Introduction to Transverse Beam Dynamics

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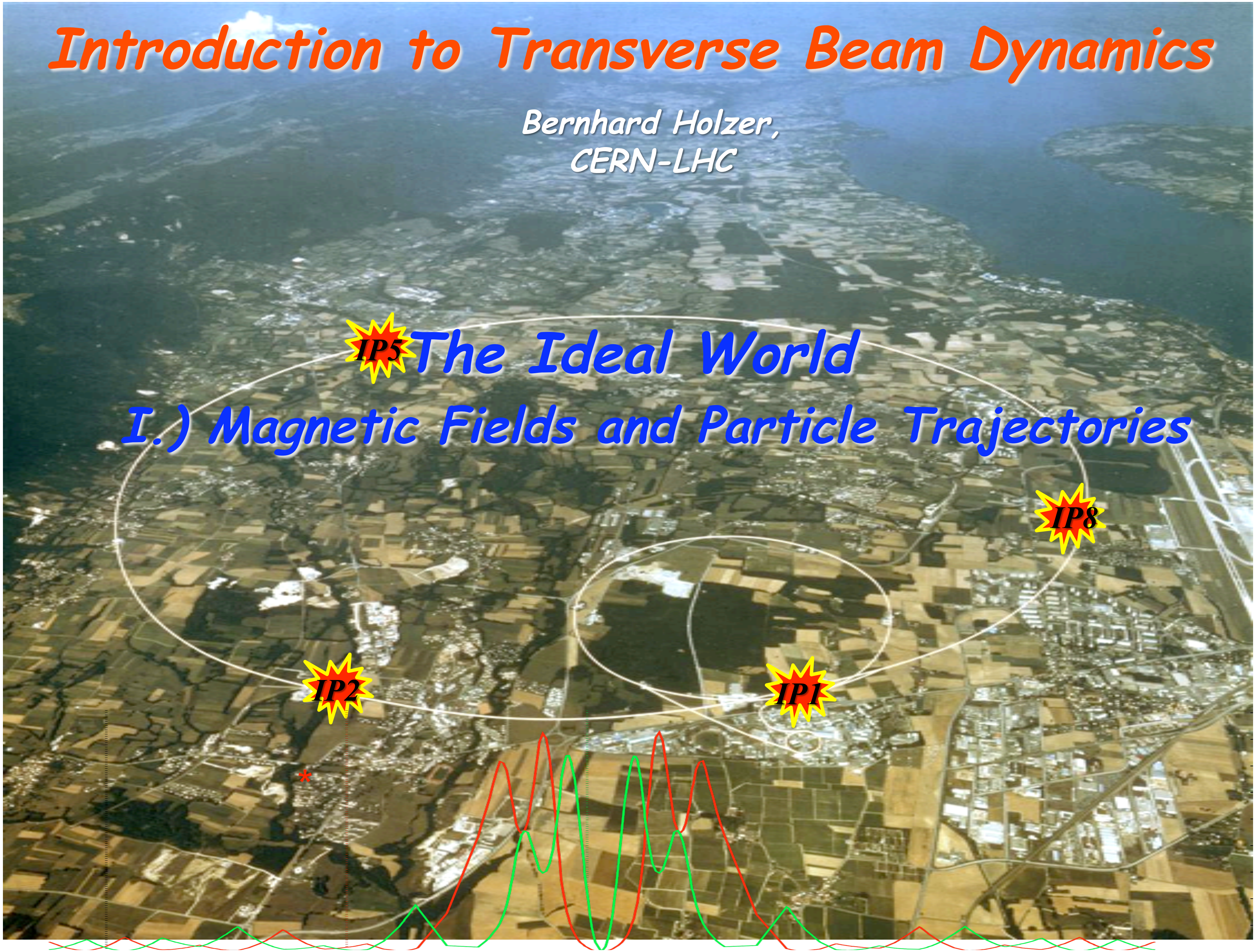
CERN-LHC

The Ideal World

I.) Magnetic Fields and Particle Trajectories



Luminosity Run of a typical storage ring:

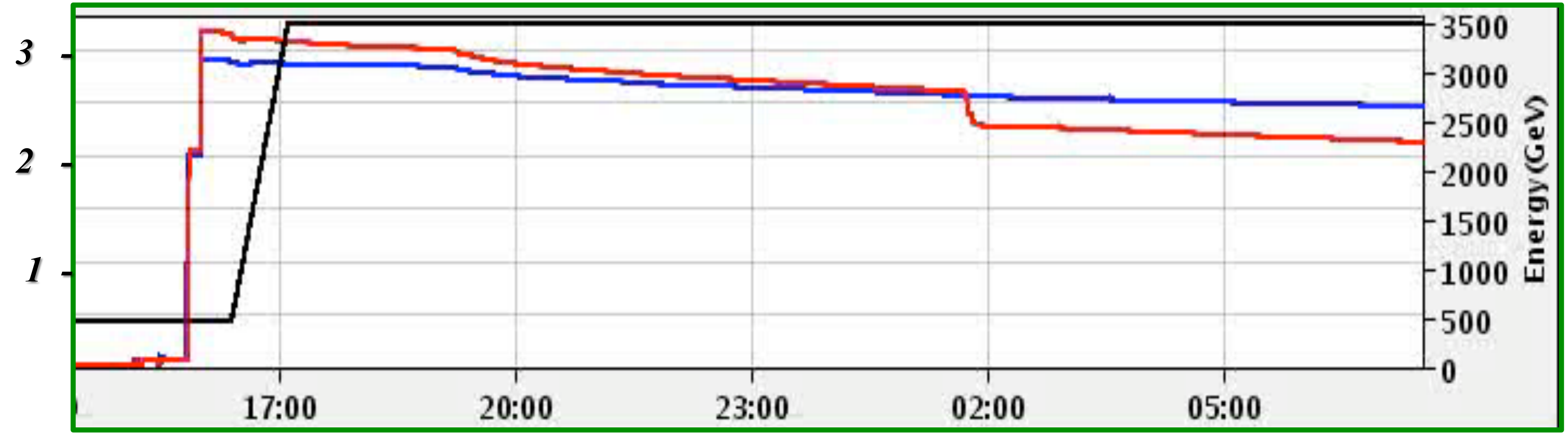
LHC Storage Ring: Protons accelerated and stored for 12 hours

distance of particles travelling at about v ≈ c

$L = 10^{10}$-10^{11} km

... several times Sun - Pluto and back

intensity (10^{11})



→ *guide the particles on a well defined orbit („design orbit")*

→ *focus the particles to keep each single particle trajectory within the vacuum chamber of the storage ring, i.e. close to the design orbit.*

1.) Introduction and Basic Ideas

„ … in the end and after all it should be a kind of circular machine“

→ need transverse deflecting force

Lorentz force

$$\vec{F} = q * (\cancel{\vec{E}} + \vec{v} \times \vec{B})$$

typical velocity in high energy machines:

$$v \approx c \approx 3 * 10^8 \, {}^{m}\!/_{s}$$

Example:

$$B = 1\,T \quad \rightarrow \quad F = q * 3 * 10^8 \frac{m}{s} * 1 \frac{Vs}{m^2}$$

$$F = q * \underbrace{300 \frac{MV}{m}}_{E}$$

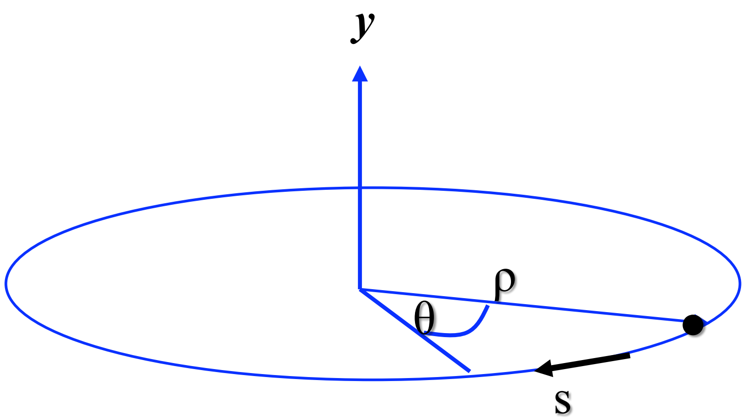
equivalent el. field …

technical limit for el. field:

$$E \le 1 \frac{MV}{m}$$

old greek dictum of wisdom:
if you are clever, you use magnetic fields in an accelerator wherever it is possible.

