Introduction to Transverse Beam Dynamics

1.) Introduction and Basic Ideas

„ ... in the end and after all it should be a kind of circular machine"

→ need transverse deflecting force

Lorentz force

$$\vec{F} = q * (\cancel{\vec{E}} + \vec{v} \times \vec{B})$$

typical velocity in high energy machines:

$$v \approx c \approx 3 * 10^8 \, m/s$$

Example:

$$B = 1\,T \quad \rightarrow \quad F = q * 3 * 10^8 \frac{m}{s} * 1 \frac{Vs}{m^2}$$

$$F = \underbrace{q * 300 \frac{MV}{m}}_{E}$$

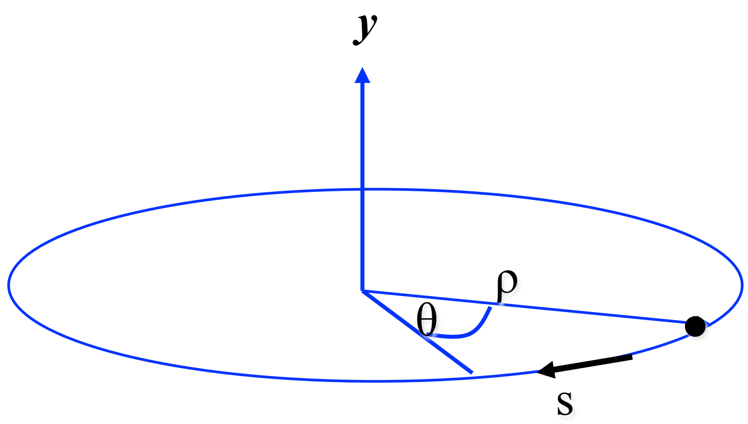
equivalent electrical field: E

technical limit for el. field:

$$E \leq 1 \frac{MV}{m}$$

old greek dictum of wisdom:
if you are clever, you use magnetic fields in an accelerator wherever it is possible.

The ideal circular orbit



circular coordinate system

condition for circular orbit:

Lorentz force $$F_L = e\,v\,B$$

centrifugal force $$F_{centr} = \frac{\gamma\, m_0\, v^2}{\rho}$$

$$\frac{\gamma\, m_0\, v^{\cancel{2}}}{\rho} = e\,\cancel{v}\,B$$

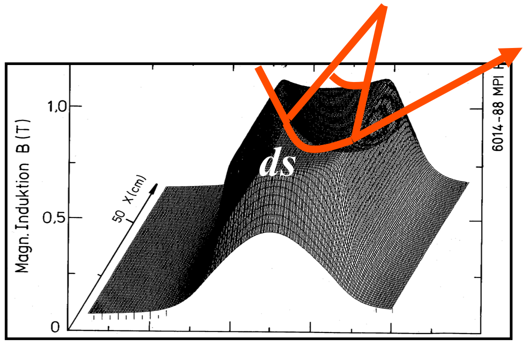
$$\boxed{\frac{p}{e} = B\,\rho}$$

$B\rho$ = "beam rigidity"

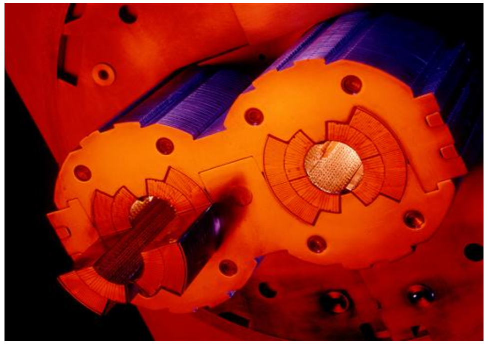
2.) The Magnetic Guide Field

Dipole Magnets:

define the ideal orbit
homogeneous field created
by two flat pole shoes



field map of a storage ring dipole magnet



s.c. LHC dipole

Normalise magnetic field to momentum:

$$\frac{p}{e} = B\rho \quad \longrightarrow \quad \frac{1}{\rho} = \frac{eB}{p}$$

convenient units:

$$B = [T] = \left[\frac{Vs}{m^2}\right] \qquad p = \left[\frac{GeV}{c}\right]$$

$$B \approx 1 \ldots 8\ T$$

Example LHC:

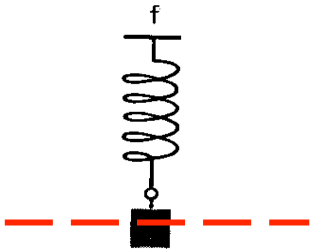
$$\left.\begin{array}{l} B = 8.3\,T \\ p = 7000\,\dfrac{GeV}{c} \end{array}\right\}$$

$$\frac{1}{\rho} = e\,\frac{8.3\ Vs/m^2}{7000 * 10^9\ eV/c} = \frac{8.3\,s\,3*10^8\ m/s}{7000*10^9\,m^2}$$

$$\rho = 2.81\ km$$

3.) Focusing Properties – Transverse Beam Optics

Classical Mechanics:
pendulum



there is a restoring force, proportional to the elongation x:

$$F = m * \frac{d^2 x}{dt^2} = -k * x$$

Solution $\omega = \sqrt{k/m}, \quad x(t) = x_0 * \cos(\sqrt{\tfrac{k}{m}}\, t + \varphi)$

general solution: free harmonic oszillation

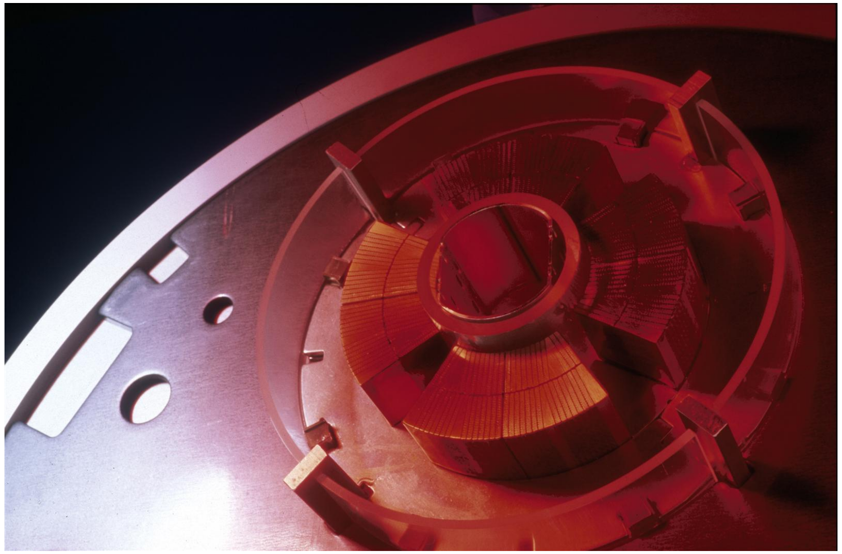
Storage Ring: *we need a Lorentz force that rises as a function of the distance to the design orbit*

$$F(x) = q * v * B(x)$$

$$B_y = g\,x \qquad B_x = g\,y \qquad k = \frac{g}{p/e}$$

LHC main quadrupole magnet

$$g \approx 25 \ldots 220 \;\; T/m$$



4.) The equation of motion:

Linear approximation:

**** ideal particle → design orbit***

**** any other particle → coordinates x, y*** ***small quantities***
$x,y \ll \rho$

→ magnetic guide field: only linear terms in x & y of B have to be taken into account

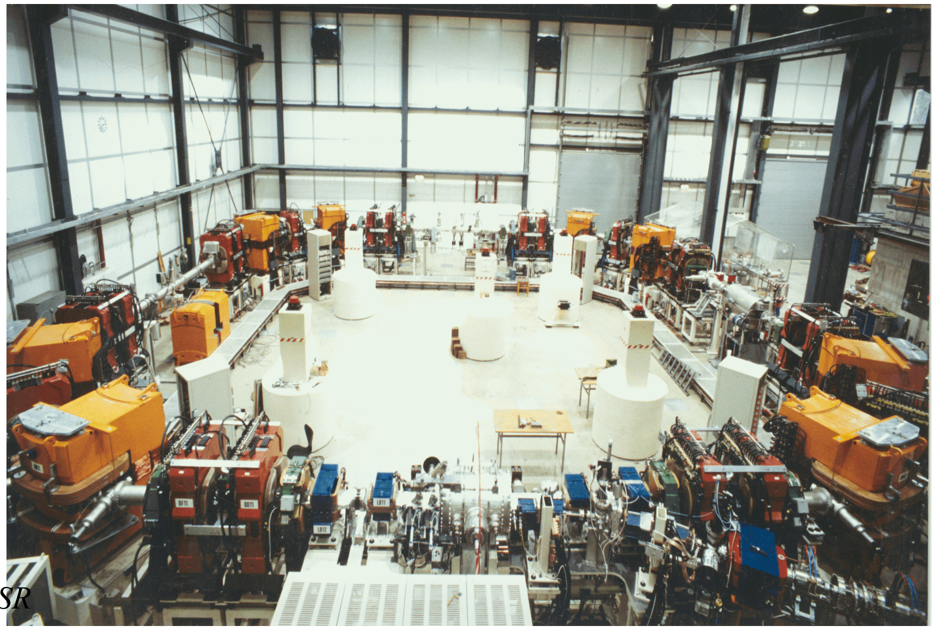
Taylor Expansion of the B field ... *normalised to momentum $p/e = B\rho$*
and only terms linear in x, y taken into account *dipole fields / quadrupole fields*

$$\frac{B(x)}{p/e} = \frac{B_0}{B_0\rho} + \frac{g*x}{p/e} + \cancel{\frac{1}{2!}\frac{eg'}{p/e}} + \cancel{\frac{1}{3!}\frac{eg''}{p/e}} + \dots$$

Separate Function Machines:

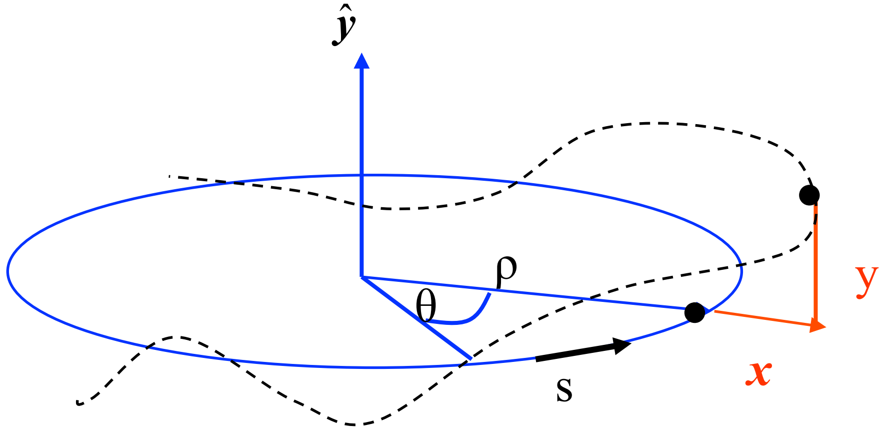
Split the magnets and optimise them according to their job:

bending, focusing etc



Example:
heavy ion storage ring TSR

Equation of Motion:



Consider local segment of a particle trajectory
... and remember the old days:
(Goldstein page 27)

radial acceleration:

$$a_r = \frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2$$

Ideal orbit: $\rho = const, \quad \frac{d\rho}{dt} = 0$

Force: $F = m\rho\left(\frac{d\theta}{dt}\right)^2 = m\rho\omega^2$

$$F = mv^2/\rho$$

general trajectory: $\rho \rightarrow \rho + x$

$$\boldsymbol{F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v}$$

independent variable: $t \rightarrow s$

$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt}$$

$$B_y = B_0 + x\frac{\partial B_y}{\partial x}$$

In linear aproximation (x,y << ρ and only dipole & quadruple fields)
we can derive a differential equation for the transverse motion of the particles

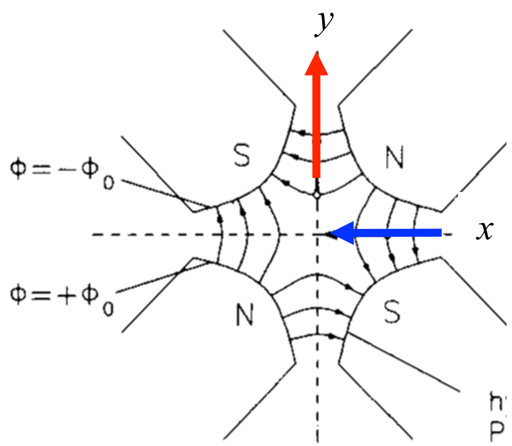
$$x'' + x\left(\frac{1}{\rho^2} - k\right) = 0$$

Under the influence of the focusing fields from the quadrupoles „k" and dipoles 1/ ρ² the transverse movement of the particles inside looks like a harmonic oscillation

* ***Equation for the vertical motion:***

$$\frac{1}{\rho^2} = 0$$ ***no dipoles … in general …***

$$k \quad \leftrightarrow \quad -k$$ ***quadrupole field changes sign***



$$y'' + k\,y = 0$$

... mmmppfff ... Just another differential equation .... but it does not look sooo comfortable.

5.) Solution of Trajectory Equations

Define … hor. plane: $K = 1/\rho^2 - k$

… vert. Plane: $K = k$

$$\boldsymbol{x}'' + \boldsymbol{K}\ \boldsymbol{x} = 0$$

Differential Equation of harmonic oscillator … with spring constant K

Ansatz: $x(s) = a_1 \cdot \cos(\omega s) + a_2 \cdot \sin(\omega s)$

general solution: linear combination of two independent solutions

$$x'(s) = -a_1\omega \sin(\omega s) + a_2\omega \cos(\omega s)$$

$$x''(s) = -a_1\omega^2 \cos(\omega s) - a_2\omega^2 \sin(\omega s) = -\omega^2 x(s) \longrightarrow \omega = \sqrt{K}$$

general solution:

$$x(s) = a_1 \cos(\sqrt{K}s) + a_2 \sin(\sqrt{K}s)$$

determine a_1 , a_2 by boundary conditions:

$$s = 0 \quad \longrightarrow \quad \begin{cases} x(0) = x_0 \quad , \quad a_1 = x_0 \\ x'(0) = x'_0 \quad , \quad a_2 = \dfrac{x'_0}{\sqrt{K}} \end{cases}$$

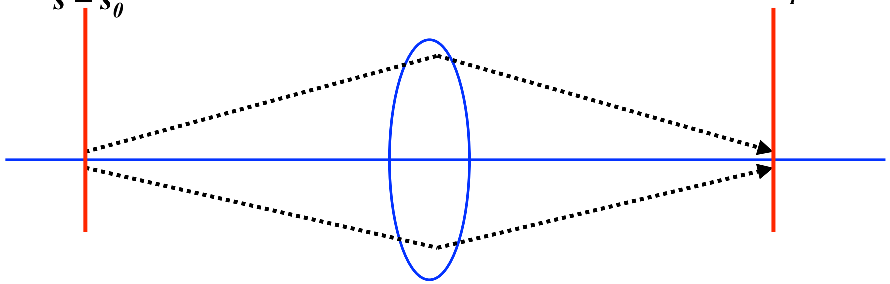
Hor. Focusing Quadrupole $K > 0$:

$$x(s) = x_0 \cdot \cos(\sqrt{|K|}s) + x'_0 \cdot \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s)$$

$$x'(s) = -x_0 \cdot \sqrt{|K|} \cdot \sin(\sqrt{|K|}s) + x'_0 \cdot \cos(\sqrt{|K|}s)$$

For convenience expressed in matrix formalism:

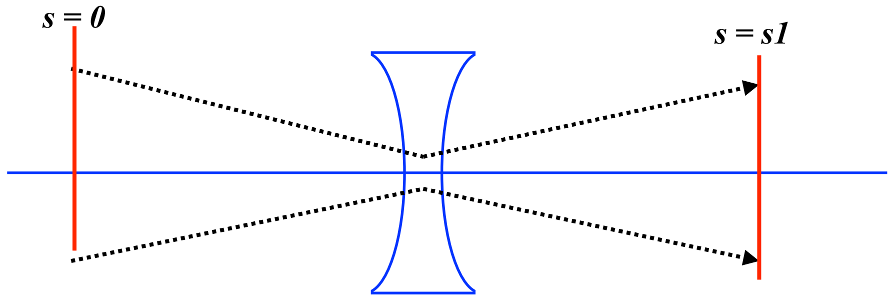
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M_{foc} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$



$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \dfrac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s \\ -\sqrt{|K|} \sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0$$

hor. defocusing quadrupole:

$$x'' - K\,x = 0$$



Remember from school:

$$f(s) = \cosh(s) \quad , \qquad f'(s) = \sinh(s)$$

Ansatz: $$x(s) = a_1 \cdot \cosh(\omega s) + a_2 \cdot \sinh(\omega s)$$

$$M_{defoc} = \begin{pmatrix} \cosh \sqrt{|K|}\,l & \frac{1}{\sqrt{|K|}} \sinh \sqrt{|K|}\,l \\ \sqrt{|K|} \sinh \sqrt{|K|}\,l & \cosh \sqrt{|K|}\,l \end{pmatrix}$$

drift space:

$$K = 0$$

$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

! with the assumptions made, the motion in the horizontal and vertical planes are independent „ ... the particle motion in x & y is uncoupled“

Combining the two planes:

Clear enough (hopefully ... ?) : a quadrupole magnet that is focussing o-in one plane acts as defocusing lens in the other plane ... et vice versa.

hor foc. quadrupole lens

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}s \\ -\sqrt{|K|}\sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}$$

matrix of the same magnet in the vert. plane:

$$M_{defoc} = \begin{pmatrix} \cosh\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}l \\ \sqrt{|K|}\sinh\sqrt{|K|}l & \cosh\sqrt{|K|}l \end{pmatrix}$$

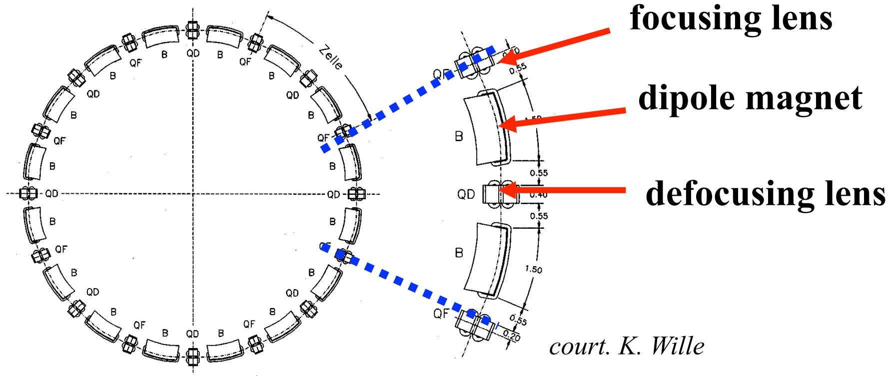
$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_f = \begin{pmatrix} \cos(\sqrt{|k|}s) & \frac{1}{\sqrt{|k|}}\sin(\sqrt{|k|}s) & 0 & 0 \\ -\sqrt{|k|}\sin(\sqrt{|k|}s) & \cos(\sqrt{|k|}s) & 0 & 0 \\ 0 & 0 & \cosh(\sqrt{|k|}s) & \frac{1}{\sqrt{|k|}}\sinh(\sqrt{|k|}s) \\ 0 & 0 & \sqrt{|k|}\sinh(\sqrt{|k|}s) & \cosh(\sqrt{|k|}s) \end{pmatrix} * \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_i$$

