Materials & Properties II:
Thermal & Electrical Characteristics

Sergio Calatroni - CERN
Outline (we will discuss mostly metals)

• Electrical properties
  - Electrical conductivity
    o Temperature dependence
    o Limiting factors
  - Surface resistance
    o Relevance for accelerators
    o Heat exchange by radiation (emissivity)

• Thermal properties
  - Thermal conductivity
    o Temperature dependence, electron & phonons
    o Limiting factors
The electrical resistivity of metals changes with temperature.
All pure metals…

Electrical resistivity of Be

Electrical resistivity of Al

Electrical resistivity of Ag
Alloys?

Resistivity of Fe Alloys

- AISI 304 L
- AISI 316 L
- Invar 36
Some resistivity values (in µΩ.cm) (pure metals)

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<tr>
<th>ELEMENT</th>
<th>77 K</th>
<th>273 K</th>
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<tbody>
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<td>Sb</td>
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Variation of a factor ~70 for pure metals at room temperature

Even alloys have seldom more than a few 100s of µΩcm
We will not discuss semiconductors (or in general effects not due to electron transport)
Definition of electrical resistivity $\rho$

\[ R = \frac{\rho \times \text{length}}{\text{section}} \]

The electrical resistance of a real object (for example, a cable)

\[ \sigma = \frac{1}{\rho} \]

The electrical resistivity is measured in Ohm.m

Its inverse is the conductivity measured in S/m

\[ \rho = \frac{m_e v_F}{n e^2} = \frac{m_e}{n e^2} (\tau) \]

Changes with: temperature, impurities, crystal defects

Electron relaxation time

Electron mean free path

Constant for a given material
Basics (simplified free electron Drude model)

Electrical current = movement of conduction electrons
Defects

Defects in metals result in electron-defect collisions. They lead to a reduction in mean free path $\ell$, or equivalently in a reduced relaxation time $\tau$.

They are at the origin of electrical resistivity $\rho$. 
Possible defects: phonons

Crystal lattice vibrations: phonons
Temperature dependent
Possible defects: phonons

Crystal lattice vibrations: phonons
Temperature dependent
Possible defects: impurities

Can be inclusions of foreign atoms, lattice defects, dislocations
Not dependent on temperature
Possible defects: grain boundaries

Grain boundaries, internal or external surfaces
Not dependent on temperature
The two components of electrical resistivity

Proportional to:
- Impurity content
- Crystal defects
- Grain boundaries

Does not depend on temperature

Temperature dependent part

It is characteristic of each metal, and can be calculated

Varies of several orders of magnitude between room temperature and “low” temperature
Temperature dependence: Bloch-Grüneisen function

\[ \rho_{ph}(T) = \left( \frac{\Theta_d}{T} \right)^5 \int_0^{\Theta_d} \frac{x^5}{(e^x - 1)(1 - e^{-x})} \, dx \]

Debye temperature:
\(~\text{maximum frequency of crystal lattice vibrations (phonons)}\)

\[ \Theta_d = \frac{h \nu_s}{2 \pi k_B} \left( \frac{6 \pi^2 N}{V} \right)^{1/3} \]

Given by total number of high-energy phonons proportional \(~T\)

\[ \rho \approx T \quad T >> \Theta_d \]

Given by total number of phonons at low energy \(~T^3\) and their scattering efficiency \(~T^2\)

\[ \rho \approx T^5 \quad T << \Theta_d \]
Low-temperature limits: Matthiessen’s rule

\[ \rho_{\text{total}}(T) = \rho_{\text{phonons}}(T) + \rho_{\text{impurities}} + \rho_{\text{grain boundaries}} + \ldots \]

Or in other terms

\[ \ell_{\text{total}}(T) = \left( \frac{1}{\ell_{\text{phonons}}} + \frac{1}{\ell_{\text{impurities}}} + \frac{1}{\ell_{\text{grain boundaries}}} + \ldots \right)^{-1} \]

Every contribution is additive. Physically, it means that the different sources of scattering for the electrons are independent.
### Effect of added impurities (copper)

