1.2.5 Temporal coherence

We have seen that spatial coherence measures the correlation of the field at two separate spatial locations. In a similar manner, temporal coherence specifies the extent to which the radiation maintains a definite phase relationship at two different times. Temporal coherence is characterized by the coherence time, which can be experimentally determined by measuring the path length difference over which fringes can be observed in a Michelson interferometer. A simple representation of a coherent wave in time is given by

$$E_0(t) = e_0 \exp\left(-\frac{t^2}{4\sigma_\tau^2} - i\omega_1 t\right). \tag{1.100}$$

Here σ_τ is the rms temporal width of the intensity profile $|E_0(t)^2|$. The coherence time t_{coh} can be defined as

$$t_{\text{coh}} \equiv \int d\tau \ |\mathcal{C}(\tau)|^2, \tag{1.101}$$

where $\mathcal{C}(\tau)$ is the normalized, first order correlation function (or complex degree of temporal coherence) given by

$$\mathcal{C}(\tau) \equiv \frac{\left\langle \int dt \ E(t)E^*(t+\tau)\right\rangle}{\left\langle \int dt \ |E(t)|^2\right\rangle}, \tag{1.102}$$

and the brackets denote ensemble averaging. In the simple Gaussian model of Eq. (1.100), the coherence time $t_{\text{coh}} = 2\sqrt{\pi}\sigma_\tau$.

In the frequency domain, we have

$$E_\omega^0 = \int dt \ e^{i\omega t} E_0(t) = \frac{e_0\sqrt{\pi}}{\sigma_\omega} \exp\left[-\frac{(\omega-\omega_1)^2}{4\sigma_\omega^2}\right], \tag{1.103}$$

where $\sigma_\omega = (2\sigma_\tau)^{-1}$ is the rms width of the frequency profile $|E_\omega|^2$. Let us introduce the temporal (longitudinal) phase space variables ct and $(\omega-\omega_1)/\omega_1 = \Delta\omega/\omega_1$. The Gaussian wave packet then satisfies

$$c\sigma_\tau \cdot \frac{\sigma_\omega}{\omega_1} = \frac{\lambda_1}{4\pi}, \tag{1.104}$$

which is the same phase space area relationship as (1.58) obtained for a transversely coherent Gaussian beam.

Most radiation observed in nature, however, is temporally incoherent. Sunlight, fluorescent light bulbs, black-body radiation, and undulator radiation (which we study in the next chapter) are all temporally incoherent, and are often referred to as chaotic light or as a partially coherent wave. As a mathematical model of such chaotic light, we consider a collection of coherent Gaussian pulses that are displaced randomly in time with respect to each other:

$$E(t) = \sum_{j=1}^{N_e} E_0(t-t_j) = e_0 \sum_{j=1}^{N_e} \exp\left[-\frac{(t-t_j)^2}{4\sigma_\tau^2} - i\omega_1(t-t_j)\right]. \tag{1.105}$$

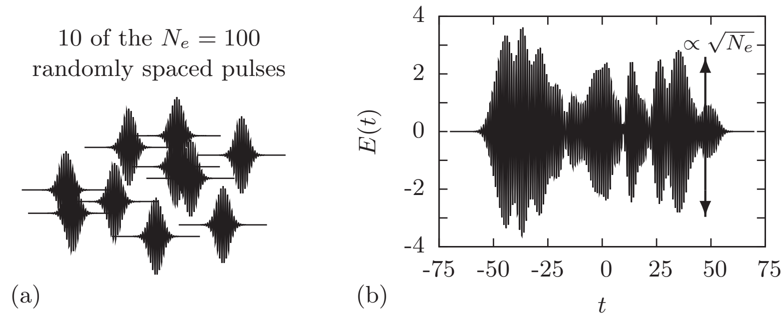

Figure 1.12 (a) Representation of the randomly phased wave packets that chooses 10 out of the 100 total waves. The individual waves are shown transversely displaced for illustration purpose only. (b) Total electric field, given by the incoherent sum of the 100 wave packets. The field consists of order $T/4\sigma_\tau \approx 10$ regular regions (i.e., $M_L \approx 10$ longitudinal modes).