Transformation through a system of lattice elements

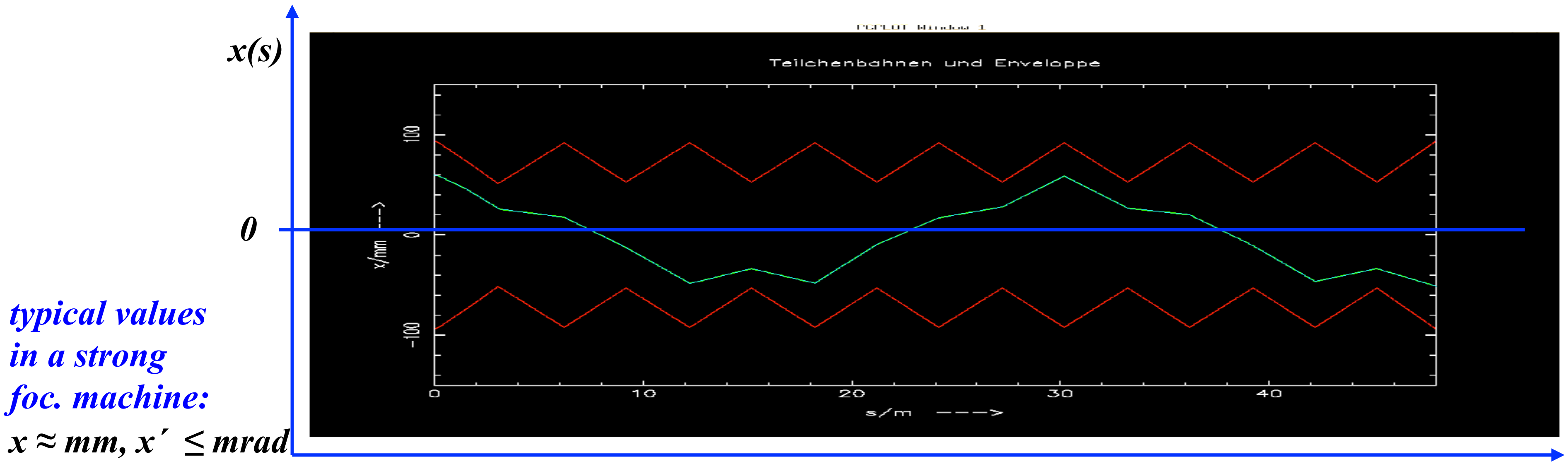
combine the single element solutions by multiplication of the matrices

$$M_{total} = M_{QF} * M_D * M_{QD} * M_{Bend} * M_{D^*}.....$$

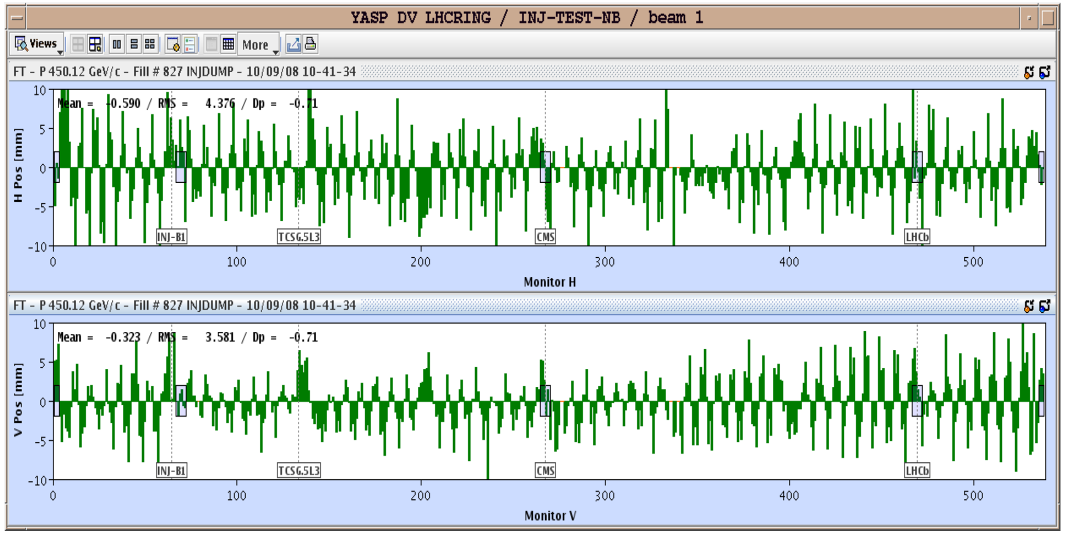
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s_2, s_1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$



in each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator „



6.) Orbit & Tune:



Tune: number of oscillations per turn

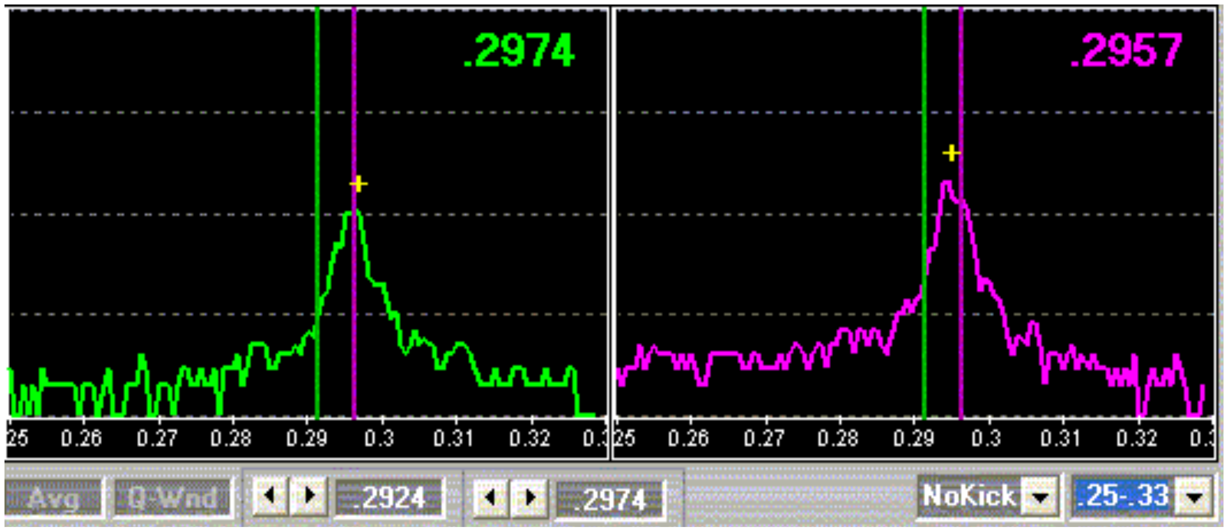
64.31
59.32

Relevant for beam stability:

non integer part

LHC revolution frequency: 11.3 kHz

$$0.31*11.3 = 3.5kHz$$

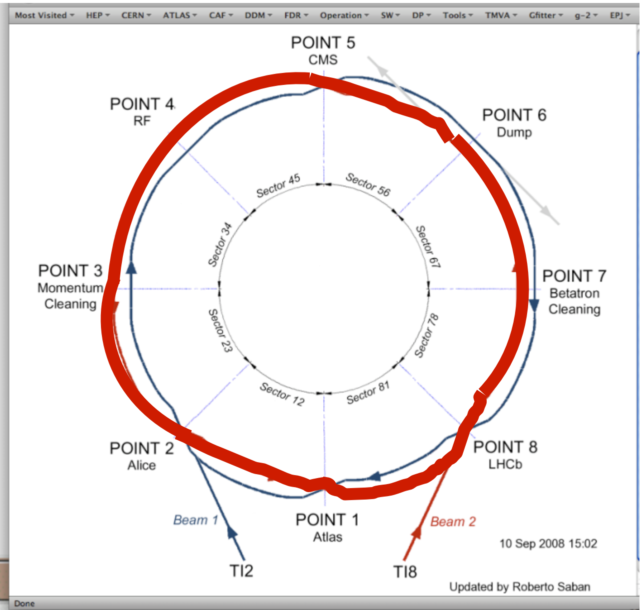


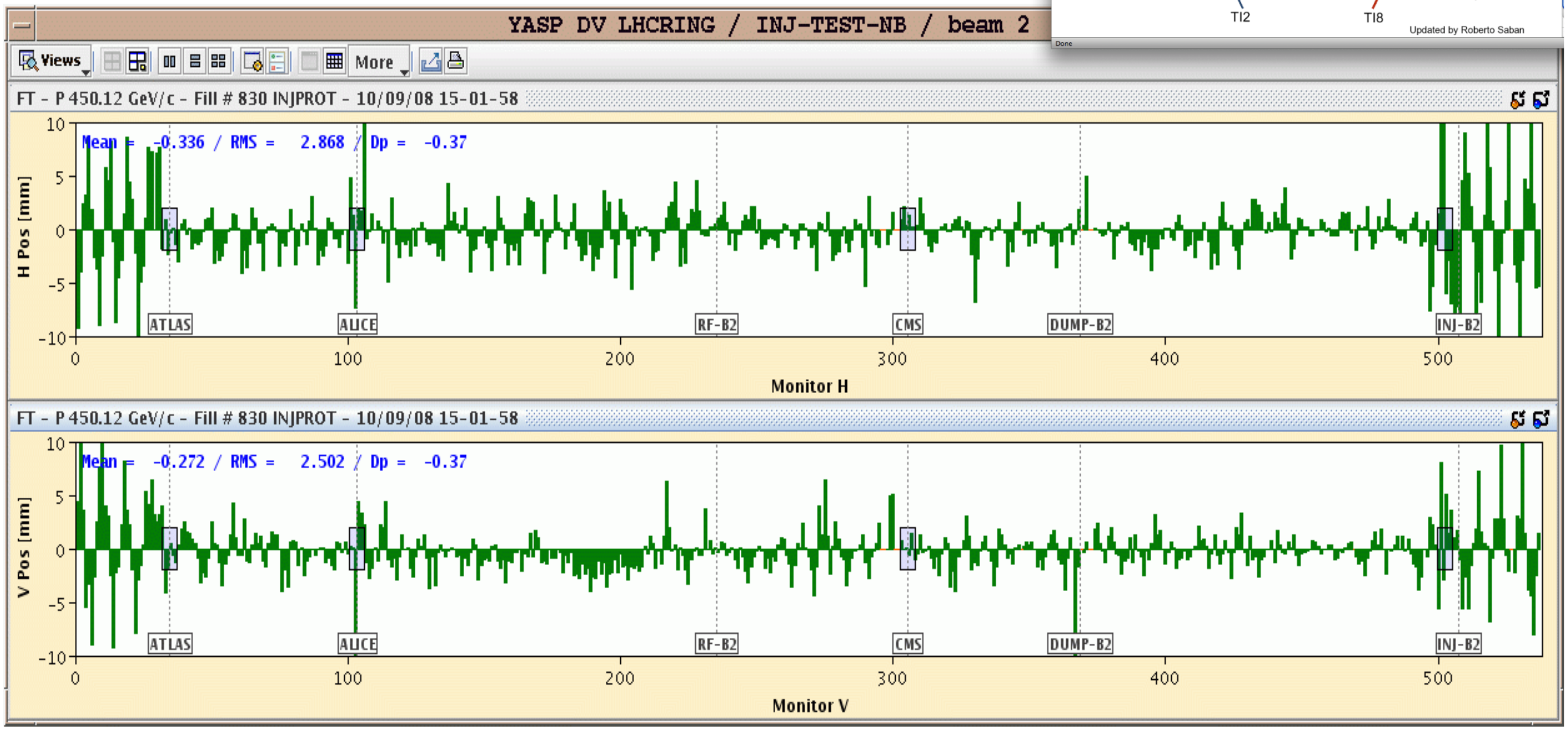
LHC Operation: Beam Commissioning

First turn steering "by sector:"

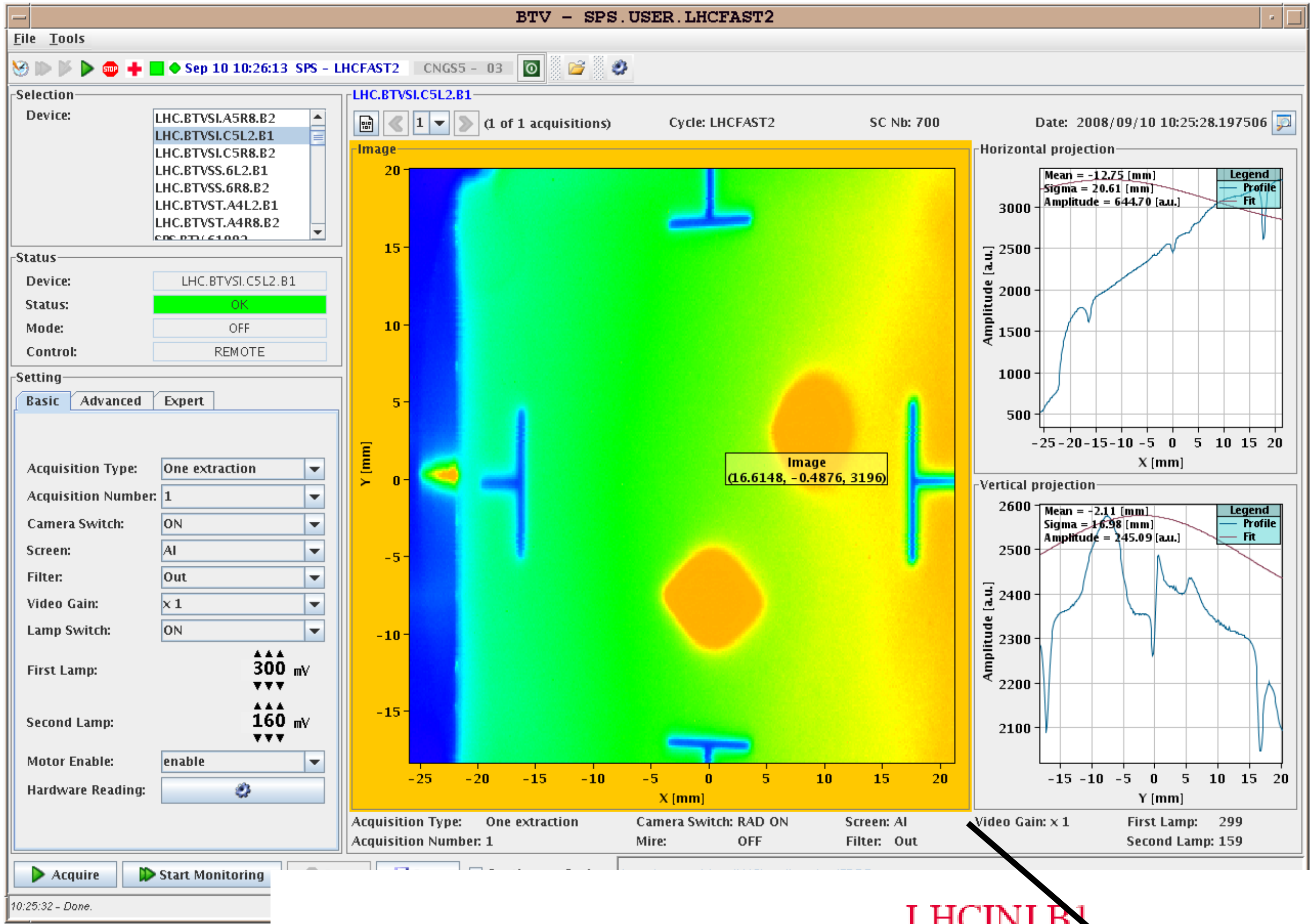
- One beam at the time
- Beam through 1 sector (1/8 ring),

correct trajectory, open collimator and move on.