#### CAS Vacuum 2017 - S.C.

<table>
<thead>
<tr>
<th>Beimengung</th>
<th>Löslichkeit in Kupfer in Gew.-% bei Raumtemp.</th>
<th>$\Delta g/a$ (m$\Omega$·cm) je Atom-%</th>
<th>beobachteter Streubereich von $\Delta g/a$</th>
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<td>Zn</td>
<td>30</td>
<td>0,3</td>
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</tr>
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</table>

$\rho_{(Cu)}(300K)=1.65\ \mu\Omega.cm$

Note: alloys behave as having a very large amount of impurities embedded in the material.
An useful quantity: RRR

\[ \rho_{\text{total}}(300K) = \rho_{\text{phonons}}(300K) + \rho_{\text{impurities}} + \rho_{\text{grain boundaries}} \]

\[ \rho_{\text{total}}(4.2K) = \rho_{\text{impurities}} + \rho_{\text{grain boundaries}} \]

\[ RRR = \frac{\rho_{\text{total}}(300K)}{\rho_{\text{total}}(4.2K)} = \frac{\rho_{\text{phonons}}(300K)}{\rho_0} \]

\[ \rho_0 = \frac{\rho_{\text{phonons}}(300K)}{(RRR - 1)} \]

Experimentally, we have a very neat feature remembering that

\[ R = \frac{\rho \times \text{length}}{\text{section}} \]

Independent of the geometry of the sample.
Final example: copper RRR 100

\[
RRR = \frac{\rho_{300\,K}}{\rho_{<10\,K}} = 100
\]

\[
\rho_{300K} = \rho_{\text{phonon.}} = 1.55 \times 10^{-8} \, \Omega m
\]

\[
\Rightarrow \rho_{10K} = 1.55 \times 10^{-10} \, \Omega m
\]

If this is due only to oxygen:

\[
\rho_{\text{imp.}} = 5.3 \times 10^{-8} \, \Omega m / \text{at\% of O}
\]

\[
\frac{1.55 \times 10^{-10}}{5.3 \times 10^{-8}} = 0.003 \text{ at \% of O}
\]

\[
\Rightarrow 30 \text{ ppm atomic !}
\]

This is Cu-OFE
Estimates of mean free path

$$\rho = \frac{m}{ne^2 \tau} = \frac{m_e v_F}{ne^2 \ell}$$

Typical values? Example of Cu at room temperature

- Let’s assume one conduction electron per atom.
- $$\rho = 1.55 \times 10^{-8} \ \Omega \text{m}.$$
- density = 89400 kg/m$^3$
- $$m = 9.11 \times 10^{-31} \ \text{kg}, \ e = 1.6 \times 10^{-19} \ \text{C}, \ A = 63.5, \ N_A = 6.022 \times 10^{23}$$

Exercise! Solution:

- $$\tau \approx 2.5 \times 10^{-14} \ \text{s. Knowing that} \ v_F = 1.6 \times 10^6 \ \text{m/s we have}$$
- $$\ell \approx 4 \times 10^{-8} \ \text{m at room temperature. It can be x100 ÷ x1000 larger at low temperature}$$
Interlude: LHC

- **8.33 T dipoles (nominal field) @ 1.9 K**
- Beam screen operating from 4 K to 20 K
- SS + Cu colaminated, RRR ≈ 60
Magnetoresistance

Electron trajectories are bent due to the magnetic field

Cyclotron radius: \( r = \frac{mv_F}{eB} \)

\( \ell \rightarrow \ell/4 \)
\( \tau \rightarrow \tau/4 \)

\( \ell_{\text{eff}} = r \left[ \sin \left( \frac{\ell}{r} \right) \right] = r \left[ \sin \left( \frac{\pi}{2} \right) \right] \)

\( \ell_{\text{eff}} = r \frac{2}{\pi} \)

\( \ell_{\text{eff}} \approx \ell \)

\( \ell_{\text{eff}} = r \left[ \sin \left( \frac{\pi}{8} \right) \right] \)