In Eq. (1.105), t_j is a random number, and the sum extends to N_e to suggest that these wave packets have been created by electrons. We illustrate this partially coherent wave (chaotic light) in Fig. 1.12, which we obtained by using $N_e = 100$ wave packets with $\lambda_1 = 2\pi/\omega_1 = 1$ and $\sigma_\tau = 2$ ($\sigma_\omega = 0.25$), assuming that the t_j's are randomly distributed with equal probability over the bunch length duration $T = 100$. Panel (a) shows 10 randomly chosen such wave packets; plotting many more than this results in a jumbled disarray. Fig. 1.12(b) shows the $E(t)$ that results by summing over all 100 waves. The remarkable feature of this plot is that the resultant wave is a relatively regular oscillation that is interrupted only a few times, much fewer than one might have naively guessed based on the fact that it is a random superposition of 100 wave packets. In fact, the duration of each regular region is independent of the number of wave packets, and is instead governed by the time over which the wave maintains a definite phase relationship, namely, the coherence time. Note that the coherence time of a random collection of Gaussian waves (1.105) equals that of the single mode (1.100). Thus, each regular region can be identified with a coherent mode whose temporal width is of order the coherence time t_{coh}. The number of regular regions equals the number of coherent longitudinal modes M_L, which is roughly the ratio of the bunch length to the coherence length. Approximately, we have

$$M_L \approx \frac{T}{t_{\text{coh}}} = \frac{T}{2\sqrt{\pi}\sigma_\tau} \approx \frac{T}{4\sigma_\tau}. \tag{1.106}$$

The average field intensity scales linearly with the number of sources, while the instantaneous intensity fluctuates as a function of time. Associated with this intensity variation will be a fluctuation in the observed number of photons $\mathcal{N}_{\text{ph}}$ over a given time. Denoting the average photon number by $\langle\mathcal{N}_{\text{ph}}\rangle$, the rms

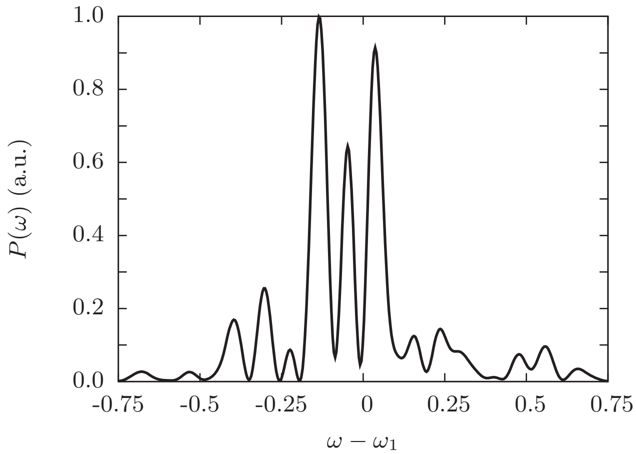

Figure 1.13 Intensity spectrum of Eq. (1.110) using identical parameters as Fig. 1.12(b). The spectrum consists of $M \sim 10$ sharp frequency spikes of approximate width $2/T \approx 0.02$, which are distributed within a Gaussian envelope of rms width $\sigma_\omega \sim 0.25$. The height and placement of the spectral peaks fluctuate by 100% for different sets of random numbers.

squared fluctuation in the number of photons observed is

$$\sigma^2_{\mathcal{N}_{\rm ph}} = \frac{\langle \mathcal{N}_{\rm ph} \rangle^2}{M_L}, \tag{1.107}$$

where M_L is the number of longitudinal modes in the observation time T.