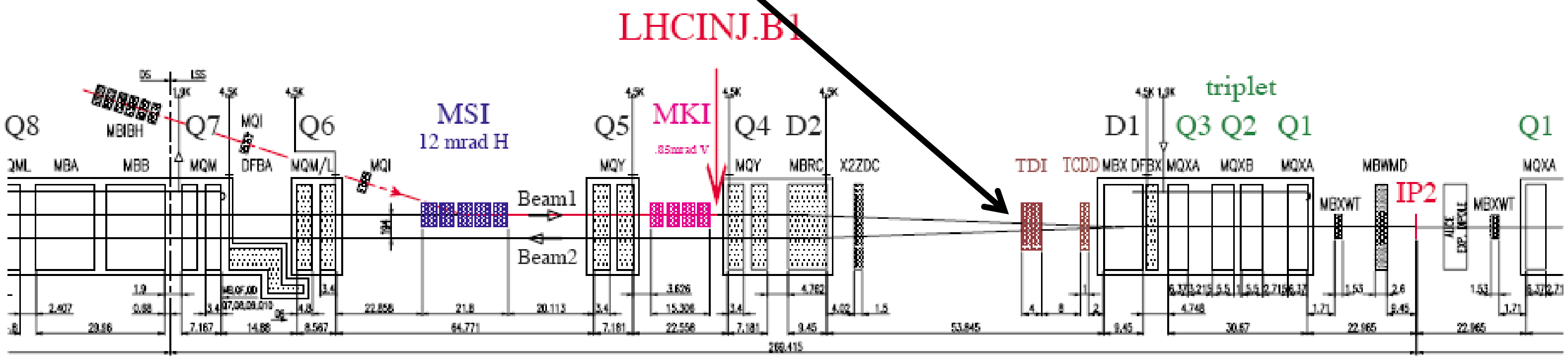




LHC Operation: the First Beam



Beam 1 on OTR screen
1st and 2nd turn

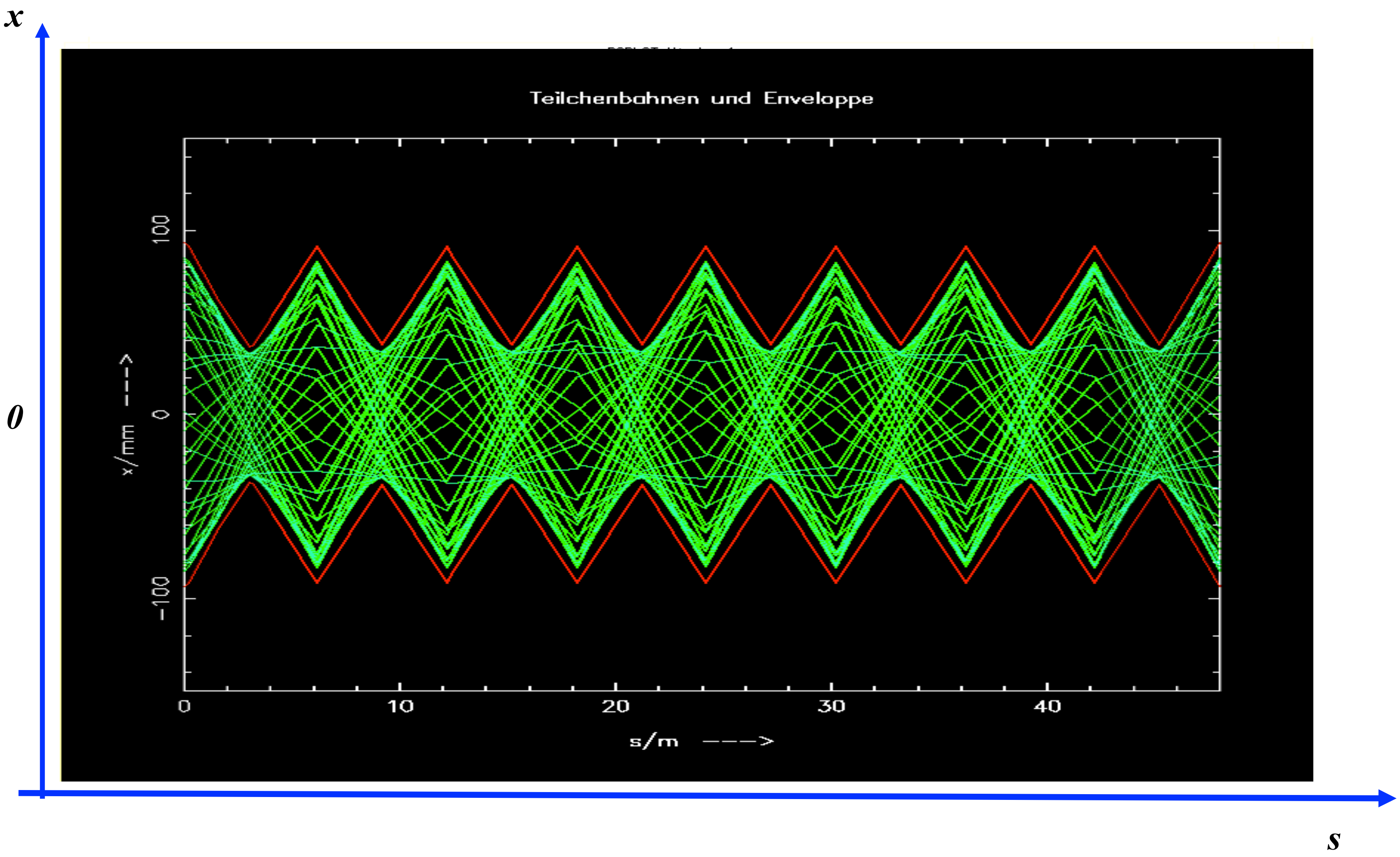


There is a fundamental difference between a circular machine

.... and a linear accelerator or transfer line.

Question: ***what will happen, if the particle performs a second turn ?***

... or a third one or ... 10^{10} turns



Astronomer Hill:

differential equation for motions with periodic focusing properties
„Hill 's equation"

Example: particle motion with periodic coefficient

equation of motion: $$x''(s) - k(s)x(s) = 0$$

restoring force $\neq$ const,
k(s) = depending on the position s
k(s+L) = k(s), periodic function

we expect a kind of quasi harmonic oscillation: amplitude & phase will depend on the position s in the ring.

7.) The Beta Function

General solution of Hill´s equation:

(i)

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi)$$

ε, Φ = integration constants determined by initial conditions

β(s) periodic function given by focusing properties of the lattice ↔ quadrupoles

$$\beta(s + L) = \beta(s)$$

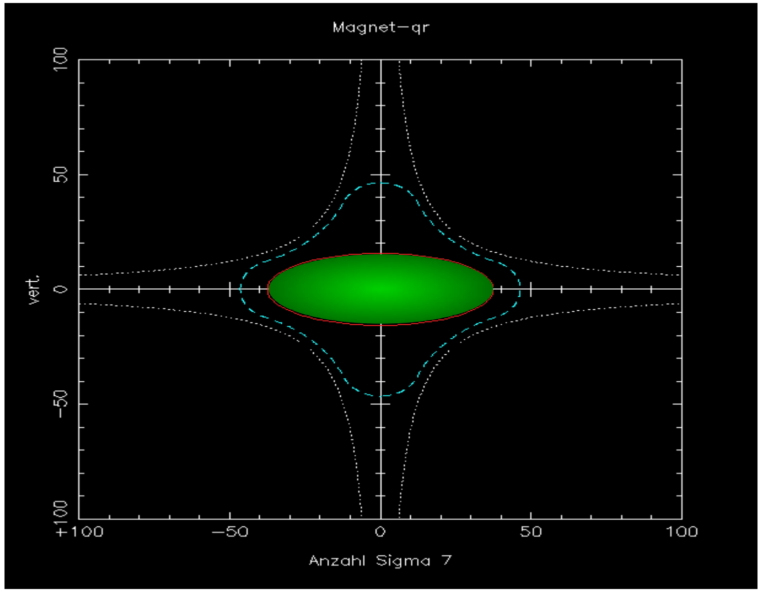
Inserting (i) into the equation of motion …

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

Ψ(s) = „phase advance“ of the oscillation between point „0“ and „s“ in the lattice.
For one complete revolution: number of oscillations per turn „Tune“

$$Q_y = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}$$

The Beta Function

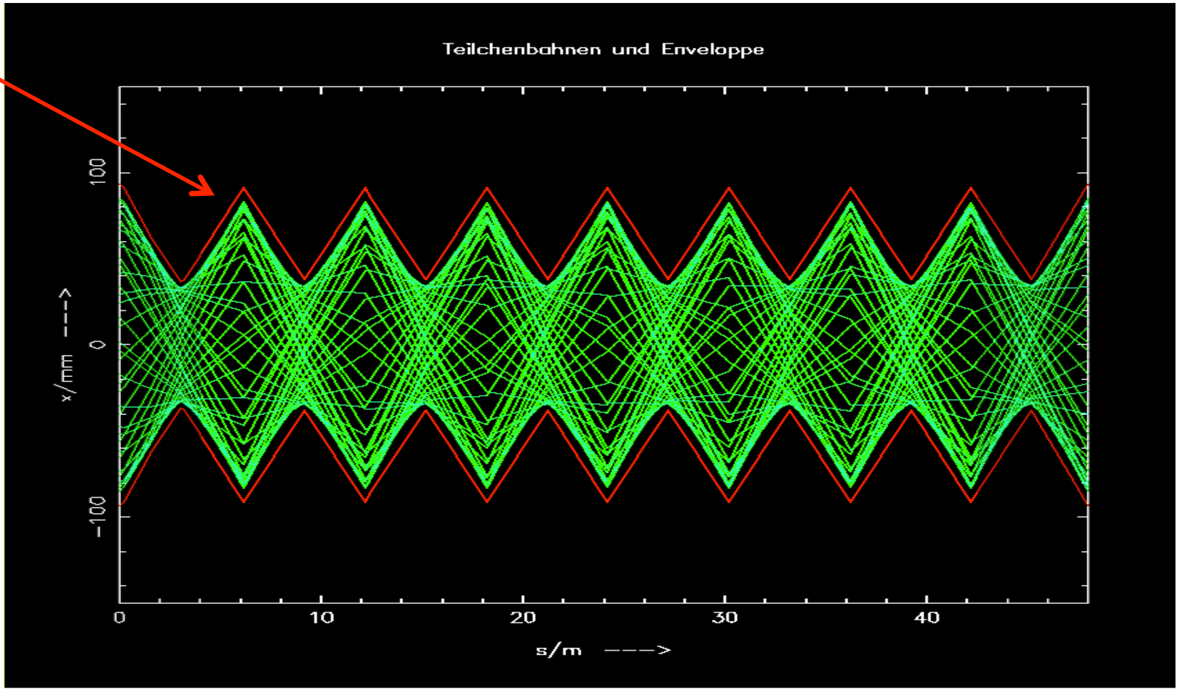


Amplitude of a particle trajectory:

$$x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s) + \varphi)$$

Maximum size of a particle amplitude

$$\hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$



β determines the beam size
(... the envelope of all particle trajectories at a given position "s" in the storage ring.

It reflects the periodicity of the magnet structure.

8.) Beam Emittance and Phase Space Ellipse

general solution of Hill equation

$$\begin{cases} (1) \quad x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\,\cos(\psi(s)+\phi) \\ (2) \quad x'(s) = -\dfrac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\,\{\alpha(s)\cos(\psi(s)+\phi)+\sin(\psi(s)+\phi)\} \end{cases}$$

from (1) we get

$$\cos(\psi(s)+\phi) = \frac{x(s)}{\sqrt{\varepsilon}\,\sqrt{\beta(s)}}$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

Insert into (2) and solve for ε

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$

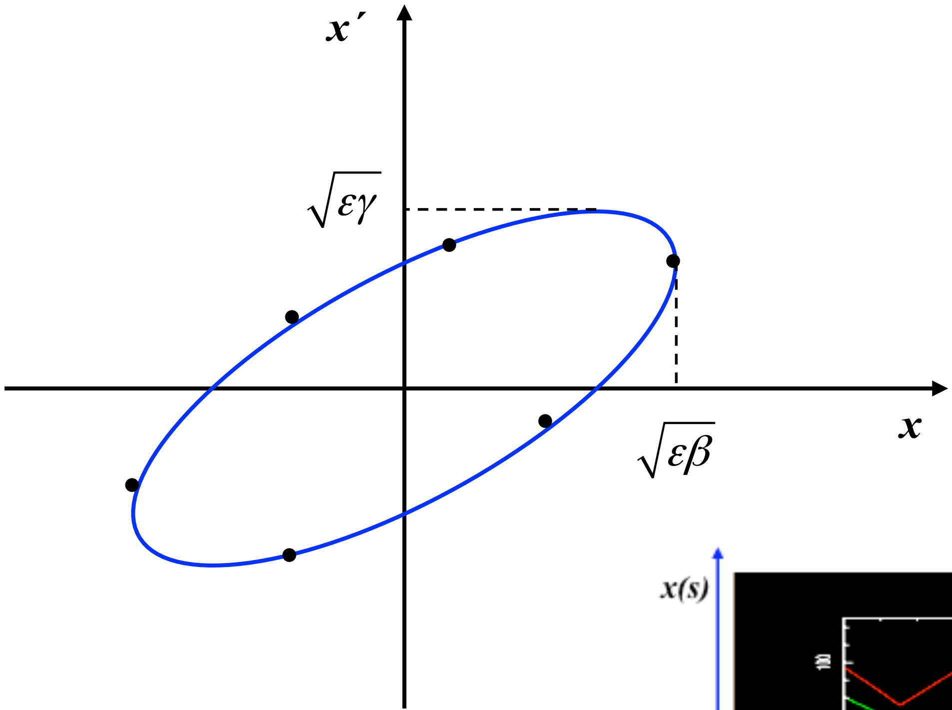
***** *ε* **is a** ***constant*** ***of the motion … it is independent of „s"***

**** parametric representation of an ellipse in the x x ' space***

**** shape and orientation of ellipse are given by α, β, γ***

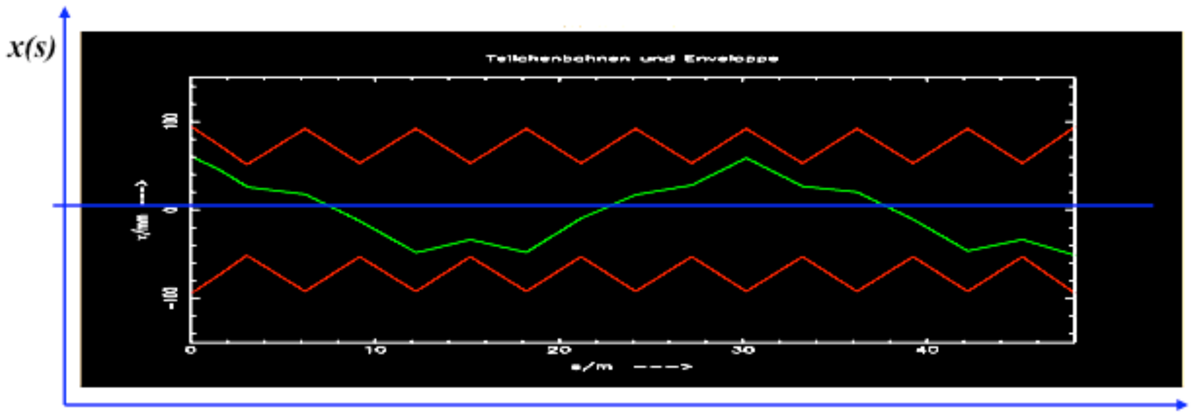
Beam Emittance and Phase Space Ellipse

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s)\, x'^2(s)$$



***Liouville:* in reasonable storage rings**
area in phase space is constant.

A = π*ε=const

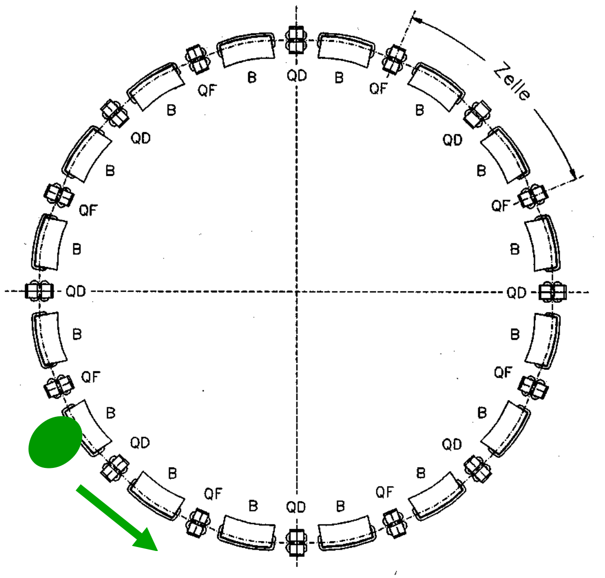


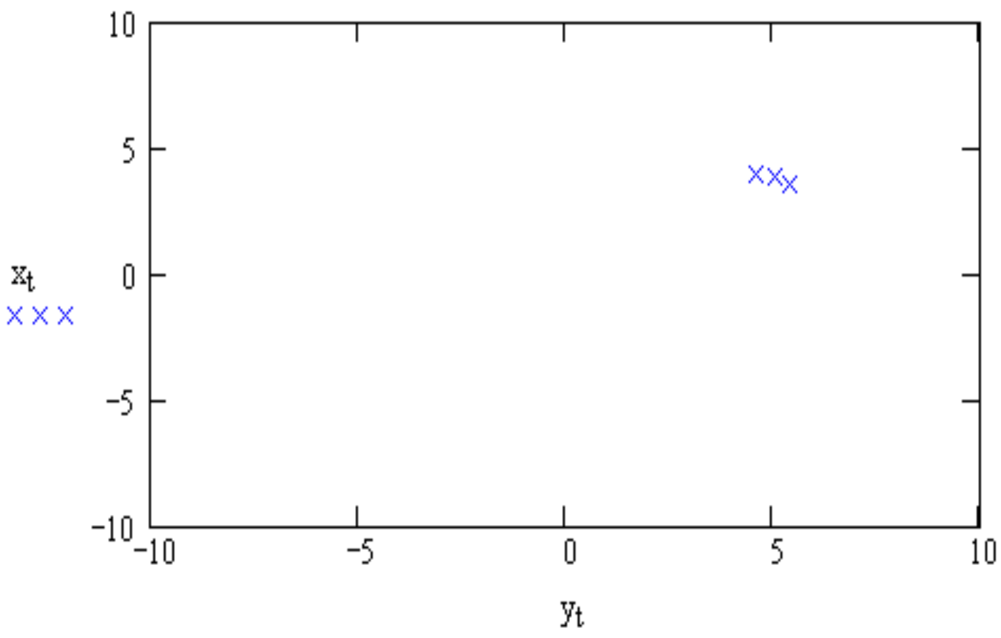
ε beam emittance = woozilycity of the particle ensemble, intrinsic beam parameter,
cannot be changed by the foc. properties.
Scientifiquely speaking: area covered in transverse x, x´ phase space … and it is constant !!!

Particle Tracking in a Storage Ring

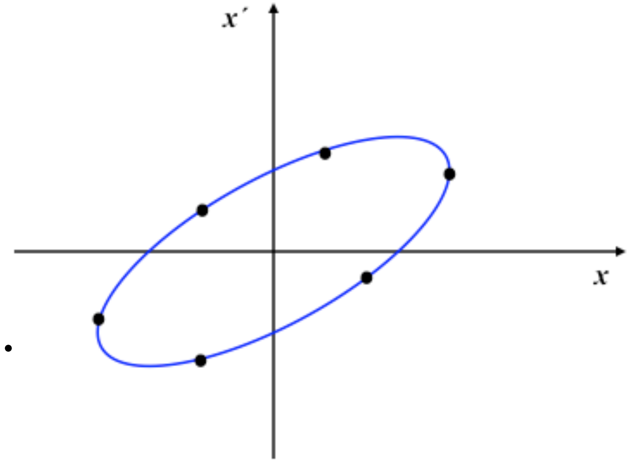
Calculate x, x´ for each linear accelerator element according to matrix formalism

plot x, x´as a function of „s"





Phase Space Ellipse



particel trajectory: $x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\}$

max. Amplitude: $\hat{x}(s) = \sqrt{\varepsilon\beta}$ $\longrightarrow$ ***x´ at that position …***

… put $\hat{x}(s)$ ***into*** $\boldsymbol{\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\, x'^2(s)}$ ***and solve for x´***

$$\varepsilon = \gamma \cdot \varepsilon\beta + 2\alpha\sqrt{\varepsilon\beta}\cdot x' + \beta x'^2$$

$$\longrightarrow \quad x' = -\alpha \cdot \sqrt{\varepsilon / \beta}$$

* ***A high β-function means a large beam size and a small beam divergence.*** ***!***
… et vice versa !!!

