Short Review/Refresher

Classical Electrodynamics (CED)

(.. and applications to accelerators)

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Recommended Reading Material (in this order)

[1] **R.P. Feynman,** *Feynman lectures on Physics*, **Vol2.**

[2] **Proceedings of CAS:** *RF for accelerators*, **Ebeltoft, Denmark, 8-17 June 2010, Edited by R. Bailey, CERN-2011-007.**

[3] **J.D. Jackson,** *Classical Electrodynamics* **(Wiley, 1998 ..)**

[4] **L. Landau, E. Lifschitz,** *The Classical Theory of Fields*, **Vol2. (Butterworth-Heinemann, 1975)**

[5] **J. Slater, N. Frank,** *Electromagnetism*, **(McGraw-Hill, 1947, and Dover Books, 1970)**

Some refresher on required vector calculus in backup slides (Gauss, Stoke ..)

OUTLINE

This does not replace a full course (i.e. $\approx$ 60 hours, some additional material in backup slides, details in bibliography)

Also, it cannot be treated systematically without special relativity.

The main topics discussed:

- **Basic electromagnetic phenomena**
- **Maxwell's equations**
- **Lorentz force and motion of particles in electromagnetic fields**
- **Electromagnetic waves in vacuum**
- **Electromagnetic waves in conducting media, waves in RF cavities and wave guides**

Variables and units used in this lecture

Formulae use SI units throughout.

$\vec{E}(\vec{r},t)$	=	**electric field [V/m]**
$\vec{H}(\vec{r},t)$	=	**magnetic field [A/m]**
$\vec{D}(\vec{r},t)$	=	**electric displacement [C/m^2]**
$\vec{B}(\vec{r},t)$	=	**magnetic flux density [T]**
q	=	**electric charge [C]**
$\rho(\vec{r},t)$	=	**electric charge density [C/m^3]**
$\vec{I}$, $\vec{j}(\vec{r},t)$	=	**current [A], current density [A/m^2]**
μ_0	=	**permeability of vacuum, 4** $\pi \cdot 10^{-7}$ **[H/m or N/A^2]**
ϵ_0	=	**permittivity of vacuum, 8.854** $\cdot 10^{-12}$ **[F/m]**

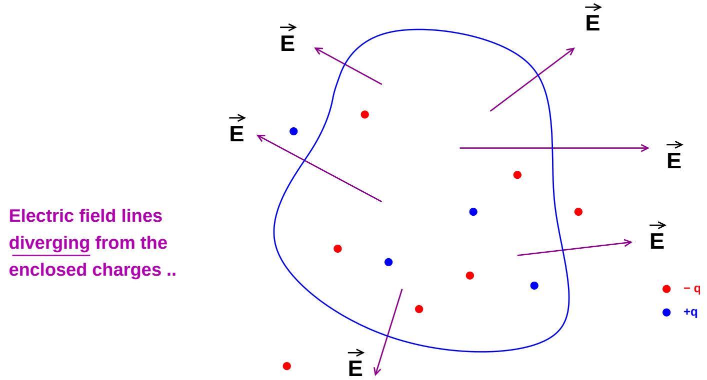
To save typing and space where possible (e.g. equal arguments):

$\vec{E}(\vec{r},t)$ → $\vec{E}$ **same for other variables ..**

- ELECTROSTATICS -

Gauss' theorem in the simplest form:

Surface S enclosing a volume V within which are charges: $q_1, q_2, \ldots$

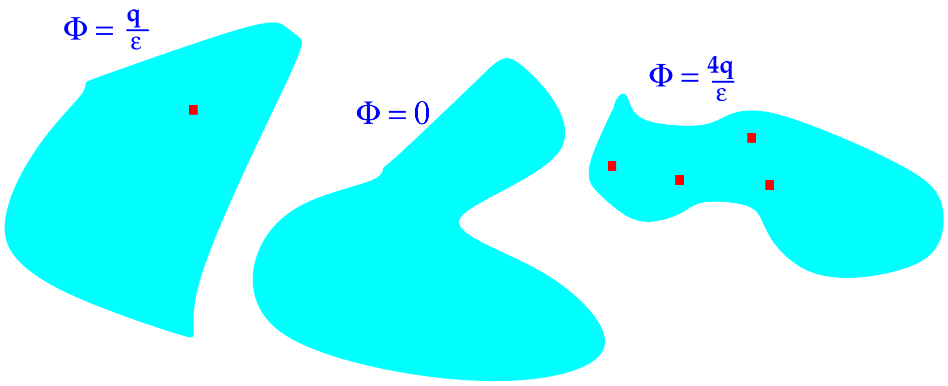


Sum up the fields passing through the surface → flux Φ

$$\Phi = \int_S \vec{E} \cdot \vec{n} \; dA = \sum_i \frac{q_i}{\epsilon_0} = \frac{Q}{\epsilon_0}$$

$\vec{n}$ is the normal unit vector and $\vec{E}$ the electric field at an area element dA of the surface

Surface integral of $\vec{E}$ equals total charge Q inside enclosed volume



Essence:

This holds for <u>any</u> arbitrary (closed) surface S, and:

- **Does not matter <u>how</u> the particles are distributed inside the volume**
- **Does not matter <u>whether</u> the particles are moving**
- **Does not matter <u>whether</u> the particles are in vacuum or material**

If we have not discrete charges but a continuous*) distribution:

Replace charge by charge density q_i ➡ ρ = charge per unit volume dV.

For a charge density it is replaced by a volume integral:

$$\int_S \vec{E} \cdot \vec{n}\; dA \quad = \quad \underbrace{\int_V \frac{\rho}{\epsilon_0} dV \quad = \quad \frac{Q}{\epsilon_0}}_{\text{this part is trivial}}$$

The volume V is the one enclosed by the surface S

*) **obviously does not exist ...**

With some vector calculus:

$$\int_S \vec{E} \cdot d\vec{A} = \int_V \frac{\rho}{\epsilon_0} \cdot dV = \frac{Q}{\epsilon_0} = \Phi_E$$

$$\underbrace{\int_S \vec{E} \cdot d\vec{A} = \int_V \nabla \vec{E} \cdot dV}_{\text{Gauss' formula}}$$ (relates surface and volume integrals)

→ $\nabla \vec{E} = \frac{\rho}{\epsilon_0}$ written as divergence : div $\vec{E}$

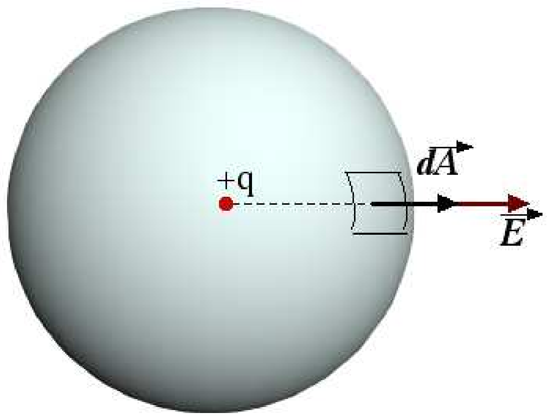
Flux of electric field $\vec{E}$ through any closed surface is proportional to net electric charge Q enclosed in the region (Gauss' Theorem).

Written with charge density ρ we get Maxwell's first equation:

$$\text{div}\vec{E} = \nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon_0}$$

Divergence: "measures" outward flux Φ_E of the field ...

Simplest possible example: flux from a charge q



A charge q generates a field $\vec{E}$ according to (Coulomb):

$$\vec{E} \; = \; \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}$$

Enclose it by a sphere: $\vec{E} \; = \; const.$ on a sphere (area is $4\pi \cdot r^2$):

$$\int\int_{sphere} \vec{E} \cdot d\vec{A} \; = \; \frac{q}{4\pi\epsilon_0}\int\int_{sphere}\frac{dA}{r^2} \; = \; \frac{q}{\epsilon_0}$$

Surface integral through sphere A is charge inside the sphere (any radius)

We can derive the field $\vec{E}$ from a scalar electrostatic potential $\phi(x,y,z)$, i.e.:

$$\vec{E} = -\,\text{grad}\ \phi = -\nabla\phi = -\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

then we have

$$\nabla\vec{E} = -\nabla^2\phi = -\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) = \frac{\rho(x,y,z)}{\epsilon_0}$$

This is <u>Poisson's equation</u>

All we need is to find ϕ Example →

Simplest possible charge distribution: point charge

$$\phi(r) = \frac{q}{4\pi\epsilon_0 \; r}$$

$$\vec{E} = -\nabla\phi(r) = -\frac{q}{4\pi\epsilon_0} \cdot \frac{\vec{r}}{r^3}$$

A very important example: 3D Gaussian distribution

$$\rho(x,y,z) = \frac{Q}{\sigma_x \sigma_y \sigma_z \sqrt{2\pi}^3} \exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{z^2}{2\sigma_z^2}\right)$$

($\sigma_x, \sigma_y, \sigma_z$ r.m.s. sizes)

$$\phi(x,y,z,\sigma_x,\sigma_y,\sigma_z) = \frac{Q}{4\pi\epsilon_0} \int_0^\infty \frac{\exp(-\frac{x^2}{2\sigma_x^2+t} - \frac{y^2}{2\sigma_y^2+t} - \frac{z^2}{2\sigma_z^2+t})}{\sqrt{(2\sigma_x^2+t)(2\sigma_y^2+t)(2\sigma_z^2+t)}} dt$$

For the interested: Fields given in the backup slides

For a derivation, see e.g. W. Herr, *Beam-Beam Effects*,
in Proceedings CAS Zeuthen, 2003, CERN-2006-002, and references therein.

Very important in practice:

Poisson's equation in Polar coordinates (r, φ)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) \;+\; \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \varphi^2} \;=\; -\frac{\rho}{\epsilon_0}$$

Poisson's equation in Cylindrical coordinates (r, φ, z)

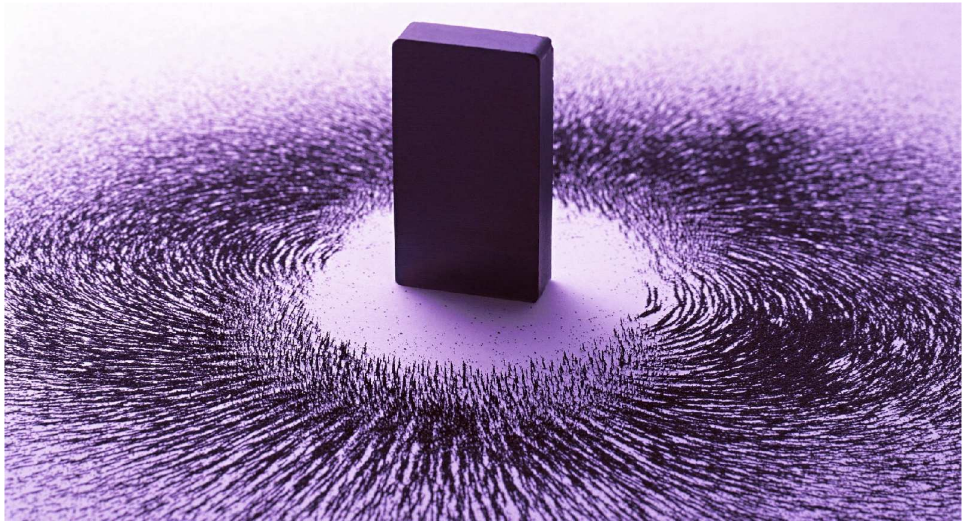
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) \;+\; \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \varphi^2} \;+\; \frac{\partial^2 \phi}{\partial z^2} \;=\; -\frac{\rho}{\epsilon_0}$$

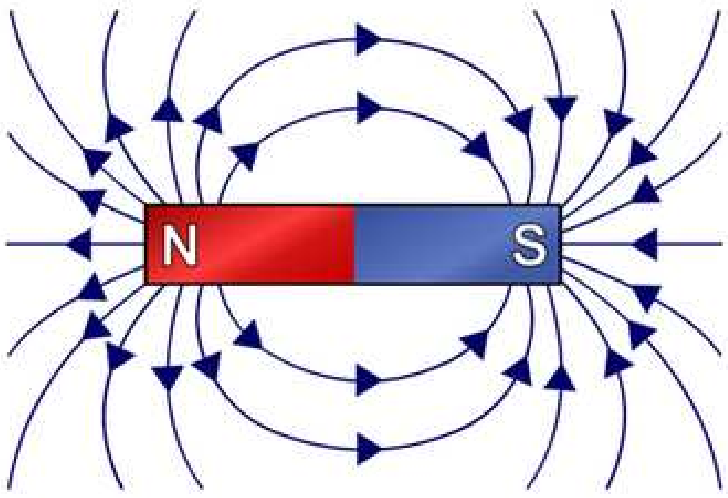
Poisson's equation in Spherical coordinates (r, θ, φ)

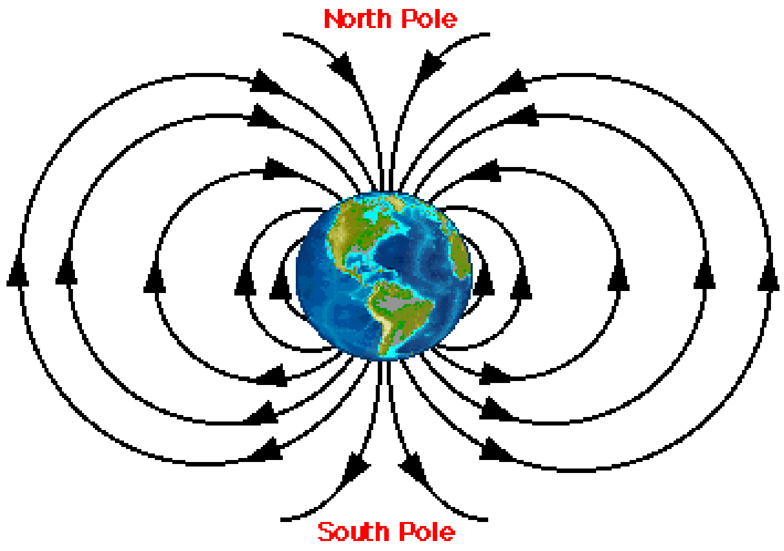
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \phi}{\partial r}\right) \;+\; \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta\frac{\partial \phi}{\partial \theta}\right) \;+\; \frac{1}{r^2 sin\theta}\frac{\partial^2 \phi}{\partial \varphi^2} \;=\; -\frac{\rho}{\epsilon_0}$$

