PAUL SCHERRER INSTITUT

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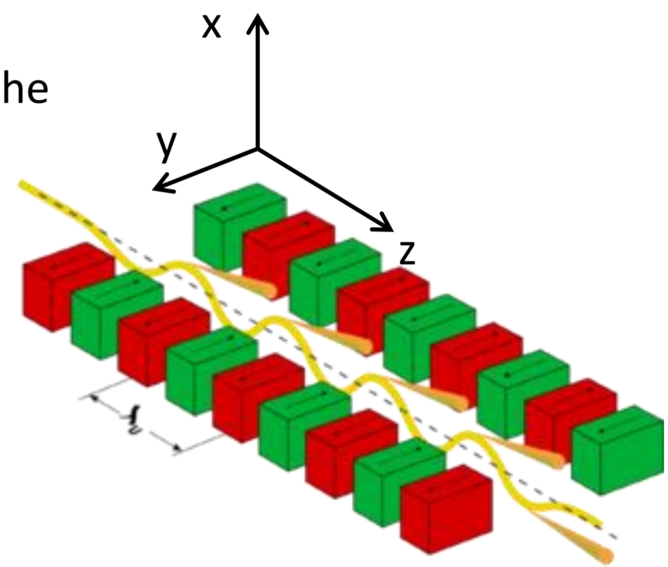
Motion in an Undulator

CERN Accelerator School – FELs and ERLs

On-axis Field of Planar Undulator

- For planar undulator with only one transverse magnetic field component, the field is given on-axis:



$$\vec{B} = B_0 \vec{e}_y \sin(k_u z)$$

with $$k_u = \frac{2\pi}{\lambda_u}$$

- Using Lorentz Force equation with the assumptions:
  - Relativistic electron beam moves primarily into *z*-direction
  - The energy of the electrons is preserved in the magnetic field.

$$\vec{F} = e\vec{v} \times \vec{B} \quad \Rightarrow \quad \gamma mc \frac{d}{dt} \beta_x = -ecB_0 \beta_z \sin(k_u z)$$

- The equation cannot be solved directly because the time-dependence of the longitudinal position *z(t)* and velocity *β(t)* is unknown.

Dominant Motion On-Axis I

- We assume that the deflection strength per module is small so that the electron still moves predominantly in the z-direction.
- The transverse motion and the resulting modulation of the longitudinal motion can be regarded as small.

$$z(t) = c\bar{\beta}_z t + \varepsilon f(t), \quad \beta_z(t) = \bar{\beta}_z + (\varepsilon / c) f'(t)$$

- The parameter and function ε and f(t) are undefined but we assume that they are sufficiently small to treat them as a perturbation.
- Later we will justify this assumption when the explicit form of ε is known.
- Because the motion in a magnetic field does not change the energy the longitudinal and transverse velocity are linked by

$$\gamma = \sqrt{\frac{1}{1-\left|\vec{\beta}\right|^2}}$$

Dominant Motion On-Axis II

- Integration of leading term in Lorentz Force equation:

$$\gamma mc \frac{d}{dt}\beta_x = -ecB_0\beta_z \sin(k_u z) \quad \Rightarrow \quad \frac{d}{dt}\beta_x \approx -\frac{eB_0}{\gamma m}\bar{\beta}_z \sin(k_u c\bar{\beta}_z t)$$

$$\Rightarrow \quad \beta_x = \frac{eB_0}{\gamma mck_u}\cos(k_u z)$$

- The physical constants and the undulator parameters are combined into the so-called **undulator parameter**

$$\boxed{K = \frac{eB_0}{mck_u}} = 0.93 \cdot B_0\left[\mathrm{T}\right] \cdot \lambda_u\left[\mathrm{cm}\right] \quad \Rightarrow \quad \beta_x = \frac{K}{\gamma}\cos(k_u z)$$

- Two comments:
  - The value of K is typically around unity
  - For a relativistic beam the maximum angle in the orbit is $x' = \beta_x / \beta_z \approx K / \gamma$