\( \ell_{\text{eff}} \approx \ell \frac{2}{8 \pi} \approx \ell/4 \)
Fermi sphere

- The real picture: the whole Fermi sphere is displaced from equilibrium under the electric field $E$, the force $F$ acting on each electron being $-eE$
- This displacement in steady state results in a net momentum per electron $\delta k = F\tau/h$ thus a net speed increment $\delta v = F\tau/m = -eE\tau/m$
- $j = ne\delta v = ne^2E\tau/m$ and from the definition of Ohm’s law $j = \sigma E$ we have $\sigma = \frac{ne^2\tau}{m}$
The speed of conduction electrons

- Fermi velocity \( v_F = 1.6 \times 10^6 \text{ m/s} \)
- \( \delta v = j / ne \) thus \( \delta v = \frac{\sigma E}{ne} = \frac{e \tau E}{m} \)

As an order-of-magnitude, in a common conductor, we may have a potential drop of \(~1\text{V} \) over \(~1\text{m} \)

- \( E = \frac{V}{d} \approx 1 \text{ V/m} \) and as a consequence \( \delta v \approx 4 \times 10^{-3} \text{ m/s} \)
- The drift velocity of the conduction electrons is orders of magnitude smaller than the Fermi velocity

(Repeat the same exercise with 1 A of current, in a copper conductor of 1 cm\(^2\) cross section)
Square resistance and surface resistance

Consider a square sheet of metal and calculate its resistance to a transverse current flow:

This is the so-called square resistance often indicated as $R$.
Square resistance and surface resistance

And now imagine that instead of DC we have RF, and the RF current is confined in a skin depth: \( \delta = \sqrt{\frac{2 \rho}{\omega \mu_0}} \)

\[
R = \frac{\rho}{d} = \frac{\rho}{d}
\]

\[
R_s = \frac{\rho}{\delta} = \sqrt{\frac{\rho \mu_0 \omega}{2}}
\]

This is a (simplified) definition of surface resistance \( R_s \)

(We will discuss this in more details at the tutorials)
Surface impedance in normal metals

- The Surface Impedance $Z_s$ is a complex number defined at the interface between two media.

- The real part $R_s$ contains all information about power losses (per unit surface)

\[\bar{P} = \frac{1}{2} R_s I^2 \]

\[= \frac{1}{d^2} R_s H_0^2\]

- The imaginary part $X_s$ contains all information about the field penetration in the material

\[\delta = \frac{2}{\mu_0 \omega} X_s\]

- For copper ($\rho = 1.75 \times 10^{-8} \, \mu\Omega.\text{cm}$) at 350 MHz:

- $R_s = X_s = 5 \, \text{m}\Omega$ and $\delta = 3.5 \, \mu\text{m}$
Why the surface resistance (impedance)?

- It is used for all interactions between E.M. fields and materials

- In RF cavities: quality factor

  \[ Q_0 = \frac{\Gamma}{R_s} \]

- In beam dynamics (more at the tutorials):

  - Longitudinal impedance and power dissipation from wakes is

    \[ P_{\text{loss}} = M I_b^2 \text{Re}\left| Z_s^{\text{eff}} \right| \]

    where \( Z_s^{\text{eff}} \) is a summation of \( (2\pi R/2\pi b) Z_s \) over the bunch frequency spectrum

  - Transverse impedance:

    \[ Z_T = \frac{2\pi R c}{\pi b^3 \omega} Z_s \]
From RF to infrared: the blackbody

Thermal exchanges by radiation are mediated by EM waves in the infrared regime.

Schematization of a blackbody

Peak ≈ 3000 µm x K
Blackbody radiation

• A blackbody is an idealized perfectly emitting and absorbing body (a cavity with a tiny hole)

• Stefan-Boltzmann law of radiated power density:

\[
\frac{P}{A} = \sigma T^4 \quad \sigma \approx 5.67 \times 10^{-8} \text{W/(m}^2\text{K}^4)\]

• At thermal equilibrium: \( \varepsilon = \alpha \)

• \( \varepsilon \) is the emissivity (blackbody=1)

• A “grey” body will obey: \( 1 = r + \alpha (t) \)

• Thus for a grey body: \( \frac{P}{A} = \varepsilon \sigma T^4 \)
From RF to infrared in metals

- Thermal exchanges by radiation are mediated by EM waves in the infrared regime.