Examples for solutions in [3]

- MAGNETOSTATICS -







Definitions:

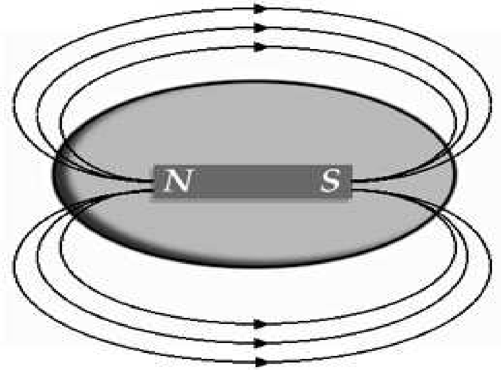
Magnetic field lines from North to South

Properties:

Described as vector fields

All field lines are closed lines →

Gauss' second law ...



$$\int_S \vec{B} \;\; d\vec{A} = \int_V \nabla\vec{B} \;\; dV \;\; = \;\; 0$$

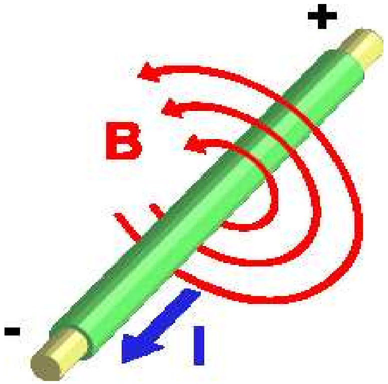
$$\nabla\vec{B} = 0$$

Closed field lines of magnetic flux density ($\vec{B}$): What goes out ANY closed surface also goes in, Maxwell's second equation:

$$\nabla\vec{B} = \mu_0\nabla\vec{H} = 0$$

→ **Physical significance: no Magnetic Charges (Monopoles)**

From Ampere/Oersted law, for example current density $\vec{j}$:



<u>Static</u> electric current induces encircling (curling) magnetic field

$$\text{curl}\vec{B} \;=\; \nabla \times \vec{B} = \mu_0 \vec{j}$$

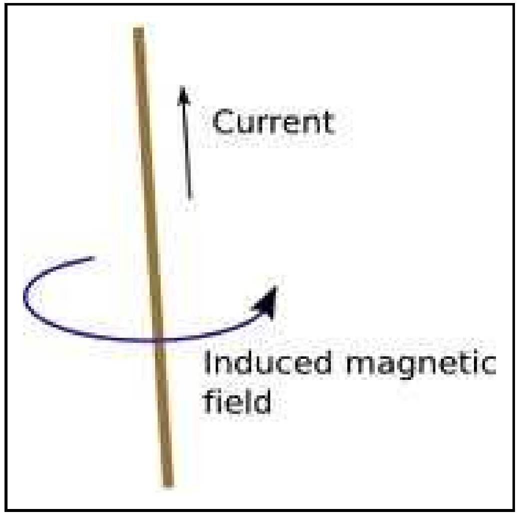
or in integral form the current density becomes the current I:

$$\int\int_A \nabla \times \vec{B} \;\; d\vec{A} \;=\; \int\int_A \mu_0 \vec{j} \;\; d\vec{A} \;\;=\;\; \mu_0 \vec{I}$$

Curl: "measures" <u>directional</u> strength <u>along</u> the field lines ...

Application (derivation see [1 - 5]):

For a static electric current I in a single wire we get Biot-Savart law (we have used Stoke's theorem and area of a circle $A = r^2 \cdot \pi$):



$$\vec{B} = \frac{\mu_0}{4\pi} \oint \vec{I} \cdot \frac{\vec{r} \times d\vec{s}}{r^3}$$

$$\vec{B} = \frac{\mu_0}{2\pi} \frac{\vec{I}}{r}$$

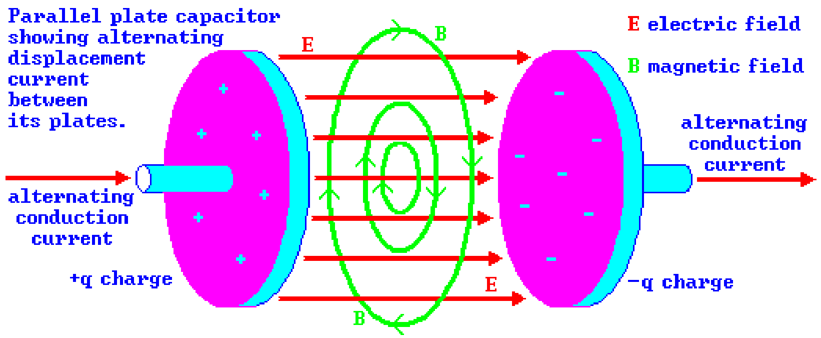
For magnetic field calculations in wires ..

- THIS IS NOT THE WHOLE STORY -

- enter Maxwell -

Do we need an electric current ?

Maxwell's displacement current, e.g. a charging capacitor $\vec{j}_d$:



Defining a Displacement Current $\vec{I}_d$:

$$\vec{I}_d = \frac{dq}{dt} = \epsilon_0 \cdot \frac{d\Phi}{dt} = \epsilon_0 \frac{d}{dt} \int\int_{area} \vec{E} \cdot d\vec{A}$$

Not a current from moving charges

But a current from time varying electric fields

Displacement current I_d produces magnetic field, just like "actual currents" do ...

→ **Time varying electric field induce magnetic field (using the current density $\vec{j}_d$**

$$\nabla \times \vec{B} = \mu_0 \vec{j}_d = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

Bottom line:

Magnetic fields $\vec{B}$ can be generated in two ways:

$\nabla \times \vec{B} = \mu_0 \vec{j}$ **(electric current, Ampere)**

$\nabla \times \vec{B} = \mu_0 \vec{j}_d = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$ **(changing electric field, Maxwell)**

or putting them together:

$$\nabla \times \vec{B} = \mu_0(\vec{j} + \vec{j}_d) = \mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

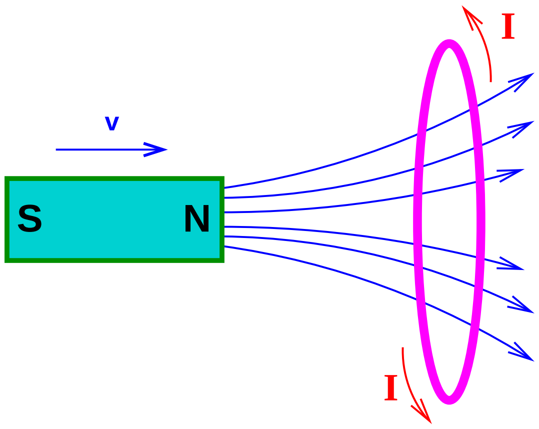
or as integral equations (using Stoke's formula):

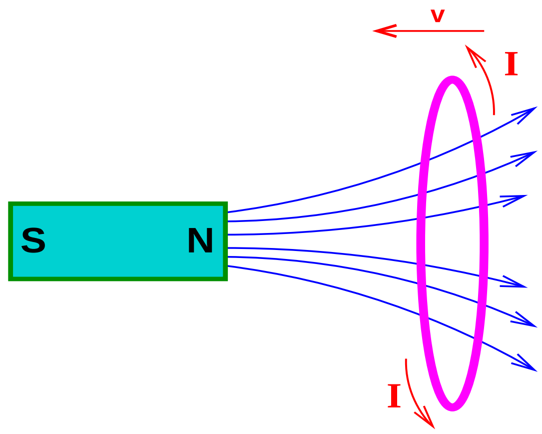
$$\underbrace{\oint_C \vec{B} \cdot d\vec{r} = \int_A \nabla \times \vec{B} \cdot d\vec{A}}_{\text{Stoke's formula}} = \int_A \left(\mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{A}$$

- enter Faraday -

- first unification -

Faraday's law (electromagnetic induction):





A changing flux Ω through an area A produces "electromotive force" (EMF) ➡ in a conducting coil: current $\color{red}{I}$

(Can move magnet or coil: any relative motion will do ..)

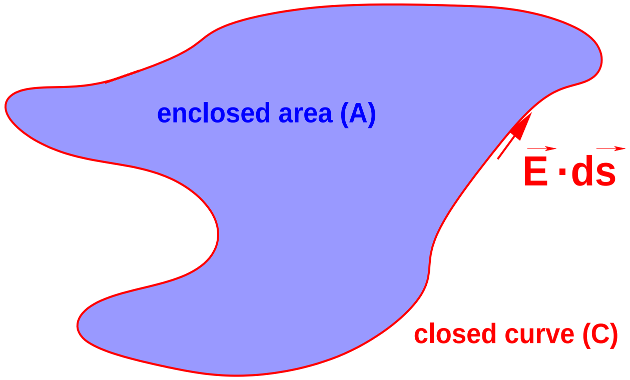
$$\text{flux} \;=\; \Omega \;=\; \int_A \vec{B} d\vec{A} \qquad\qquad \text{EMF} \;=\; \oint_C \vec{E} \cdot d\vec{s}$$

$$-\frac{\partial \Omega}{\partial t} \;=\; -\frac{\partial}{\partial t} \underbrace{\int_A \vec{B} d\vec{A}}_{flux\ \Omega} = \oint_C \vec{E} \cdot d\vec{s}$$

$$-\frac{\partial \Omega}{\partial t} \; = \; -\int_A \frac{\partial}{\partial t}\vec{B} d\vec{A} = \oint_C \vec{E} \cdot d\vec{s}$$

- **In a conducting coil: changing flux induces circulating current**
- **Flux can be changed by:**
  - **Change of magnetic field $\vec{B}$ with time t (e.g. transformers)**
  - **Change of area A with time t (e.g. dynamos)**
- **Electromotive force (EMF):**
  1. **Energy of a unit charge after one loop**
  2. **Voltage if the loop is cut, i.e. open circuit**

$$-\int_A \frac{\partial \vec{B}}{\partial t} d\vec{A} = \underbrace{\int_A \nabla \times \vec{E}\ d\vec{A} = \oint_C \vec{E} \cdot d\vec{s}}_{Stoke's formula}$$

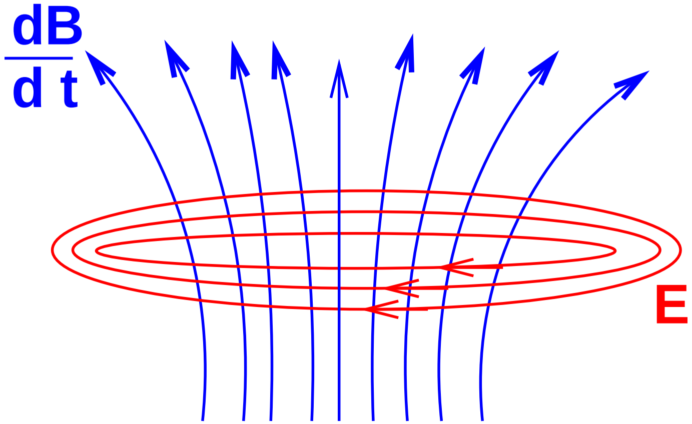


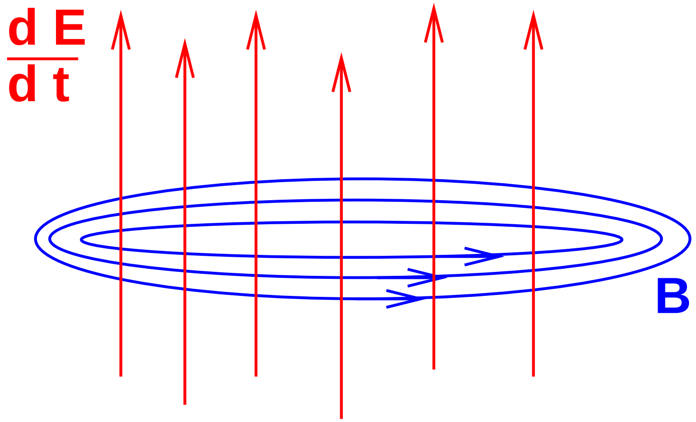
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Changing field through <u>any</u> closed area induces electric field in the (arbitrary) boundary

→ **becomes Maxwell-Faraday law**

Summary: Time Varying Fields (most significant for RF systems !)