The ideal circular orbit



circular coordinate system

condition for circular orbit:

Lorentz force $$F_L = e\,v\,B$$

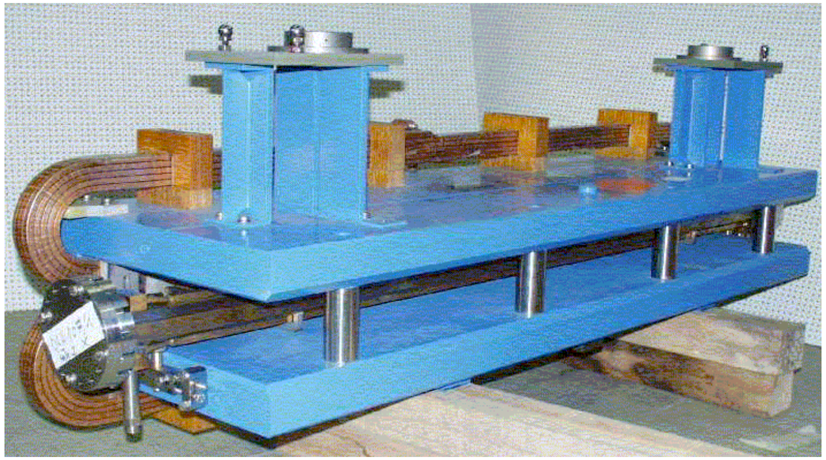
centrifugal force $$F_{centr} = \frac{\gamma\, m_0\, v^2}{\rho}$$

$$\frac{\gamma\, m_0\, v^{\cancel{2}}}{\rho} = e\,\cancel{v}\,B$$

$$\boxed{\frac{p}{e} = B\,\rho}$$

Bρ = "beam rigidity"

1.) The Magnetic Guide Field



Dipole Magnets:

define the ideal orbit
homogeneous field created
by two flat pole shoes

$$B = \frac{\mu_0 \, n \, I}{\boxed{h}}$$

Normalise magnetic field to momentum:

$$\frac{p}{e} = B\rho \quad \longrightarrow \quad \frac{1}{\rho} = \frac{e\,B}{p}$$

convenient units:

$$B = [T] = \left[\frac{Vs}{m^2}\right] \qquad p = \left[\frac{GeV}{c}\right]$$

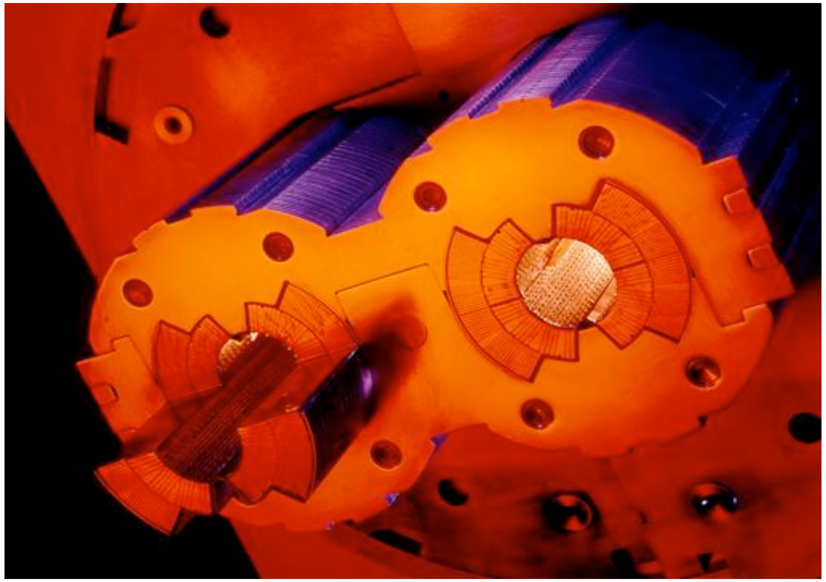
Example LHC:

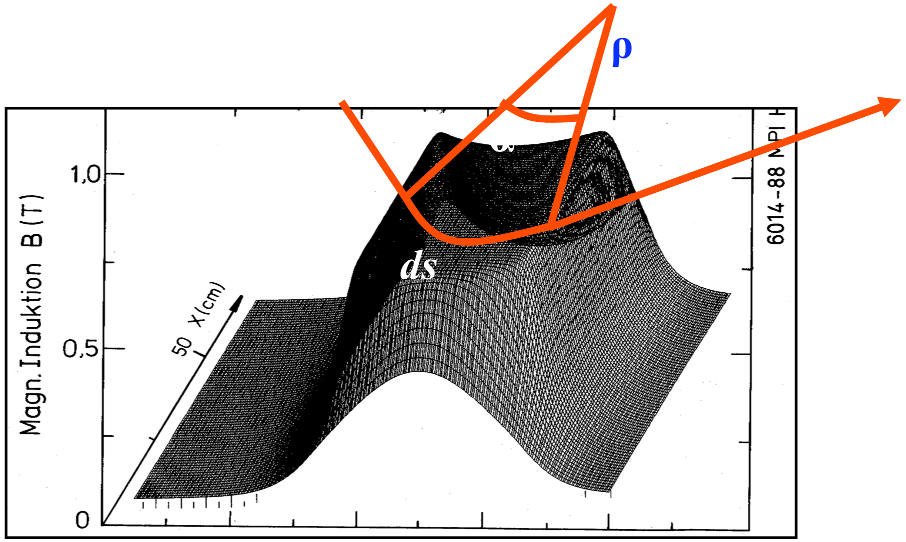
$$\left.\begin{aligned} B &= 8.3\,T \\ p &= 7000\,\frac{GeV}{c} \end{aligned}\right\}$$

$$\frac{1}{\rho} = e\,\frac{8.3\,{}^{Vs}\!/_{m^2}}{7000*10^9\,{}^{eV}\!/_{c}} = \frac{8.3\,s\,3*10^8\,{}^{m}\!/_{s}}{7000*10^9\,m^2}$$

$$\frac{1}{\rho} = 0.333\,\frac{8.3}{7000}\,{}^{1}\!/_{m}$$

The Magnetic Guide Field





field map of a storage ring dipole magnet

$$\rho = 2.53\ \boldsymbol{km} \longrightarrow 2\pi\rho = 17.6\ \boldsymbol{km}$$

$$\approx 66\%$$

$$\boldsymbol{B} \approx 1 \ldots 8\ \boldsymbol{T}$$

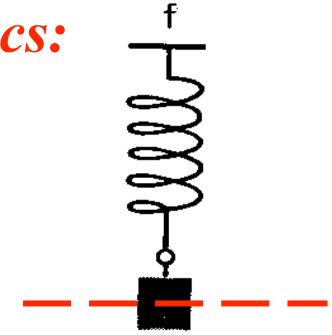
rule of thumb:

$$\boxed{\frac{1}{\rho} \approx 0.3 \frac{\boldsymbol{B}\left[\boldsymbol{T}\right]}{\boldsymbol{p}\left[\boldsymbol{GeV/c}\right]}}$$

„normalised bending strength“

Focusing Properties – Transverse Beam Optics

Classical Mechanics:
pendulum



there is a restoring force, proportional to the elongation x:

$$F = m * \frac{d^2 x}{dt^2} = -k * x$$

general solution: free harmonic oszillation

Ansatz $x(t) = A * \cos(\omega t + \varphi)$

$$\dot{x} = -A\omega * \sin(\omega t + \varphi)$$

$$\ddot{x} = -A\omega^2 * \cos(\omega t + \varphi)$$

Solution $\omega = \sqrt{k/m}, \quad x(t) = x_0 * \cos(\sqrt{\tfrac{k}{m}} t + \varphi)$

Storage Ring: we need a Lorentz force that rises as a function of the distance to .... ?