- At 300 K, $\lambda_{\text{peak}} \approx 10 \, \mu\text{m}$ of wavelength $\rightarrow \approx 10^{13}$ Hz or $\tau_{\text{RF}} \approx 10^{-13} \, \text{s}$

- The theory of normal skin effect is usually applied for: $\omega_{\text{RF}} \tau < 1$

- But it can be applied also for: $\omega_{\text{RF}} \tau > 1$

- In the latter case it means: $\tau > \tau_{\text{RF}}$

- For metals at moderate $T$ we can then use the standard skin effect theory to calculate emissivity
Emissivity of metals

• From: \( 1 = \rho + \alpha \Rightarrow \varepsilon = 1 - r \)

• Thus we can calculate emissivity from reflectivity:

\[
\varepsilon = 1 - r \approx 4 \frac{R_s}{R_{\text{vacuum}}}
\]

\[
R_s = \sqrt{\frac{\rho \mu_0 \omega}{2}} \approx \Omega
\]

\[
R_{\text{vacuum}} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.7\Omega
\]

• The emissivity of metals is small

• The emissivity of metals depends on resistivity

• Thus, the emissivity of metals depends on temperature and on frequency
Practical case: 316 LN

Emissivité hémisphérique absolue après différents traitements de surface

\[ \rho_t = \prod \rho_i \]

\( \rho_i \): Réflectance
\( \rho_t \): Réflectance totale
Thermal conductivity of metals

![Graph showing thermal conductivity of copper as a function of temperature. The graph indicates a peak and a constant region.]

Copper

$k(W/mK)$

$T(K)$

$T^{-1}$

Peak

Constant
Thermal conductivity: insulators

Determined by phonons (lattice vibrations). Phonons behave like a “gas”
Thermal conductivity: insulators

Thermal conductivity $K_{ph}$ from heat capacity $C_{ph}$ (as in thermodynamics of gases)

\[ K_{ph} = \frac{1}{3} C_{ph} v_s \ell = \frac{1}{3} C_{ph} v_s^2 \tau \]

\[ C_{ph} = 3Nk_B T = 3RT \quad T >> \Theta_d \]

\[ C_{ph} = \frac{12 \pi^4}{5} Nk_B \left( \frac{T}{\Theta_d} \right)^3 \quad T << \Theta_d \]

\[ \tau = \frac{1}{T} \quad T >> \Theta_d \]

\[ \tau = \text{const.} \quad T << \Theta_d \]

$K_{ph} \approx \text{const} \quad T >> \Theta_d$

$K_{ph} \approx \left( \frac{T}{\Theta_d} \right)^3 \quad T << \Theta_d$

for ultra-pure crystals

\[ K_{peak} = \frac{1}{3} C_{ph} v_s \Theta \]

$\Theta = \text{max dimension of specimen}$
Thermal conductivity: metals

Determined by both electrons and phonons.

Thermal conductivity $K_{el}$ from heat capacity $C_{el}$

\[ K_{el} = \frac{1}{3} C_{el} v_F \ell = \frac{\pi^2 n k_B^2 T}{3 m v_F^2} v_F \ell = \frac{\pi^2 n k_B^2 T}{3 m} \tau \]

\[ \tau = \frac{1}{T} \quad T >> \Theta_d \]

\[ \tau = \text{const.} \quad T << \Theta_d \]

impurities
Thermal conductivity of metals: total

Copper
Wiedemann-Franz

Proportionality between thermal conductivity and electrical conductivity

\[
\frac{K_{el}}{\sigma} = \frac{\pi^2}{3} \left( \frac{k_B}{e} \right)^2 T = LT \quad T > \Theta_d
\]

\[L = 2.45 \times 10^{-8} \text{ W} \Omega K^{-2}\]

(Lorentz number)

Useful for simple estimations, if one or the other quantity are known

Useful also (very very approximately) to estimate contact resistances
The LHC collimator