The formula (1.107) for the photon number variation can be generalized in two respects. First, the mode counting must include the number of transverse modes M_T in both the x and y directions, so that the total number of modes

$$M = M_L M_T^2. \tag{1.108}$$

Second, there are inherent intensity fluctuations arising from quantum mechanical uncertainty in the form of photon shot noise. This number uncertainty is attributable to the discrete quantum nature of electromagnetic radiation, and, like any shot noise, it adds a contribution to $\sigma^2_{\mathcal{N}_{\rm ph}}$ equal to the average number $\langle \mathcal{N}_{\rm ph} \rangle$. Thus, the rms squared photon number fluctuation is

$$\sigma^2_{\mathcal{N}_{\rm ph}} = \frac{\langle \mathcal{N}_{\rm ph} \rangle^2}{M} + \langle \mathcal{N}_{\rm ph} \rangle = \frac{\langle \mathcal{N}_{\rm ph} \rangle^2}{M} \left(1 + \frac{1}{\delta_{\rm degen}} \right). \tag{1.109}$$

The second term in parentheses is the inverse of the number of photons per mode, which is also known as the degeneracy parameter. In the classical devices that we consider there are many photons per mode, $\langle \mathcal{N}_{\rm ph} \rangle / M \equiv \delta_{\rm degen} \gg 1$, and the fluctuations due to quantum uncertainty are negligible. In this classical limit the length of the radiation pulse can be determined by measuring its intensity fluctuations, from which the source electron beam length may be deduced [11].

It is interesting to note that the mode counting we performed in the time domain can also be done in the frequency domain. Figure 1.13 shows the intensity spectrum $P(\omega) \propto |E_\omega|^2$, where

$$E_\omega = \frac{e_0\sqrt{\pi}}{\sigma_\omega} \sum_{j=1}^{N_e} \exp\left[-\frac{(\omega-\omega_1)^2}{4\sigma_\omega^2} + i\omega t_j\right] \tag{1.110}$$

using the same wave parameters as in Fig. 1.12. The spectrum consists of sharp peaks of width $\Delta\omega \sim 2/T$ that are randomly distributed within the radiation bandwidth $\sigma_\omega = (2\sigma_\tau)^{-1}$. In other words, the frequency bandwidth $\Delta\omega$ of each mode is set by the duration of the entire radiation pulse T, while the frequency range over which the modes exist is given by the inverse coherence time. Thus, the total number of spectral peaks is the same as the number of the coherent modes in the time domain.

1.2.6 Bunching and intensity enhancement

Let us calculate the average electric field intensity generated by many electrons as expressed in Eq. (1.110). Defining $\left|E_\omega^0\right|^2$ to be the intensity due to a single electron, we have

$$\left\langle \left|E(\omega)\right|^2 \right\rangle = \left|E_\omega^0\right|^2 \left\langle \left| \sum_{j=1}^{N_e} e^{i\omega t_j} \right|^2 \right\rangle, \tag{1.111}$$

where the angular bracket denotes an ensemble average over many instances of the initial particle distribution. Dividing the double sum into the piece where the particles are identical (the phase factor being unity) and the remaining terms, we obtain

$$\left\langle \left| \sum_{j=1}^{N_e} e^{i\omega t_j} \right|^2 \right\rangle = N_e + \left\langle \sum_{j\neq k}^{N_e} e^{i\omega(t_j - t_k)} \right\rangle. \tag{1.112}$$