.... the design orbit

$$F(x) = q * v * B(x)$$

2.) Quadrupole Magnets:

required: *focusing forces to keep trajectories in vicinity of the ideal orbit*

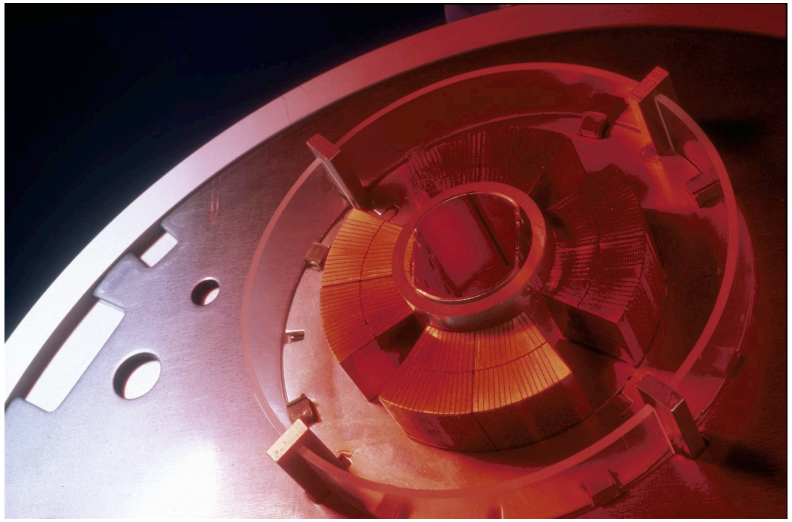
linear increasing Lorentz force

linear increasing magnetic field $B_y = g\,x \qquad B_x = g\,y$

normalised quadrupole field:

gradient of a quadrupole magnet: $g = \dfrac{2\mu_0 nI}{\boxed{r^2}}$

$\longrightarrow$ $k = \dfrac{g}{p/e}$



LHC main quadrupole magnet

$$g \approx 25 \ldots 220 \;\; T/m$$

simple rule: $$\boxed{k = 0.3\,\frac{g(T/m)}{p(GeV/c)}}$$

what about the vertical plane:
... Maxwell

$$\vec{\nabla} \times \vec{B} = \cancel{\vec{j}} + \cancel{\frac{\partial \vec{E}}{\partial t}} = 0 \qquad \Rightarrow \qquad \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$$

3.) The equation of motion:

Linear approximation:

* ideal particle → design orbit

* any other particle → coordinates x, y small quantities
$x,y << \rho$

→ magnetic guide field: only linear terms in x & y of B
have to be taken into account

Taylor Expansion of the B field:

$$B_y(x) = B_{y0} + \frac{dB_y}{dx} x + \frac{1}{2!}\frac{d^2B_y}{dx^2} x^2 + \frac{1}{3!}\frac{eg''}{dx^3} + \ldots$$

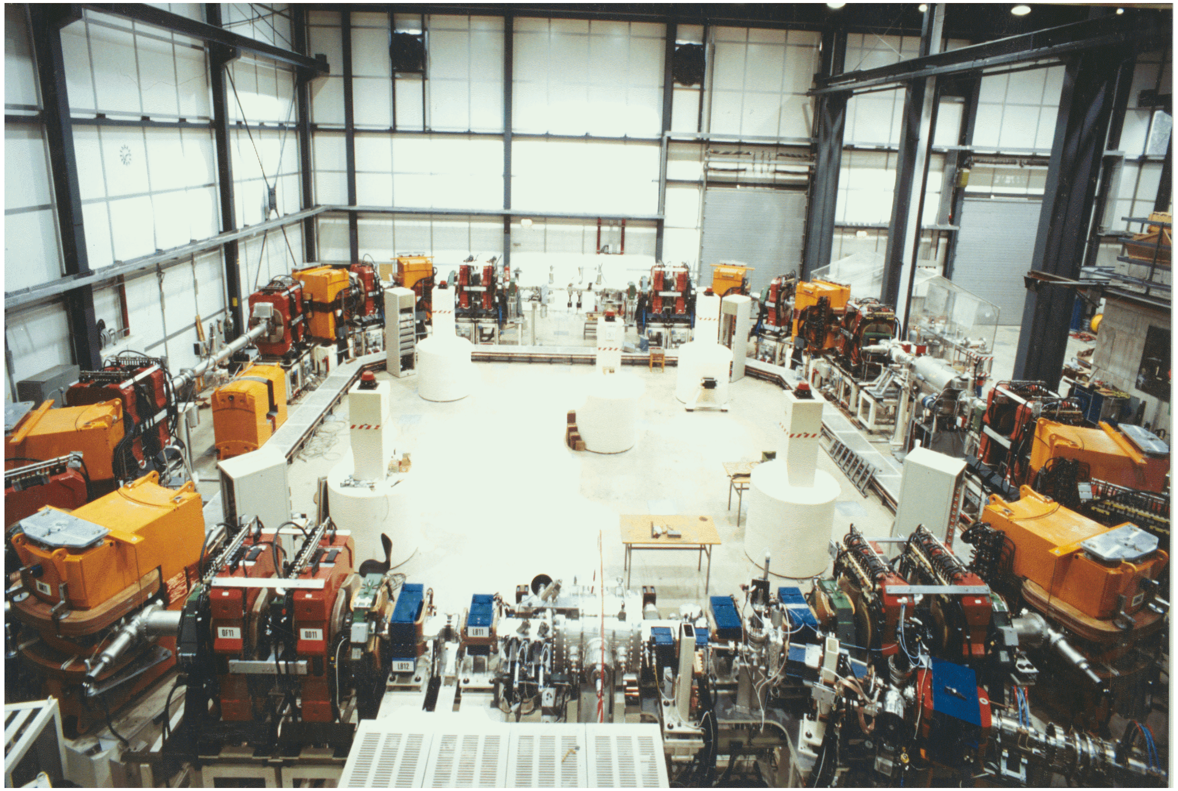
normalise to momentum
$p/e = B\rho$

$$\frac{B(x)}{p/e} = \frac{B_0}{B_0\rho} + \frac{g * x}{p/e} + \frac{1}{2!}\frac{eg'}{p/e} + \frac{1}{3!}\frac{eg''}{p/e} + \ldots$$

The Equation of Motion:

$$\frac{B(x)}{p/e} = \frac{1}{\rho} + k\,x + \xcancel{\frac{1}{2!}m\,x^{2}} + \xcancel{\frac{1}{3!}n\,x^{3}} + \ldots$$

only terms linear in x, y taken into account *dipole fields*
quadrupole fields



Separate Function Machines:

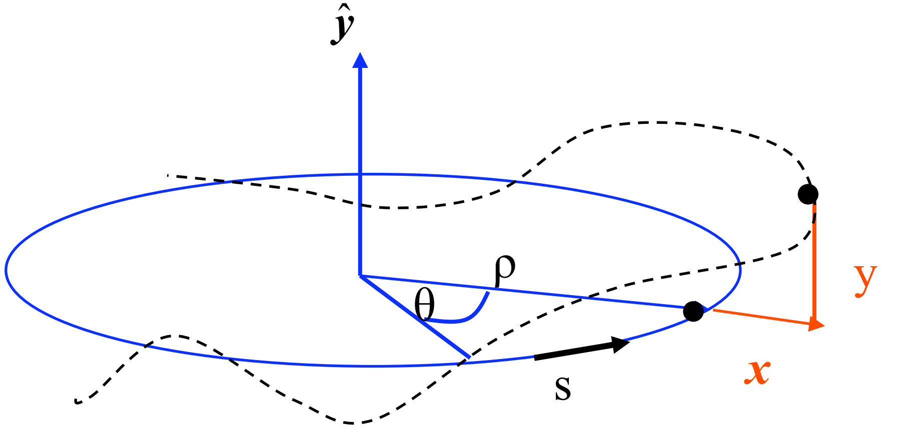
Split the magnets and optimise them according to their job:

bending, focusing etc

Example:
heavy ion storage ring TSR

* **man sieht nur**
dipole und quads → linear

Equation of Motion:



Consider local segment of a particle trajectory
... and remember the old days:

(Goldstein page 27)

radial acceleration:

$$a_r = \frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2$$

Ideal orbit: $\rho = const, \quad \frac{d\rho}{dt} = 0$

Force: $F = m\rho\left(\frac{d\theta}{dt}\right)^2 = m\rho\omega^2$

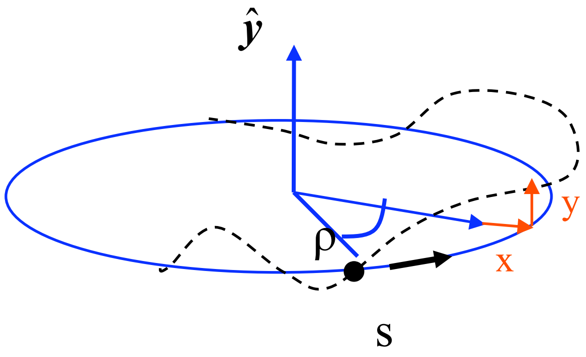
$$F = mv^2 / \rho$$

general trajectory: $\rho \rightarrow \rho + x$

$$\boldsymbol{F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} \;=\; e\,B_y\,v}$$

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v$$

① ②



① $$\frac{d^2}{dt^2}(x+\rho) = \frac{d^2}{dt^2}x$$ *… as ρ = const*

② *remember: x ≈ mm , ρ ≈ m … → develop for small x*

$$\frac{1}{x+\rho} \approx \frac{1}{\rho}(1-\frac{x}{\rho})$$

Taylor Expansion

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!}f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \quad \dots$$

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1-\frac{x}{\rho}) = eB_y v$$

guide field in linear approx.

$$B_y = B_0 + x\frac{\partial B_y}{\partial x}$$

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1-\frac{x}{\rho}) = ev\left\{B_0 + x\frac{\partial B_y}{\partial x}\right\} \qquad \Bigg| \quad : m$$

$$\frac{d^2x}{dt^2} - \frac{v^2}{\rho}(1-\frac{x}{\rho}) = \frac{e\,v\,B_0}{m} + \frac{e\,v\,x\,g}{m}$$

independent variable: t → s

$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt}$$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{ds}\frac{ds}{dt}\right) = \frac{d}{ds}\left(\underbrace{\frac{dx}{ds}}_{x'}\underbrace{\frac{ds}{dt}}_{v}\right)\frac{ds}{dt}$$

$$\frac{d^2x}{dt^2} = x''\,v^2 + \cancel{\frac{dx}{ds}\frac{dv}{ds}}\,v$$

$$x''v^2 - \frac{v^2}{\rho}(1-\frac{x}{\rho}) = \frac{e\,v\,B_0}{m} + \frac{e\,v\,x\,g}{m} \qquad \Bigg| \quad : v^2$$

$$x'' - \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right) = \frac{e\ B_0}{mv} + \frac{e\ x\,g}{mv}$$

$m\,v = p$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = \frac{B_0}{p/e} + \frac{x\,g}{p/e}$$

normalize to momentum of particle

$$\frac{B_0}{p/e} = -\frac{1}{\rho}$$

$$\frac{g}{p/e} = k$$

$$x'' - \cancel{\frac{1}{\rho}} + \frac{x}{\rho^2} = -\cancel{\frac{1}{\rho}} + k\,x$$

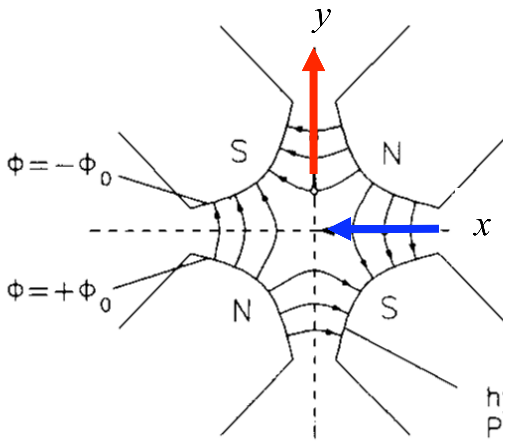
$$x'' + x\left(\frac{1}{\rho^2} - k\right) = 0$$

- *Equation for the vertical motion:*

$\frac{1}{\rho^2} = 0$ *no dipoles … in general …*

$k \leftrightarrow -k$ *quadrupole field changes sign*

$$y'' + k\ y = 0$$



Remarks:

* $$x'' + (\frac{1}{\rho^2} - k) \cdot x = 0$$

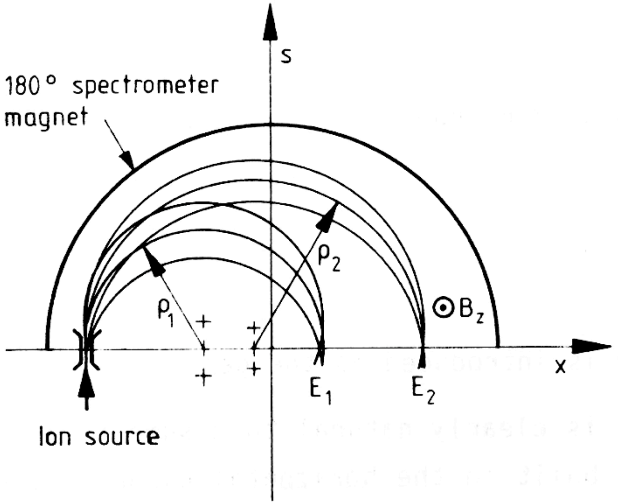
… there seems to be a focusing even without a quadrupole gradient

„weak focusing of dipole magnets“

$$\boldsymbol{k} = 0 \quad \Rightarrow \quad \boldsymbol{x}'' = -\frac{1}{\rho^2}\boldsymbol{x}$$

even without quadrupoles there is a retriving force (i.e. focusing) in the bending plane of the dipole magnets

… in large machines it is weak. (!)



Mass spectrometer: particles are separated according to their energy and focused due to the 1/ρ effect of the dipole

* *Hard Edge Model:*

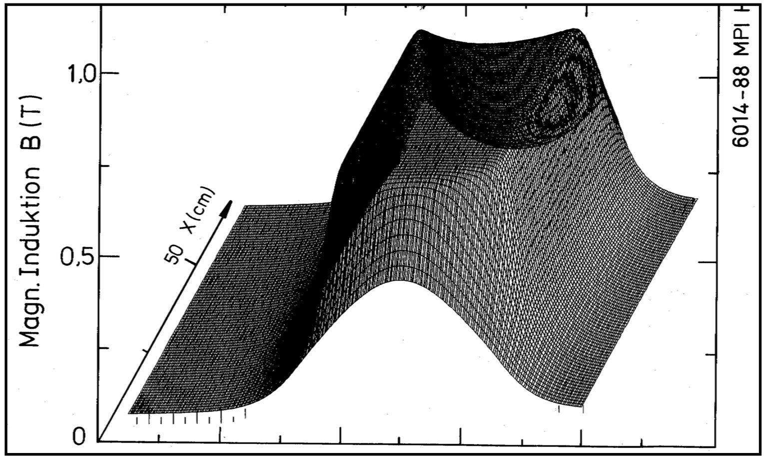
$$x'' + \left\{\frac{1}{\rho^2} - k\right\} x = 0$$

… this equation is not correct !!!

$$x''(s) + \left\{\frac{1}{\rho^2(s)} - k(s)\right\} x(s) = 0$$

bending and focusing fields … are functions of the independent variable „s“

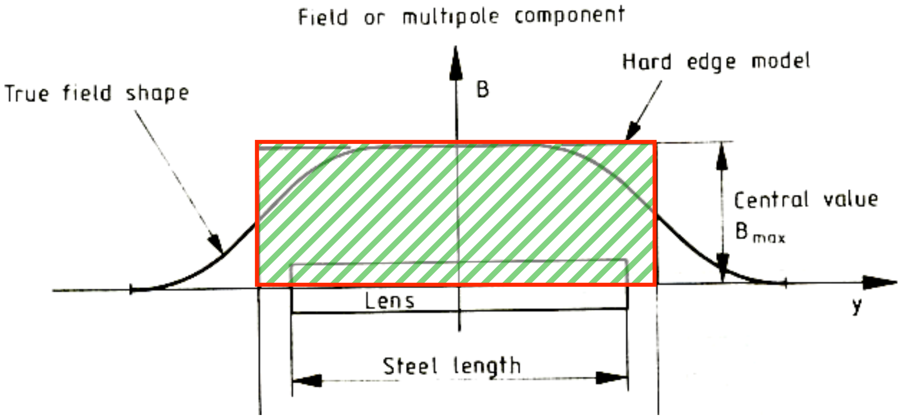
!



