

\[ E_{y,n} = -\frac{\rho_L y}{4\pi \epsilon_0 h^2} \frac{(-1)^n}{n^2} \]

Total electric field

\[ E_y = \sum_{n=1}^{\infty} E_{y,n} = \frac{\rho_L y}{\pi \epsilon_0} \frac{\pi^2}{48h^2} \]
Equation of motion

In the equation of motion

\[
\frac{d^2 y}{ds^2} + k_y y = \frac{e}{m\gamma^3 v_0^2} E_{b,y} + \frac{e}{m\gamma v_0^2} E_{i,y}
\]

\[
\frac{d^2 y}{ds^2} + k_y y = \frac{2K}{Y(X+Y)} y + K\gamma^2 \frac{\pi^2}{24\hbar^2} y
\]

as \( \nabla \cdot \vec{E} = 0 \)

\[
\frac{\partial E_x}{\partial x} = -\frac{\partial E_y}{\partial y}
\]
\[
\frac{d^2y}{ds^2} + k_y y - \frac{2K}{Y(X+Y)} \left[ 1 + \gamma^2 \frac{\pi^2}{48} \frac{Y(X+Y)}{h^2} \right] y = 0
\]

\[
\frac{d^2x}{ds^2} + k_x x - \frac{2K}{X(X+Y)} \left[ 1 - \gamma^2 \frac{\pi^2}{48} \frac{X(X+Y)}{h^2} \right] x = 0
\]

Laslett Tuneshift

\[
\Delta Q_y \approx -\frac{R_m^2}{Q_{y0}} \frac{K}{Y(X+Y)} \left[ 1 + \gamma^2 \frac{\pi^2}{48} \frac{Y(X+Y)}{h^2} \right]
\]

\[
\Delta Q_x \approx -\frac{R_m^2}{Q_{x0}} \frac{K}{X(X+Y)} \left[ 1 - \gamma^2 \frac{\pi^2}{48} \frac{X(X+Y)}{h^2} \right]
\]
Image currents
Ferromagnetic Boundaries

\[ B_{1n} = B_{2n}, \quad \frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2}, \quad \mu_1 \ll \mu_2, \quad B_{1,t} \approx 0 \]
Ferromagnetic Boundaries

image current

beam current
The magnetic field component $B_x$ can be calculated using the following equation:

$$B_x = \frac{\mu_0 I}{2\pi} \frac{1}{2g - y}$$
In the equation of motion

\[ B_x = \frac{\mu_0 I}{2\pi} \sum_{n=1}^{\infty} \left( \frac{1}{2ng - y} - \frac{1}{2ng + y} \right) \]

for \( g \gg y \)

\[ B_x = \frac{\mu_0 Iy}{4\pi g^2} \frac{\pi^2}{6} \]

In the equation of motion

\[ \frac{d^2 y}{ds^2} + k_y y = \frac{2K}{Y(X + Y)}y - \frac{1}{m\gamma v_0^2} v_z B_x \]
therefore

\[ \frac{d^2 y}{ds^2} + k_y y = \frac{2K}{Y(X + Y)} y + K \frac{2\gamma^2 \beta^2 \pi^2}{24g^2} y \]

incoherent SC

ferromagnetic induced image current (coherent force)

Tune-shift !
Coherent Motion
Coherent motion

2D beam
charge density $\rho_L$
all image charges moves
Coherent motion
all image charges moves

Force on the beam
NO FORCE CREATES ON THE BEAM
all image charges moves

Force on the beam

\[ E_{y,n} = -\frac{\rho L}{2\pi \varepsilon_0} \left[ \frac{1}{2nh + 2y} - \frac{1}{2nh - 2y} \right] \]

\( n = 1, 3, 5, 7, \ldots \)

\[ E_{y,n} = \frac{\rho L}{2\pi \varepsilon_0} \frac{4y}{(2nh)^2 - (2y)^2} \]

\[ E_{y,n} = \frac{\rho L}{2\pi \varepsilon_0 h^2} y \frac{1}{n^2} \]