* ***In the middle of a quadrupole β = maximum,***
α = zero $\Big\}$ $x' = 0$

… and the ellipse is flat

Phase Space Ellipse

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\, x'^2(s)$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

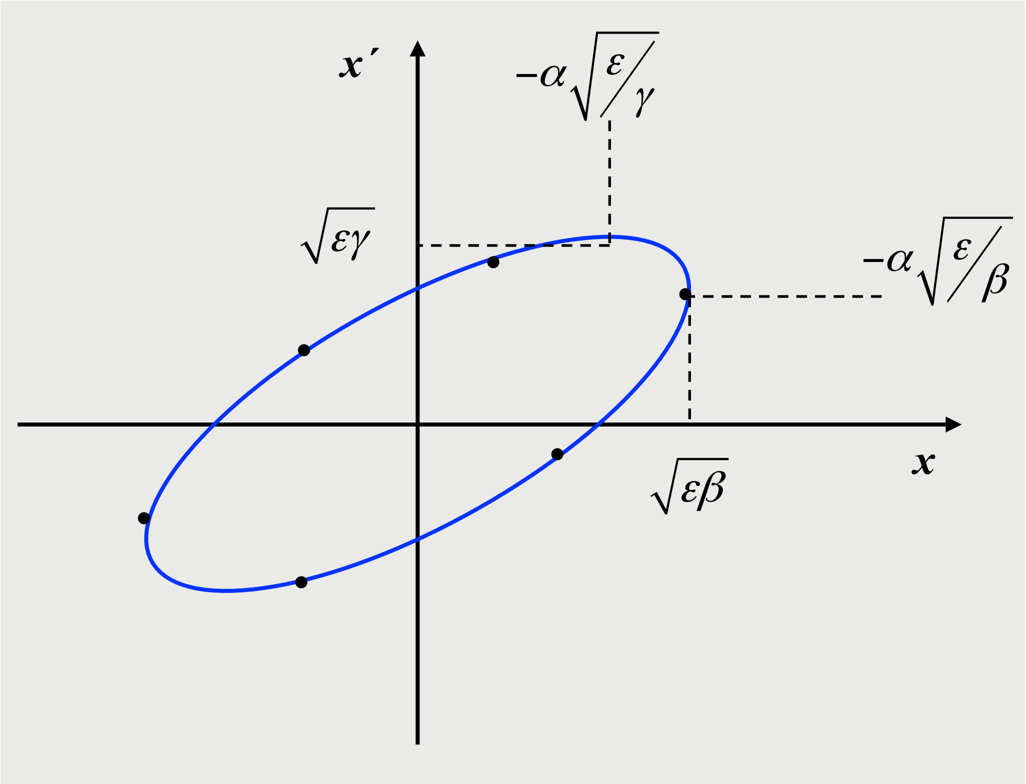
$$\longrightarrow \quad \varepsilon = \frac{x^2}{\beta} + \frac{\alpha^2 x^2}{\beta} + 2\alpha \cdot xx' + \beta \cdot x'^2$$

… solve for x´ $\quad x'_{1,2} = \dfrac{-\alpha \cdot x \pm \sqrt{\varepsilon\beta - x^2}}{\beta}$

… and determine $\hat{x}'$ *via:* $\quad \dfrac{dx'}{dx} = 0$

$$\longrightarrow \quad \hat{x}' = \sqrt{\varepsilon\gamma}$$

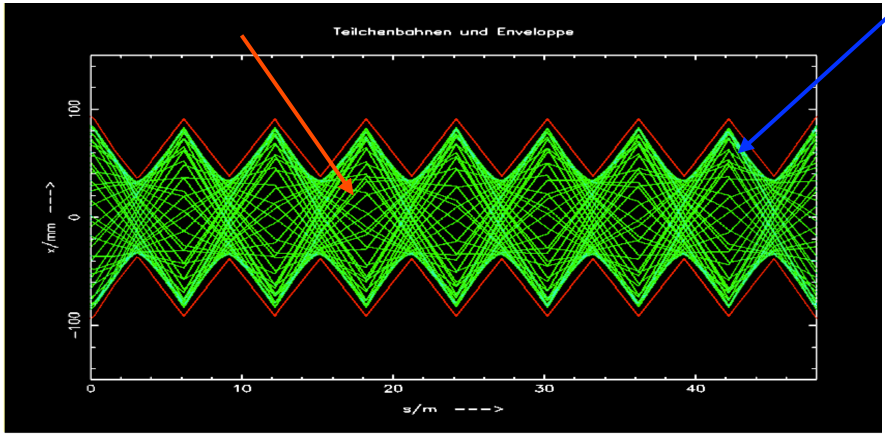
$$\longrightarrow \quad \hat{x} = \pm\alpha\sqrt{\varepsilon/\gamma}$$



shape and orientation of the phase space ellipse depend on the Twiss parameters β α γ

Emittance of the Particle Ensemble:

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\Psi(s)+\phi) \qquad \hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$



single particle trajectories, N ≈ 10 [11] per bunch

Gauß Particle Distribution:

$$\rho(x) = \frac{N \cdot e}{\sqrt{2\pi}\sigma_x} \cdot e^{-\frac{1}{2}\frac{x^2}{\sigma_x{}^2}}$$

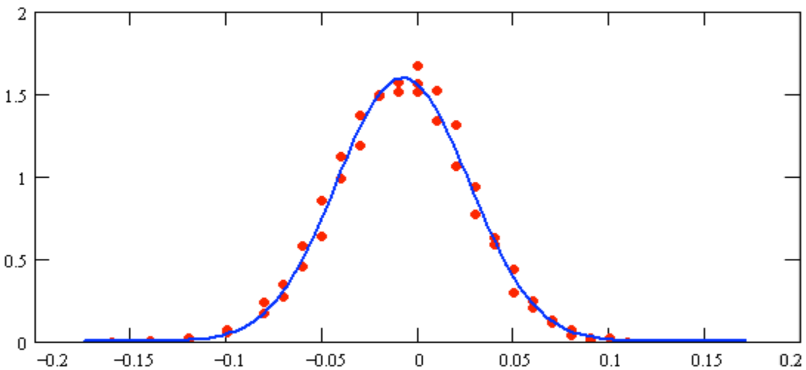
particle at distance 1 σ from centre
↔ 68.3 % of all beam particles

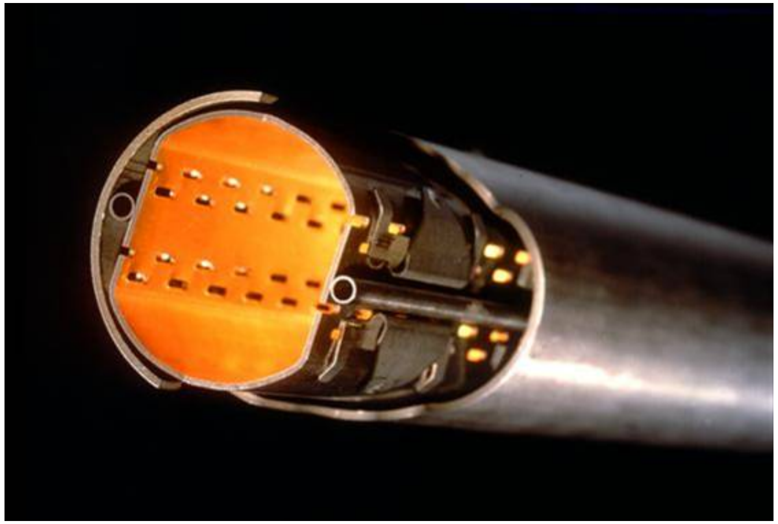
LHC:

$$\beta = 180\, m$$

$$\varepsilon = 5*10^{-10}\, m\, rad$$

$$\sigma = \sqrt{\varepsilon*\beta} = \sqrt{5*10^{-10}\, m*180\, m} = 0.3\, mm$$





aperture requirements: $r_0 = 12 * \sigma$

9.) Transfer Matrix M

… yes we had the topic already

general solution of Hill´s equation

$$\begin{cases} x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\} \\ x'(s) = \dfrac{-\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left[\alpha(s)\cos\{\psi(s)+\phi\}+\sin\{\psi(s)+\phi\}\right] \end{cases}$$

remember the trigonometrical gymnastics: sin(a + b) = … etc

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta_s}\left(\cos\psi_s\cos\phi - \sin\psi_s\sin\phi\right)$$

$$x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta_s}}\left[\alpha_s\cos\psi_s\cos\phi - \alpha_s\sin\psi_s\sin\phi + \sin\psi_s\cos\phi + \cos\psi_s\sin\phi\right]$$

starting at point s(0) = s_0, where we put $\Psi(0) = 0$

$$\left.\begin{aligned} \cos\phi &= \frac{x_0}{\sqrt{\varepsilon\beta_0}} \quad , \\ \sin\phi &= -\frac{1}{\sqrt{\varepsilon}}\left(x_0'\sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}}\right) \end{aligned}\right\}$$

inserting above …

$$x(s)=\sqrt{\frac{\beta_s}{\beta_0}}\left\{\cos\psi_s+\alpha_0\sin\psi_s\right\}x_0+\left\{\sqrt{\beta_s\beta_0}\sin\psi_s\right\}x_0'$$

$$x'(s)=\frac{1}{\sqrt{\beta_s\beta_0}}\left\{(\alpha_0-\alpha_s)\cos\psi_s-(1+\alpha_0\alpha_s)\sin\psi_s\right\}x_0+\sqrt{\frac{\beta_0}{\beta_s}}\left\{\cos\psi_s-\alpha_s\sin\psi_s\right\}x_0'$$

which can be expressed ... for convenience ... in matrix form

$$\begin{pmatrix}x\\x'\end{pmatrix}_s=M\begin{pmatrix}x\\x'\end{pmatrix}_0$$

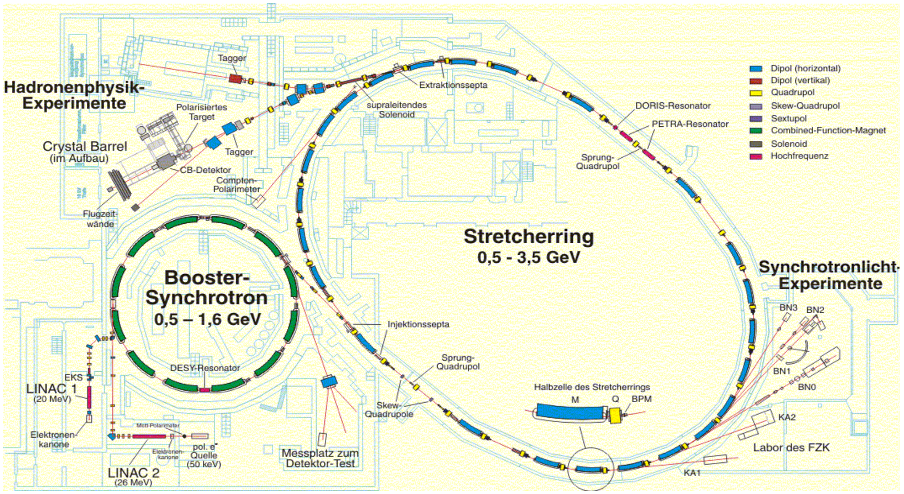
$$M=\begin{pmatrix}\sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s+\alpha_0\sin\psi_s\right) & \sqrt{\beta_s\beta_0}\sin\psi_s\\ \frac{(\alpha_0-\alpha_s)\cos\psi_s-(1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s-\alpha_s\sin\psi_s\right)\end{pmatrix}$$

* ***we can calculate the single particle trajectories between two locations in the ring, if we know the α β γ at these positions.***
* ***and nothing but the α β γ at these positions.***
* ***... !***

** Äquivalenz der Matrizen*

10.) Periodic Lattices

$$M=\begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s+\alpha_0\sin\psi_s\right) & \sqrt{\beta_s\beta_0}\,\sin\psi_s \\ \frac{(\alpha_0-\alpha_s)\cos\psi_s-(1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s-\alpha_s\sin\psi_s\right) \end{pmatrix}$$



ELSA Electron Storage Ring

„This rather formidable looking matrix simplifies considerably if we consider one complete revolution …"

$$M(s)=\begin{pmatrix} \cos\psi_{turn}+\alpha_s\sin\psi_{turn} & \beta_s\sin\psi_{turn} \\ -\gamma_s\sin\psi_{turn} & \cos\psi_{turn}-\alpha_s\sin\psi_{turn} \end{pmatrix}$$

$$\psi_{turn}=\int_s^{s+L}\frac{ds}{\beta(s)}$$

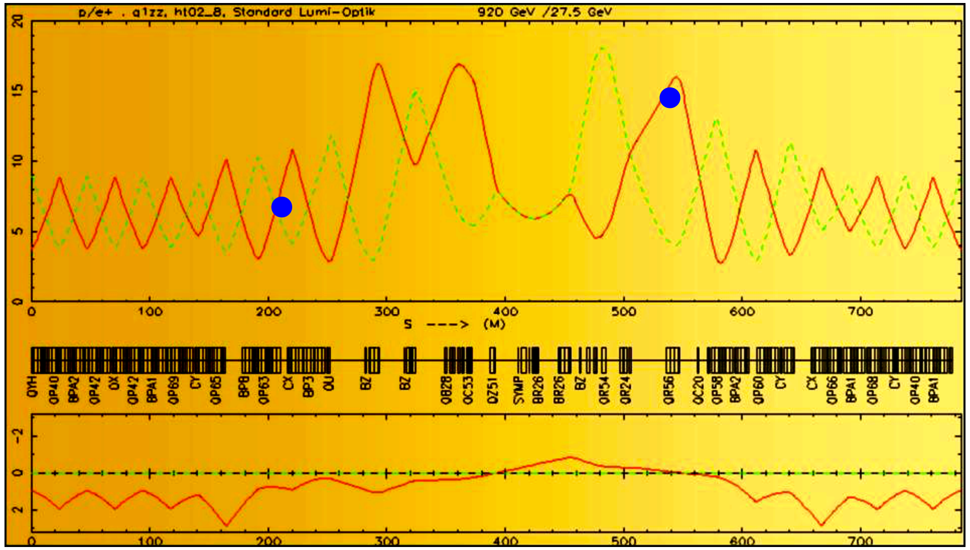
ψ_{turn} = phase advance per period

$$Q=\frac{1}{2\pi}\oint\frac{ds}{\beta(s)}$$

Tune:* *Phase advance per turn in units of 2π

11.) Transformation of α, β, γ

consider two positions in the storage ring: s_0 , s



Betafunction in a storage ring

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

since ε = const (Liouville):

$$\varepsilon = \beta_s x'^2 + 2\alpha_s x x' + \gamma_s x^2$$

$$\varepsilon = \beta_0 x_0'^2 + 2\alpha_0 x_0 x_0' + \gamma_0 x_0^2$$

... remember W = CS´-SC´ = 1

$$\begin{pmatrix} x \\ x' \end{pmatrix}_0 = M^{-1} * \begin{pmatrix} x \\ x' \end{pmatrix}_s$$

$$M^{-1} = \begin{pmatrix} S' & -S \\ -C' & C \end{pmatrix}$$

$$\rightarrow \quad x_0 = S'x - Sx'$$

$$x_0' = -C'x + Cx'$$

... inserting into ε

$$\varepsilon = \beta_0 (Cx' - C'x)^2 + 2\alpha_0 (S'x - Sx')(Cx' - C'x) + \gamma_0 (S'x - Sx')^2$$

sort via x, x´and compare the coefficients to get ....

$$\beta(s) = C^2\beta_0 - 2SC\alpha_0 + S^2\gamma_0$$

$$\alpha(s) = -CC'\beta_0 + (SC' + S'C)\alpha_0 - SS'\gamma_0$$

$$\gamma(s) = C'^2\beta_0 - 2S'C'\alpha_0 + S'^2\gamma_0$$

in matrix notation:

$$\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC'+CS' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \cdot \begin{pmatrix}\beta_0\\ \alpha_0\\ \gamma_0\end{pmatrix}$$

!

1.) this expression is important

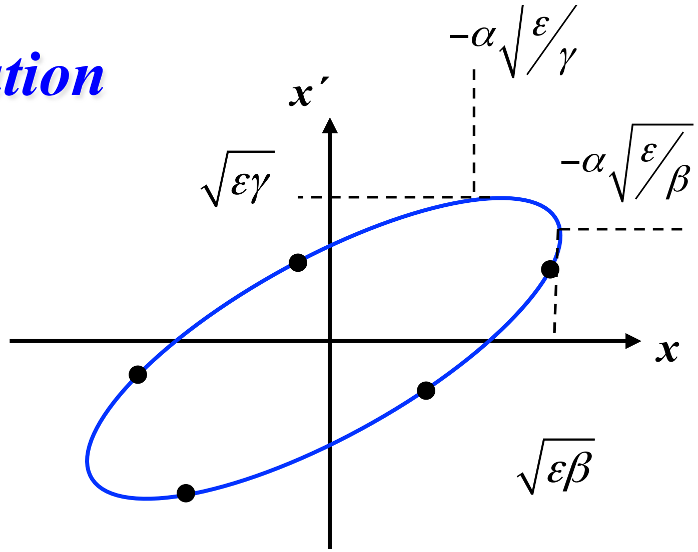
2.) it is true for rings as well as for transfer lines

2.) given the twiss parameters α, β, γ at any point in the lattice we can transform them and calculate their values at any other point in the ring.

3.) the transfer matrix is given by the focusing properties of the lattice elements, the elements of M are just those that we used to calculate single particle trajectories.

4.) go back to point 1.)

12.) Liouville during Acceleration



$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s)\, x'^2(s)$$

According to Hamiltonian mechanics:
phase space diagram relates the variables q and p

$$q = position = x$$
$$p = momentum = \gamma m v = m c \gamma \beta_x$$

$$\int p\, dq = const$$

Liouville: ***Area in phase space is constant.***

But so sorry ... $\varepsilon \neq const$!

for convenience (i.e. because we are lazy bones) we use in accelerator theory:

$$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{\beta_x}{\beta}$$

where $\beta_x = \frac{\dot{x}}{c}$ $\quad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

$$\int p dq = mc \int \gamma \beta_x dx$$

$$\int p dq = mc\gamma\beta \underbrace{\int x' dx}_{\varepsilon}$$

$$\boxed{\Rightarrow \quad \varepsilon = \int x' dx \propto \frac{1}{\beta\gamma}}$$

the beam emittance shrinks during acceleration $\varepsilon \sim 1/\gamma$

Nota bene:

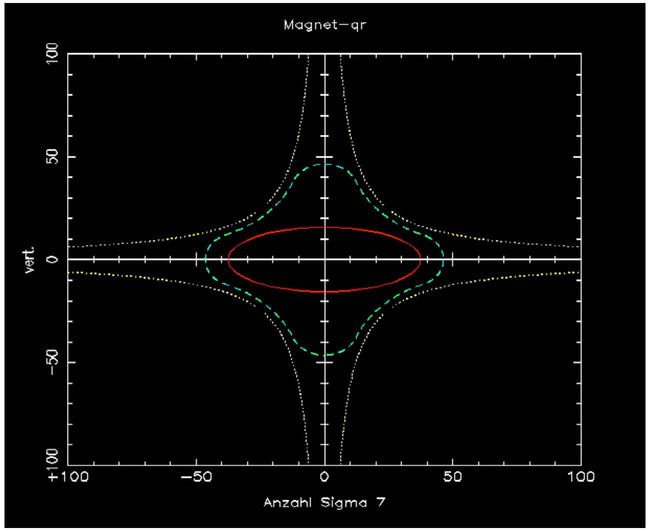
1.) A proton machine … or an electron linac … needs the highest aperture at injection energy !!!
as soon as we start to accelerate the beam size shrinks as $\gamma^{-1/2}$ in both planes.