- **Time varying magnetic fields produce curling electric field:**

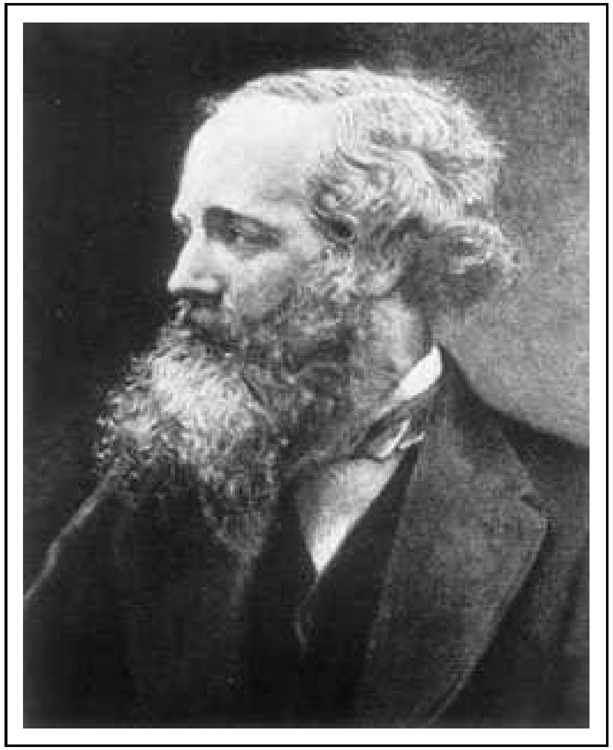
$$\text{curl}(\vec{E}) \;=\; \nabla \times \vec{E} \;=\; -\frac{d\vec{B}}{\partial t}$$

- **Time varying electric fields produce curling magnetic field:**

$$\text{curl}(\vec{B}) \;=\; \nabla \times \vec{B} \;=\; \mu_0 \epsilon_0 \frac{d\vec{E}}{\partial t}$$

because of the $\times$ they are perpendicular: $\vec{E} \perp \vec{B}$

Put together: Maxwell's Equations in vacuum (SI units)



$$\nabla \vec{E} = \frac{\rho}{\epsilon_0} = -\Delta\phi \qquad \text{(I)}$$

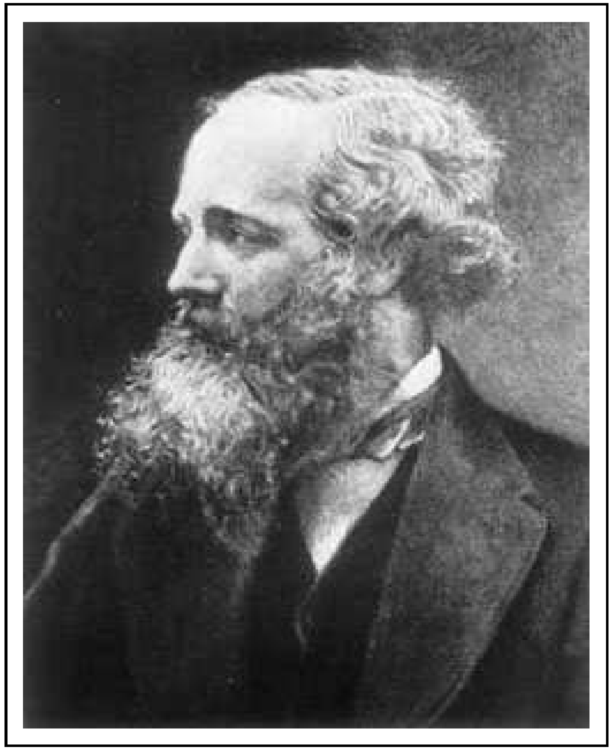
$$\nabla \vec{B} = 0 \qquad \text{(II)}$$

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt} \qquad \text{(III)}$$

$$\nabla \times \vec{B} = \mu_0 \left(\vec{j} + \epsilon_0 \frac{d\vec{E}}{dt} \right) \qquad \text{(IV)}$$

(a.k.a. Microscopic Maxwell equations)

For completeness



$$\int_A \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

$$\int_A \vec{B} \cdot d\vec{A} = 0$$

$$\oint_C \vec{E} \cdot d\vec{s} = -\int_A \left(\frac{d\vec{B}}{dt} \right) \cdot d\vec{A}$$

$$\oint_C \vec{B} \cdot d\vec{s} = \mu_0 \int_A \left(\vec{j} + \epsilon_0 \frac{d\vec{E}}{dt} \right) \cdot d\vec{A}$$

Equivalent equations written in Integral Form, (using Gauss' and Stoke's formulae)

Maxwell in Physical terms

1. **Electric fields $\vec{E}$ are generated by charges and proportional to total charge**
2. **Magnetic monopoles do <u>not</u> exist**
3. **Changing magnetic flux generates circumscribing electric fields/currents**

4.1 **Changing electric flux generates circumscribing magnetic fields**

4.2 **Static electric current generates circumscribing magnetic fields**

Frequent complaint: "I have seen them in a different form !"

The Babel of Units: →

Units:	**Gauss law**	**Ampere/Maxwell**
SI	$\nabla\vec{E} \;=\; \frac{\rho}{\epsilon_0}$	$\nabla\times\vec{B} = \;\mu_0\vec{j} \;+\; \mu_0\epsilon_0\frac{d\vec{E}}{dt}$
Electro-static ($\epsilon_0 = 1$)	$\nabla\vec{E} \;=\; 4\pi\rho$	$\nabla\times\vec{B} = \;\frac{4\pi}{c^2}\vec{j} \;+\; \frac{1}{c^2}\frac{d\vec{E}}{dt}$
Electro-magnetic ($\mu_0 = 1$)	$\nabla\vec{E} \;=\; 4\pi c^2\rho$	$\nabla\times\vec{B} = \;4\pi\vec{j} \;+\; \frac{1}{c^2}\frac{d\vec{E}}{dt}$
Gauss cgs	$\nabla\vec{E} \;=\; 4\pi\rho$	$\nabla\times\vec{B} = \;\frac{4\pi}{c}\vec{j} \;+\; \frac{1}{c}\frac{d\vec{E}}{dt}$
Lorentz	$\nabla\vec{E} \;=\; \rho$	$\nabla\times\vec{B} = \;\frac{1}{c}\vec{j} \;+\; \frac{1}{c}\frac{d\vec{E}}{dt}$

Also: $\vec{B}^{Gauss} = \sqrt{\frac{4\pi}{\mu_0}}\vec{B}^{SI}$ $\qquad \rho^{Gauss} = \frac{\rho^{SI}}{\sqrt{4\pi\epsilon_0}}$ **and so on .....**

That's not all → Electromagnetic fields in material

In vacuum:

$$\vec{D} = \epsilon_0 \cdot \vec{E}, \qquad \vec{B} = \mu_0 \cdot \vec{H}$$

In a material:

$$\vec{D} = \epsilon_r \cdot \epsilon_0 \cdot \vec{E} \;=\; \epsilon_0 \vec{E} \;+\; \vec{P}$$

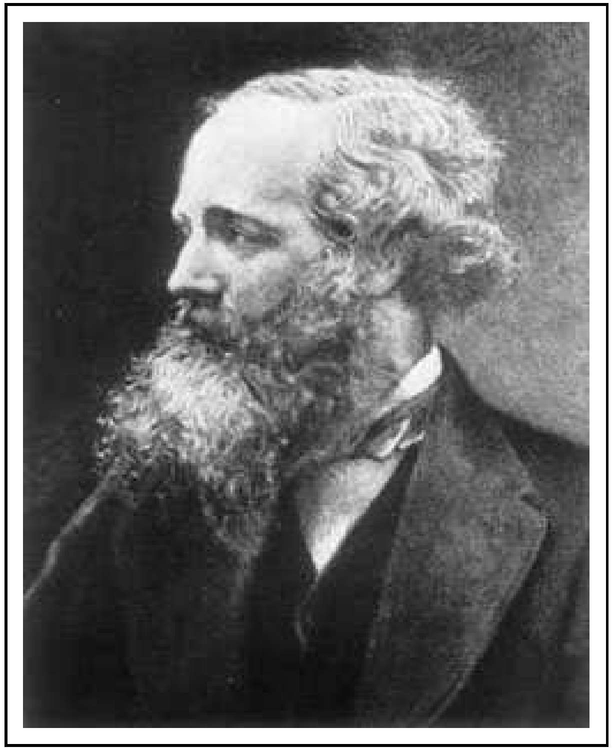
$$\vec{B} = \mu_r \cdot \mu_0 \cdot \vec{H} \;=\; \mu_0 \vec{H} \;+\; \vec{M}$$

Origin: $\vec{P}$**olarization and** $\vec{M}$**agnetization**

$\epsilon_r(\vec{E}, \vec{r}, \omega)$ → ϵ_r **is relative permittivity** $\approx \quad [1 - 10^5]$
$\mu_r(\vec{H}, \vec{r}, \omega)$ → μ_r **is relative permeability** $\approx \quad [0(!) - 10^6]$

(i.e.: linear, isotropic, non-dispersive)

Once more: Maxwell's Equations



$$\nabla \vec{D} = \rho$$
$$\nabla \vec{B} = 0$$
$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$
$$\nabla \times \vec{H} = \vec{j} + \frac{d\vec{D}}{dt}$$

(a.k.a. Macroscopic Maxwell equations)

Something on potentials (needed in lecture on Relativity):

Electric fields can be written using a (scalar) potential ϕ:

$$\vec{E} \;=\; -\vec{\nabla}\phi$$

Since div $\vec{B} = 0$, **we can write $\vec{B}$ using a (vector) potential $\vec{A}$:**

$$\vec{B} \;=\; \vec{\nabla} \times \vec{A} \;=\; \text{curl } \vec{A}$$

combining Maxwell(I) + Maxwell(III):

$$\vec{E} \;=\; -\vec{\nabla}\phi \;-\; \frac{\partial \vec{A}}{\partial t}$$

Fields can be written as derivatives of scalar and vector potentials $\Phi(x, y, z)$ and $\vec{A}(x, y, z)$

(absolute values of potentials Φ and $\vec{A}$ can not be measured ..)

The Coulomb potential of a static charge q is written as:

$$\Phi(\vec{r}) \;=\; \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{|\vec{r} - \vec{r}_q|}$$

where $\vec{r}$ is the observation point and $\vec{r}_q$ the location of the charge

The vector potential is linked to the current $\vec{j}$:

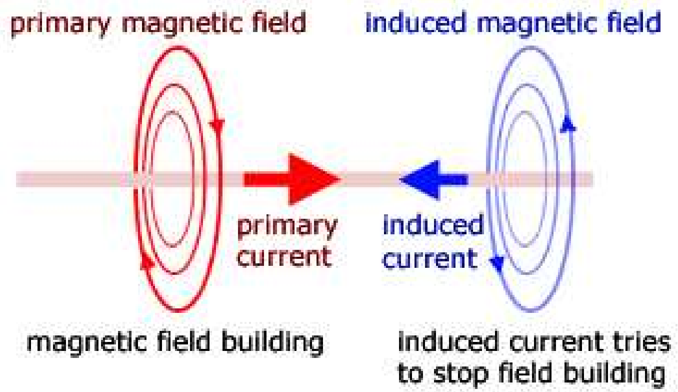
$$\nabla^2 \vec{A} \;=\; \mu_0 \vec{j}$$

The knowledge of the potentials allows the computation of the fields → see lecture on relativity (fields of moving charges)

Applications of Maxwell's Equations

- Powering of magnets
- Lorentz force, motion in EM fields
  - Motion in electric fields
  - Motion in magnetic fields
- EM waves (in vacuum and in material)
- Boundary conditions
- EM waves in cavities and wave guides

Powering and self-induction



- Induced magnetic flux $\vec{B}$ changes with changing current

➡ Induces a current and magnetic field $\vec{B}_i$ voltage in the conductor

➡ Induced current will oppose change of current (Lenz's law)

➡ We want to change a current to ramp a magnet ...

Ramp rate defines required Voltage:

$$U = -L \frac{\partial I}{\partial t}$$

Inductance L in Henry (H)

Example:

- **Required ramp rate: 10 A/s**
- **With L = 15.1 H per powering sector**

➡ **Required Voltage is $\approx$ 150 V**

Lorentz force on charged particles

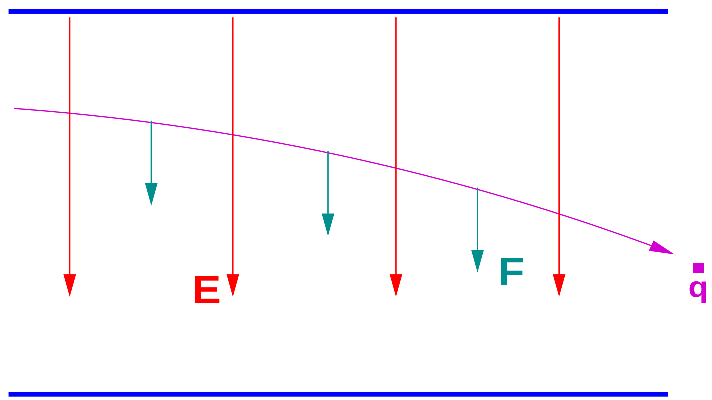
Moving ($\vec{v}$) charged (q) particles in electric ($\vec{E}$) and magnetic ($\vec{B}$) fields experience a force $\vec{f}$ (Lorentz force):

$$\vec{f} \;=\; q \cdot (\vec{E} \;+\; \vec{v} \;\times\; \vec{B})$$

Why a mysterious and incomprehensible dependence on the velocity of the charge ???

Often treated as ad hoc plugin to Maxwell's equation, but it is not (see lecture on "Special Relativity") !!