\( n = 1, 3, 5, 7, \ldots \)
therefore

\[ E_{y,n} = \frac{\rho_L}{4\pi \varepsilon_0 h^2 y} \left[ \frac{1}{n^2} - \frac{(-1)^n}{n^2} \right] \]

(trick!)

with \ n = 1, 2, 3, 4, 5, 6, ...

\[ E_y = \sum_{n=1}^{\infty} E_{y,n} = \frac{\rho_L}{4\pi \varepsilon_0 h^2 y} \left[ \frac{\pi^2}{6} + \frac{\pi^2}{12} \right] = \frac{\rho_L}{16\pi \varepsilon_0 h^2 \pi^2 y} \]

The electric field $E_x$ due to coherent shift is zero on the center of mass ☺️
equation of motion

\[ \frac{d^2 y_c}{ds^2} + k_y y_c = \frac{e}{m \gamma v_0^2} \frac{\rho_L}{16 \pi \epsilon_0 h^2} \pi^2 y_c \]

but

\[ I = v_z \rho_L \simeq v_0 \rho_L \]

\[ \frac{d^2 y_c}{ds^2} + k_y y_c = K \frac{\gamma^2 \pi^2}{8 h^2} y_c \]

\[ \frac{d^2 y_c}{ds^2} + \left( k_y - 2K \frac{\gamma^2 \pi^2}{16 h^2} \right) y_c = 0 \]
Coherent detuning

\[ \Delta Q_{y,c} \sim - \frac{R_m^2}{Q_y} K \frac{\gamma^2 \pi^2}{16\hbar^2} \]
The Collective Effects

Thanks to Oliver Boine-Frankenheim, I. Hofmann, U. Niedermayer, D. Brandt
Interaction of the beam with the environment

Direct self fields

Image self fields

Space Charge

Wake fields
Effect on the dynamics
Resistive wall effect: Finite conductivity

Narrow-band resonators: Cavity-like objects

Broad-band resonators: Tapers, other non-resonant structures
Wake Field
Cavities
Cavities
Cavities
Cavities
Cavities
Cavities
Cavities
Cavities
Cavities
Cavities
Cavities
All together

\[ \text{Beam} \]
RLC Features

- **Resonance frequency**
  \[ \omega_r = \frac{1}{\sqrt{LC}} \]

- **Quality factor**
  \[ Q = R\sqrt{\frac{C}{L}} \]

- **Damping rate**
  \[ \alpha = \frac{\omega_r}{2Q} \]
\[ V(t) = e^{-\alpha t} \left[ A \cos \left( \omega_r \sqrt{1 - \frac{1}{4Q^2} t} \right) + B \sin \left( \omega_r \sqrt{1 - \frac{1}{4Q^2} t} \right) \right] \]
Response to one particle

What happen when one particle goes through the cavity?

Before

\[ I(t) = q\delta(t) \]

\[ V(0^-) = 0 \]

\[ \dot{V}(0^-) = 0 \]
Response to one particle

\[ I(t) = q\delta(t) \]

\[ V(0^+) = \frac{q}{C} \]

\[ \dot{V}(0^+) = -\frac{2\omega_r k_{pm}}{Q} q \]
Pulse Response

\[ V(t) = 2qk_{pm}e^{-\alpha t} \left[ \cos \left( \omega_r \sqrt{1 - \frac{1}{4Q^2} t} \right) - \frac{\sin \left( \omega_r \sqrt{1 - \frac{1}{4Q^2} t} \right)}{2Q \sqrt{1 - \frac{1}{4Q^2}}} \right] \]

This is the potential in the cavity

Green or wake function

\[ G(t) = \frac{V(t)}{q} \]
\[ G(t)q = -\int_{z_1}^{z_2} E_z(z, t) \, dz \]
Summary