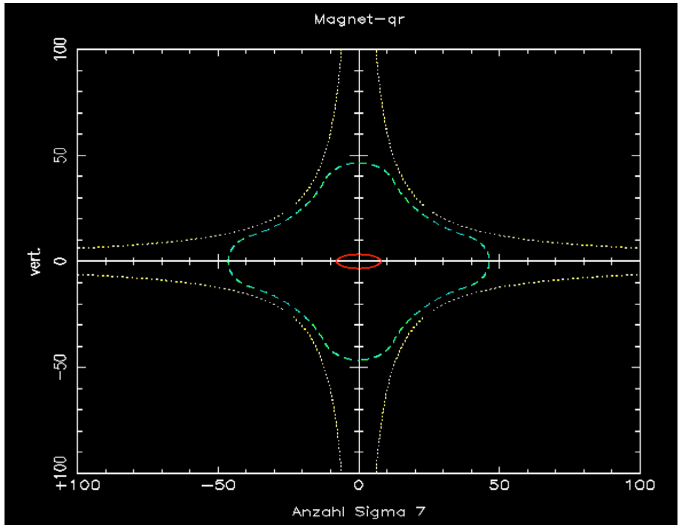
$$\sigma = \sqrt{\varepsilon\beta}$$

Example: HERA proton ring

injection energy: 40 GeV $\gamma = 43$ *emittance* ε *(40GeV)* $= 1.2 * 10^{-7}$

flat top energy: 920 GeV $\gamma = 980$ ε *(920GeV)* $= 5.1 * 10^{-9}$





7 σ beam envelope at E = 40 GeV

… and at E = 920 GeV

The „ not so ideal world “

13.) The „ $\Delta p / p \neq 0$“ Problem

ideal accelerator: all particles will see the same accelerating voltage.

$\rightarrow \Delta p / p = 0$

„nearly ideal“ accelerator: Cockroft Walton or van de Graaf

$\Delta p / p \approx 10^{-5}$

Vivitron, Straßbourg, inner structure of the acc. section

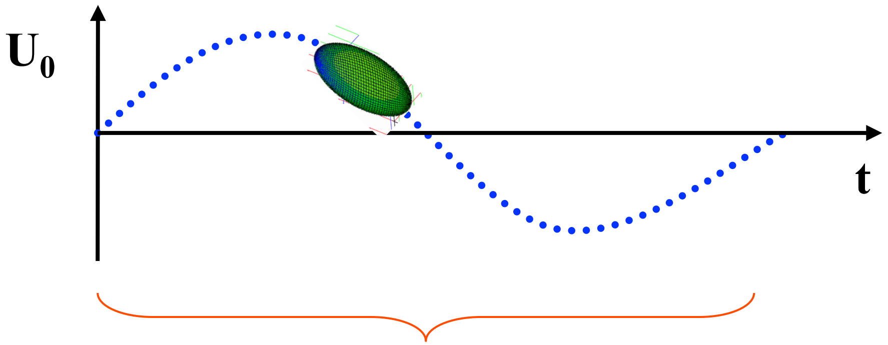
MP Tandem van de Graaf Accelerator at MPI for Nucl. Phys. Heidelberg

RF Acceleration ↔ AC voltage

Problem: panta rhei !!!
(Heraklit: 540-480 v. Chr.)

RF cavities of an electron ring

Example:



$$\nu = 500 MHz$$
$$c = \lambda \nu$$
$$\lambda = 60\ cm$$

$$\lambda = 60\ cm$$

$$\sin(90^o) = 1$$
$$\sin(84^o) = 0.994$$

$$\frac{\Delta U}{U} = 6.0\ \ 10^{-3}$$

typical momentum spread of an electron bunch:

$$\frac{\Delta p}{p} \approx 1.0\ \ 10^{-3}$$

Dispersive and Chromatic Effects: $\Delta p/p \neq 0$

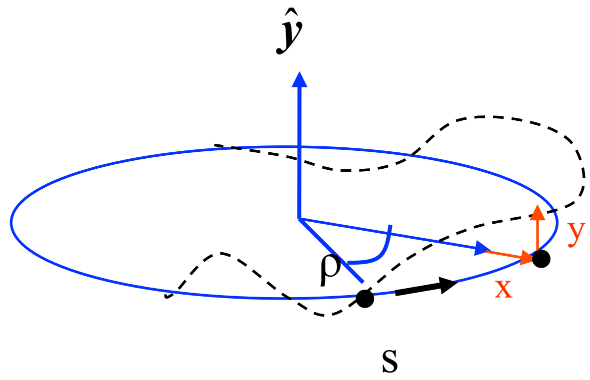
Are there any Problems ???

Sure there are !!!

font colors due to pedagogical reasons

14.) Dispersion: *trajectories for $\Delta p / p \neq 0$*

***Question:** do you remember last session, page 12 ? … sure you do*



***Force** acting on the particle*

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v$$

remember: $x \approx mm$, $\rho \approx m$ … → develop for small x

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1-\frac{x}{\rho}) = eB_y v$$

consider only linear fields, and change independent variable: $t \rightarrow s$

$$B_y = B_0 + x\frac{\partial B_y}{\partial x}$$

$$x'' - \frac{1}{\rho}(1-\frac{x}{\rho}) = \frac{e\;B_0}{mv} + \frac{e\;x\,g}{mv}$$

$p = p_0 + \Delta p$

… but now take a small momentum error into account !!!

Dispersion:

develop for small momentum error

$$\Delta p << p_0 \Rightarrow \frac{1}{p_0 + \Delta p} \approx \frac{1}{p_0} - \frac{\Delta p}{p_0^2}$$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} \approx \underbrace{\frac{e\ B_0}{p_0}}_{-\frac{1}{\rho}} - \frac{\Delta p}{p_0^2} eB_0 + \underbrace{\frac{xeg}{p_0}}_{k*x} - \underbrace{xeg\frac{\Delta p}{p_0^2}}_{\approx 0}$$

$$x'' + \frac{x}{\rho^2} \approx \frac{\Delta p}{p_0} * \underbrace{\frac{(-eB_0)}{p_0}}_{\frac{1}{\rho}} + k*x = \frac{\Delta p}{p_0} * \frac{1}{\rho} + k*x$$

$$x'' + \frac{x}{\rho^2} - kx = \frac{\Delta p}{p_0} \frac{1}{\rho} \quad \longrightarrow \quad \boxed{x'' + x(\frac{1}{\rho^2} - k) = \frac{\Delta p}{p_0} \frac{1}{\rho}}$$

***Momentum spread** of the beam adds a term on the r.h.s. of the equation of motion.*
*→ **inhomogeneous differential equation.***

Dispersion:

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$\begin{cases} x_h''(s) + K(s) \cdot x_h(s) = 0 \\ x_i''(s) + K(s) \cdot x_i(s) = \dfrac{1}{\rho} \cdot \dfrac{\Delta p}{p} \end{cases}$$

Normalise with respect to Δp/p:

$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

Dispersion function D(s)

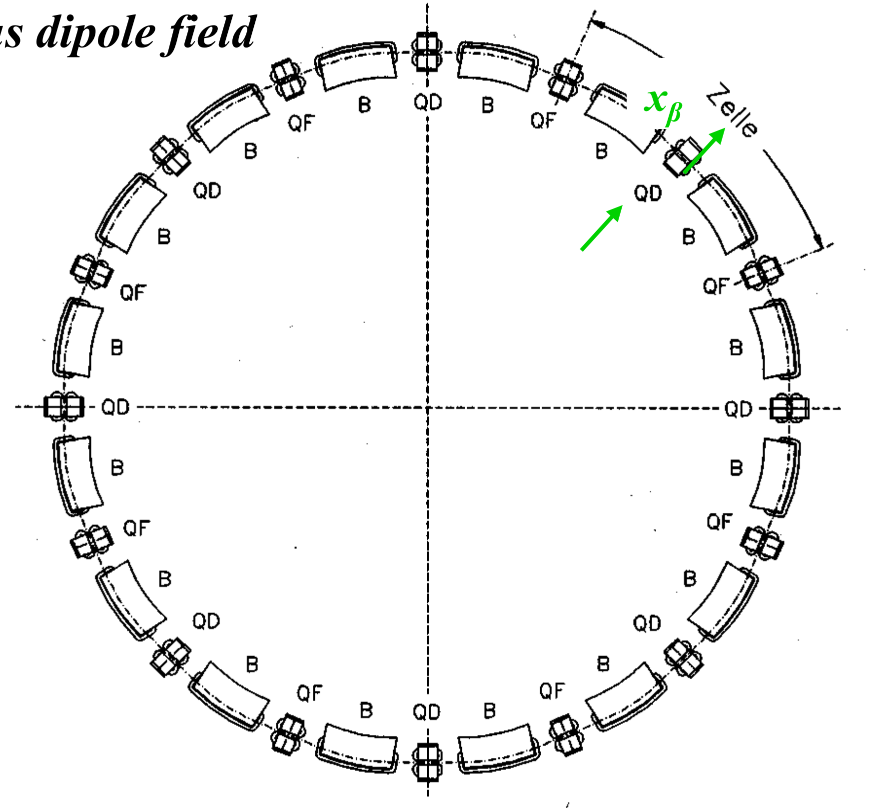
**** is that special orbit, an ideal particle would have for Δp/p = 1***

**** the orbit of any particle is the sum of the well known x_β and the dispersion***

**** as D(s) is just another orbit it will be subject to the focusing properties of the lattice***

Dispersion

Example: homogeneous dipole field



it for $\Delta p/p > 0$

$$: D(s) \cdot \frac{\Delta p}{p}$$

Matrix formalism:

$$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$

$$x(s) = C(s) \cdot x_0 + S(s) \cdot x_0' + D(s) \cdot \frac{\Delta p}{p}$$

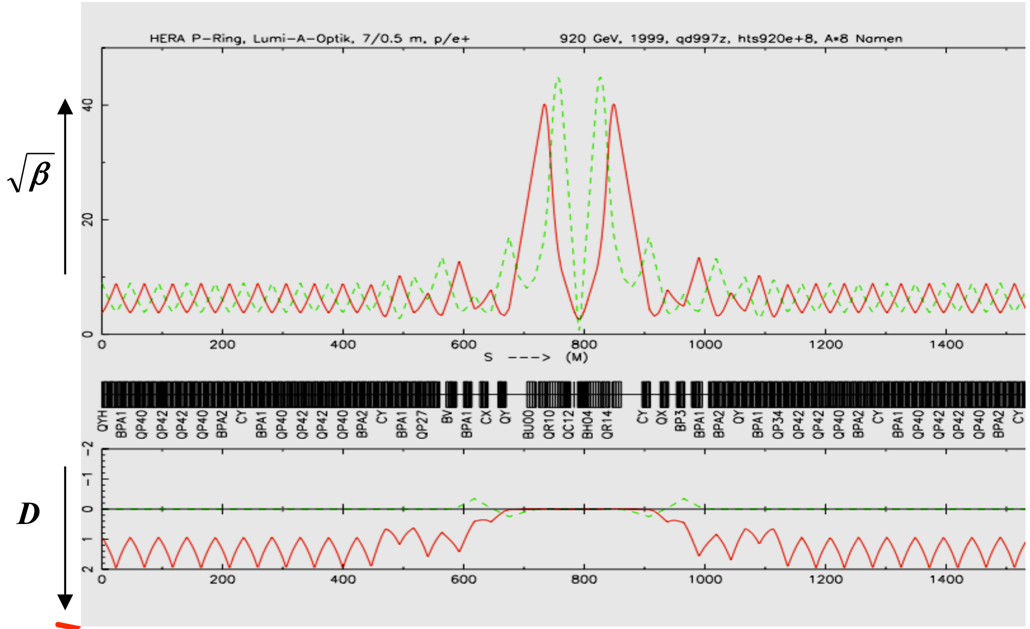
$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}$$

$$C = \cos(\sqrt{|k|}s) \qquad S = \frac{1}{\sqrt{|k|}} \sin(\sqrt{|k|}s)$$

$$C' = \frac{dC}{ds} \qquad S' = \frac{dS}{ds}$$

or expressed as 3x3 matrix

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0$$



Example

$$\left. \begin{aligned} x_\beta &= 1 \ldots 2\, mm \\ D(s) &\approx 1 \ldots 2\, m \\ \Delta p/p &\approx 1 \cdot 10^{-3} \end{aligned} \right\}$$

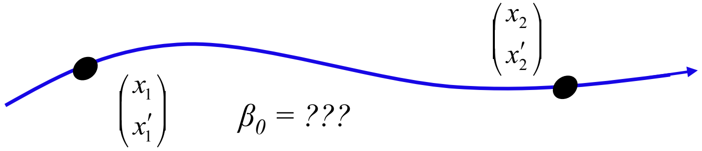
Amplitude of Orbit oscillation contribution due to Dispersion ≈ beam size

→ Whenever we want to get ‚smallest beam sizes the dispersion must vanish (e.g. at the IP)

Calculate D, D´: ... takes a couple of sunny Sunday evenings !

$$D(s) = S(s) \int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s}) d\tilde{s} - C(s) \int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s}) d\tilde{s}$$

15.) Twiss parametrisation of transfer Matrix ... for a transfer line:



$$\begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = M_{1\to 2} * \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

$$M = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\psi_{21} + \alpha_1 \sin\psi_{21}) & \sqrt{\beta_1\beta_2}\sin\psi_{21} \\ \frac{(\alpha_1 - \alpha_2)\cos\psi_{21} - (1+\alpha_1\alpha_2)\sin\psi_{21}}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\psi_{21} - \alpha_2 \sin\psi_{21}) \end{pmatrix}$$

***BUT:* Twiss parameters in s non-periodic system are NOT defined.**

--> 2 possibilities: ** twiss parameters determined by circular pre-accelerator*

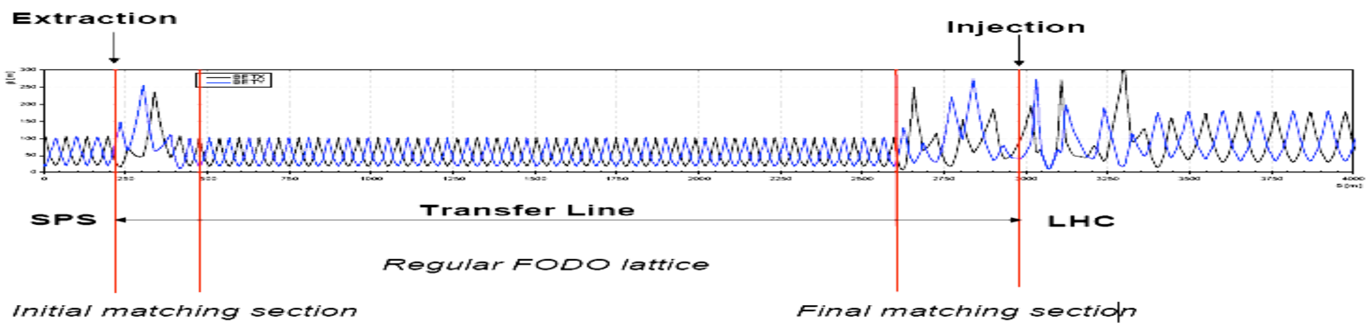
** fit α, β, ε according to the initial particle distribution*

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + CS' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

**** Parameters at start of line have to be propagated to match parameters at the end of the line***

8 variables: α_x β_x D_x D'_x and α_y β_y D_y D'_y

→ Install an equivalent a number of independent quadrupoles



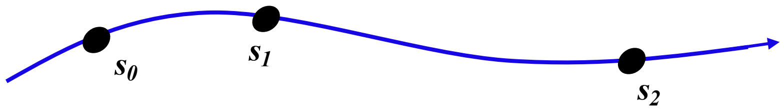
SPS-LHC transfer line optics

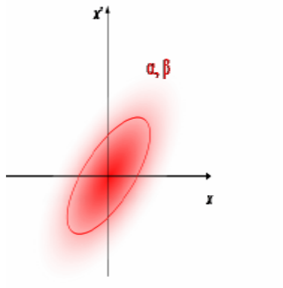
Twiss parameters for a transfer line:

yes, yes … „Twiss parameters in non-periodic system are NOT defined"

BUT they can be determined:

1.) measure the beam size at 3 locations





by definition: $\sigma = \sqrt{\varepsilon\beta}$

and $\varepsilon = const,$

$$\varepsilon = \frac{\sigma_0^2}{\beta_0} = \frac{\sigma_1^2}{\beta_1} = \frac{\sigma_2^2}{\beta_2}$$

2.) Transform the twiss babies develop through the line

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC'+CS' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

$$\left.\begin{aligned} &\rightarrow \beta_1 = C_1^2\beta_0 - 2C_1S_1\alpha_0 + \frac{S_1^2}{\beta_0}(1+\alpha_0^2) \\ &\rightarrow \beta_2 = C_2^2\beta_0 - 2C_2S_2\alpha_0 + \frac{S_2^2}{\beta_0}(1+\alpha_0^2) \end{aligned}\right\}$$

$$\alpha_0 = \frac{1}{2}\beta_0 \cdot \Gamma$$

$$\Gamma = \frac{(\sigma_2/\sigma_0)^2/S_2^2 - (\sigma_1/\sigma_0)^2/S_1^2 - (C_2/S_2)^2 + (C_1/S_1)^2}{C_1/S_1 - C_2/S_2}$$

3.) and express the start twiss as a function of the measured beam sizes

$$\beta_0 = \frac{1}{\left|\sqrt{(\sigma_1/\sigma_0)^2/S_1^2 - (C_1/S_1)^2 + (C_1/S_1)\Gamma - \Gamma^2/4}\right|}$$

… as easy as that ;-)

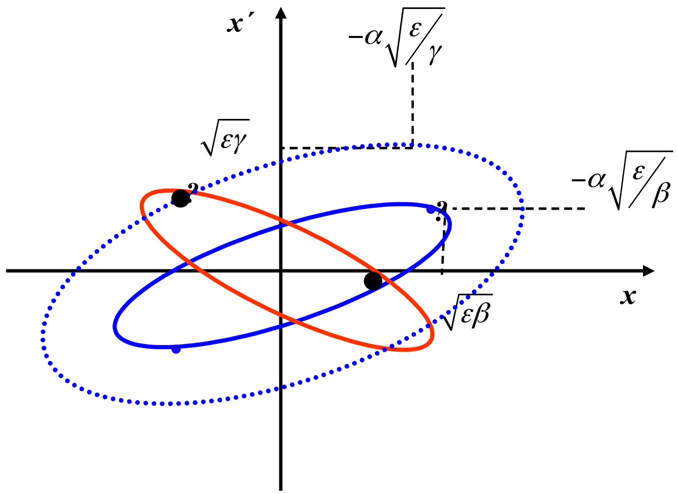
for details see e.g. P.J.Bryant in CAS 94-01

16.) Transferlines & Injection: Errors & Tolerances:

Normalised Phase Space:

$$\left.\begin{aligned} x &\rightarrow \frac{x}{\sqrt{\beta}} \\ x' &\rightarrow \frac{\alpha}{\sqrt{\beta}}x + \sqrt{\beta}x' \end{aligned}\right\}$$

Ellipse → circle



Mismatch of Beam Optics

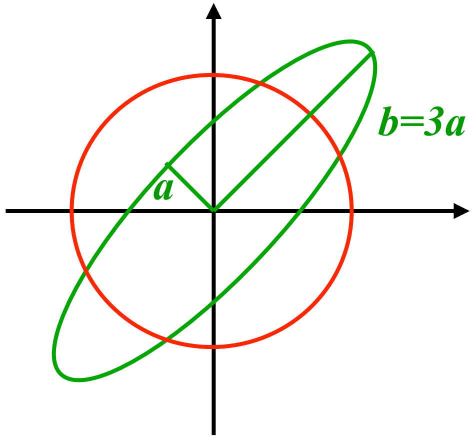
$$\lambda = \sqrt{b/a}$$

$$\varepsilon_{new} = \frac{1}{2}\varepsilon_0(\lambda^2 + \frac{1}{\lambda^2})$$

Example: b = 3a

$$\lambda = \sqrt{3}$$

$$\rightarrow \varepsilon_{new} = 1.67 * \varepsilon_0$$



Smallest beam emittances need careful optics match between the different (parts of) the machines

For details: see also: Edwards / Syphers, K. Brown, Chao, Tigner

space charge problems: the unavoidable quadrupole error:

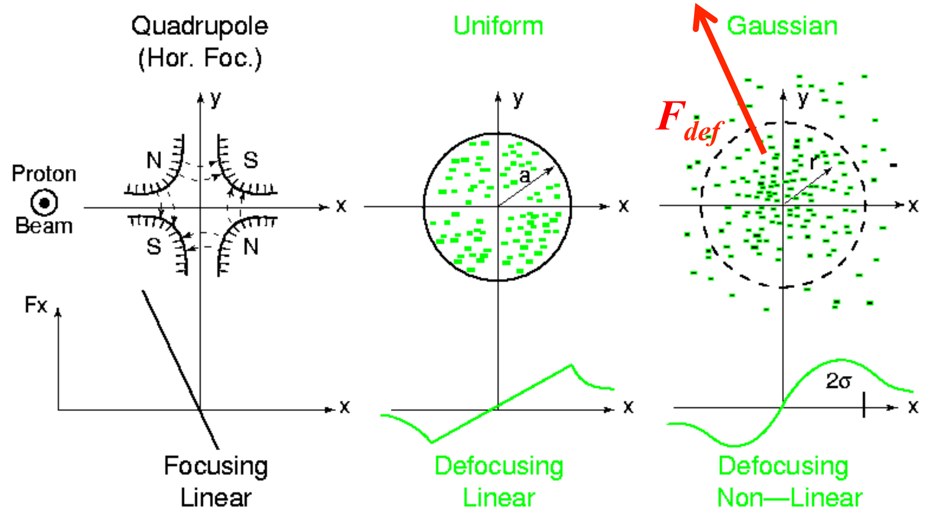
Transport line with quadrupoles

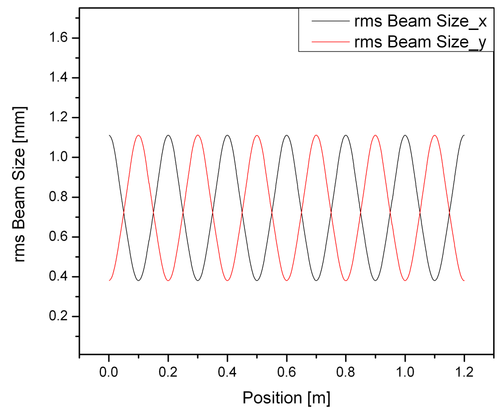
$$x'' + K(s)x = 0$$

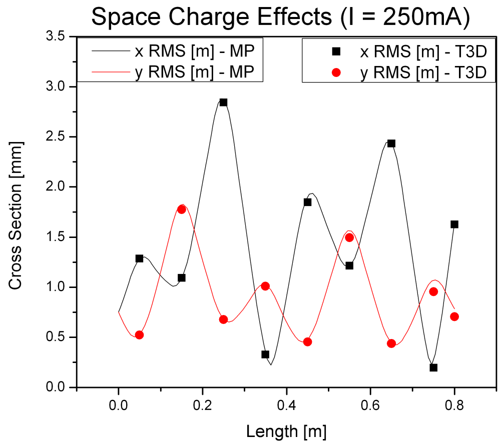
Transport line with quadrupoles and space charge

$$x'' + (K(s) + K_{SC}(s))x = 0$$

$$x'' + \left(K(s) - \frac{2r_0 I}{ea^2\beta^3\gamma^3 c} \right)x = 0$$







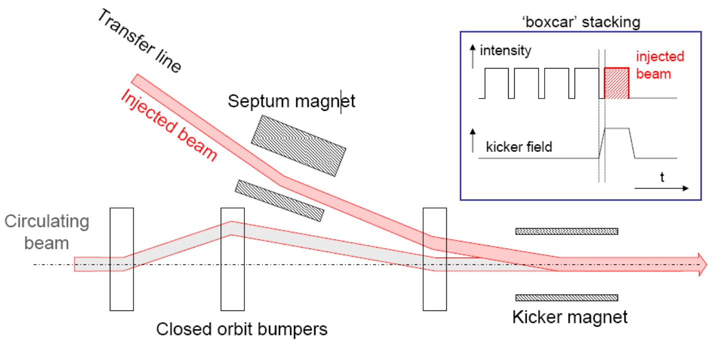
see e.g. K.H. Schindl in CAS, CERN 2006-002

no space charge ***space charge effect of I = 250mA***

The careful optics match between the different (parts of) the machines have to be done for Realistic conditions – including e.g. space charge effects

Transferlines & Injection: Errors & Tolerances

** quadrupole strengths --> "beta beat" $\Delta\beta / \beta$*
** alignment of magnets --> orbit distortion in transferline & storage ring*
** septum & kicker pulses --> orbit distortion & emittance dilution in storage ring*



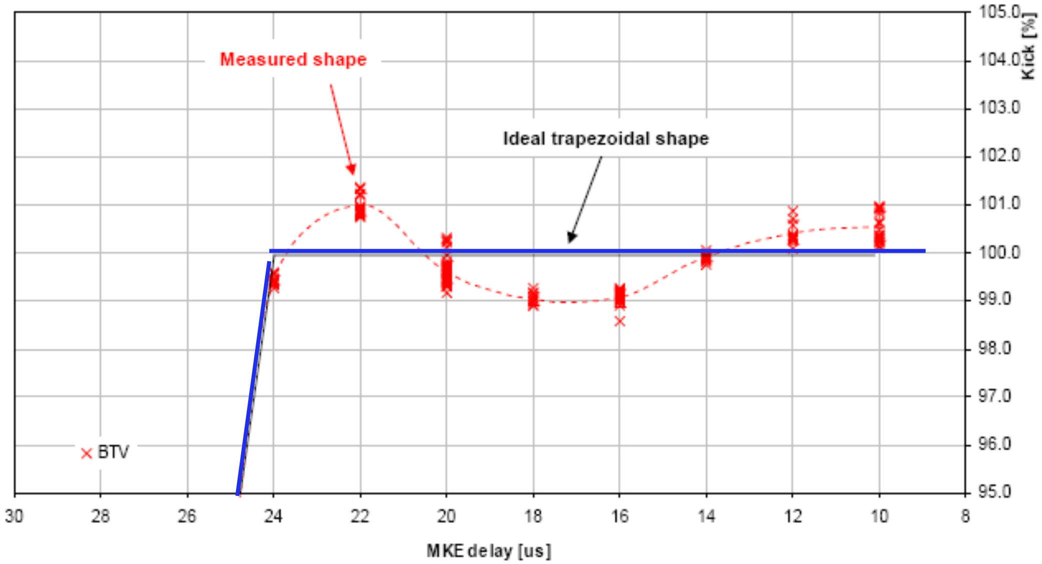
Example: Error in position Δa:

$$\varepsilon_{new} = \varepsilon_0 * (1 + \frac{\Delta a^2}{2})$$

$\Delta a = 0.5\ \sigma$

$$\rightarrow \qquad \varepsilon_{new} = 1.125 * \varepsilon_0$$

And remember: 1 σ < mm



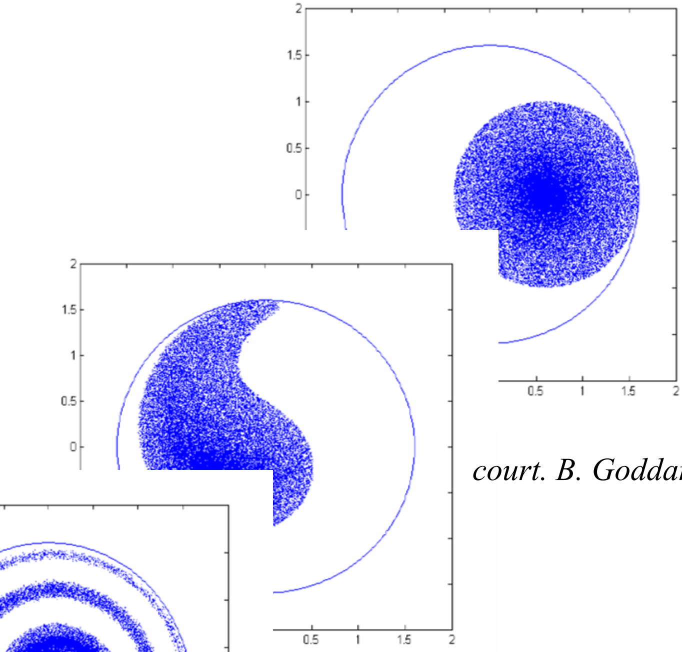
Kicker "plateau" at the end of the PS - SPS transferline measured via injection - oscillations

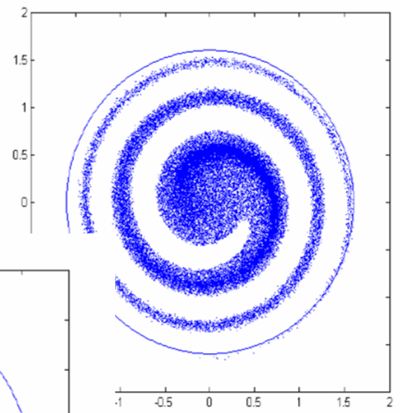
Filamentation

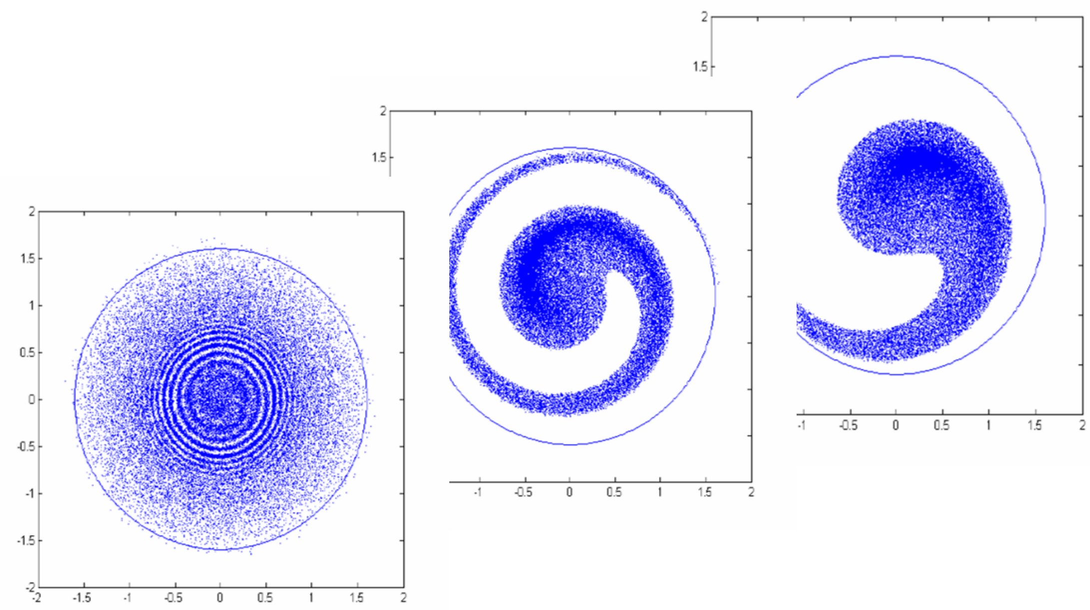
***Injection errors** (position or angle) dilute the beam emittance*

***Non-linear effects** (e.g. magnetic field multipoles) introduce distort the harmonic oscillation and lead to **amplitude dependent effects** into particle motion.*

Over many turns, a phase-space oscillation is transformed into an emittance increase.





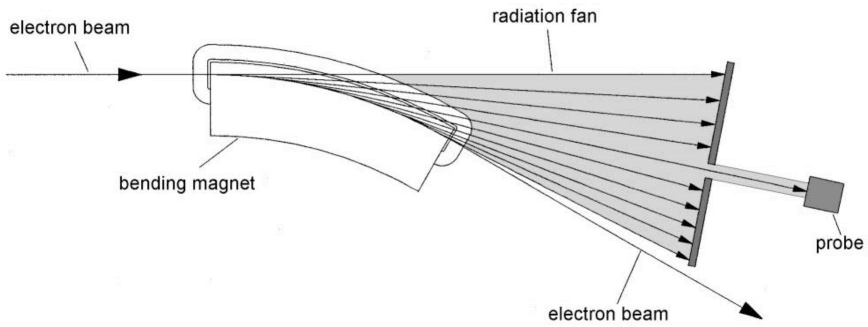


court. B. Goddard

17.) Emittance in an electron ring

ε is determined by the radiation process …

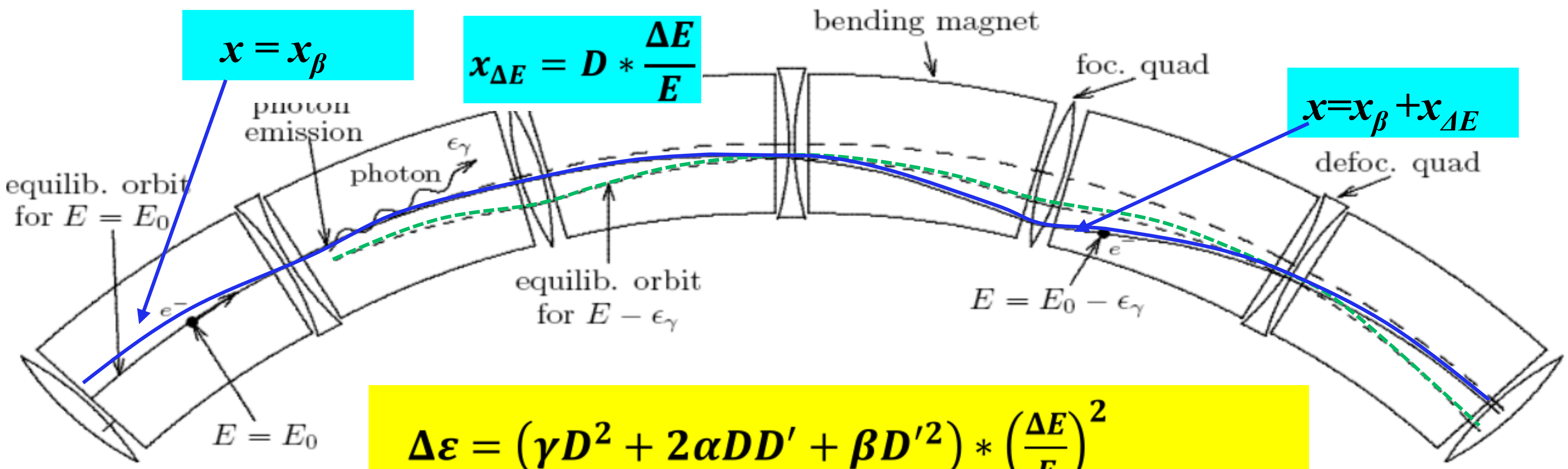
i.e. by the fact that the particle looses energy and is thus travelling on on a dispersive orbit



$$P_s = \frac{e^2 c}{6\pi\varepsilon_0} * \frac{1}{(m_0 c^2)^4} \frac{E^4}{R^4}$$ Synchrotron radiation power

$$\Delta E = \frac{e^2}{3\varepsilon_0 (m_0 c^2)^4} \frac{E^4}{R}$$ Energy loss per turn

$$\omega_c = \frac{3c\gamma^3}{2R}$$ Critical energy



$$\Delta\varepsilon = (\gamma D^2 + 2\alpha D D' + \beta D'^2) * \left(\frac{\Delta E}{E}\right)^2$$

***Avoid (vertical) Dispersion* as much as possible**

Court. Lenny Rivkin

1.) Dipole Errors / Quadrupole Misalignment

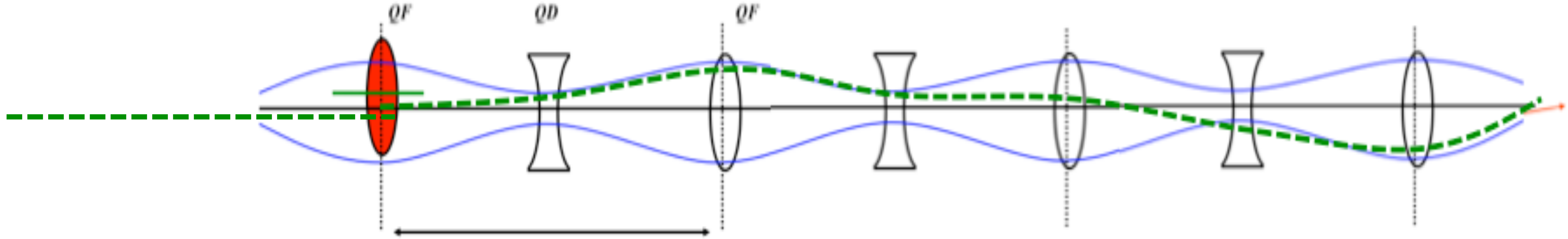
The Design Orbit is defined by the strength and arrangement of the dipoles.
Under the influence of dipole imperfections and quadrupole misalignments we obtain a "Closed Orbit" which is hopefully still closed and not too far away from the design.