Motion in an electric field



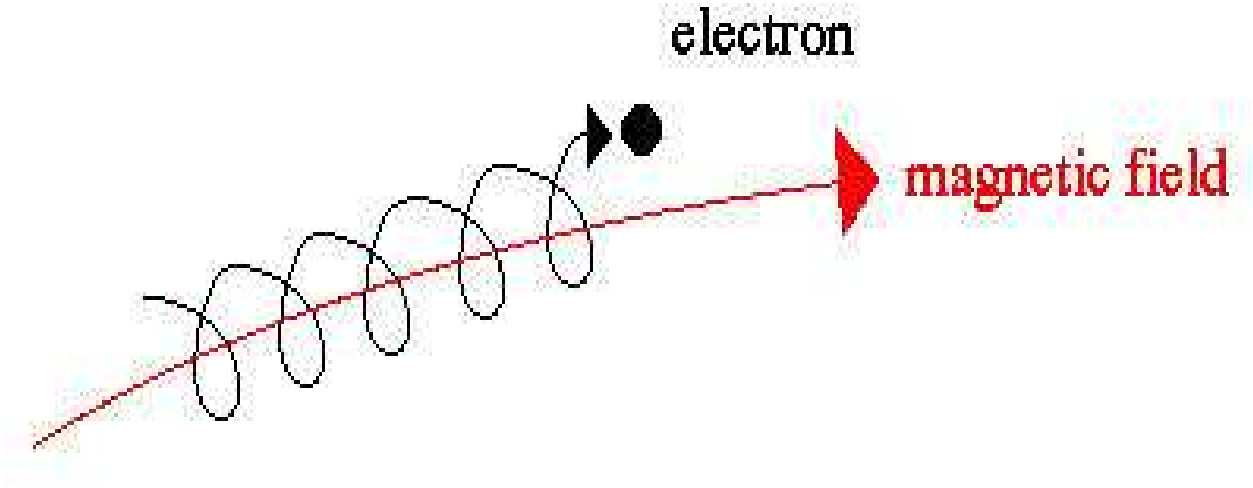
$$\frac{d}{dt}(m_0\vec{v}) \quad = \quad \vec{f} \quad = \quad q \cdot \vec{E}$$

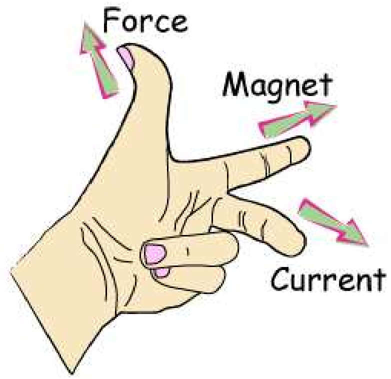
The solution is:

$$\vec{v}_{\parallel} \quad = \quad \frac{q \cdot \vec{E}}{m_0} \cdot \; t \quad \Rightarrow \quad \vec{r}_{\parallel} \quad = \quad \frac{q \cdot \vec{E}}{2m_0} \cdot \; t^2 \qquad (\textbf{parabola})$$

Constant E-field deflects beams: TV, electrostatic separators (SPS,LEP)

Motion in magnetic fields





Assume first no electric field:

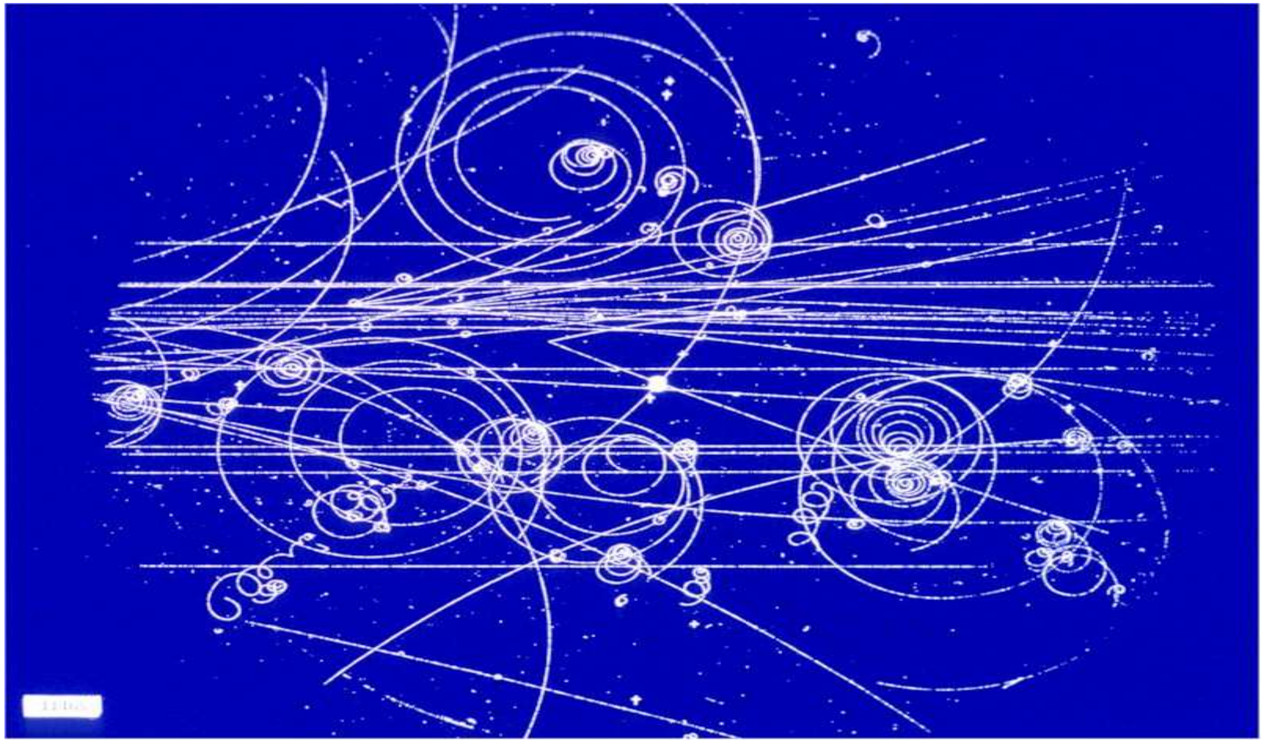
$$\frac{d}{dt}(m_0\vec{v}) = \vec{f} = q \cdot \vec{v} \times \vec{B}$$

Force is perpendicular to both, $\vec{v}$ and $\vec{B}$

No forces on particles at rest

<u>**Why: see lecture on special relativity**</u>

Important application:



Tracks from particle collisions, lower energy particles have smaller bending radius, allows determination of momenta ..

Q1: what is the direction of the magnetic field ???

Q2: what is the charge of the incoming particle ???

Example: Motion in a magnetic dipole

Practical units:

$$B[T] \cdot \rho[m] \quad = \quad \frac{p[eV/c]}{c[m/s]}$$

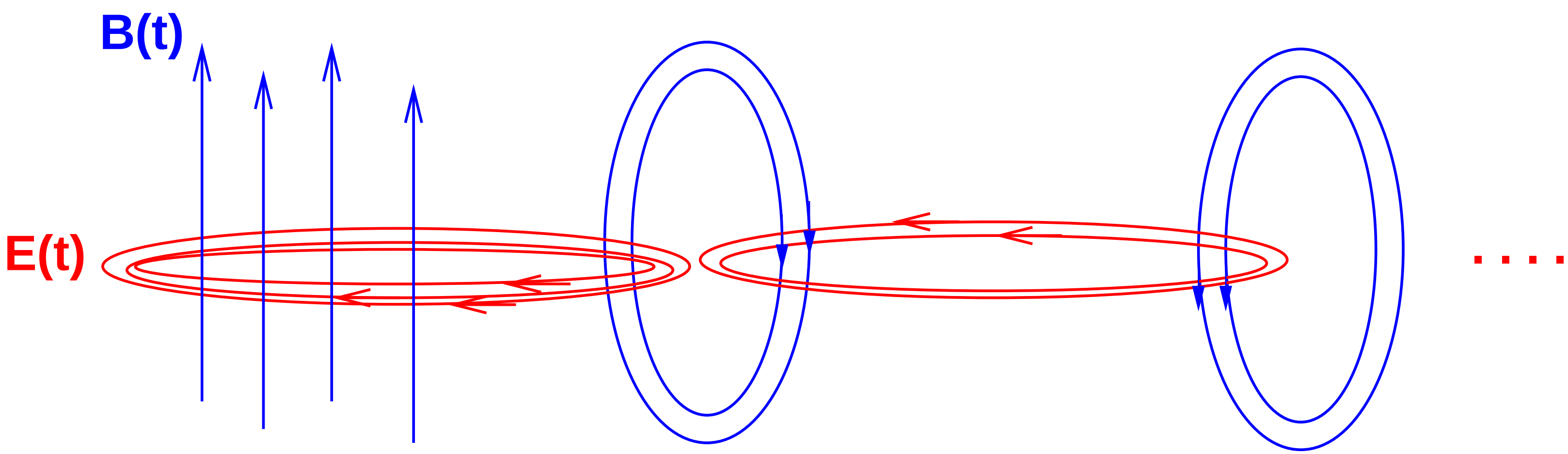
Example LHC:

B = 8.33 T, p = 7000 GeV/c → ρ **= 2804 m**

Use of static fields (some examples, incomplete)

- **Magnetic fields**
  - **Bending magnets**
  - **Focusing magnets (quadrupoles)**
  - **Correction magnets (sextupoles, octupoles, orbit correctors, ..)**
- **Electric fields**
  - **Electrostatic separators (beam separation in particle-antiparticle colliders)**
  - **Very low energy machines**
- **What about non-static, time-varying fields ?**

Time Varying Fields (very schematic)



Time varying magnetic fields produce circular electric fields

Time varying electric fields produce circular magnetic fields

➡ Can produce self-sustaining, propagating fields (i.e. waves)

➡ Example for source (classical picture): oscillating charge

Electromagnetic waves (classical picture)

Vacuum: only fields, no charges ($\rho = 0$), no current ($j = 0$) ...

$$\textbf{From:} \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

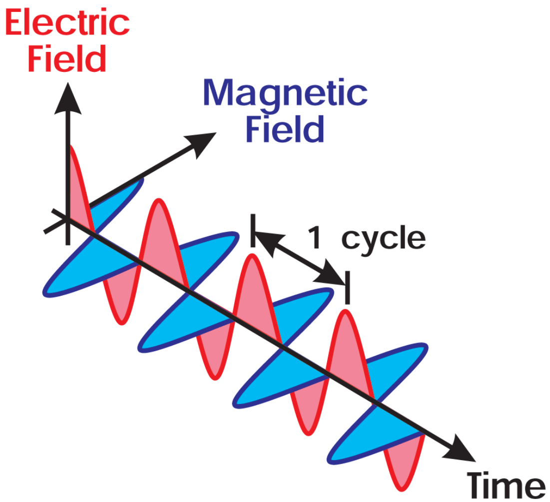
$$\begin{aligned}
\Longrightarrow \quad \nabla \times (\nabla \times \vec{E}) &= -\nabla \times (\frac{\partial \vec{B}}{\partial t}) \\
\Longrightarrow \quad -(\nabla^2 \vec{E}) &= -\frac{\partial}{\partial t}(\nabla \times \vec{B}) \\
\Longrightarrow \quad -(\nabla^2 \vec{E}) &= -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}
\end{aligned}$$

It happens to be: $\mu_0 \cdot \epsilon_0 = \frac{1}{c^2}$

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \cdot \epsilon_0 \cdot \frac{\partial^2 \vec{E}}{\partial t^2} \quad \textbf{and} \quad \nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = \mu_0 \cdot \epsilon_0 \cdot \frac{\partial^2 \vec{B}}{\partial t^2}$$

General form of a wave equation

Solutions of the wave equations:



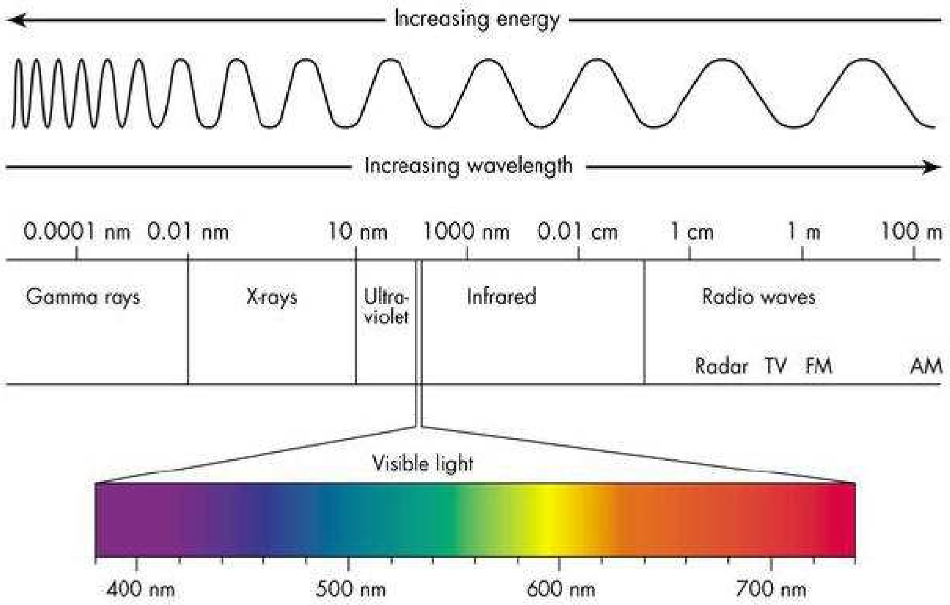
$$
\begin{aligned}
\vec{E} &= \vec{E_0} e^{i(\vec{k}\cdot\vec{r}-\omega t)} \\
\vec{B} &= \vec{B_0} e^{i(\vec{k}\cdot\vec{r}-\omega t)} \\
|\vec{k}| &= \frac{2\pi}{\lambda} = \frac{\omega}{c} \quad (\textbf{propagation vector}) \\
\lambda &= (\textbf{wave length, 1 cycle}) \\
\omega &= (\textbf{frequency} \cdot 2\pi) \\
c &= \frac{\omega}{k} = (\textbf{wave velocity})
\end{aligned}
$$

Magnetic and electric fields are transverse to direction of propagation:

$\vec{E} \perp \vec{B} \perp \vec{k}$ ➡ $\vec{k} \times \vec{E_0} = \omega \vec{B_0}$

Speed of wave in vacuum: **c = 299792458.000 m/s**

Examples: Spectrum of EM waves (we are exposed to)



Radio	➡	**as low as** 40 **Hz**	($\lesssim\ 10^{-13}$ **eV**)
CMB	➡	$\lesssim\ 3\cdot 10^{11}$ **Hz**	($\lesssim\ 10^{-3}$ **eV**)
yellow light	➡	$\approx\ 5\cdot 10^{14}$ **Hz**	($\approx$ **2 eV**)
X rays	➡	$\leq\ 1\cdot 10^{18}$ **Hz**	($\approx$ **4 keV**)
γ **rays**	➡	$\leq\ 3\cdot 10^{21}$ **Hz**	($\leq$ **12 MeV**)
$\pi^0 \ \to \gamma\gamma$	➡	$\geq\ 2\cdot 10^{22}$ **Hz**	($\geq$ **70 MeV**)

Polarization of EM waves (Classical Picture !):

The solutions of the wave equations imply monochromatic plane waves:

$$\vec{E} \;=\; \vec{E_0} e^{i(\vec{k}\cdot\vec{r}-\omega t)} \qquad\qquad \vec{B} \;=\; \vec{B_0} e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

Look now only at electric field, re-written using unit vectors in the plane transverse to propagation: $\vec{\epsilon_1} \;\perp\; \vec{\epsilon_2} \;\perp\; \vec{k}$

Two Components: $\vec{E_1} \;=\; \vec{\epsilon_1} E_1 \; e^{i(\vec{k}\cdot\vec{r}-\omega t)} \qquad \vec{E_2} \;=\; \vec{\epsilon_2} E_2 \; e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$$\Rightarrow \vec{E} \;=\; (\vec{E_1} + \vec{E_2}) \;=\; (\vec{\epsilon_1} E_1 + \vec{\epsilon_2} E_2) \; e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

With a phase shift ϕ **between the two directions:**

$$\vec{E} \;= \vec{\epsilon_1} E_1 \; e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \vec{\epsilon_2} E_2 \; e^{i(\vec{k}\cdot\vec{r}-\omega t+\phi)}$$

$\phi = 0$**: linearly polarized light**

$\phi \neq 0$**: elliptically polarized light**

$\phi = \pm\frac{\pi}{2}$ **and** $E_1 \;=\; E_2$**: circularly polarized light**

Polarized light - why interesting:

Produced (amongst others) in Synchrotron light machines (linearly and circularly polarized light, adjustable)

blue sky !