The wake function tells us what is the longitudinal field experienced by another particle passing through the cavity later.
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Impedance
Impedance

It is a quantity that relate $V$ and $I$

\[
\omega = 0 \quad \rightarrow \quad V = RI
\]

\[
\omega > 0
\]

\[
V(t) = \hat{I} R \frac{\cos(\omega t) + Q \frac{\omega^2 - \omega_r^2}{\omega_r \omega} \sin(\omega t)}{1 + Q^2 \left( \frac{\omega^2 - \omega_r^2}{\omega_r \omega} \right)^2}
\]
Impedance

\[ V(t) = Z_r(\omega) \hat{I} \cos(\omega t) - Z_i(\omega) \hat{I} \sin(\omega t) \]

\[ Z_r(\omega) = R \frac{1}{1 + Q^2 \left( \frac{\omega^2 - \omega_r^2}{\omega_r \omega} \right)^2} \]

\[ Z_i(\omega) = -R \frac{Q \frac{\omega^2 - \omega_r^2}{\omega_r \omega}}{1 + Q^2 \left( \frac{\omega^2 - \omega_r^2}{\omega_r \omega} \right)^2} \]
Properties

\[ Z(\omega) \]

\[ Z_r(\omega) \]

\[ Z_i(\omega) \]

\[ \omega_r \]
Properties

At  $\omega = \omega_r$

\[
\begin{cases}
  Z_i(\omega_r) & \text{is zero} \\
  Z_r(\omega_r) & \text{is maximum}
\end{cases}
\]

$0 < \omega < \omega_r$  \quad $Z_i(\omega) > 0$  \quad inductive

$\omega > \omega_r$  \quad $Z_i(\omega) < 0$  \quad capacitive

$Z_r(\omega) = Z_r(-\omega)$  \quad $Z_i(\omega) = -Z_i(-\omega)$
Power dissipated

\[ V(t)I(t) = \hat{I}^2 R \frac{\cos^2(\omega t) + Q \frac{\omega^2 - \omega_r^2}{\omega_r \omega} \sin(\omega t) \cos(\omega t)}{1 + Q^2 \left( \frac{\omega^2 - \omega_r^2}{\omega_r \omega} \right)^2} \]

\[ V(t)I(t) = \hat{I}^2 Z_r(\omega) \cos^2(\omega t) + \hat{I}^2 Z_i(\omega) \sin(\omega t) \cos(\omega t) \]

The power dissipated depends on the resistive impedance

\[ < V(t)I(t) >_{cycle} = \frac{1}{2} \hat{I}^2 Z_r(\omega) \]
Complex notation

\[ Z(\omega) = Z_r(\omega) + iZ_i(\omega) \]

If \( Q \) is very large only for \( \omega \) close to \( \omega_r \),

\[ \frac{\omega^2 - \omega_r^2}{\omega_r \omega} = \frac{(\omega - \omega_r)(\omega + \omega_r)}{\omega_r \omega} \approx \frac{2\Delta\omega}{\omega_r} \]

\[ Z(\omega) = R \frac{1 - iQ \frac{\omega^2 - \omega_r^2}{\omega_r \omega}}{1 + Q^2 \left(\frac{\omega^2 - \omega_r^2}{\omega_r \omega}\right)^2} = R \frac{1 - i2Q \frac{\Delta\omega}{\omega_r}}{1 + 4Q^2 \left(\frac{\Delta\omega}{\omega_r}\right)^2} \]
Green function

\[ G(t) = \frac{1}{2k_{pm}} \]

Impedance

\[ Z(\omega) = Z_R(\omega) + iZ_I(\omega) \]

\[ \frac{Z(\omega)}{R_s} \]

\[ \omega/\omega_r \]

For \( Q = 3.0 \)