Dominant Motion On-Axis III

- Because total energy is preserved, longitudinal and transverse velocities are linked by energy:

$$\beta_x^2 + \beta_z^2 = 1 - \frac{1}{\gamma^2} \quad \Rightarrow \quad \beta_z = \sqrt{1 - \frac{1}{\gamma^2} - \beta_x^2} \approx 1 - \frac{1}{2}\left(\frac{1}{\gamma^2} + \beta_x^2\right)$$

- Using the expression of β_x and the identity $\cos(x)^2=[1+\cos(2x)]/2$, we are getting:

$$\beta_x = \frac{K}{\gamma}\cos(k_u z)$$

$$\beta_z = 1 - \frac{1+K^2/2}{2\gamma^2} - \frac{K^2}{4\gamma^2}\cos(2k_u z)$$

- The mean longitudinal velocity is given:

$$\bar{\beta}_z = 1 - \frac{1+K^2/2}{2\gamma^2}$$

The integration by perturbation is well justified because the oscillating term in longitudinal velocity remains small over the entire undulator length.

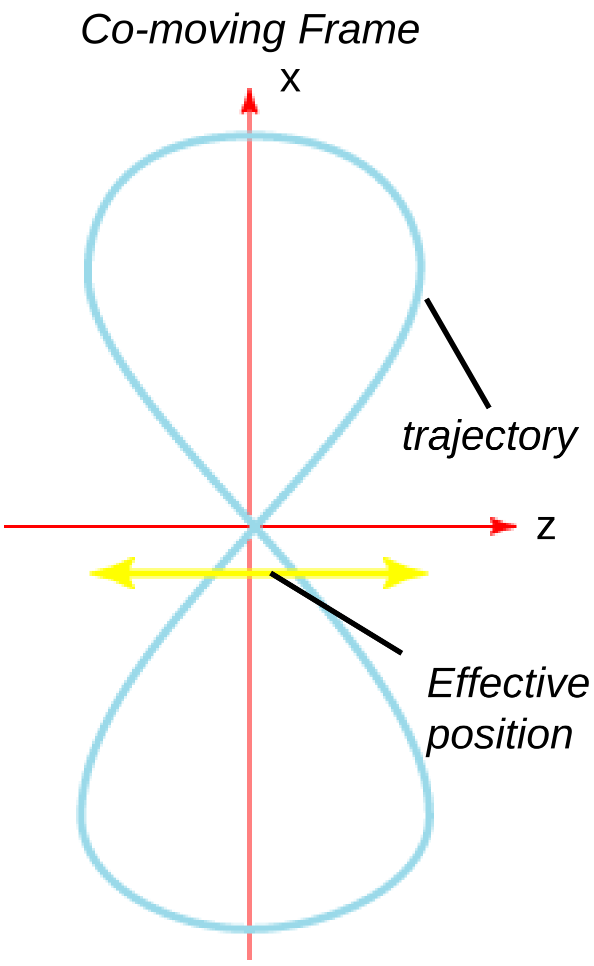
Trajectory in Planar Undulator

- Integration (by perturbation again) of the velocities in x and z yield the trajectory:

$$x(t) = \frac{K}{\gamma \bar{\beta}_z k_u} \sin(c \bar{\beta}_z k_u z(t))$$

$$z(t) = c\bar{\beta}_z t - \frac{K^2}{8\gamma^2 \bar{\beta}_z k_u} \sin(2c\bar{\beta}_z k_u t)$$

- Longitudinal wiggle motion has half period length.
- Causes a figure “8” motion in the co-moving frame.
- The longitudinal position is effectively smeared out
  – Coupling to harmonics
  – Reduced coupling to fundamental