Accelerator and other applications:

- **Polarized light reacts differently with charged particles**
- **Beam diagnostics, medical diagnostics (blood sugar, ..)**
- **Inverse FEL**
- **Material science**
- **3-D motion pictures, LCD display, outdoor activities, cameras (glare), ...**
- **...**

Energy in electromagnetic waves (in brief, details in [2, 3, 4]):

We define as the Poynting vector (SI units):

$$\vec{S} \;=\; \frac{1}{\mu_0}\vec{E} \times \vec{B} \qquad\qquad \text{(in direction of propagation)}$$

describes the "energy flux", i.e. energy crossing a unit area, per second $[\frac{J}{m^2 s}]$

In free space: energy in a plane is shared between electric and magnetic field

The energy density $\mathcal{H}$ would be:

$$\mathcal{H} \;=\; \frac{1}{2}\left(\epsilon_0 E^2 \;+\; \frac{1}{\mu_0}B^2\right)$$

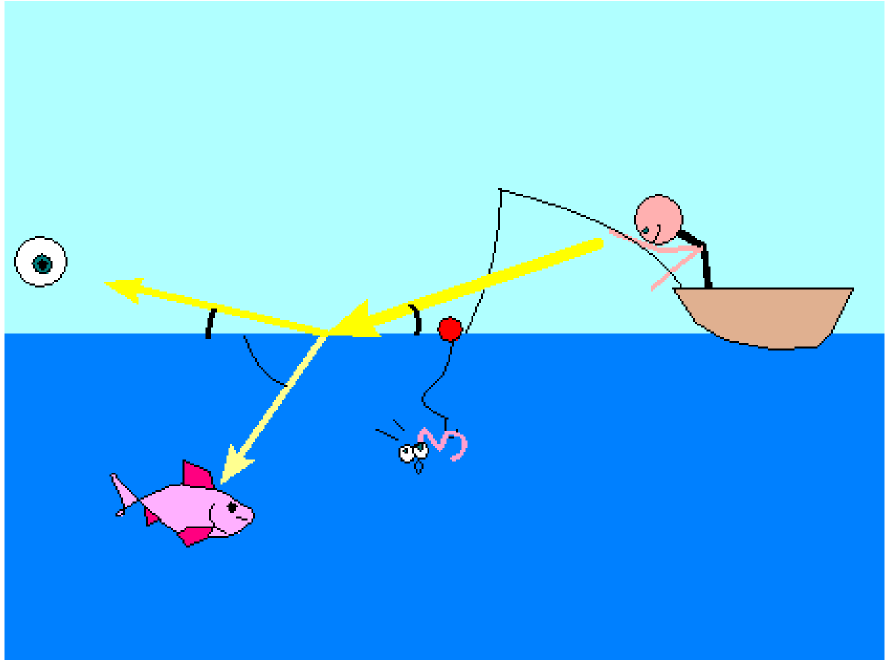
Waves interacting with material

Need to look at the behaviour of electromagnetic fields at <u>boundaries between different materials</u> (air-glass, air-water, vacuum-metal, ...).
Have to consider two particular cases:

- **Ideal conductor (i.e. no resistance), apply to:**
  - **RF cavities**
  - **Wave guides**
- **Conductor with finite resistance, apply to:**
  - **Penetration and attenuation of fields in material (skin depth)**
  - **Impedance calculations**

Can be derived from Maxwell's equations, here only the results !

Observation: between air and water



- Some of the light is reflected
- Some of the light is transmitted and refracted

→ Reason are boundary conditions for fields between two materials

Extreme case: surface of ideal conductor

For an ideal conductor (i.e. no resistance) the tangential electric field must vanish Corresponding conditions for normal magnetic fields. We must have:

$$\vec{E}_t \ = \ 0, \qquad \vec{B}_n \ = \ 0$$

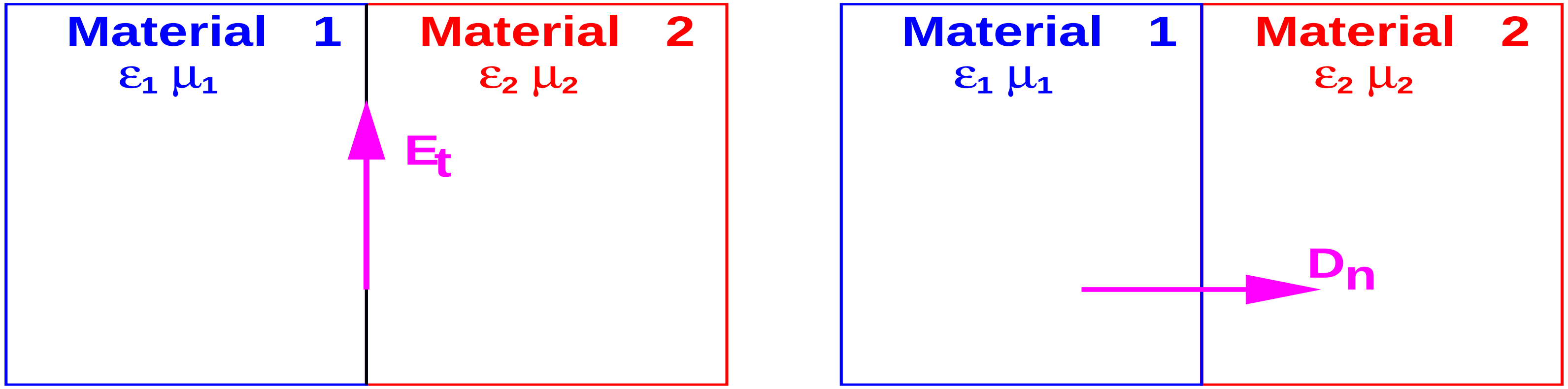
This implies:

- **Fields at any point in the conductor are zero.**
- **Only some field patterns are allowed in waveguides and RF cavities**

A very nice lecture in R.P.Feynman, Vol. II

Now for Boundary Conditions between two different regions →

Boundary conditions for <u>electric</u> fields



Assuming <u>no</u> surface charges (proof e.g. [3, 5])*:

From curl $\vec{E} = 0$**:**

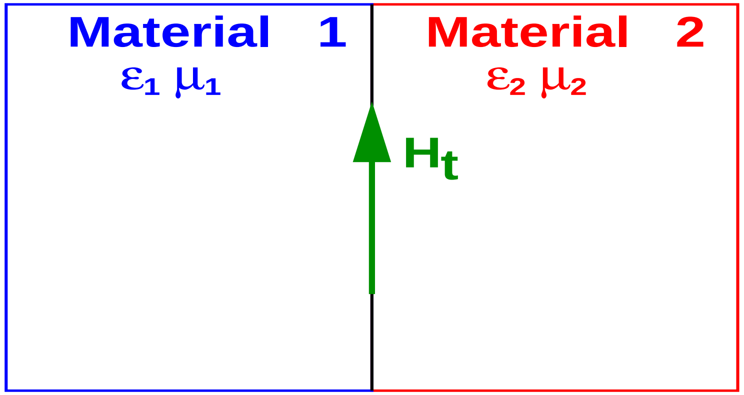
➡ **<u>tangential $\vec{E}$-field</u> continuous across boundary** $(E_t^1 \;=\; E_t^2)$

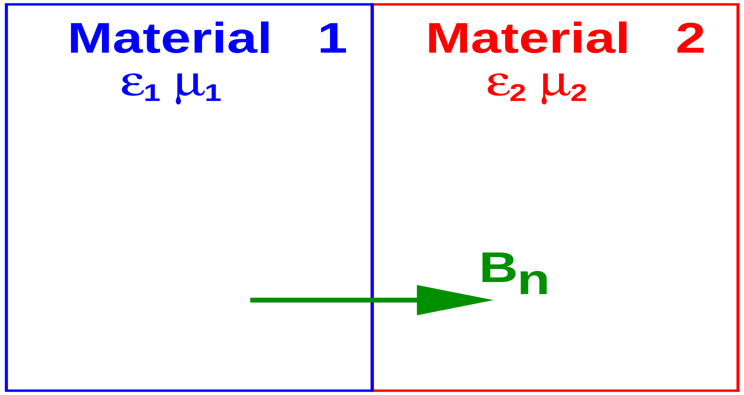
From div $\vec{D} = \rho$**:**

➡ **<u>normal $\vec{D}$-field</u> continuous across boundary** $(D_n^1 \;=\; D_n^2)$

* with surface charges, see backup slides

Boundary conditions for magnetic fields





Assuming no surface currents (proof e.g. [3, 5])*:

From curl $\vec{H} = \vec{j}$**:**

➡ **tangential $\vec{H}$-field continuous across boundary** $(H_t^1 = H_t^2)$

From div $\vec{B} = 0$**:**

➡ **normal $\vec{B}$-field continuous across boundary** $(B_n^1 = B_n^2)$

* with surface current, see backup slides

Summary: boundary conditions for fields

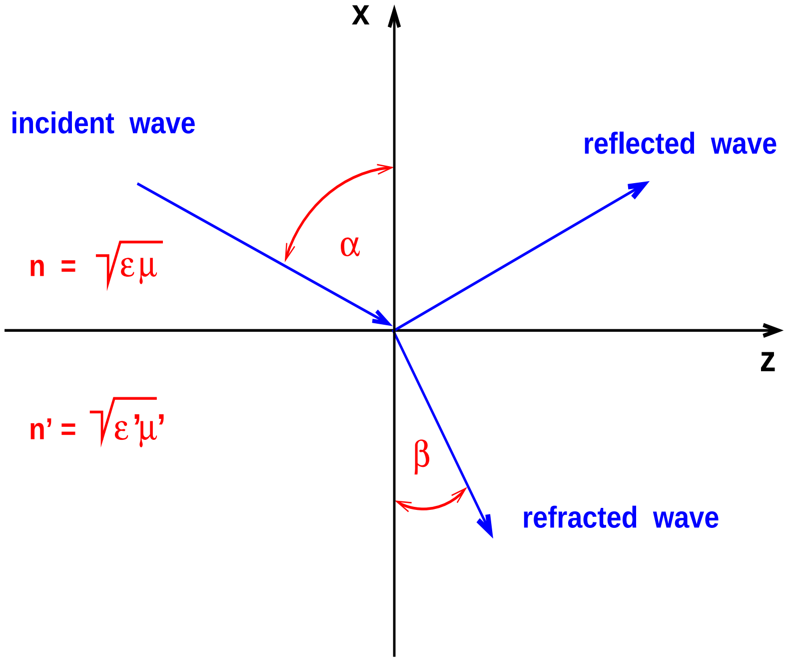
Electromagnetic fields at boundaries between different materials with different permittivity and permeability ($\epsilon_1, \epsilon_2, \mu_1, \mu_2$).

- $(E_t^1 \ = \ E_t^2), \quad (E_n^1 \ \neq \ E_n^2)$
- $(D_t^1 \ \neq \ D_t^2), \quad (D_n^1 \ = \ D_n^2)$
- $(H_t^1 \ = \ H_t^2), \quad (H_n^1 \ \neq \ H_n^2)$
- $(B_t^1 \ \neq \ B_t^2), \quad (B_n^1 \ = \ B_n^2)$

(derivation deserves its own lectures, just accept it)

They determine: reflection, refraction and refraction index n.

Reflection and refraction angles related to the refraction index n and n':



$$\frac{\sin\alpha}{\sin\beta} = \frac{n'}{n} = \tan\alpha_B$$

n depends on wave length

$$\frac{dn}{d\lambda} < 0$$

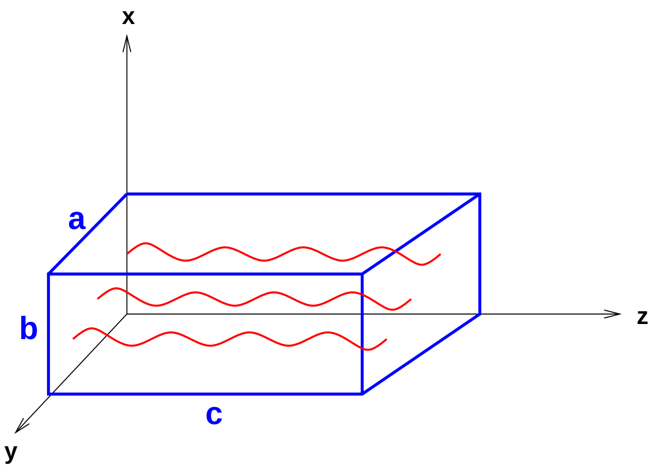
If light is incident under angle α_B [3]:

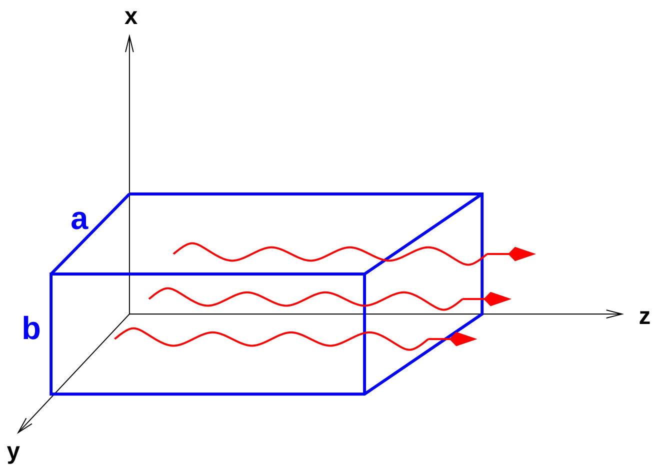
Reflected light is linearly polarized perpendicular to plane of incidence

(Application: fishing → air-water gives $\alpha_B \approx 53^o$)

Rectangular cavities and wave guides

Rectangular, conducting cavities and wave guides (schematic) with dimensions $a \times b \times c$ and $a \times b$:





- **Fields must be zero at boundary**
- **RF cavity, fields can persist and be stored (reflection !)**
- **Plane waves can propagate along wave guides, here in z-direction**

Assume a rectangular RF cavity ($a,\ b,\ c$), ideal conductor.