For \( Q = 15.0 \)
Wake potential $\leftrightarrow$ Impedance

Charge through the cavity at $t' \quad t > t' > 0 \quad dq(t') = I(t') \, dt'$

The wake of that charge at time $t$ is $G(t - t')$

The potential in the cavity at time $t$ due to the charge passing at $t'$ is $dq(t')G(t - t')$

The total potential due to all charges passing through the cavity is

$$V(t) = \int_{0}^{t} dq(t')G(t - t')$$
If now the current $I$ is

$$I(t') = \hat{I} e^{i\omega t'}$$

then

$$V(t) = \int_0^t \hat{I} e^{i\omega t'} G(t - t') dt'$$

with some change of variable

$$V(t) = I(t) \int_0^t e^{-i\omega \tau} G(\tau) d\tau$$

We wait long enough that transient effect disappears, hence

$$Z(\omega) = \frac{V(t)}{I(t)} = \int_0^\infty e^{-i\omega \tau} G(\tau) d\tau$$
Complicated geometries of the vacuum chamber give an effect on the beam which is described by the impedance \( Z(\omega) \)
Consequences of impedances

Energy loss on pipes → heating (important if you have a superconducting machine!)
Consequences of impedances

Feed-back to the beam as a hole: collective effects

Impedance

Dynamics of the all beam is affected

We have seen the longitudinal impedance in a cavity

More types of impedances ...
Longitudinal dynamics
Longitudinal dynamics

synchronous orbit

\[ R_0 \]

\[ T_0 \]
\[ \omega_0 \]
\[ p_0 \]
\[ E_0 \]
Longitudinal dynamics

This property comes from the magnets

\[
\frac{\delta C}{C'} = \alpha_c \frac{\delta p}{p}
\]
Longitudinal dynamics

\[ C + \delta C \]

\[ \rho + \delta \rho \]

\[ v + dv \]
Nobody can go faster than light

If this is large, this velocity will always be less than “c”

Therefore at a certain point the circumference growth but the particle speed remain “c”

It takes longer to make one turn!
\[
\frac{\delta T}{T_0} = \frac{1}{T_0} \delta \left( \frac{L}{v} \right) = \left( \alpha_c - \frac{1}{\gamma^2} \right) \frac{\delta p}{p} = \eta \frac{\delta p}{p}
\]

If \( \alpha_c = \frac{1}{\gamma^2} \) we are at the transition energy \( E_T \)

If \( E < E_T \) increasing energy \( \rightarrow \) revolution time shorter

If \( E > E_T \) increasing energy \( \rightarrow \) revolution time longer !!
synchronous orbit

\[ R_0 \]

\[ \begin{align*}
& T_0 \\
& \omega_0 \\
& p_0 \\
& E_0
\end{align*} \]
The synchronous particle has energy $E$ and goes through the cavity at time $t_s$

Voltage in the cavity $V = \hat{V} \sin(h\omega_0 t_s)$

$\phi_s = h\omega_0 t_s$ this is the phase of the synchronous particle

This is a phase we know each time the particle goes through the cavity
Non synchronous particle

slower particle (if below transition)
Voltage on the particle

\[ V = \hat{V} \sin(\phi_s + h\omega_0 \tau) \]

Gain of energy

\[ \delta E = e\hat{V} \sin(\phi_s + h\omega_0 \tau) \]

Now we include an energy loss per turn an per particle \( U \)

\[ \delta E = e\hat{V} \sin(\phi_s + h\omega_0 \tau) - U \]

Define relative energy \( \epsilon = \frac{\Delta E}{E_0} \)

\[ \delta \epsilon = \frac{e\hat{V}}{E_0} \sin(\phi_s + h\omega_0 \tau) - \frac{U}{E_0} \]
\[ \frac{\delta \varepsilon}{T_0} = \frac{e \hat{V}}{T_0 E_0} \sin(\phi_s + h\omega_0 \tau) - \frac{U}{T_0 E_0} \]