- Cooling Pipes
- Glidcop bar
- Glidcop plates
- Clamping springs
- Graphitic jaw
- Rack and pinion
- Stepper motor
- Return spring
Contact resistance (both electrical and thermal)

- Complicated… and no time left 😓

Contact area: \( A \sim P^n \quad n \approx O(1) \)

- Contacts depend also on oxidation, material(s) properties, temperature…

Example for electric contacts:

- Theoretically:
  - \( R \propto P^{-1/3} \) in elastic regime
  - \( R \propto P^{-1/2} \) in plastic regime

- Experimentally:
  - \( R \propto P^{1.5-1/2} \) (same as for thermal contacts)

\( n \) depends on:

- Plastic deformation
- Elastic deformation

Roughness “height” and “shape”
References

• Charles Kittel, “Introduction to solid state physics”
• Ashcroft & Mermin, “Solid State Physics”
• S. W. Van Sciver, “Helium Cryogenics”
• M. Hein, “HTS thin films at µ-wave frequencies”
• J.A. Stratton, “Electromagnetic Theory”
• Touloukian & DeWitt, “Thermophysical Properties of Matter”
The end. Questions?
Plane waves in vacuum

Plane wave solution of Maxwell’s equations in vacuum:

\[ E = E_0 e^{i(kz-\omega t)} \quad H = H_0 e^{i(kz-\omega t)} \quad H = E_0 \frac{k}{\omega \mu_0} e^{i(kz-\omega t)} \]

Where (in vacuum):

\[ k = \frac{2\pi}{\lambda} = \frac{\omega}{c} = \omega \sqrt{\varepsilon_0 \mu_0} = \sqrt{\frac{\varepsilon_0}{\mu_0}} \]

So that:

\[ E = E_0 e^{i(kz-\omega t)} \quad H = E_0 \sqrt{\frac{\varepsilon_0}{\mu_0}} e^{i(kz-\omega t)} \]

The ratio \( Z = \frac{|E|}{|H|} = \frac{\sqrt{\mu_0}}{\sqrt{\varepsilon_0}} = 376.7 \Omega \) is often called impedance of the free space and the above equations are valid in a continuous medium.
Plane waves in normal metals

More generally, in metals:

\[ k^2 = \omega^2 \varepsilon \mu + i \omega \sigma \mu \]

\[ Z = \frac{|E|}{|H|} = \sqrt{\frac{\mu}{\varepsilon}} = \frac{\omega \mu}{k} \]

This results from taking the full Maxwell's equations, plus a supplementary equation which relates locally current density and field:

\[ \mathbf{J}(\mathbf{x}, t) = \sigma \mathbf{E}(\mathbf{x}, t) \quad \sigma_0 = \frac{ne^2 \ell}{m_e v_F} = \frac{ne^2 \tau}{m_e} \]

In metals \( \omega \sigma \mu \gg \omega^2 \varepsilon \mu \Rightarrow k^2 = i \omega \sigma \mu \quad k = \sqrt{i \omega \sigma \mu} = \alpha + i \beta \)

and the wave equations become:

\[ E = E_0 e^{i(kz - \omega t)} = E_0 e^{i(\alpha z - \omega t)} e^{-\beta z} \]

With \( \delta = \frac{1}{\beta} = \sqrt{\frac{\sigma \omega \mu}{2}} \) is the damping coefficient of the wave inside a metal, and \( \delta \) is also called the field penetration depth.
Surface impedance

\[ Z_s = \frac{V}{I} \]

\[ V = dE_z(0) \; ; \; I = d \int_0^{+\infty} J_z(y) dy \]

\[ \delta = \frac{1}{J_z(0)} \int_0^{+\infty} J_z(y) dy \]

\[ Z_s = R_s + iX_s = \frac{E_z(0)}{+\infty} \int_0^y J_z(y) dy = \frac{E_z(0)}{\delta J_z(0)} = \frac{E_z(0)}{H_x(0)} \]

\[ P_{tot}(t) = \frac{1}{2} R_s I^2 = \frac{1}{2} R_s d^2 H_x^2 \]

\[ \bar{P} / S = P_{rf} = \frac{1}{2} R_s H_{rf}^2 = \frac{1}{2} R_s \left( \frac{B_{rf}}{\mu_o} \right)^2 \]
Normal metals in the local limit