Dipole field error: $$\theta = \frac{dl}{\rho} = \frac{\int B\,dl}{B\rho}$$

Quadrupole offset: $$g = \frac{dB}{dx} \rightarrow \quad \Delta x \cdot g \quad = \Delta x \frac{dB}{dx} = \Delta B$$

misaligned quadrupoles (or orbit offsets in quadrupoles) create dipole effects that lead to a distorted "closed orbit"

normalised to p/e: $$\Delta x \cdot k = \Delta x \cdot \frac{g}{B\rho} = \frac{1}{\rho} \qquad \begin{pmatrix} x \\ x' \end{pmatrix}_i = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ x' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{l}{\rho} \end{pmatrix}$$



In a Linac – starting with a perfect orbit – the misaligned quadrupole creates an oscillation that is transformed from now on downstream via

$$\begin{pmatrix} x \\ x' \end{pmatrix}_f = M \begin{pmatrix} x \\ x' \end{pmatrix}_i$$

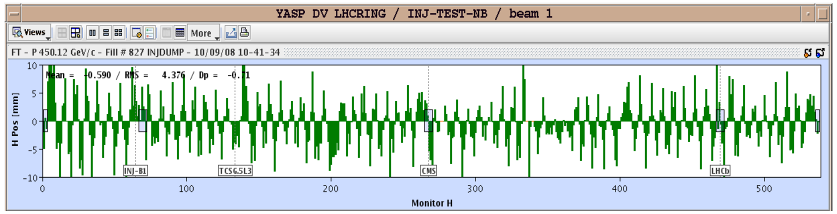
... and in a circular machine ??

we have to obey the periodicity condition.
The orbit is closed !! ... even under the influence of a orbit kick.

Calculation of the new closed orbit:
the general orbit will always be a solution of Hill, so ...

$$x(s) = a \cdot \sqrt{\beta} \cos(\psi(s) + \varphi$$

We set at the location of the error s=0, Ψ(s)=0
and require as 1st boundary condition:
periodic amplitude



$$x(s+L) = x(s)$$

$$\cancel{a} \cdot \sqrt{\cancel{\beta(s+L)}} \cdot \cos(\psi(s) + 2\pi Q - \varphi) = \cancel{a} \cdot \sqrt{\cancel{\beta(s)}} \cdot \cos(\psi(s) - \varphi)$$

$$\cos(2\pi Q - \varphi) = \cos(-\varphi) = \cos(\varphi)$$

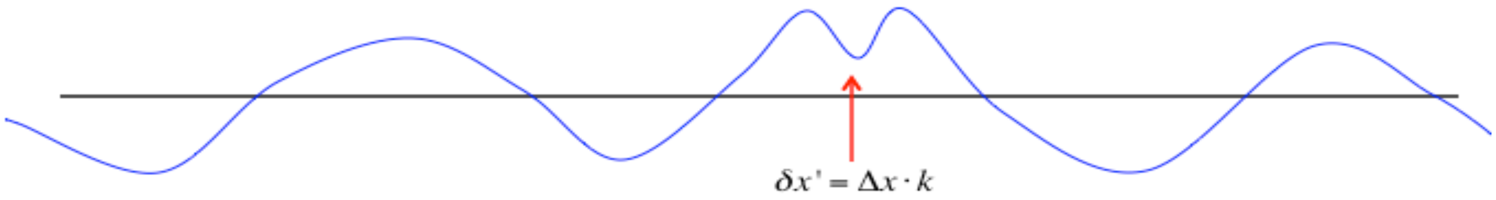
$$\boxed{\rightarrow \varphi = \pi Q}$$

$$\beta(s+L) = \beta(s)$$
$$\psi(s=0) = 0$$
$$\psi(s+L) = 2\pi Q$$

Misalignment error in a circular machine

2^{nd} boundary condition: x' (s+L) + δx'= x'(s)

we have to close the orbit



$$x(s) = a \cdot \sqrt{\beta} \cos(\psi(s) - \varphi)$$

$$x'(s) = a \cdot \sqrt{\beta} \left(-\sin(\psi(s) - \varphi)\right) \psi' + \frac{\beta'(s)}{2\sqrt{\beta}} a \cdot \cos(\psi(s) - \varphi)$$

$$x'(s) = -a \cdot \frac{1}{\sqrt{\beta}} \left(\sin(\psi(s) - \varphi)\right) + \frac{\beta'(s)}{2\sqrt{\beta}} a \cdot \cos(\psi(s) - \varphi)$$

$$\psi(s) = \int \frac{1}{\beta(s)} ds$$

$$\psi'(s) = \frac{1}{\beta(s)}$$

boundary condition: x' (s+L) + δx'= x'(s)

$$-a \cdot \frac{1}{\sqrt{\beta(\tilde{s}+L)}} \left(\sin(2\pi Q - \varphi)\right) + \frac{\beta'(\tilde{s}+L)}{2\beta(\tilde{s}+L)} \sqrt{\beta(\tilde{s}+L)}\, a \cdot \cos(2\pi Q - \varphi) + \frac{\Delta \tilde{s}}{\rho} =$$

$$= -a \cdot \frac{1}{\sqrt{\beta(\tilde{s})}} \left(\sin(-\varphi)\right) + \frac{\beta'(\tilde{s})}{2\beta(\tilde{s})} \sqrt{\beta(\tilde{s})}\, a \cdot \cos(-\varphi)$$

Nota bene: $\tilde{s}$ *refers to the location of the kick*

Misalignment error in a circular machine

Now we use: $\beta(s+L) = \beta(s)$, $\varphi=\pi Q$

$$\frac{-a}{\sqrt{\beta(\tilde{s})}}\left(\sin(\pi Q)\right)+\frac{\beta'(\tilde{s})}{2\beta(\tilde{s})}\sqrt{\beta(\tilde{s})}\,a\cdot\cos(\pi Q)+\frac{\Delta\tilde{s}}{\rho} = \frac{a}{\sqrt{\beta(\tilde{s})}}\left(\sin(\pi Q)\right)+\frac{\beta'(\tilde{s})}{2\beta(\tilde{s})}\sqrt{\beta(\tilde{s})}\,a\cdot\cos(\pi Q)$$

$$\Rightarrow 2\,a\cdot\frac{\sin(\pi Q)}{\sqrt{\beta(\tilde{s})}}=\frac{\Delta\tilde{s}}{\rho} \quad \Rightarrow$$

$$a=\frac{\Delta\tilde{s}}{\rho}\cdot\sqrt{\beta(\tilde{s})}\,\frac{1}{2\sin(\pi Q)}$$

! this is the amplitude of the orbit oscillation resulting from a single kick

inserting in the equation of motion

$$x(s)=a\cdot\sqrt{\beta}\cos(\psi(s)+\varphi$$

$$x(s)=\frac{\Delta\tilde{s}}{\rho}\cdot\frac{\sqrt{\beta(\tilde{s})}\sqrt{\beta(s)}\cos(\psi(s)-\varphi)}{2\sin(\pi Q)}$$

! the distorted orbit depends on the kick strength,
! the local β function
! the β function at the observation point

!!! there is a resoncance denominator
→ watch your tune !!!

Misalignment error in a circular machine

For completeness:

if we do not set $\psi(s=0)=0$ *we have to write a bit more but finally we get:*

$$x(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi Q)} * \oint \sqrt{\beta(\tilde{s})}\,\frac{1}{\rho(\tilde{s})}\cos(\left|\psi(\tilde{s})-\psi(s)\right| - \pi Q)\,d\tilde{s}$$

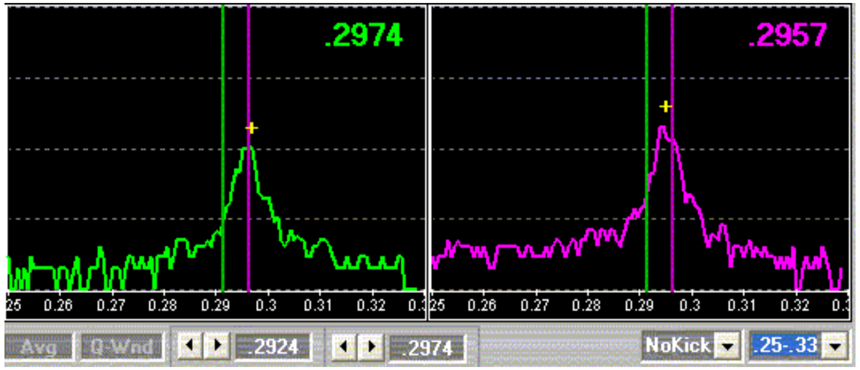
Reminder: LHC

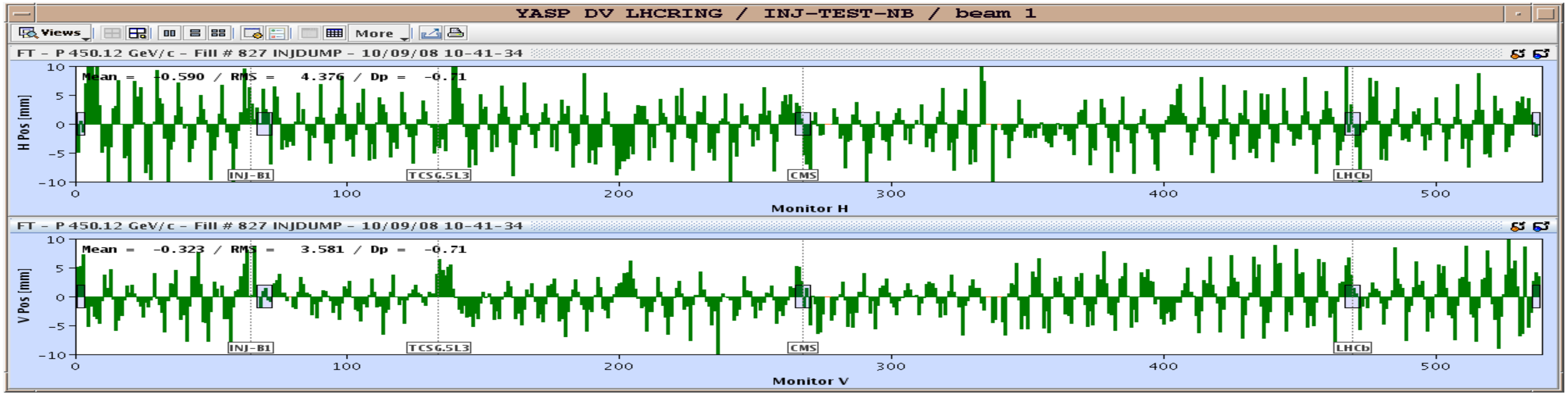
Tune: $Q_x = 64.31$, $Q_y = 59.32$

Relevant for beam stability:

non integer part

avoid integer tunes

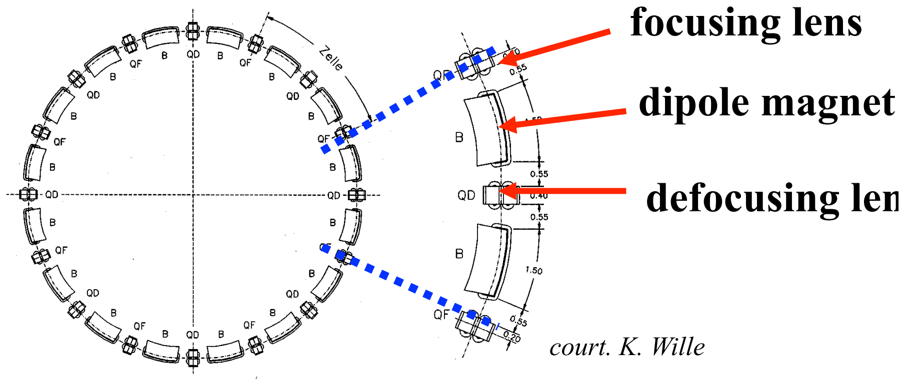




LHC First Turn Steering

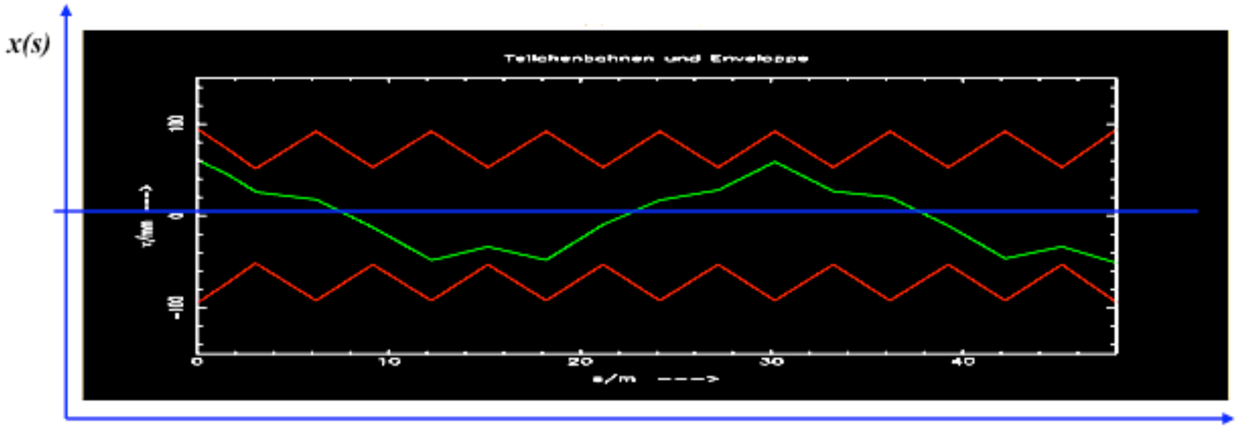
$$M_{total} = M_{QF} * M_D * M_{QD} * M_{Bend} * M_{D*.....}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s_2, s_1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$



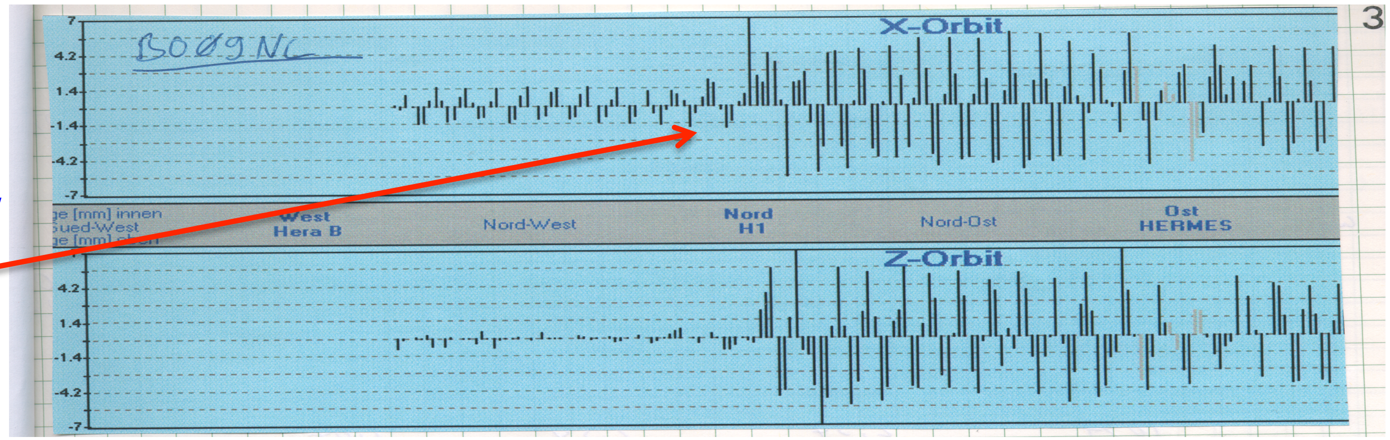
court. K. Wille

in theory
nice harmonic oscillation



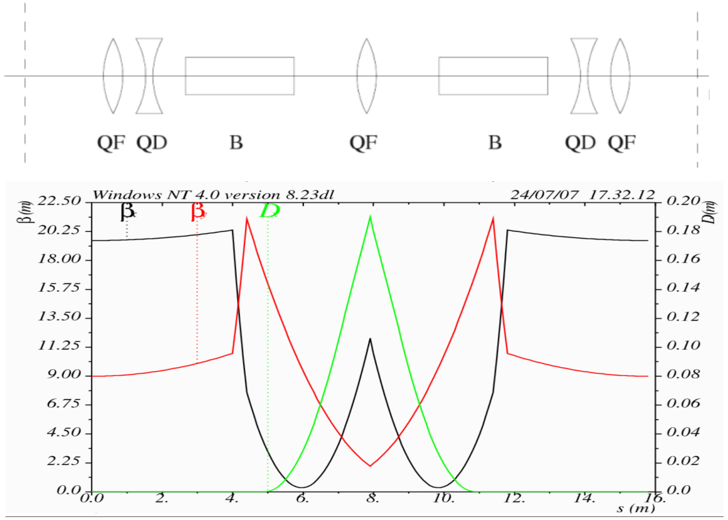
in reality:
effect of many localised orbit distortions

-> correct



18.) Emittance in an electron ring / or linear machine

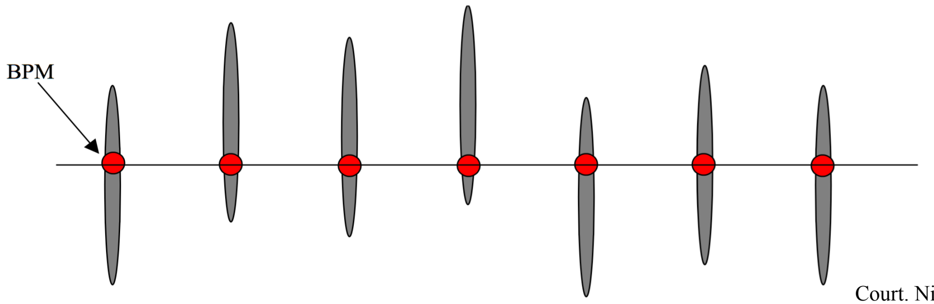
Hor. Plane: *localise the non-zero dispersion e.g. by applying a DBA lattice*



Example: PETRA 3
Lattice: FODO & 9 DBA cells
$\varepsilon_x = 1.2\ nm, \ \ \varepsilon_y / \varepsilon_x = 1\%$

Vert. Plane: *ε is determined by the radiation process & the (spurious) dispersion…*

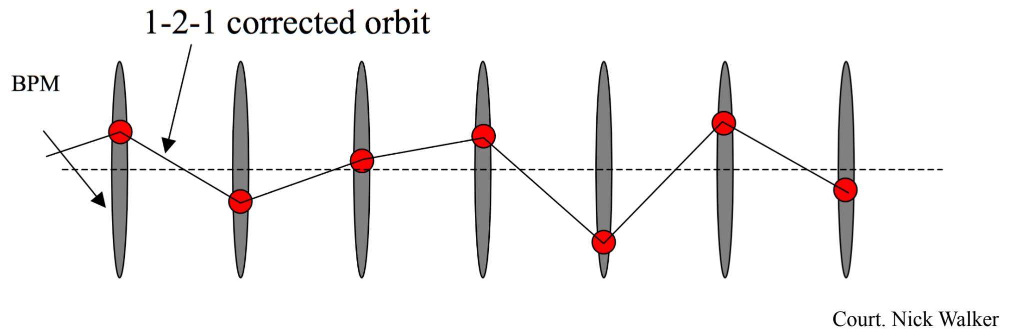
Example: Misaligned quadrupoles,
but BPMs ok
→Apply local kick compensation
→null problemo