Harmonics in Planar Undulator

- Including longitudinal oscillation term in transverse oscillation:

$$\beta_x = \frac{K}{\gamma}\cos(k_u z) = \frac{K}{\gamma}\Re e\left(e^{ik_u\bar{z}} \cdot e^{-i\chi\sin(2k_u\bar{z})}\right) \qquad \bar{z} = c\bar{\beta}_z t \qquad \chi = \frac{K^2}{8\gamma^2\bar{\beta}_z}$$

$$= \frac{K}{\gamma}\Re e\left(e^{ik_u\bar{z}}\sum_{m=-\infty}^{\infty}(-1)^m J_m(\chi)e^{i2mk_u\bar{z}}\right)$$

$$= \sum_{m=-\infty}^{\infty}(-1)^m J_m(\chi)\frac{K}{\gamma}\cos([2m+1]k_u\bar{z})$$

$$= \sum_{m=0}^{\infty}(-1)^m[J_m(\chi) - J_{m+1}(\chi)]\frac{K}{\gamma}\cos([2m+1]k_u\bar{z})$$

Identities of Bessel Function

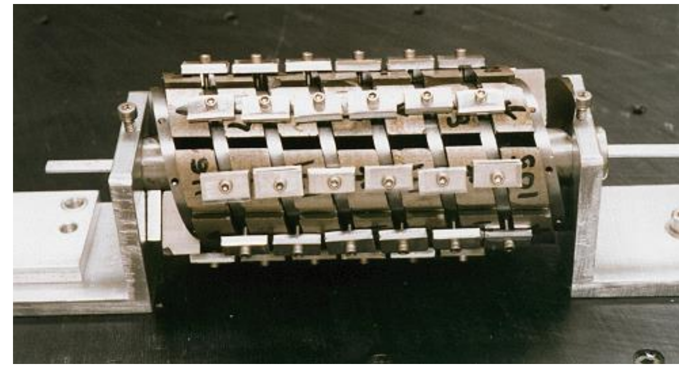
$$e^{ia\sin b} = \sum_{m=-\infty}^{\infty} J_m(a)e^{imb}$$

$$J_m(-a) = (-1)^m J_m(a)$$

- Motion has:
  - Reduced amplitude of fundamental oscillation $\qquad J_m(\chi) \ll 1 \text{ for } m \neq 0$
  - Occurrence of odd harmonics.
  - On the scale of the undulator period the harmonics are hardly noticeable (Though it becomes important with respect to a given radiation wavelength)

On-axis Motion in Helical Undulator

- Helical undulators has a transverse magnetic field, which rotates along the undulator axis:



$$\vec{B} = B_0\left[\vec{e}_y \sin(k_u z) + \vec{e}_x \cos(k_u z)\right]$$

- From the Lorentz force we obtain:

$$\frac{d}{dt}\beta_x = -\frac{ecB_0}{\gamma mc}\beta_z \sin(k_u z) \qquad \frac{d}{dt}\beta_y = \frac{ecB_0}{\gamma mc}\beta_z \cos(k_u z)$$

- Integration similar to planar undulator case:

$$\beta_x = \frac{K}{\gamma}\cos(k_u z) \qquad \beta_y = \frac{K}{\gamma}\sin(k_u z)$$

- Longitudinal velocity

$$\beta_z = \sqrt{1 - \frac{1}{\gamma^2} - \beta_x^2 - \beta_y^2} \approx 1 - \frac{1}{2}\left(\frac{1}{\gamma^2} + \beta_x^2 + \beta_y^2\right) = 1 - \frac{1+K^2}{2\gamma^2}$$

Note that there is no longitudinal oscillation

→ no harmonics are excited

Comparison Planar and Helical Undulator

- Excluding higher harmonics in case of planar undulator
- Using average position $\bar{z} \rightarrow z$ (for convenience)

	Planar	Helical
β_x	$[J_0(\chi) - J_1(\chi)] \frac{K}{\gamma} \cos(k_u z)$	$\frac{K}{\gamma} \cos(k_u z)$
β_y	0	$\frac{K}{\gamma} \sin(k_u z)$
β_z	$1 - \frac{1 + K^2/2}{2\gamma^2} - \frac{K^2}{4\gamma^2} \cos(2k_u z)$	$1 - \frac{1 + K^2}{2\gamma^2}$
$\bar{\beta}_z$	$1 - \frac{1 + K^2/2}{2\gamma^2}$	$1 - \frac{1 + K^2}{2\gamma^2}$