Inside a magnet we assume constant focusing properties !

$$\frac{1}{\rho} = const \qquad k = const$$

$$B\, l_{eff} = \int_0^{l_{mag}} B\, ds$$



4.) Solution of Trajectory Equations

Define … hor. plane: $K = 1/\rho^2 - k$

… vert. Plane: $K = k$

$$\left.\begin{array}{l} \\ \\ \end{array}\right\} \qquad \boxed{\boldsymbol{x}'' + \boldsymbol{K}\ \boldsymbol{x} = 0}$$

Differential Equation of harmonic oscillator … with spring constant K

Ansatz: $\boxed{x(s) = a_1 \cdot \cos(\omega s) + a_2 \cdot \sin(\omega s)}$

general solution: linear combination of two independent solutions

$$x'(s) = -a_1\omega \sin(\omega s) + a_2\omega \cos(\omega s)$$

$$x''(s) = -a_1\omega^2 \cos(\omega s) - a_2\omega^2 \sin(\omega s) = -\omega^2 x(s) \qquad \longrightarrow \qquad \omega = \sqrt{K}$$

general solution:

$$x(s) = a_1 \cos(\sqrt{K}s) + a_2 \sin(\sqrt{K}s)$$

determine a_1 , a_2 by boundary conditions:

$$s = 0 \quad \longrightarrow \quad \begin{cases} x(0) = x_0 \quad , \quad a_1 = x_0 \\ x'(0) = x'_0 \quad , \quad a_2 = \dfrac{x'_0}{\sqrt{K}} \end{cases}$$

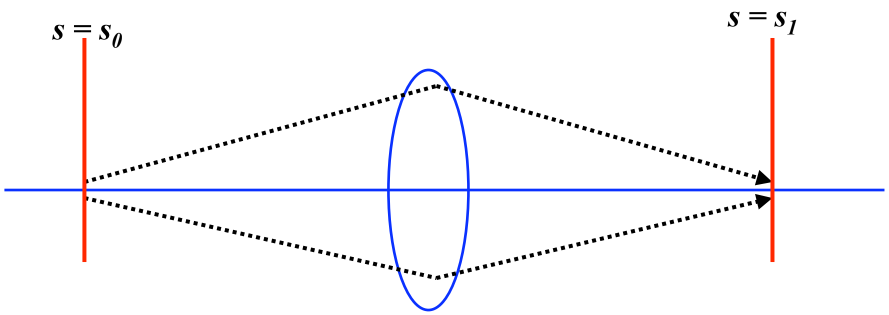
Hor. Focusing Quadrupole K > 0:

$$x(s) = x_0 \cdot \cos(\sqrt{|K|}s) + x'_0 \cdot \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s)$$

$$x'(s) = -x_0 \cdot \sqrt{|K|} \cdot \sin(\sqrt{|K|}s) + x'_0 \cdot \cos(\sqrt{|K|}s)$$

For convenience expressed in matrix formalism:

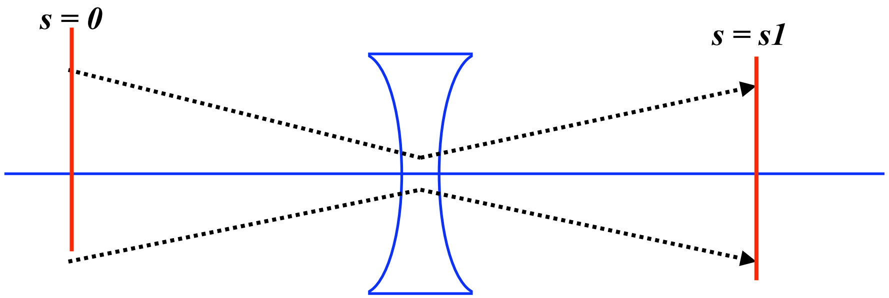
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M_{foc} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$



$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \dfrac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s \\ -\sqrt{|K|}\sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0$$

hor. defocusing quadrupole:

$$x'' - K\ x = 0$$



Remember from school:

$$f(s) = \cosh(s) \quad , \qquad f'(s) = \sinh(s)$$

Ansatz: $$x(s) = a_1 \cdot \cosh(\omega s) + a_2 \cdot \sinh(\omega s)$$

$$M_{defoc} = \begin{pmatrix} \cosh\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}l \\ \sqrt{|K|}\sinh\sqrt{|K|}l & \cosh\sqrt{|K|}l \end{pmatrix}$$

drift space:

$$K = 0$$

$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

! *with the assumptions made, the motion in the horizontal and vertical planes are independent „ ... the particle motion in x & y is uncoupled“*

Thin Lens Approximation:

matrix of a quadrupole lens

$$M = \begin{pmatrix} \cos\sqrt{|k|}l & \frac{1}{\sqrt{|k|}}\sin\sqrt{|k|}l \\ -\sqrt{|k|}\sin\sqrt{|k|}l & \cos\sqrt{|k|}l \end{pmatrix}$$

in many practical cases we have the situation:

$$f = \frac{1}{kl_q} >> l_q$$

... focal length of the lens is much bigger than the length of the magnet

limes: $l_q \rightarrow 0$ ***while keeping*** $k\, l_q$ ***= const***

$$M_x = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \qquad M_z = \begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix}$$

... useful for fast (and in large machines still quite accurate) „back on the envelope calculations“ ... and for the guided studies !

Combining the two planes:

Clear enough (hopefully ... ?) : a quadrupole magnet that is focussing o-in one plane acts as defocusing lens in the other plane ... et vice versa.

hor foc. quadrupole lens

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}s \\ -\sqrt{|K|}\sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}$$

matrix of the same magnet in the vert. plane:

$$M_{defoc} = \begin{pmatrix} \cosh\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}l \\ \sqrt{|K|}\sinh\sqrt{|K|}l & \cosh\sqrt{|K|}l \end{pmatrix}$$

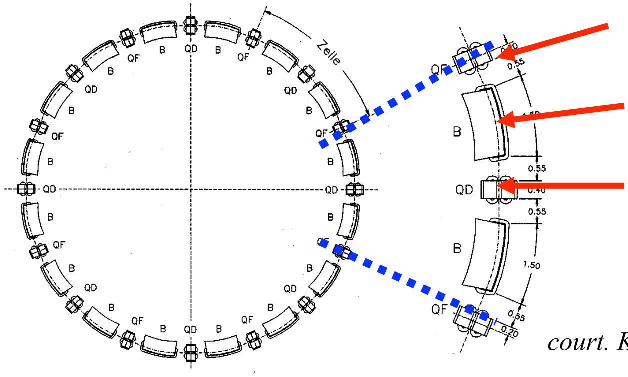
$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_f = \begin{pmatrix} \cos(\sqrt{|k|}s) & \frac{1}{\sqrt{|k|}}\sin(\sqrt{|k|}s) & 0 & 0 \\ -\sqrt{|k|}\sin(\sqrt{|k|}s) & \cos(\sqrt{|k|}s) & 0 & 0 \\ 0 & 0 & \cosh(\sqrt{|k|}s) & \frac{1}{\sqrt{|k|}}\sinh(\sqrt{|k|}s) \\ 0 & 0 & \sqrt{|k|}\sinh(\sqrt{|k|}s) & \cosh(\sqrt{|k|}s) \end{pmatrix} * \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_i$$

Transformation through a system of lattice elements

combine the single element solutions by multiplication of the matrices

$$M_{total} = M_{QF} * M_{D} * M_{QD} * M_{Bend} * M_{D*.....}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s_2, s_1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$