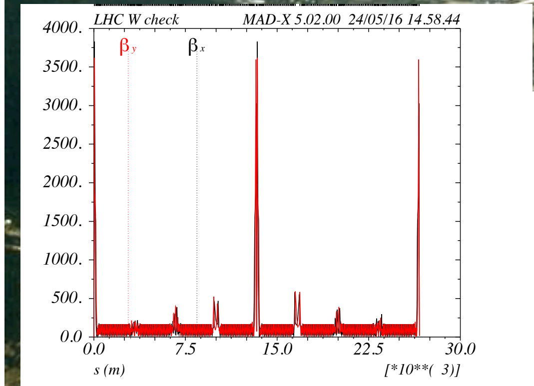
Court. Nick Walker

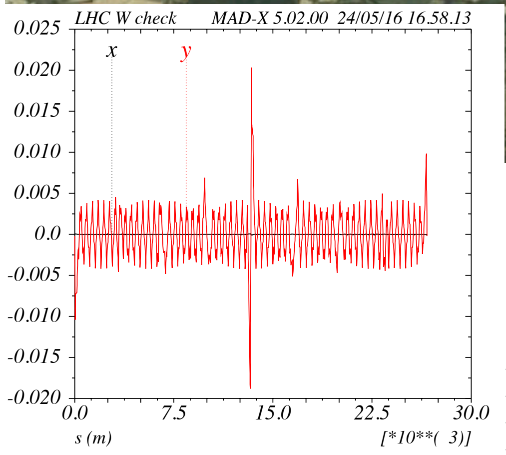
Avoid (vertical) Dispersion as much as possible
… in the end ... any (!) non-zero D_y will inevitably lead to ε_y

Emittance in an electron ring / or linear machine

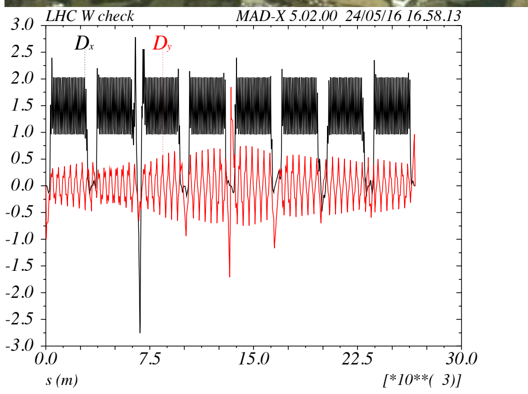


Quadrupoles ok, but BPM offset
→local orbit "correction" → bad orbit
→Dispersion increases
→Emittance increases





one single BPM offset, 200 μm
Corrected blindly → Δy = 5 … 20 mm
Vert. Dispersion → D= 0.5 ... 2m



$$\Delta\varepsilon = (\gamma D^2 + 2\alpha D D' + \beta D'^2) * \left(\frac{\Delta E}{E}\right)^2$$

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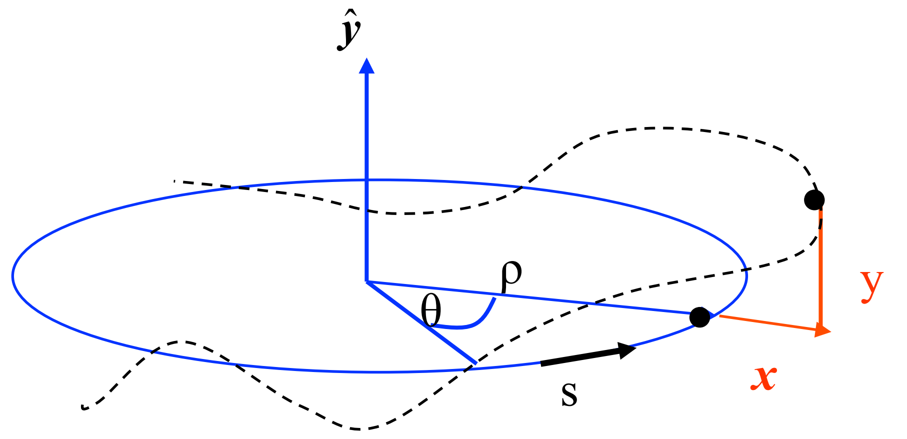
Appendix:

Equation of Motion:



Consider local segment of a particle trajectory
... and remember the old days:
(Goldstein page 27)

radial acceleration:

$$a_r = \frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2$$

Ideal orbit: $\rho = const, \quad \frac{d\rho}{dt} = 0$

Force: $F = m\rho\left(\frac{d\theta}{dt}\right)^2 = m\rho\omega^2$

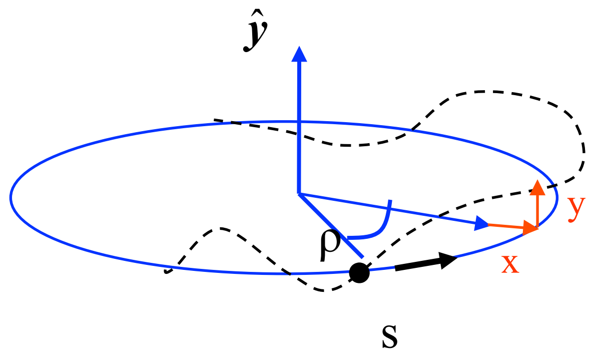
$$F = mv^2/\rho$$

general trajectory: $\rho \rightarrow \rho + x$

$$\boldsymbol{F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v}$$

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v$$

1 2



1 $$\frac{d^2}{dt^2}(x+\rho) = \frac{d^2}{dt^2}x \qquad \text{... as } \rho = const$$

2 *remember:* $x \approx mm$, $\rho \approx m$ ... → *develop for small x*

$$\frac{1}{x+\rho} \approx \frac{1}{\rho}\left(1-\frac{x}{\rho}\right)$$

Taylor Expansion

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!}f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + $$

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}\left(1-\frac{x}{\rho}\right) = eB_y v$$

guide field in linear approx.

$$B_y = B_0 + x\frac{\partial B_y}{\partial x} \qquad m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1-\frac{x}{\rho}) = ev\left\{B_0 + x\frac{\partial B_y}{\partial x}\right\} \qquad : m$$

$$\frac{d^2x}{dt^2} - \frac{v^2}{\rho}(1-\frac{x}{\rho}) = \frac{e\,v\,B_0}{m} + \frac{e\,v\,x\,g}{m}$$

independent variable: $t \rightarrow s$

$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt}$$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{ds}\frac{ds}{dt}\right) = \frac{d}{ds}\left(\underbrace{\frac{dx}{ds}}_{x'}\underbrace{\frac{ds}{dt}}_{v}\right)\frac{ds}{dt}$$

$$\frac{d^2x}{dt^2} = x''\,v^2 + \cancel{\frac{dx}{ds}\frac{dv}{ds}}\,v$$

$$x''v^2 - \frac{v^2}{\rho}(1-\frac{x}{\rho}) = \frac{e\,v\,B_0}{m} + \frac{e\,v\,x\,g}{m} \qquad : v^2$$

$$x'' - \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right) = \frac{e\ B_0}{mv} + \frac{e\ x\, g}{mv}$$

$$m\,v = p$$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = \frac{B_0}{p/e} + \frac{x\, g}{p/e}$$

normalize to momentum of particle

$$\frac{B_0}{p/e} = -\frac{1}{\rho}$$

$$\frac{g}{p/e} = k$$

$$x'' - \cancel{\frac{1}{\rho}} + \frac{x}{\rho^2} = -\cancel{\frac{1}{\rho}} + k\,x$$

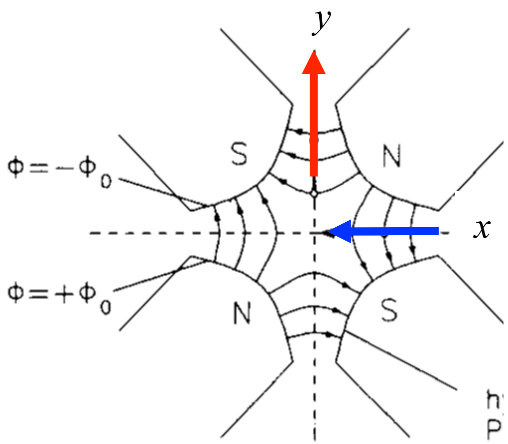
$$\boxed{x'' + x\left(\frac{1}{\rho^2} - k\right) = 0}$$

* *Equation for the vertical motion:*

$\frac{1}{\rho^2} = 0$ *no dipoles … in general …*

$k \leftrightarrow -k$ *quadrupole field changes sign*

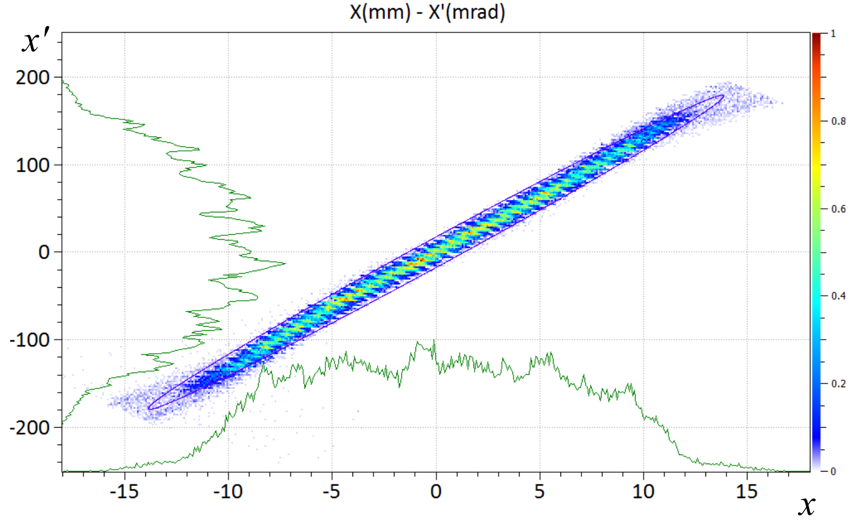
$$y'' + k\ y = 0$$



Statistical Interpretation of the beam emittance

Sometimes it is not so easy to draw a bare ellipse around the particle distribution an phase space.

And a more careful fitting of phase space distributions is needed to obtain the Twiss family members



court. L.Hein

We interpret the projections of the phase space distribution such, that they corespond to the rms values of the particle distribution:

$$\sigma_x = \sqrt{\epsilon_{x,rms} \cdot \beta_x}$$

$$\sigma'_x = \sqrt{\epsilon_{x,rms} \cdot \gamma_x}$$

With the statistical interpretation:

$$\sigma_x^2(s) = \langle x^2 \rangle, \qquad \sigma_{x'}^2(s) = \langle x'^2 \rangle$$

And the correlation function $\sigma_{xx'} = \langle xx' \rangle = -\alpha_x \epsilon_{x,rms}$

We obtain for the rms emittance

$$\varepsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}^2} = \sqrt{\langle x \rangle^2 \langle x' \rangle^2 - \langle xx' \rangle^2}$$

... for clarity:
each and every partile will still follow an ellipse in x,x' spce as long as we have to consider conservative forces only.

for details see e.g. N.Walker in CAS 2005

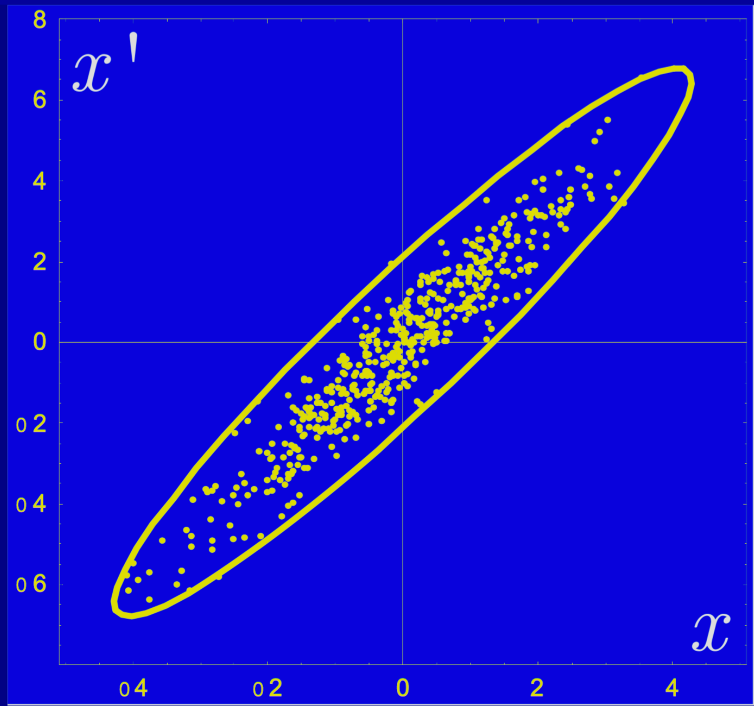
Sometimes we express the rms emittance as determinant of a "beam or sigma matrix

2nd-order moments:

$$\sigma = \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle x'^2 \rangle \end{pmatrix}$$

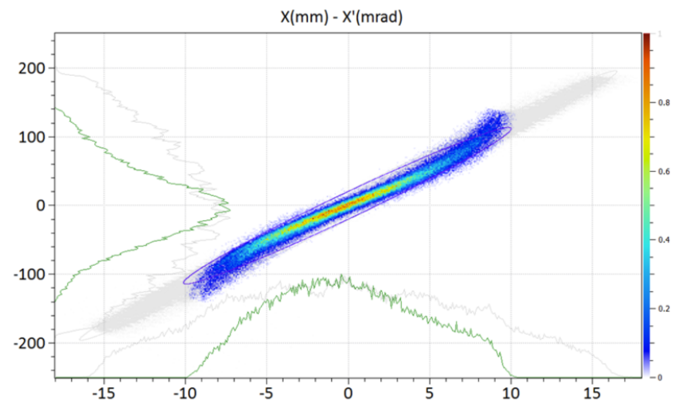
$$\varepsilon = \sqrt{|\sigma|}$$

$$= \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$



Whith the obvious connection to the Twiss paraetrisation

$$\sigma = \begin{pmatrix} \varepsilon\beta & -\varepsilon\alpha \\ -\varepsilon\alpha & \varepsilon(1+\alpha^2)/\beta \end{pmatrix}$$

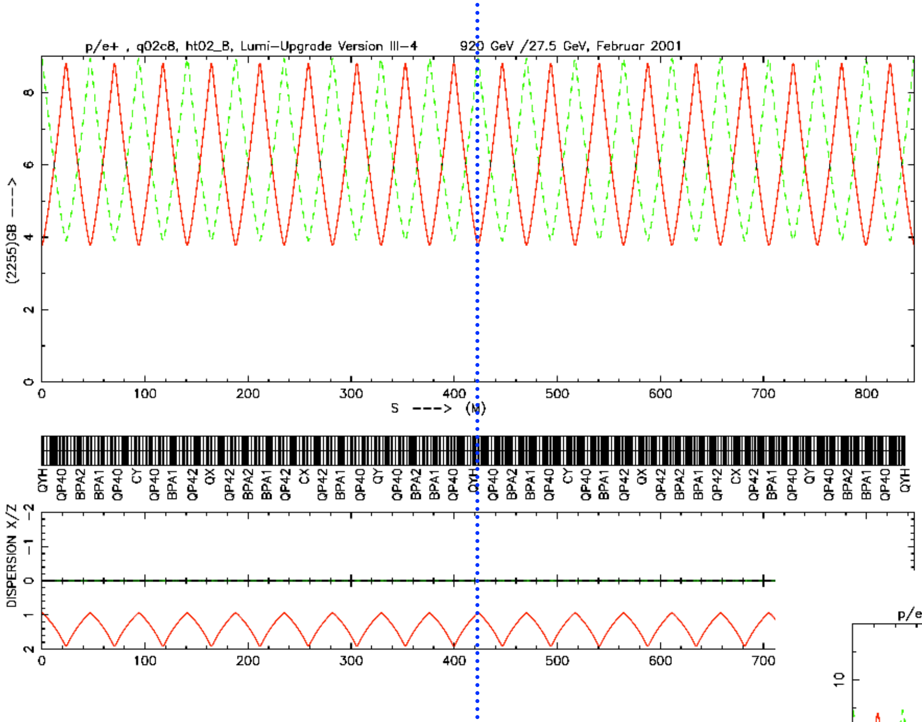


The phase space area ca differ considerably from the ideal ellipse in case of non-linear fields or special initial distributions

for details see e.g. N.Walker in CAS 2005

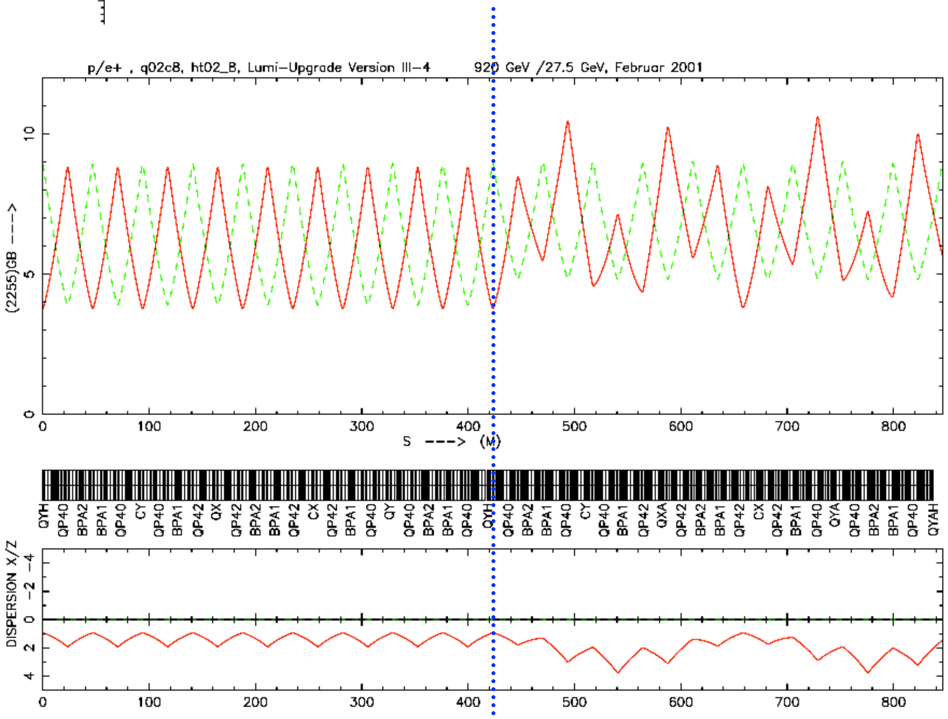
Matched & unmatched Transferline

Example: HERA Arc, FoDo structure



Transferline: matched beam optics.

twiss parameters at start correspond to periodic Twiss



Transferline: un-matched beam optics
at half the way:

twiss parameters at start correspond to periodic Twiss
quadstrengths reduced by 20 % for second part