Off-Axis Field Components I

- The simple field dependence $\vec{B} = B_0 \vec{e}_y \sin(k_u z)$ cannot be used for the entire transverse plane because it violates Maxwell condition of free space:

$$\vec{\nabla} \times \vec{B} = 0$$

- We assume a vector potential to derive B:

$\vec{\nabla} \times \vec{A} = \vec{B}$ ⇨ Condition I: $\Delta \vec{A} = 0$ Condition II: $\vec{\nabla} \vec{A} = 0$

- Dominant vector component is in x:

$$A_x = -\frac{B_0}{k_u} \cosh(k_x x) \cosh(k_y y) \cos(k_u z)$$

Condition I: $\Delta A_x = (k_x^2 + k_y^2 - k_u^2) A_x = 0 \quad \Rightarrow \quad k_x^2 + k_y^2 = k_u^2$

Meaning of k_x and k_y will be explained later

Condition II: $\partial_x A_x = -\partial_y A_y \quad \Rightarrow \quad A_y = \frac{B_0}{k_u} \frac{k_x}{k_y} \sinh(k_x x) \sinh(k_y y) \cos(k_u z)$

Off-Axis Field Components II

- With the valid vector potential the field is:

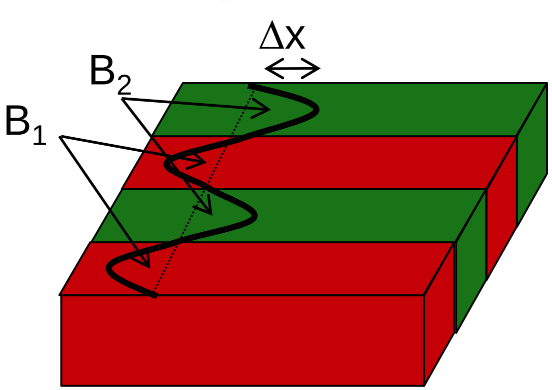
$$\vec{B} = B_0 \begin{pmatrix} \frac{k_x}{k_y} \sinh(k_x x)\sinh(k_y y)\sin(k_u z) \\ \cosh(k_x x)\cosh(k_y y)\sin(k_u z) \\ \frac{k_u}{k_y}\cosh(k_x x)\sinh(k_y y)\cos(k_u z) \end{pmatrix}$$

- It provide focusing of the electron if not injected on-axis:

Horizontal Focusing:

- Effective Field B_1 for $[x, x+\Delta x]$
- Effective Field B_2 for $[x-\Delta x, x]$
- Off-Axis: $B_1 > B_2$

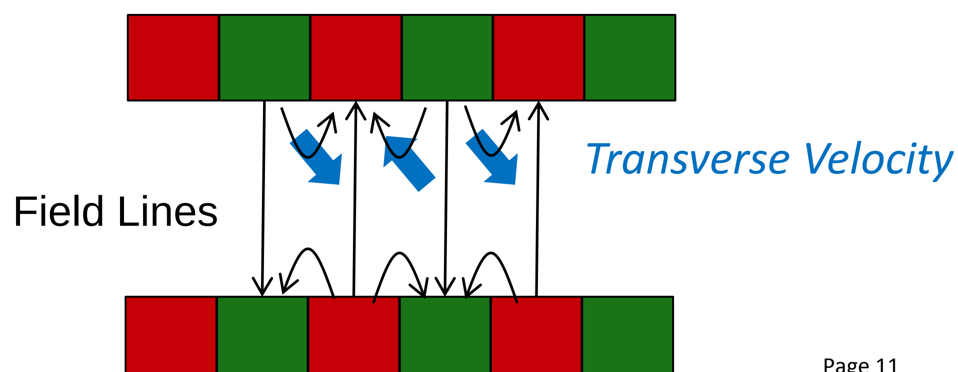
⇨ Net kick inwards



Vertical Focusing:

- 1^{st} half-period: $F_y = -ev_x B_z$
- 2^{nd} half-period: $F_y = -e(-v_x)(-B_z)$
- Off-Axis: $B_z \sim k_y y$

⇨ Net kick inwards



Curved Poles – Meaning of k_x and k_y

- Note that the magnetic field can also be derived from a scalar potential

$$\phi = -\frac{B_0}{k_y}\cosh\left(k_x x\right)\sinh\left(k_y y\right)\sin\left(k_u z\right)$$

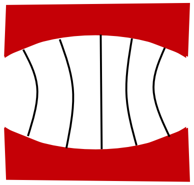
$$-\vec{\nabla}\phi = \vec{B} = B_0 \begin{pmatrix} \frac{k_x}{k_y}\sinh(k_x x)\sinh(k_y y)\sin(k_u z) \\ \cosh(k_x x)\cosh(k_y y)\sin(k_u z) \\ \frac{k_u}{k_y}\cosh(k_x x)\sinh(k_y y)\cos(k_u z) \end{pmatrix}$$

- Setting the scalar potential to a constant values defines an equipotential plane with the dependence for small transverse extensions:

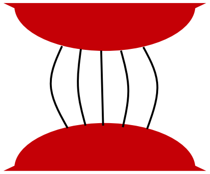
$$\sinh(k_y\, y) = \frac{c}{\cosh(k_x\, x)} \quad \Rightarrow \quad \Delta y \approx c_1 + c_2\left[1 - \frac{1}{2}k_x^2 x^2\right]$$

- The parameter k_x describes the transverse dependence on the pole surface:
  - $k_x^2 > 0$ → poles are curved inwards
  - $k_x^2 < 0$ → cosh is replaced with cos → poles are curved outwards → defocusing

$k_x^2 > 0$



$k_x^2 < 0$



Helical Undulator

- In an ideal helical undulator the focusing is symmetric with: $k_x^2 = k_y^2 = k_u^2 / 2$
- The simplest vector potential and magnetic field is:

$$\vec{A} = \begin{pmatrix} A_r \\ A_\phi \\ A_z \end{pmatrix} = \frac{B_0}{k_u} \begin{pmatrix} [I_0(k_u r) - I_2(k_u r)]\cos(\phi - k_u z) \\ [I_0(k_u r) + I_2(k_u r)]\sin(\phi - k_u z) \\ 0 \end{pmatrix} \quad \vec{B} = B_0 \begin{pmatrix} [I_0(k_u r) + I_2(k_u r)]\cos(\phi - k_u z) \\ [I_0(k_u r) - I_2(k_u r)]\sin(\phi - k_u z) \\ 2I_1(k_u r)\sin(\phi - k_u z) \end{pmatrix}$$

- However a helical undulator field is also obtained by superposition of two planar fields. If the symmetry is broken (e.g. APPLE Undulator) the roll-off parameters k_x and k_y are different for the two polarization planes.

- The sum of all coefficient in square still have still to be the square of ku.
- For APPLE type undulator the resulting net constants can be significantly larger, e.g.:

$$k_x^2 = -5k_u^2,\ k_y^2 = 6k_u^2$$

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The Hamilton Function and Electron Motion I

- The Hamilton function is a constant of motion because there is no explicit time-dependence in the vector potential (here planar undulator):

$$H=\sqrt{\left(\vec{P}-e\vec{A}\right)^2c^2+m^2c^4}=\gamma mc^2$$

$$\vec{A}=\frac{B_0}{k_u}\begin{pmatrix}-\cosh(k_x x)\cosh(k_y y)\cos(k_u z)\\ \frac{k_x}{k_y}\sinh(k_x x)\sinh(k_y y)\cos(k_u z)\\ 0\end{pmatrix}$$