If \( \frac{\delta \varepsilon}{T_0} \) is small, than this term is equal to the time derivative of \( \varepsilon \)

\[ \dot{\varepsilon} = \frac{e \hat{V} \omega_0}{2\pi E_0} \sin(\phi_s + h\omega_0 \tau) - \frac{\omega_0 U}{2\pi E_0} \]

but \( U \), depends on \( \varepsilon \), and \( \tau \) \text{  } \rightarrow \text{  } U(\varepsilon, \tau) \]
If $\epsilon$, and $\tau$ are small we can expand

$$\dot{\epsilon} = \frac{e\hat{V}\omega_0}{2\pi E_0} \sin(\phi_s) + \frac{e\hat{V}\omega_0}{2\pi E_0} \cos(\phi_s) h\omega_0 \tau - \frac{\omega_0 U_0}{2\pi E_0} \tau - \frac{\omega_0}{2\pi E_0} \frac{\partial U}{\partial E} \epsilon - \frac{\omega_0}{2\pi E_0} \frac{\partial U}{\partial t} \tau$$

These two terms are equal for the synchronous particle.

We remain with the equation

$$\dot{\epsilon} = \frac{e\hat{V} h\omega_0^2}{2\pi E_0} \cos(\phi_s) \tau - \frac{\omega_0}{2\pi E_0} \frac{\partial U}{\partial E} \epsilon - \frac{\omega_0}{2\pi E_0} \frac{\partial U}{\partial t} \tau$$

In addition at high energy

$$\frac{\delta T}{T} \approx \eta \frac{\delta E}{E}$$

$$\dot{\tau} = \eta \epsilon$$
\[
\ddot{\tau} = \eta \frac{e \hbar \omega_0^2}{2\pi E_0} \cos(\phi_s) \tau - \eta \frac{\omega_0}{2\pi E_0} \frac{\partial U}{\partial \epsilon} \tau - \eta \frac{\omega_0}{2\pi E_0} \frac{\partial U}{\partial t} \tau
\]

\[
\omega_{s0}^2 = -\eta \frac{e \hbar \omega_0^2}{2\pi E_0} \cos(\phi_s)
\]

\[
\alpha_s = \frac{1}{2} \frac{\omega_0}{2\pi E} \frac{\partial U}{\partial E}
\]

Final equation of motion (in tau)

\[
\ddot{\tau} + 2\alpha_s \dot{\tau} + \left[ \omega_{s0}^2 + \eta \frac{\omega_0}{2\pi E_0} \frac{\partial U}{\partial t} \right] \tau = 0
\]
Solution

\[ \tau \propto e^{\lambda t} \quad \rightarrow \quad \lambda^2 + 2\alpha_s \lambda + \left[ \omega_{s0}^2 + \eta \frac{\omega_0}{2\pi E_0} \frac{\partial U}{\partial t} \right] = 0 \]

Solving for \( \lambda \):

\[ \lambda = -\alpha_s \pm \sqrt{\alpha_s^2 - (\omega_{s0}^2 + \ldots)} \]

that is

\[ \lambda = -\alpha_s \pm i\omega_{s1} \quad \text{with} \quad \omega_{s1}^2 = \omega_{s0}^2 + \eta \frac{\omega_0}{2\pi E_0} \frac{\partial U}{\partial t} - \alpha_s^2 \]

\[ \tau = \hat{\tau} e^{-\alpha_s t} \cos(\omega_{s1} t) \quad \rightarrow \quad \text{if} \quad \alpha_s > 0 \quad \text{Solution stable} \]
Interpretation

$E < E_T$  
No Energy Loss  
$E > E_T$
Interpretation

With Energy Loss

\[ E < E_T \]

\[ \frac{\partial U}{\partial E} \epsilon > 0 \]

\[ \frac{\partial U}{\partial E} \epsilon < 0 \]

\[ E > E_T \]
Bunch Lengthening
Bunch lengthening

\[ V = -L \frac{dI_b}{dz} \]