\[ J_z(y) = J_z(0)e^{\frac{y}{\delta}} \quad \delta = \sqrt{\frac{2}{\omega \mu_o \sigma_n}} \quad \left( \vec{J}(t) = \vec{J}(0)e^{i\omega t} \right) \]

\[ Z_n = R_n + iX_n = \frac{E_z(0)}{\delta J_z(0)} = \frac{1}{\delta \sigma_n} (1 + i) = \sqrt{\frac{\mu_o \omega}{2 \sigma_n}} (1 + i) \]

\[ R_n = X_n = \frac{1}{\delta \sigma_n} = \frac{\rho_n}{\delta} = \sqrt{\frac{\mu_o \omega}{2 \sigma_n}} = \sqrt{\frac{\mu_o \omega}{2}} \rho_n \quad (R_n \propto \sqrt{\omega}) \]
Limits for conductivity and skin effect

\[ \delta = \sqrt{\frac{2\rho}{\omega \mu_0}} \]

\[ \ell \sim \frac{1}{\rho} \]

1. Normal skin effect if: \( \ell \ll \delta \)  e.g.: high temperature, low frequency

2. Anomalous skin effect if: \( \ell \gg \delta \)  e.g.: low temperature, high frequency

Note: 1 & 2 valid under the implicit assumption \( \omega \tau \ll 1 \)

1 & 2 can also be rewritten (in advanced theory) as:

\[ \frac{\ell}{\delta} \ll \left(1 + \omega^2 \tau^2\right)^{3/4} \]

It derives that 1 can be true for \( \omega \tau \ll 1 \)  and also for \( \omega \tau \gg 1 \)
Mean free path and skin depth

Frequency = 1 GHz, copper at various temperatures

Mean free path

\[ \ell \sim \frac{1}{\rho} \]

Skin depth

\[ \delta = \sqrt{\frac{2\rho}{\mu_0\omega}} \]

\[ n_{\text{eff}} \approx n_0 \frac{\delta}{\ell} \]

\[ \sigma_0 = \frac{ne^2\ell}{m_e v_F} \Rightarrow \sigma_{\text{eff}} \approx \frac{ne^2\delta}{m_e v_F} \]

\[ \delta = \sqrt{\frac{2}{\sigma_0 \omega \mu_0}} \Rightarrow \delta \xrightarrow{\tau \to \infty} \text{const.} \]
Anomalous skin effect

Debye temperatures

<table>
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<th>Element</th>
<th>$\Theta_D$ (K)</th>
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<tr>
<td>Au</td>
<td>165</td>
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<tr>
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<td>Hf</td>
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<tr>
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<td>380</td>
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<tr>
<td>Zn</td>
<td>327</td>
</tr>
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</table>
Heat capacity of solids: Dulong-Petit law

[Graph showing heat capacity vs. temperature ratio]

[Graph showing heat capacity in J mol⁻¹ K⁻¹ vs. temperature ratio]
Low-temperature heat capacity of phonon gas

(simplified plot in 2D)

Figure 10  To obtain a qualitative explanation of the Debye $T^3$ law, we suppose that all phonon modes of wavevector less than $K_T$ have the classical thermal energy $k_BT$ and that modes between $K_T$ and the Debye cutoff $K_D$ are not excited at all. Of the $3N$ possible modes, the fraction excited is $(K_T/K_D)^3 = (T/\theta)^3$, because this is the ratio of the volume of the inner sphere to the outer sphere. The energy is $U = k_BT \cdot 3N(T/\theta)^3$, and the heat capacity is $C_V = \partial U/\partial T \approx 12Nk_B(T/\theta)^3$. 
Phonon spectrum and Debye temperature

Density of states $D(\omega)$: How many elemental oscillators of frequency $\omega$

Assuming constant speed of sound

Figure 14 Density of states as a function of frequency for (a) the Debye solid and (b) an actual crystal structure. The spectrum for the crystal starts as $\omega^2$ for small $\omega$, but discontinuities develop at singular points.