Without derivations (e.g. [2, 3, 6]), the components of the fields are:

$$E_x = E_{x0} \cdot cos(k_x x) \cdot sin(k_y y) \cdot sin(k_z z) \cdot e^{-i\omega t}$$

$$E_y = E_{y0} \cdot sin(k_x x) \cdot cos(k_y y) \cdot sin(k_z z) \cdot e^{-i\omega t}$$

$$E_z = E_{z0} \cdot sin(k_x x) \cdot sin(k_y y) \cdot cos(k_z z) \cdot e^{-i\omega t}$$

with: $\nabla \times \vec{E} \ = -\dfrac{\partial \vec{B}}{\partial t}$**:**

$$B_x = \frac{i}{\omega}(E_{y0}k_z - E_{z0}k_y) \cdot sin(k_x x) \cdot cos(k_y y) \cdot cos(k_z z) \cdot e^{-i\omega t}$$

$$B_y = \frac{i}{\omega}(E_{z0}k_x - E_{x0}k_z) \cdot cos(k_x x) \cdot sin(k_y y) \cdot cos(k_z z) \cdot e^{-i\omega t}$$

$$B_z = \frac{i}{\omega}(E_{x0}k_y - E_{y0}k_x) \cdot cos(k_x x) \cdot cos(k_y y) \cdot sin(k_z z) \cdot e^{-i\omega t}$$

Consequences for RF cavities

No fields outside: field must be zero at conductor boundary !
Only possible under the condition:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

and for k_x, k_y, k_z **we can write:**

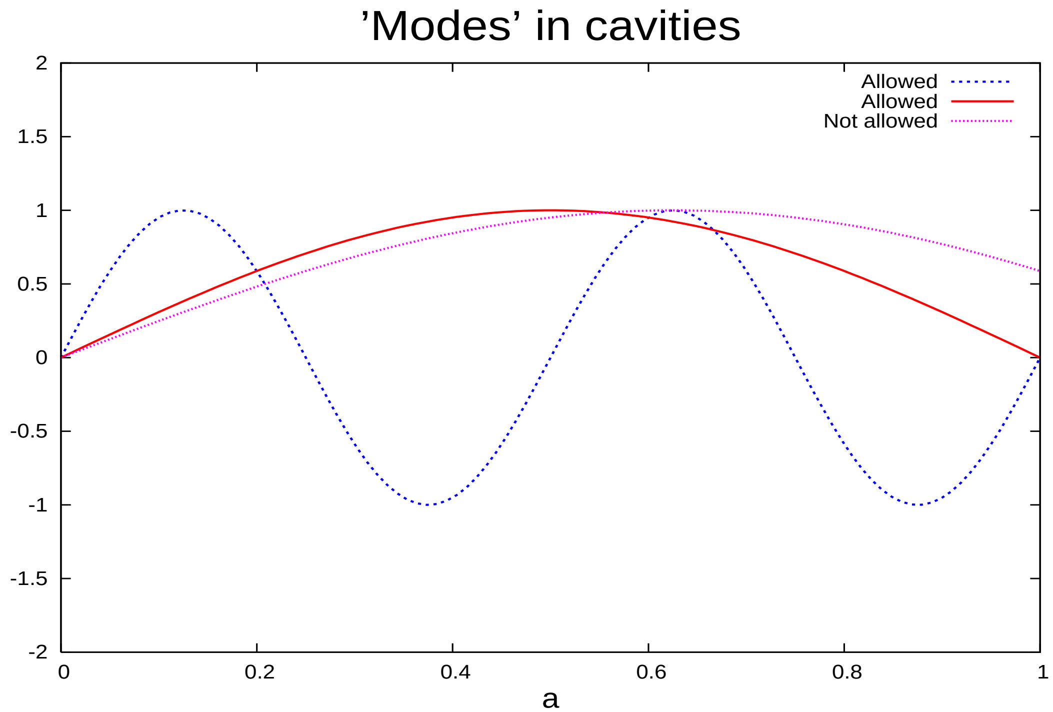
$$k_x = \frac{m_x \pi}{a}, \qquad k_y = \frac{m_y \pi}{b}, \qquad k_z = \frac{m_z \pi}{c},$$

The integer numbers m_x, m_y, m_z **are called mode numbers**

→ **number of half-wave patterns across width and height**

It means that a half wave length $\lambda/2$ **must always fit exactly the size of the cavity.**

Allowed modes



- **Only modes which 'fit' into the cavity are allowed**
- $\frac{\lambda}{2} = \frac{a}{4}$, $\frac{\lambda}{2} = \frac{a}{1}$, $\frac{\lambda}{2} = \frac{a}{0.8}$
- **No electric field at boundaries, wave must have "nodes" at the boundaries**

Similar considerations lead to (propagating) solutions in (rectangular) wave guides:

$$E_x = E_{x0} \cdot cos(k_x x) \cdot sin(k_y y) \cdot e^{i(k_z z - \omega t)}$$

$$E_y = E_{y0} \cdot sin(k_x x) \cdot cos(k_y y) \cdot e^{i(k_z z - \omega t)}$$

$$E_z = i \cdot E_{z0} \cdot sin(k_x x) \cdot sin(k_y y) \cdot e^{i(k_z z - \omega t)}$$

$$B_x = \frac{1}{\omega}(E_{y0} k_z - E_{z0} k_y) \cdot sin(k_x x) \cdot cos(k_y y) \cdot e^{i(k_z z - \omega t)}$$

$$B_y = \frac{1}{\omega}(E_{z0} k_x - E_{x0} k_z) \cdot cos(k_x x) \cdot sin(k_y y) \cdot e^{i(k_z z - \omega t)}$$

$$B_z = \frac{1}{i \cdot \omega}(E_{x0} k_y - E_{y0} k_x) \cdot cos(k_x x) \cdot cos(k_y y) \cdot e^{i(k_z z - \omega t)}$$

Consequences for wave guides

Similar considerations as for cavities, no field at boundary.

We must satisfy again the condition:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

This leads to modes like (no boundaries in direction of propagation z):

$$k_x = \frac{m_x \pi}{a}, \qquad k_y = \frac{m_y \pi}{b},$$

The numbers m_x, m_y are called mode numbers for planar waves in wave guides !

Re-writing the condition as:

$$k_z^2 = \frac{\omega^2}{c^2} - k_x^2 - k_y^2 \quad \Rightarrow \quad k_z = \sqrt{\frac{\omega^2}{c^2} - k_x^2 - k_y^2}$$

Propagation without losses requires k_z to be real, i.e.:

$$\frac{\omega^2}{c^2} > k_x^2 + k_y^2 = (\frac{m_x \pi}{a})^2 + (\frac{m_y \pi}{b})^2$$

which defines a cut-off frequency ω_c. For lowest order mode:

$$\omega_c = \frac{\pi \cdot c}{a}$$

- **Above cut-off frequency: propagation without loss**
- **At cut-off frequency: standing wave**
- **Below cut-off frequency: attenuated wave (means it does not "really fit" and k is complex).**

Classification of wave guide modes:

TE: no E-field in z-direction

TM: no B-field in z-direction

TEM: no B-field nor E-field in z-direction

What is special:

TEM modes cannot propagate in a single conductor*) !

Need two concentric conducting "cylinders": i.e. a coaxial cable ...

(for the field lines: see backup slides)

*) curl $\vec{E} = 0$, div $\vec{E} = 0$, $\vec{E} = 0$ at boundaries → zero field

Circular cavities

Wave guides and cavities are more likely to be circular.

Derivation using the Laplace equation in cylindrical coordinates, example for modes, for the derivation see e.g. [2, 3]:

$$
\begin{aligned}
E_r &= E_0 \frac{k_z}{k_r} J_n'(k_r) \cdot cos(n\theta) \cdot sin(k_z z) \cdot e^{-i\omega t} \\
E_\theta &= E_0 \frac{n k_z}{k_r^2 r} J_n(k_r) \cdot sin(n\theta) \cdot sin(k_z z) \cdot e^{-i\omega t} \\
E_z &= E_0 J_n(k_r r) \cdot cos(n\theta) \cdot sin(k_z z) \cdot e^{-i\omega t}
\end{aligned}
$$

$$
\begin{aligned}
B_r &= i E_0 \frac{\omega}{c^2 k_r^2 r} J_n(k_r r) \cdot sin(n\theta) \cdot cos(k_z z) \cdot e^{-i\omega t} \\
B_\theta &= i E_0 \frac{\omega}{c^2 k_r r} J_n'(k_r r) \cdot cos(n\theta) \cdot cos(k_z z) \cdot e^{-i\omega t} \\
B_z &= 0
\end{aligned}
$$

Homework: write it down for wave guides ..

Accelerating circular cavities

For accelerating cavities we need longitudinal electric field component $E_z \neq 0$ and magnetic field purely transverse.

$$\begin{aligned} E_r &= 0 \\ E_\theta &= 0 \\ E_z &= E_0 J_0(p_{01}\frac{r}{R}) \cdot e^{-i\omega t} \end{aligned}$$

$$\begin{aligned} B_r &= 0 \\ B_\theta &= -i\frac{E_0}{c} J_1(p_{01}\frac{r}{R}) \cdot e^{-i\omega t} \\ B_z &= 0 \end{aligned}$$

(p_{nm} is the mth zero of J_n, e.g. $p_{01} \approx$ 2.405)

This would be a cavity with a TM$_{010}$ mode: $\omega_{010} = p_{01} \cdot \dfrac{c}{R}$

Other case: finite conductivity

Starting from Maxwell equation:

$$\nabla \times \vec{H} = \vec{j} + \frac{d\vec{D}}{dt} = \underbrace{\overbrace{\sigma \cdot \vec{E}}^{\vec{j}}}_{Ohm's\ law} + \epsilon \frac{d\vec{E}}{dt}$$

Wave equations:

$$\vec{E} = \vec{E_0} e^{i(\vec{k}\cdot\vec{r}-\omega t)}, \qquad \vec{H} = \vec{H_0} e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

We want to know k, applying the calculus to the wave equations we have:

$$\frac{d\vec{E}}{dt} = -i\omega \cdot \vec{E}, \qquad \frac{d\vec{H}}{dt} = -i\omega \cdot \vec{H}, \qquad \nabla \times \vec{E} = i\vec{k} \times \vec{E}, \qquad \nabla \times \vec{H} = i\vec{k} \times \vec{H}$$

Put together:

$$\vec{k} \times \vec{H} = i\sigma \cdot \vec{E} - \omega\epsilon \cdot \vec{E} = (-i\sigma + \omega\epsilon) \cdot \vec{E}$$

Starting from:

$$\vec{k} \times \vec{H} \;=\; -\,i\sigma \cdot \vec{E} \;+\; \omega\epsilon \cdot \vec{E} \;=\; (-i\sigma + \omega\epsilon) \cdot \vec{E}$$

With $\vec{B} \;=\; \mu \; \vec{H}$**:**

$$\nabla \times \vec{E} \;=\; i\vec{k} \times \vec{E} \;=\; -\,\frac{\partial \vec{B}}{\partial t} \;=\; -\,\mu \frac{\partial \vec{H}}{\partial t} \;=\; i\omega\mu\vec{H}$$

Multiplication with $\vec{k}$**:**

$$\vec{k} \times (\vec{k} \times \vec{E}) \;=\; \omega\mu(\vec{k} \times \vec{H}) \;=\; \omega\mu(-i\sigma + \omega\epsilon) \cdot \vec{E}$$

After some calculus and $\vec{E} \perp \vec{H} \perp \vec{k}$**:**

$$k^2 \;=\; \omega\mu(-i\sigma \;+\; \omega\epsilon)$$

Skin Depth

Using $k^2 = \omega\mu(-i\sigma + \omega\epsilon)$**:**

For a good conductor $\sigma \gg \omega\epsilon$**:**

$$k^2 \approx -i\omega\mu\sigma \quad \rightarrow \quad k \approx \sqrt{\frac{\omega\mu\sigma}{2}}(1+i) = \frac{1}{\delta}(1+i)$$

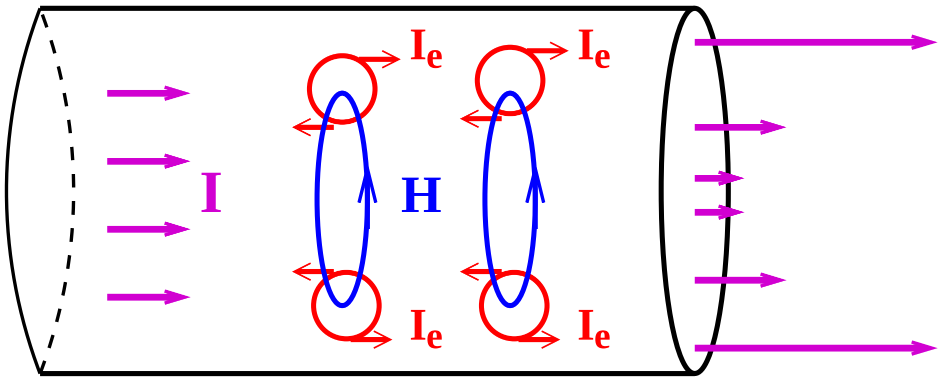
$$\boxed{\delta = \sqrt{\frac{2}{\omega\mu\sigma}}}$$

is the Skin Depth

High frequency currents "avoid" penetrating into a conductor, flow near the surface

(Note: $\sqrt{i} = e^{i\pi/4} = [(1+i)/2]\sqrt{2}$**)**

"Explanation" - inside a conductor (very schematic)



eddy currents from changing $\vec{H}$-field:

Cancel current flow in the centre of the conductor

Enforce current flow at the skin (surface)

Q: Why are high frequency cables thin ??