- The velocities are:

$$\dot{x}=\frac{\partial}{\partial p_x}H=\frac{P_x-eA_x}{\gamma m},\quad \dot{y}=\frac{\partial}{\partial p_y}H=\frac{P_y-eA_y}{\gamma m},\quad \dot{z}=\frac{\partial}{\partial p_z}H=\frac{P_z}{\gamma m}$$

Note that the velocity term proportional A_x is exactly the fast oscillation term but now with the transverse dependence on the undulator field:

$$\beta_x=\frac{K}{\gamma}\cosh(k_x x)\cosh(k_y y)\cos(k_u z)$$

- The canonical momentum P_x describes mostly the slow betatron-oscillation

The Hamilton Function and Electron Motion II

$$H = \sqrt{\left(\vec{P} - e\vec{A}\right)^2 c^2 + m^2 c^4} = \gamma m c^2$$

$$\vec{A} = \frac{B_0}{k_u}\begin{pmatrix} -\cosh(k_x x)\cosh(k_y y)\cos(k_u z) \\ \frac{k_x}{k_y}\sinh(k_x x)\sinh(k_y y)\cos(k_u z) \\ 0 \end{pmatrix}$$

- The transverse momenta are given by:

$$\vec{F} = \dot{\vec{P}} = -\vec{\nabla} H = -\frac{1}{2\gamma m}\vec{\nabla}\left(\vec{P} - e\vec{A}\right)^2$$

$$\approx -\frac{e^2 B_0^2}{4\gamma m k_u^2}\vec{\nabla}\left(\cosh(k_x x)^2 \cosh(k_y\ \mathrm{y})^2 + (k_x / k_y)^2 \sinh(k_x x)^2 \sinh(k_y\ \mathrm{y})^2\right)$$

- Some terms have been dropped or simplified for the evaluation of the “slow” betatron motion:

$$\frac{\partial}{\partial x} p_x = 0 \qquad \left\langle \cos\left(k_u z\right)^2 \right\rangle = \frac{1}{2} \qquad p_x \frac{\partial}{\partial x} A_x \propto \cos\left(k_u z\right) \Rightarrow \left\langle p_x \frac{\partial}{\partial x} A_x \right\rangle = 0$$

Electron Motion for Small Amplitudes

- The resulting betatron equations of motion become:

$$\ddot{x} = \frac{\dot{P}_x}{\gamma m} = -\frac{e^2 B_0^2}{2\gamma^2 m^2 k_u^2} k_x \sinh(k_x x)\cosh(k_x x)\left(\cosh(k_y\, y)^2 + (k_x / k_y)^2 \sinh(k_y\, y)^2\right)$$

- For small amplitudes in *x*:

$$\ddot{x} \approx -c^2 \frac{K^2}{2\gamma^2} k_x^2 x$$

- Similar calculation for *y*:

$$\ddot{y} \approx -c^2 \frac{K^2}{2\gamma^2} k_y^2 y$$

Natural focusing of undulators

Total focusing strength is given by undulator period:

$$k_x^2 + k_y^2 = k_u^2$$

Betatron Motion for Natural Focusing

- The transport matrix for quadrupole focusing is given by

$$M = \begin{pmatrix} \cos(\Omega_x z) & \frac{1}{\Omega_x}\sin(\Omega_x z) \\ -\Omega_x \sin(\Omega_x z) & \cos(\Omega_x z) \end{pmatrix}$$

$$\Omega_x = \frac{K}{\sqrt{2}\gamma\bar{\beta}_z} k_x$$

$$x'' = \frac{\ddot{x}}{c^2 \bar{\beta}_z^2}$$

Note: beam energy and long. velocity

- Matching condition is:

$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = M \cdot \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} \cdot M^T \Rightarrow \beta = \frac{2M_{12}}{\sqrt{2 - M_{11}^2 - M_{22}^2 - 2M_{12}M_{21}}} = \frac{1}{\Omega_x}, \quad \alpha = 0, \quad \gamma = \frac{1}{\beta}$$