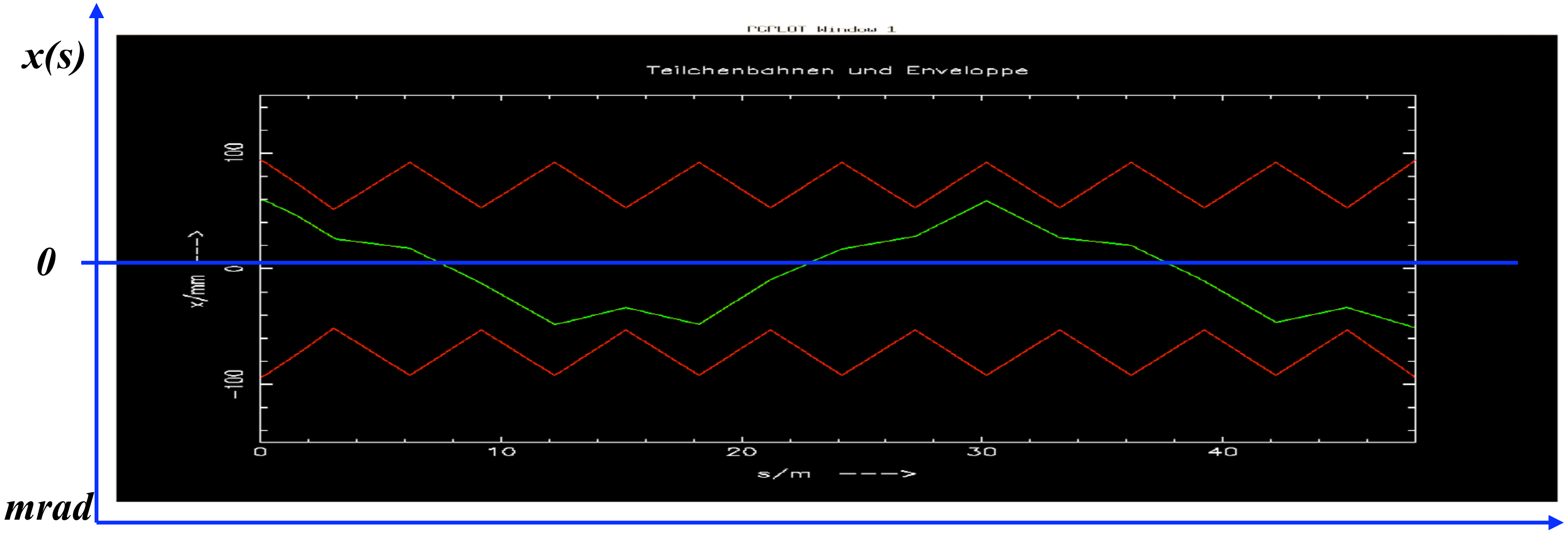
focusing lens

dipole magnet

defocusing lens

court. K. Wille

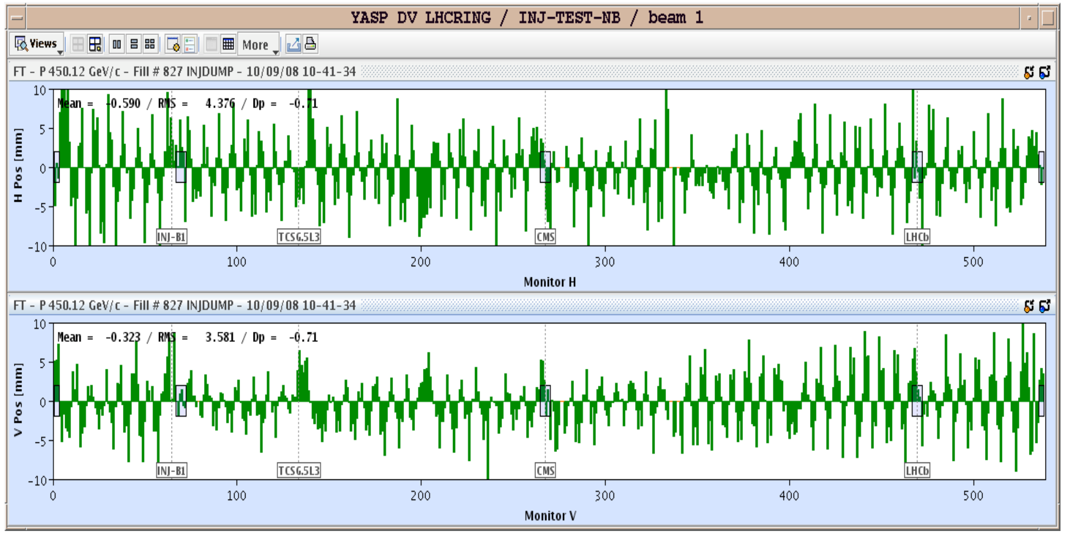
in each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator „



typical values in a strong foc. machine:

x ≈ mm, x´ ≤ mrad

5.) Orbit & Tune:



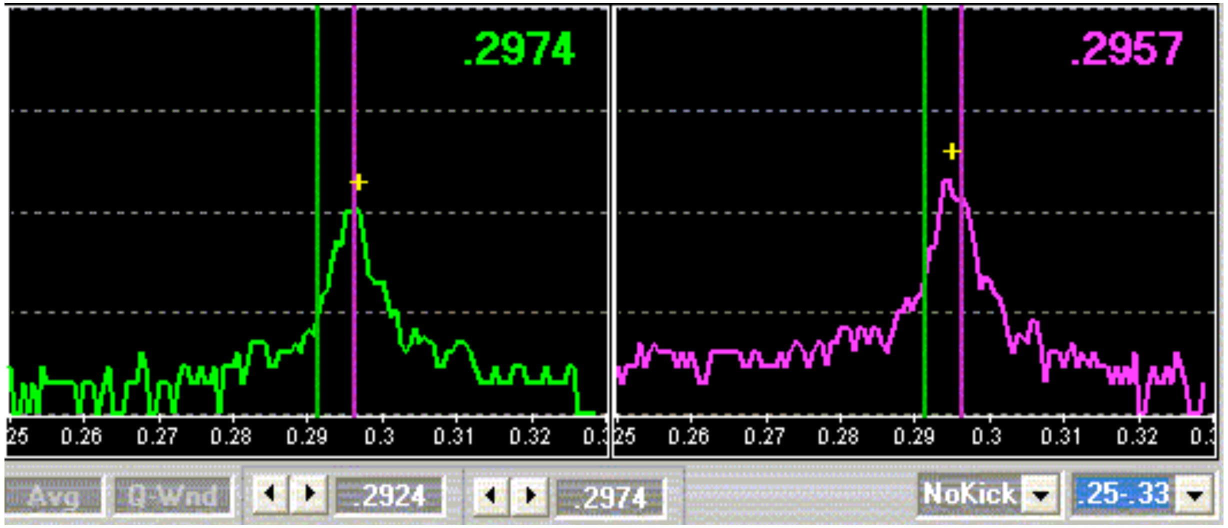
Tune: number of oscillations per turn

64.31
59.32

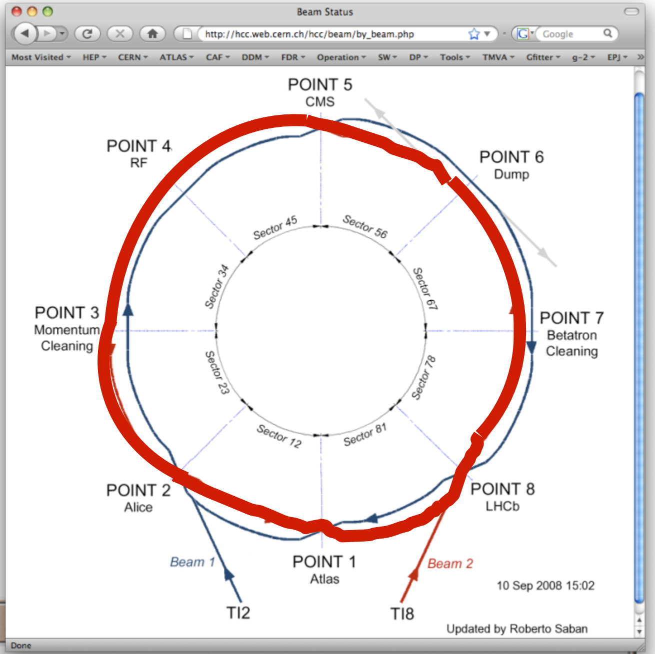
Relevant ***for beam stability:***
non integer part