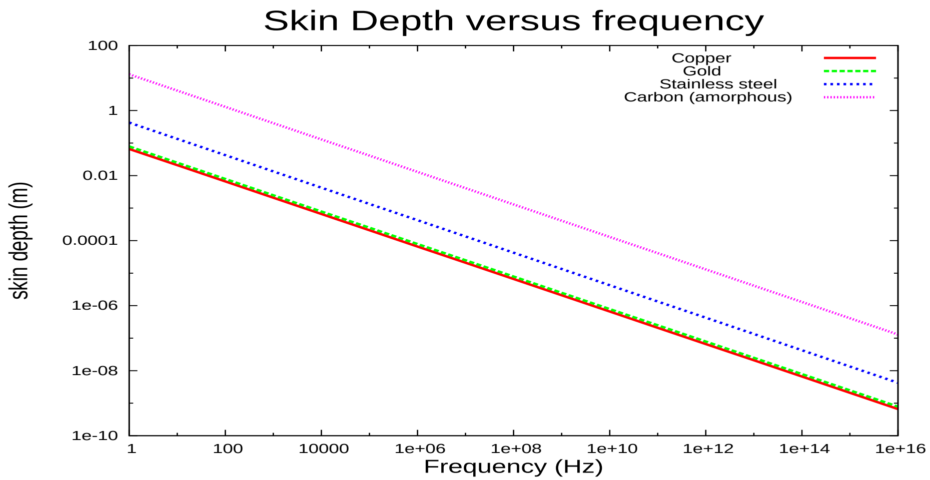
Attenuated waves - penetration depth

- **Waves incident on conducting material are attenuated**
- **Is basically Skin depth, (attenuation to 1/e)**

Wave form:

$$e^{i(kz-\omega t)} \;=\; e^{i((1+i)z/\delta-\omega t)} \;=\; e^{\frac{-z}{\delta}} \cdot e^{i(\frac{z}{\delta}-\omega t)}$$

Examples and applications



- **Skin depth Copper:**

  1 GHz: $\delta \approx$ 2.1 μm, 50 Hz: $\delta \approx$ 10 mm

  (there is an easy way to waste your money ...)

- **Penetration depth Seawater:**

  to get $\delta \approx$ 25 m you need → $\approx$ 76 Hz

  inefficient ($10^{-5} - 10^{-6}$) and very low bandwidth (0.03 bps)

Skin Depth - beam dynamics

For metal walls thicker than δ:

Resistive Wall Impedances, see later on collective effects.

Currents penetrate into the wall, depending on the frequency and conductivity.
For the transverse impedance we get the dependence:

$$Z_t(\omega) \;\propto\; \delta \;\propto\; \omega^{-1/2}$$

- **Largest impedance at low frequencies**
- **Cause instabilities (see later)**

We are done ...

- Review of basics and Maxwell's equations
- Lorentz force
- Motion of particles in electromagnetic fields
- Electromagnetic waves in vacuum
- Electromagnetic waves in media
  - Waves in RF cavities
  - Waves in wave guides
  - Penetration of waves in material

However ...

- ! Have to deal with moving charges
- ! Electromagnetic "wave" concept fuzzy: no medium
- ! Lorentz force depends on frame of reference
- ! Mutual interactions between charges and fields
- ! Cannot explain details of Cherenkov and Transition Radiation

To sort it out in a systematic framework (but ignoring Quantum Effects):

→ **"Special Relativity" ...**

- BACKUP SLIDES -

Boundary conditions in the presence of surface charges and currents

Assuming surface charges σ_s and currents j_s, we get the boundary conditions:

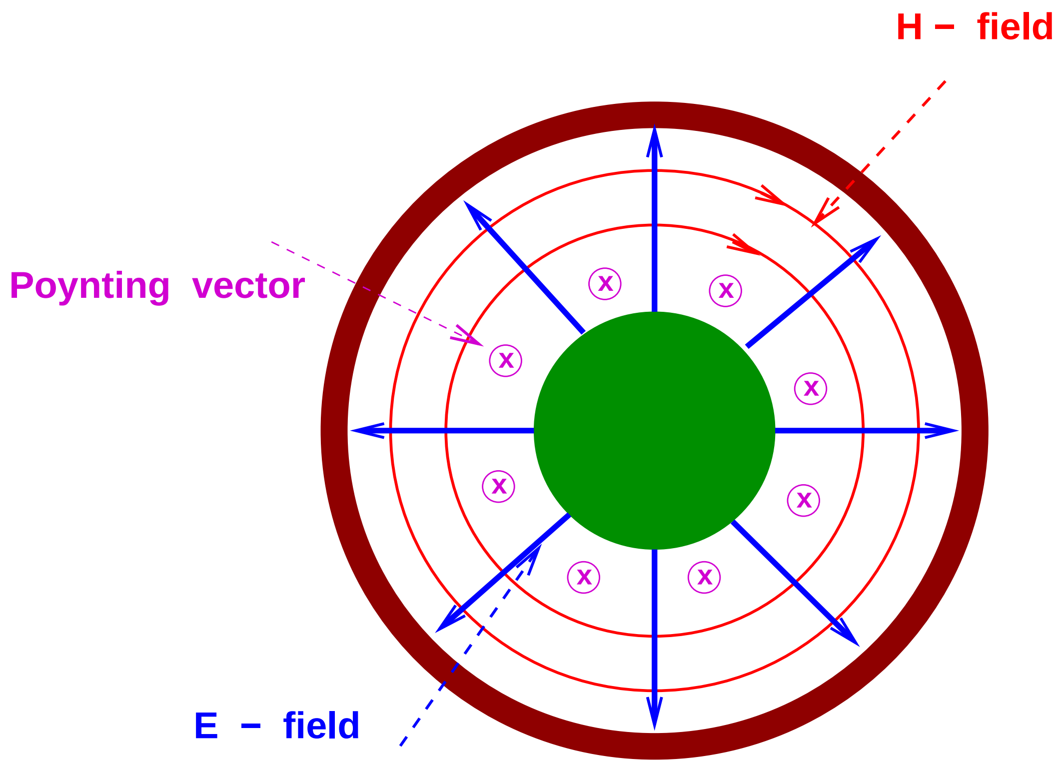
$$\mu_1 \vec{H}_n^{(1)} = \mu_2 \vec{H}_n^{(2)} \qquad \epsilon_1 \vec{E}_n^{(1)} - \epsilon_2 \vec{E}_n^{(2)} = \sigma_s$$

$$\frac{\vec{D}_t^{(1)}}{\epsilon_1} = \frac{\vec{D}_t^{(2)}}{\epsilon_2} \qquad \frac{\vec{B}_t^{(1)}}{\mu_1} - \frac{\vec{B}_t^{(2)}}{\mu_2} = j_s$$

Another assumption: both media are <u>linear</u> and <u>isotropic</u>, i.e.

$$\vec{B} = \mu \vec{H} \qquad \vec{D} = \epsilon \vec{E}$$

Coaxial cable:



Field lines and Poynting vector in a coaxial cable

Side notes:

Remark 1:

$\nabla \times \vec{B}$ **often written** curl $\vec{B}$ **or** rot $\vec{B}$ **(mostly Europe)**

Remark 2:

On very few occasions one can see it written as: $\nabla \wedge \vec{B} \; = \; \mu_0 \vec{j}$
Sometimes used in France, but usually it refers to a different algebra. If interested, see backup slides for the meaning and relevance, happy reading ..

Vector calculus ...

We can define a special vector ∇ (sometimes written as $\vec{\nabla}$):

$$\nabla = (\frac{\partial}{\partial x}, \ \frac{\partial}{\partial y}, \ \frac{\partial}{\partial z})$$

It is called the "gradient" and invokes "partial derivatives".
It can operate on a scalar function $\phi(x, y, z)$:

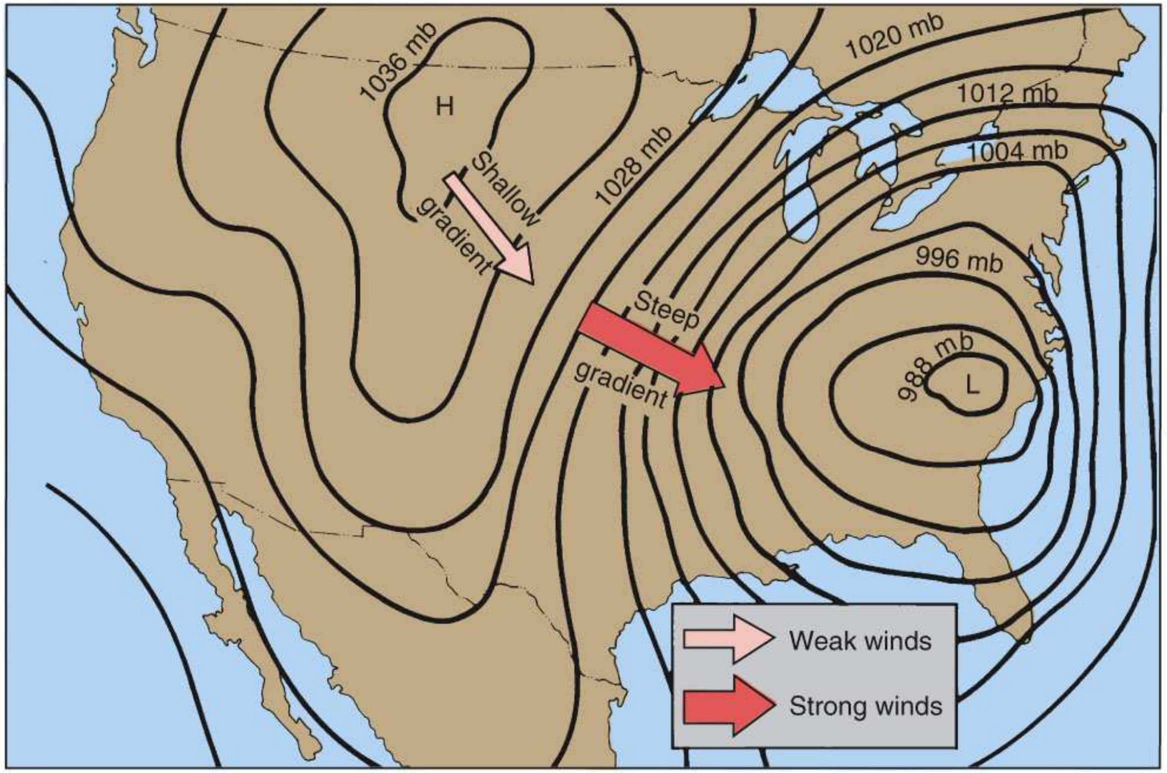
$$\nabla\phi = (\frac{\partial\phi}{\partial x}, \ \frac{\partial\phi}{\partial y}, \ \frac{\partial\phi}{\partial z}) \quad = \quad \vec{G} = (G_x, G_y, G_z)$$

and we get a vector $\vec{G}$. It is a kind of "slope" (steepness ..) in the 3 directions.

Example: $\phi(x, y, z) \ = \ C \cdot \mathbf{ln}(r^2)$ **with** $r \ = \ \sqrt{x^2 + y^2 + z^2}$

➡ $\nabla\phi \ = \ (G_x, G_y, G_z) \ = \ (\frac{2C \cdot x}{r^2}, \ \frac{2C \cdot y}{r^2}, \ \frac{2C \cdot z}{r^2})$

Gradient (slope) of a scalar field



Lines of pressure (isobars)

Gradient is large (steep) where lines are close (fast change of pressure)

Vector calculus ...

The gradient ∇ can be used as scalar or vector product with a vector $\vec{F}$, sometimes written as $\vec{\nabla}$

Used as:

$$\nabla \cdot \vec{F} \qquad \text{or} \qquad \nabla \times \vec{F}$$

Same definition for products as before, ∇ treated like a "normal" vector, but results depends on how they are applied:

$\nabla\Phi$ is a vector

$\nabla \cdot \vec{F}$ is a scalar

$\nabla \times \vec{F}$ is a pseudo-vector

$\nabla \wedge \vec{F}$ is <u>not</u> a vector

What about the $\wedge$ operation ?

- **In general dimensions:**
  - **No analogue of a cross product to yield a vector**
  - **The $\wedge$ product is not a "normal" vector, but a 2-vector (or bi-vector)**
  - **Can be interpreted as a "normal" cross product by mapping 2-vectors to "normal" vectors by using the Hodge dual:**

    $a \times b \quad = \quad *(a \wedge b)$ **(aha ...)**

Operations on vector fields ...

Two operations of ∇ have special names:

Divergence (scalar product of gradient with a vector):

$$\text{div}(\vec{F}) \quad = \quad \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Physical significance: "amount of density", (see later)

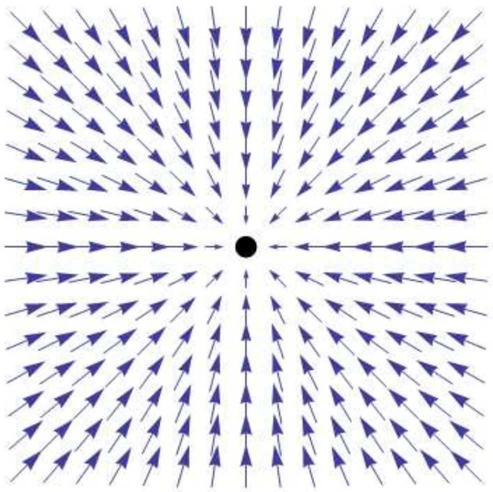
Curl (vector product of gradient with a vector):

$$\text{curl}(\vec{F}) \quad = \quad \nabla \times \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z},\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x},\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

Physical significance: "amount of rotation", (see later)

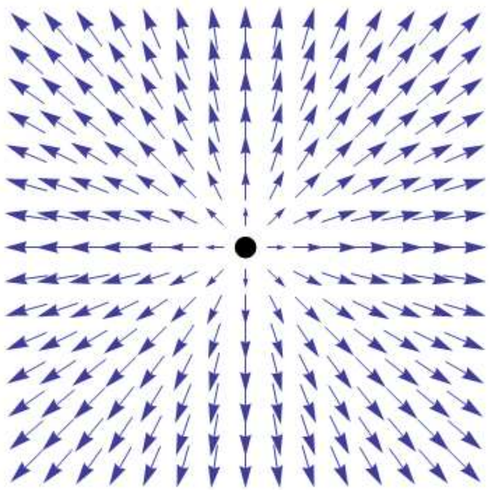
Meaning of Divergence of fields ...