We assume that the electrons are uncorrelated, so that the probability of finding any electron at position t_j is independent of the positions of all the other electrons. Thus, the temporal statistics are completely specified by the single particle probability distribution function $f(t)$, and the sum in Eq. (1.112) can be expressed as $N_e(N_e-1)$ identical integrals over f:

$$\left\langle \left| \sum_{j\neq k}^{N_e} e^{i\omega(t_j - t_k)} \right|^2 \right\rangle = N_e(N_e-1) \left| \int dt\, f(t) e^{i\omega t} \right|^2 \tag{1.113}$$

$$= N_e(N_e-1) \left|f(\omega)\right|^2. \tag{1.114}$$

This expression is general for an arbitrary collection of electrons that are independently distributed in time according to $f(t)$. To get a physical understanding of (1.113), we consider a Gaussian distributed electron bunch,

$$f(t) = \frac{1}{\sqrt{2\pi}\,\sigma_e} \exp\left(-\frac{t^2}{2\sigma_e^2}\right),$$

where σ_e is the bunch length. Carrying out the Gaussian integral, we find

$$\left\langle \left|E(\omega)\right|^2 \right\rangle = N_e \left|E_\omega^0\right|^2 \left[1 + (N_e - 1)e^{-\omega^2\sigma_e^2}\right]. \tag{1.115}$$

Typically, we have

$$(N_e - 1)e^{-\omega^2\sigma_e^2} \ll 1$$

for the frequency range we are interested in. As an example, one nano-Coulomb of charge has $N_e \sim 10^{10}$, in which case $N_e e^{-\omega^2\sigma_e^2} \sim 1$ when $c\sigma_e \sim \lambda$. Therefore, at wavelengths much shorter than the electron bunch length the second term in (1.115) is negligible, and the average radiation intensity due to N_e electrons is simply N_e times the intensity calculated for a single electron. This follows the usual rule of intensity addition for incoherent radiation arising from unbunched electron beams.

If, however, the bunch length becomes comparable to the radiation wavelength, we may have

$$(N_e - 1)e^{-\omega^2\sigma_e^2} \geq 1,$$

resulting in a significant intensity enhancement over the incoherent case. In the extreme case where $c\sigma_e \ll \lambda$, i.e., in the limit of a vanishing bunch length, the intensity equals $N_e^2 \left|E_\omega^0\right|^2$, leading to an enhancement over the incoherent case by a factor of the number of electrons N_e. Two examples of situations in which such an intensity enhancement is often observed include coherent transition radiation, produced when an electron bunch traverses an interface between two media with differing indices of refraction, and coherent synchrotron radiation, generated when the beam trajectory is bent in a magnetic field. However, the coherent radiation produced through such processes is limited to the optical region of the spectrum $\lambda \gtrsim 300$ nm even for very high-charge, single femto-second electron beams.

For these reasons, typical synchrotron radiation sources generate temporally incoherent x-rays whose average intensity scales as the number of electrons in the bunch. We will discuss the properties of such incoherent radiation in the next chapter, both from simple bending magnets and from undulators, which are a periodic series of magnets with alternating polarities. There is, however, another mechanism to produce coherent radiation at wavelengths much shorter than the electron bunch. If we again consider either expression (1.113) or (1.114), we find that the coherent term scales as the absolute square of the Fourier transform of the distribution function. Thus, one can observe a coherent intensity enhancement of $\left|E_\omega^0\right|^2$ if the distribution function $f(t)$ has structure (microbunching) at the frequency ω:

$$\left\langle \left|E(\omega)\right|^2 \right\rangle = N_e \left|E_\omega^0\right|^2 \left(1 + (N_e - 1)\left|f(\omega)\right|^2\right). \tag{1.116}$$

Free-electron lasers (FELs) are devices in which the electron beam distribution develops a periodic density modulation on the scale of the radiation wavelength, resulting in a coherent enhancement of the intensity. The density modulation arises from the resonant interaction of an electron beam with the x-rays in a periodic undulator; we will see that if the undulator is sufficiently long, the bunch current is sufficiently high, and the e-beam phase space is of sufficient quality (small emittance and energy spread), than the radiation acts on the particles to generate a periodic density modulation whose length scale is near the resonant x-ray wavelength. This leads to a significant intensity enhancement as compared to the incoherent undulator radiation, even though the electron bunch length is much longer than the radiation wavelength.