LHC revolution frequency: 11.3 kHz

$$0.31*11.3=3.5kHz$$



LHC Operation: Beam Commissioning

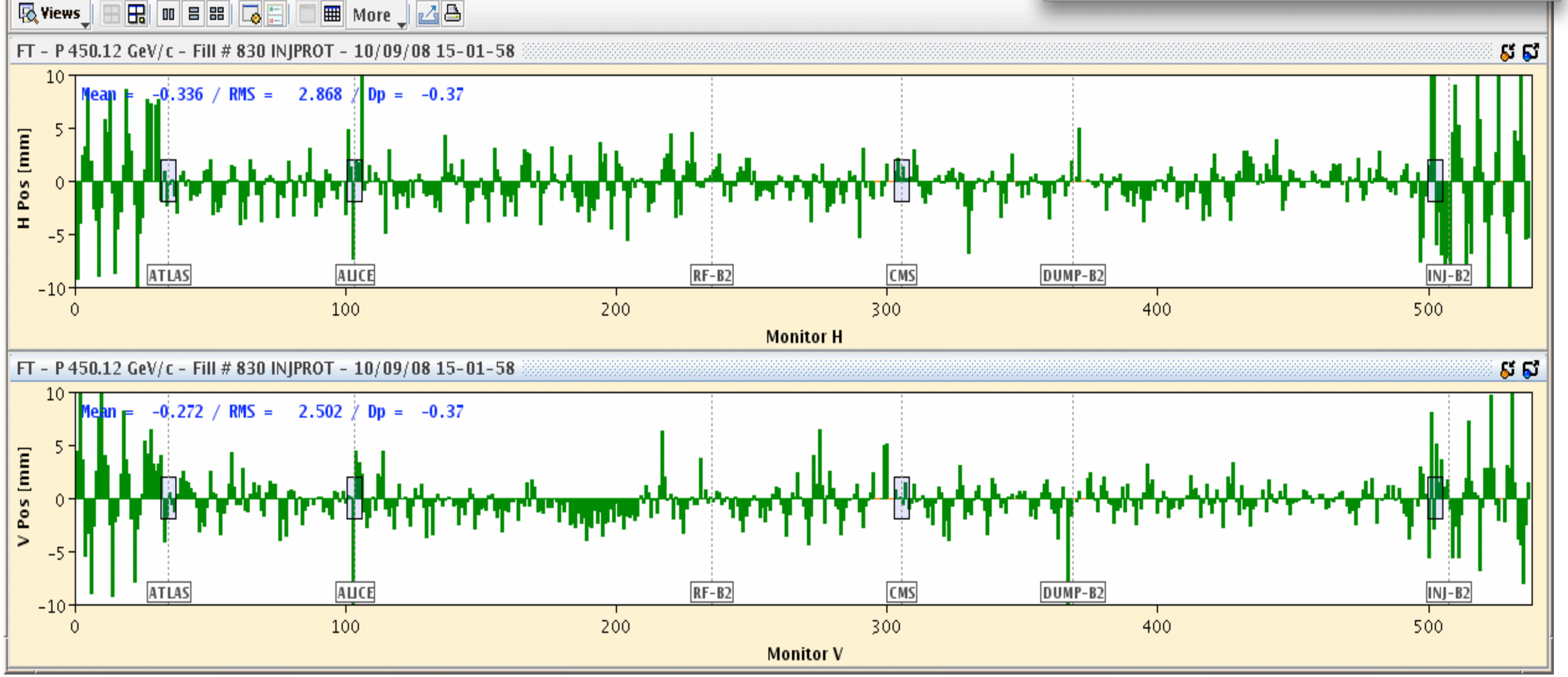


First turn steering "by sector:"

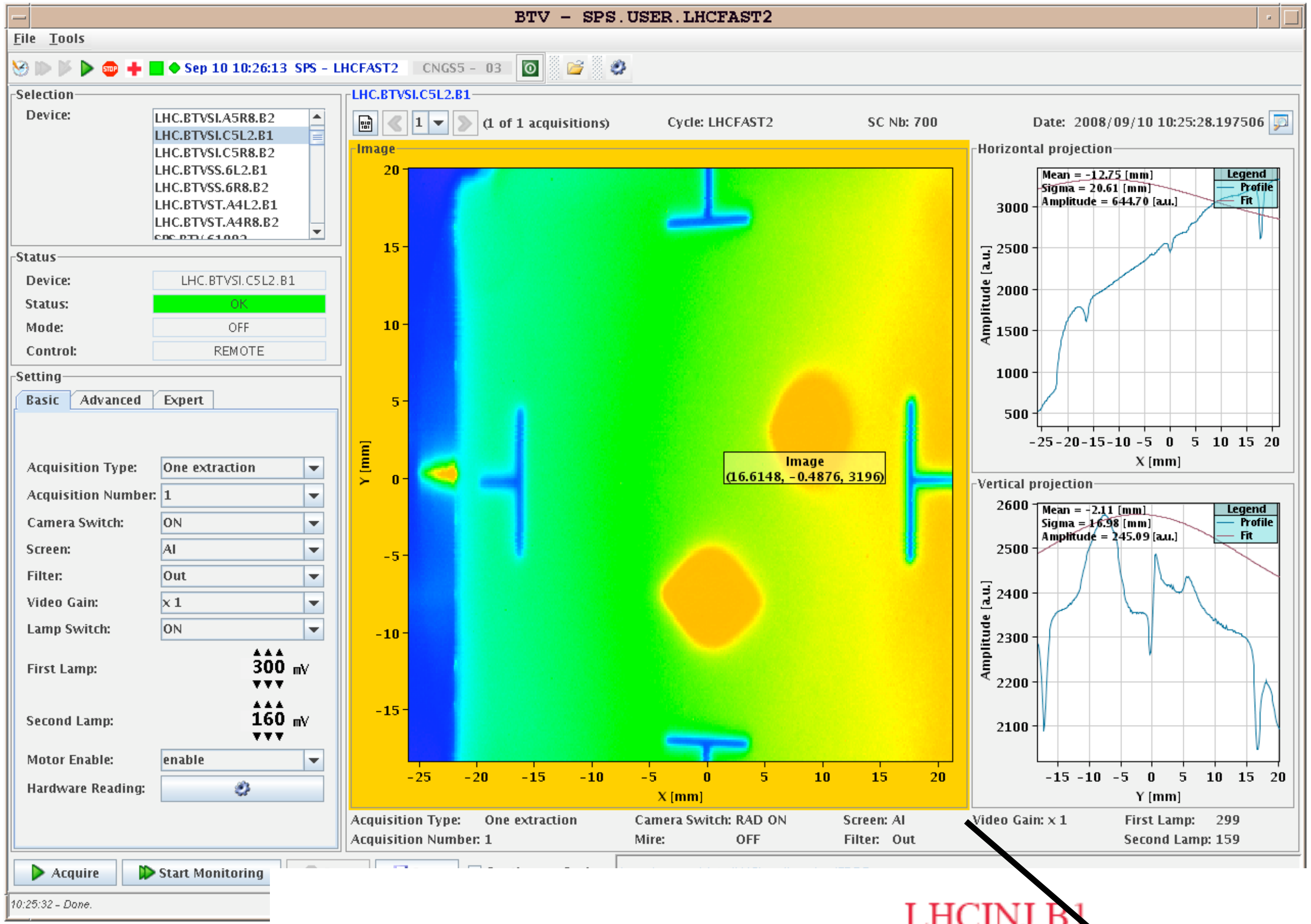
- ❑One beam at the time
- ❑Beam through 1 sector (1/8 ring),

correct trajectory, open collimator and move on.