Field lines of a vector field $\vec{F}$ seen from some origin:



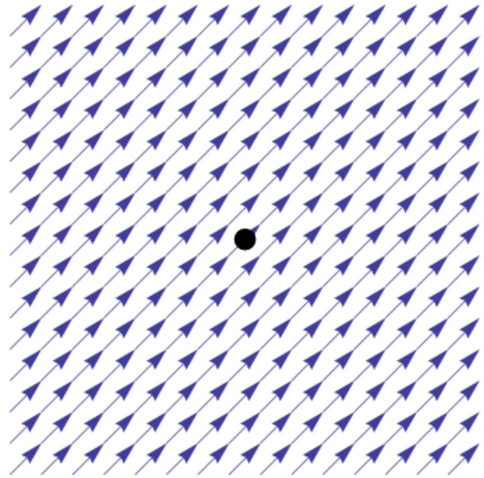
$\nabla \vec{F} < 0$

(sink)



$\nabla \vec{F} > 0$

(source)

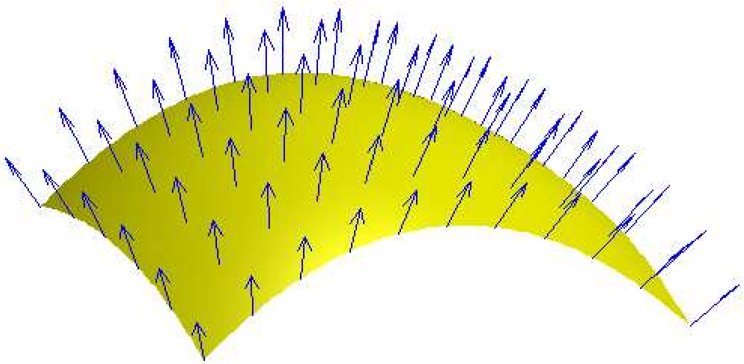


$\nabla \vec{F} = 0$

(fluid)

The divergence (scalar, a single number) characterizes what comes from (or goes to) the origin

How much comes out ?



Surface integrals: integrate field vectors passing (perpendicular) through a surface S (or area A), we obtain the Flux:

$$\longrightarrow \quad \int\int_A \vec{F} \cdot d\vec{A}$$

Density of field lines through the surface

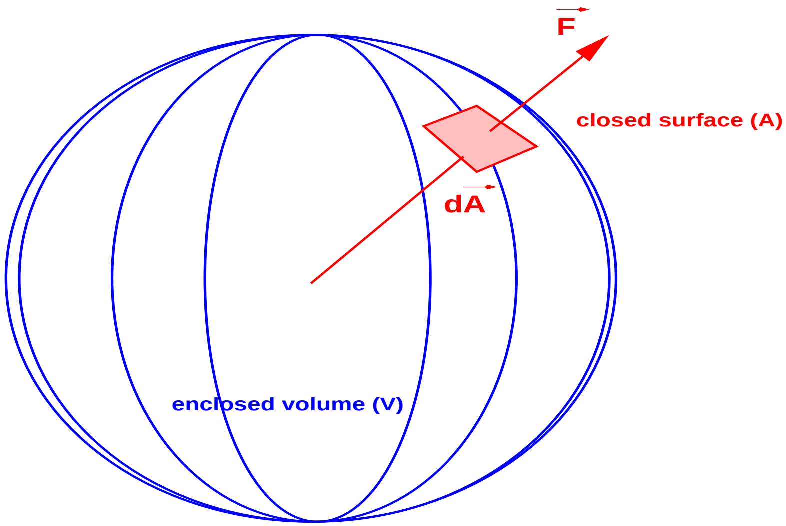
(e.g. amount of heat passing through a surface)

Surface integrals made easier ...

Gauss' Theorem:

Integral through a closed surface (flux) is integral of divergence in the enclosed volume

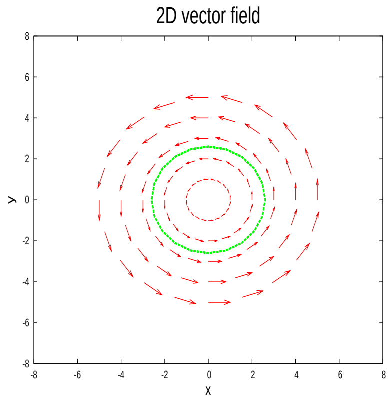
$$\int\int_A \vec{F} \cdot d\vec{A} = \int\int\int_V \nabla \cdot \vec{F} \cdot dV$$

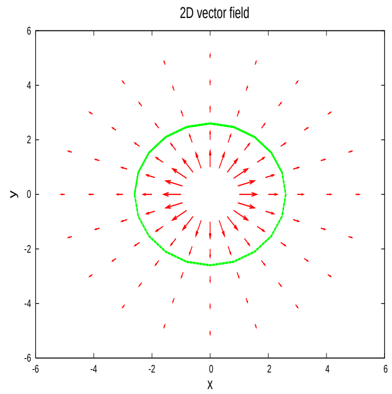


Relates surface integral to divergence

Meaning of curl of fields

The <u>curl</u> quantifies a rotation of vectors:





Line integrals: integrate field vectors along a line C:

$$\Rightarrow \oint_C \vec{F} \cdot d\vec{r}$$

"sum up" vectors (length) in <u>direction</u> of line C

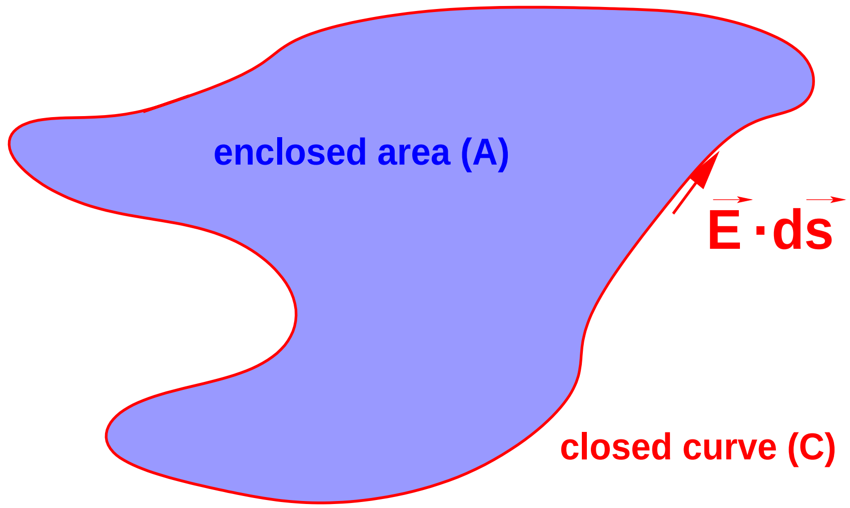
(e.g. work performed along a path ...)

Line integrals made easier ...

Stokes' Theorem:

Integral along a closed line is integral of curl in the enclosed area

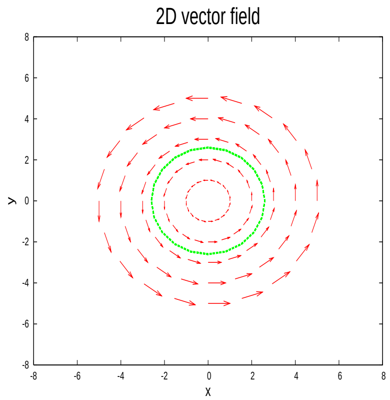
$$\oint_C \vec{F} \cdot d\vec{s} = \int\int_A \nabla \times \vec{F} \cdot d\vec{A}$$

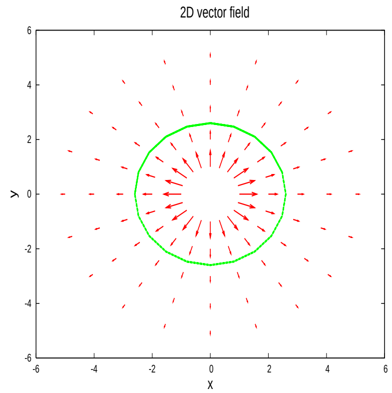


Relates line integral to curl

Integration of (vector-) fields

Two vector fields:





$$\nabla \vec{F} = 0 \qquad \nabla \times \vec{F} \neq 0 \qquad\qquad \nabla \vec{F} \neq 0 \qquad \nabla \times \vec{F} = 0$$

$$\oint_C \vec{F} \cdot d\vec{r} \;=\; \int\int_A \nabla \times \vec{F} \cdot d\vec{A}$$

Line integral for second vector field vanishes ...

Scalar products

Define a scalar product for (usual) vectors like: $\vec{a} \cdot \vec{b}$,

$$\vec{a} \ = \ (x_a, y_a, z_a) \qquad\qquad \vec{b} \ = \ (x_b, y_b, z_b)$$

$$\vec{a} \cdot \vec{b} \ \ = \ \ (x_a, y_a, z_a) \ \cdot \ (x_b, y_b, z_b) = (x_a \cdot x_b \ + y_a \cdot y_b \ + \ z_a \cdot z_b)$$

This product of two vectors is a <u>scalar</u> (number) not a vector.

(on that account: Scalar Product)

Example:

$$(-2, 2, 1) \ \cdot \ (2, 4, 3) \ = \ -2 \cdot 2 + 2 \cdot 4 + 1 \cdot 3 \ \ = \ \ 7$$

Vector products (sometimes cross product)

Define a vector product for (usual) vectors like: $\vec{a} \times \vec{b}$,

$$\vec{a} \ = \ (x_a, y_a, z_a) \qquad \vec{b} \ = \ (x_b, y_b, z_b)$$

$$\vec{a} \times \vec{b} \ = \ (x_a, y_a, z_a) \ \times \ (x_b, y_b, z_b)$$
$$= (\underbrace{y_a \cdot z_b - z_a \cdot y_b}_{x_{ab}}, \ \underbrace{z_a \cdot x_b - x_a \cdot z_b}_{y_{ab}}, \ \underbrace{x_a \cdot y_b - y_a \cdot x_b}_{z_{ab}})$$

This product of two vectors is a <u>vector</u>, not a scalar (number), (on that account: Vector Product)

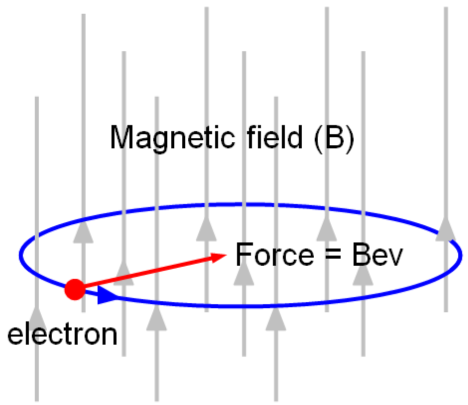
Example 1:

$$(-2, 2, 1) \ \times \ (2, 4, 3) \ = \ (2, 8, -12)$$

Example 2 (two components only in the $x - y$ plane):

$$(-2, 2, 0) \ \times \ (2, 4, 0) \ = \ (0, 0, -12)$$

Is that the full truth ?



If we have a circulating E-field along the circle of radius R ?

➡ **should get acceleration !**

Remember Maxwell's third equation:

$$\oint_C \vec{E} \cdot d\vec{r} = -\frac{d}{dt}\int_A \vec{B} \cdot d\vec{A}$$

$$\Rightarrow 2\pi R E_\theta = -\frac{d\Phi}{dt}$$

Motion in magnetic fields

- This is the principle of a Betatron
  - Time varying magnetic field creates circular electric field !
  - Time varying magnetic field deflects the charge !

For a constant radius we need:

$$-\frac{m \cdot v^2}{R} = e \cdot v \cdot B \quad \rightarrow \quad B = -\frac{p}{e \cdot R}$$

$$\frac{\partial}{\partial t} B(r,t) = -\frac{1}{e \cdot R}\frac{dp}{dt}$$

$$\rightarrow \quad B(r,t) = \frac{1}{2}\frac{1}{\pi R^2}\int\int B dS$$

B-field on orbit must be half the average over the circle → Betatron condition

Fields from Gaussian distribution - 2D

$$\Phi(x, y, \sigma_x, \sigma_y) = \frac{Q}{4\pi\epsilon_0}\int_0^\infty \frac{e^{(-\frac{x^2}{2\sigma_x^2+t}-\frac{y^2}{2\sigma_y^2+t})}}{\sqrt{(2\sigma_x^2+t)(2\sigma_y^2+t)}}dt$$

$$E_x = \frac{Q}{2\epsilon_0\sqrt{2\pi(\sigma_x^2-\sigma_y^2)}}\, Im\left[w\left(\frac{x+iy}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right) - e^{-\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2}}\, w\left(\frac{x\frac{\sigma_y}{\sigma_x}+iy\frac{\sigma_x}{\sigma_y}}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right)\right]$$

$$E_y = \frac{Q}{2\epsilon_0\sqrt{2\pi(\sigma_x^2-\sigma_y^2)}}\, Re\left[w\left(\frac{x+iy}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right) - e^{-\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2}}\, w\left(\frac{x\frac{\sigma_y}{\sigma_x}+iy\frac{\sigma_x}{\sigma_y}}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right)\right]$$

here w(z) is the complex error function

From: M. Basetti and G. Erskine, CERN-ISR-TH/80-06