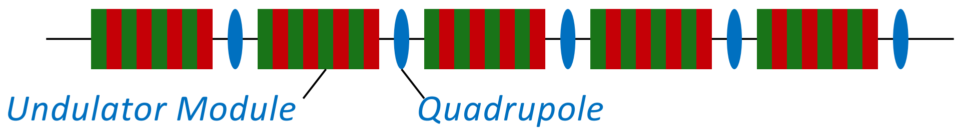
Note: twiss parameter

– At about 100 MeV the matched betatron function is around 1 m
– At 10 GeV it is 100 m

For X-ray FELs there is the need for external focusing to reduced the electron beam size

External Focusing

- If the natural focusing is not sufficient, superimposed quadrupole fields can provide more focusing.



- Equations of motions for the slow betatron-oscillation are:

$$x'' = \left(\Omega_Q^2(z) + \Omega_x^2(z)\right)x, \quad y'' = \left(-\Omega_Q^2(z) + \Omega_y^2(z)\right)y$$

Formal solution of x:

$$x(z) = \sqrt{I_x \beta_x(z)} \cos\left(\Psi_x(z) + \phi_x\right)$$

$$x'(z) = \frac{p_x}{p_z} = -\sqrt{\frac{I_x}{\beta_x(z)}}\left[\alpha(z)\cos\left(\Psi_x(z) + \phi_x\right) + \sin\left(\Psi_x(z) + \phi_x\right)\right]$$

Special case: Natural focusing only

$$x(z) = \sqrt{\frac{I_x}{\Omega_x}} \cos\left(\Omega_x z + \phi_x\right)$$

$$x'(z) = -\sqrt{I_x \Omega_x} \sin\left(\Omega_x z + \phi_x\right)$$

Longitudinal Velocity for Natural Focusing

- Including betatron-motion in longitudinal velocity:

$$\beta_z = \sqrt{1 - \frac{1}{\gamma^2} - \beta_x^2 - \beta_y^2} \approx 1 - \frac{1}{2\gamma^2} - \frac{1}{2}\left[\beta_x^2 + \beta_y^2\right]$$

$$= 1 - \frac{1 + K^2/2}{2\gamma^2} - \frac{K^2}{4\gamma^2}\cos(2k_u z) - \frac{1}{2}(x')^2 \beta_z^2 + \frac{Kx'\beta_z}{\gamma}\sin(k_u z) - \frac{1}{2}(y')^2 \beta_z^2$$

Betatron motion

$$x' = \frac{p_x}{p_z} = \frac{\beta_x}{\beta_z}$$

- Averaging out all fast oscillating terms

$$\beta_z = 1 - \frac{1 + K^2/2}{2\gamma^2} - \frac{K^2 k_x^2}{4\gamma^2}\frac{I_x}{\Omega_x}\sin\left(\Omega_x z + \phi_x\right)^2 - \frac{K^2 k_y^2}{4\gamma^2}\frac{I_y}{\Omega_y}\sin\left(\Omega_y z + \phi_y\right)^2$$

- The K-value should be evaluated at the position of the electron with:

$$K(x,y) = K\cosh(k_x x)\cosh(\mathrm{k}_y\, y) \quad \Rightarrow$$

$$K(x,y)^2 \approx K^2\left(1 + k_x^2 x^2 + k_y^2 y^2\right) = K^2 + K^2 k_x^2 \frac{I_x}{\Omega_x}\cos(\Omega_x z + \phi_x)^2 + K^2 k_y^2 \frac{I_y}{\Omega_y}\cos(\Omega_y z + \phi_y)^2$$

$$\beta_z = 1 - \frac{1 + K^2/2}{2\gamma^2} - \frac{K^2}{4\gamma^2}\left[k_x^2 \frac{I_x}{\Omega_x} + k_y^2 \frac{I_y}{\Omega_y}\right]$$

No dependence on the betatron-phase