L is the integrated inductance

In one turn change of energy per charge
Parabolic bunch

\[ \rho \propto Q \left( 1 - \frac{z^2}{\hat{z}^2} \right) \]
Parabolic bunch

\[ I = \frac{3\pi I_0}{2\omega\hat{T}} \left( 1 - \frac{\tau^2}{\hat{T}^2} \right) \]
Voltage induced

\[ V = -L \frac{dI}{dz} \]
If we compare with RF

This creates a “longitudinal detuning”

$V = -L \frac{dI}{dz}$
By using a bunch with the same longitudinal emittance a reduction of longitudinal focusing strength produces a bunch lengthening.

The bunch becomes matched with the effective voltage slope.
Effective voltage

\[ V = \hat{V} \sin(\phi_s + h\omega_0 \tau) + \frac{3\pi I_0 L}{\omega_0 \hat{\tau}^3} \tau \]

Linearizing in \( \tau \)

\[ V = \hat{V} \sin(\phi_s) + \hat{V} \cos(\phi_s) h\omega_0 \tau + \frac{3\pi I_0 L}{\omega_0 \hat{\tau}^3} \tau \]

- Induced voltage
- Focusing from RF
- Defocusing from impedance
\[ \dot{\epsilon} = \frac{e \hat{V} \omega_0}{2\pi E_0} \cos(\phi_s)\hbar \omega_0 \tau + e \frac{\omega_0}{2\pi E_0} \frac{3\pi I_0 L}{\omega_0 \hat{\tau}^3} \tau \]

But \[ \dot{\tau} = \eta \epsilon \] therefore

\[ \ddot{\tau} = \frac{\eta e \hat{V} \omega_0}{2\pi E_0} \cos(\phi_s)\hbar \omega_0 \tau + e \frac{\eta \omega_0}{2\pi E_0} \frac{3\pi I_0 L}{\omega_0 \hat{\tau}} \tau \]
Therefore

\[
\ddot{\tau} = \frac{\eta \epsilon h \dot{V} \omega_0^2}{2\pi E_0} \cos(\phi_s) \left[ 1 + \frac{1}{\dot{V} \cos(\phi_s)} \frac{3\pi I_0}{h \omega_0^3 \hat{r}^3} \left| \frac{Z}{n} \right|_0 \right] \tau
\]

\[
\omega_{s0}^2 = -\frac{\eta \epsilon h \dot{V} \omega_0^2}{2\pi E_0} \cos(\phi_s)
\]

is the longitudinal strength in absence of impedance
\[ \omega_s^2 = \omega_{s0}^2 \left[ 1 + \frac{1}{\hat{V} \cos(\phi_s)} \frac{3\pi I_0}{h\omega_0^3 \hat{\tau}^3} \left| \frac{Z}{n} \right|_0 \right] \]

Therefore the relative change in omega is

\[ \frac{\Delta \omega_s}{\omega_{s0}} = \frac{1}{2} \frac{1}{\hat{V} \cos(\phi_s)} \frac{3\pi I_0}{h\omega_0^3 \hat{\tau}^3} \left| \frac{Z}{n} \right|_0 \]

For protons \( \hat{\tau} \hat{\epsilon} = \text{constant} \)

\[ \frac{\Delta \hat{\tau}}{\tau} \approx - \frac{\Delta \omega_s}{2\omega_s} \]
The effect of the impedance is local, hence the voltage induced by impedance do not effect the center of mass (like for the space charge)
Summary

1) Wall charges creates detuning $\rightarrow$ incoherent tunes
2) Ferromagnetic material creates image currents:
   Coherent motion $\rightarrow$ coherent tunes
3) Concept of Wake field
4) Impedance of a cavity, Wake $\leftrightarrow$ impedance
5) Energy loss
6) Longitudinal dynamics, effect of energy loss
7) Bunch lengthening