YASP DV LHCRING / INJ-TEST-NB / beam 2



LHC Operation: the First Beam



BTV – SPS.USER.LHCFAST2

File Tools

Sep 10 10:26:13 SPS – LHCFAST2 CNGS5 – 03

Selection

Device:
LHC.BTVSI.A5R8.B2
LHC.BTVSI.C5L2.B1
LHC.BTVSI.C5R8.B2
LHC.BTVSS.6L2.B1
LHC.BTVSS.6R8.B2
LHC.BTVST.A4L2.B1
LHC.BTVST.A4R8.B2

Status

Device: LHC.BTVSI.C5L2.B1
Status: OK
Mode: OFF
Control: REMOTE

Setting

Basic Advanced Expert

Acquisition Type: One extraction
Acquisition Number: 1
Camera Switch: ON
Screen: Al
Filter: Out
Video Gain: x 1
Lamp Switch: ON
First Lamp: 300 mV
Second Lamp: 160 mV
Motor Enable: enable
Hardware Reading:

Acquire Start Monitoring

10:25:32 - Done.

LHC.BTVSI.C5L2.B1

1 (1 of 1 acquisitions) Cycle: LHCFAST2 SC Nb: 700 Date: 2008/09/10 10:25:28.197506

Image (16.6148, -0.4876, 3196)

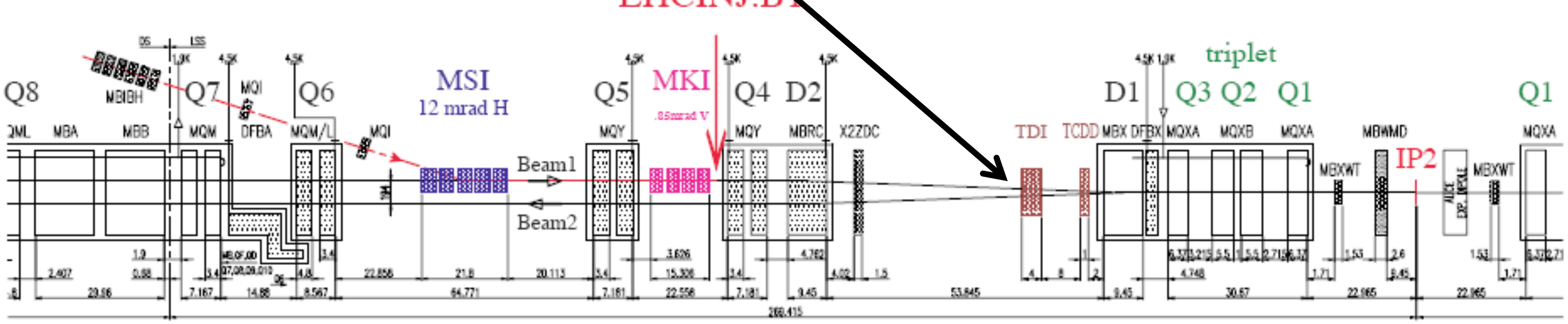
Horizontal projection
Mean = -12.75 [mm]
Sigma = 20.61 [mm]
Amplitude = 644.70 [a.u.]

Vertical projection
Mean = -2.11 [mm]
Sigma = 16.98 [mm]
Amplitude = 245.09 [a.u.]

Acquisition Type: One extraction
Acquisition Number: 1
Camera Switch: RAD ON
Mire: OFF
Screen: Al
Filter: Out
Video Gain: x 1
First Lamp: 299
Second Lamp: 159

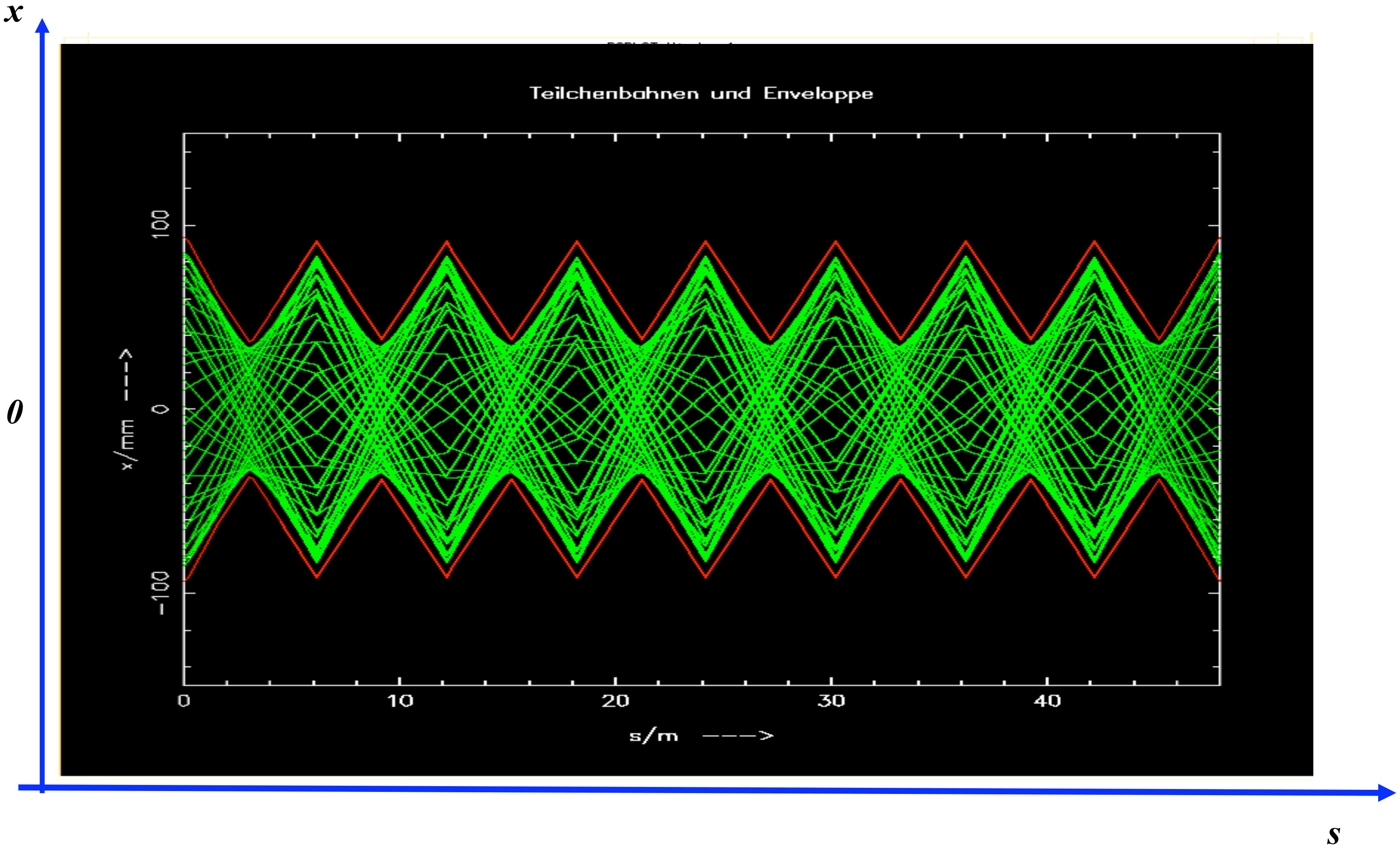
Beam 1 on OTR screen
1st and 2nd turn

LHCINJ.B1



Question: ***what will happen, if the particle performs a second turn ?***

... or a third one or ... 10^{10} turns



Résumé:

beam rigidity: $B \cdot \rho = {}^{p}\!/_{q}$

bending strength of a dipole: $\frac{1}{\rho}\left[m^{-1}\right] = \frac{0.2998 \cdot B_0(T)}{p(GeV/c)}$

focusing strength of a quadrupole: $k\left[m^{-2}\right] = \frac{0.2998 \cdot g}{p(GeV/c)}$

focal length of a quadrupole: $f = \frac{1}{k \cdot l_q}$

equation of motion: $x'' + Kx = \frac{1}{\rho}\frac{\Delta p}{p}$

matrix of a foc. quadrupole: $x_{s2} = M \cdot x_{s1}$

$$M = \begin{pmatrix} \cos\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sin\sqrt{|K|}l \\ -\sqrt{|K|}\sin\sqrt{|K|}l & \cos\sqrt{|K|}l \end{pmatrix} , \qquad M = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

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