CERN

Numerical Methods I

Kevin Li

With acknowledgements to:

H. Bartosik, X. Buffat, L.R. Carver, S. Hegglin, G. Iadarola, L. Mether, E. Metral, N. Mounet, A. Oeftiger, A. Romano, G. Rumolo, B. Salvant, M. Schenk

Outline

Introduction to macroparticle models – implementations, applications and examples

- Part 1 – numerical modelling
  - Initialisation
  - Simple tracking
  - Chromaticity and detuning
  - Wakefields with examples
    - Constant wakes
    - Dipole wakes
    - TMCI & headtail modes
- Part 2 – electron cloud
  - Modelling of e-cloud interactions
  - PIC solvers
  - Application for e-cloud instabilities

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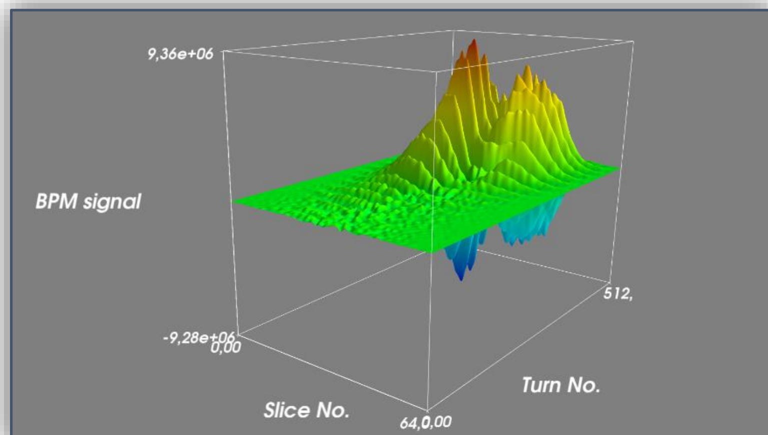
Why numerical methods?

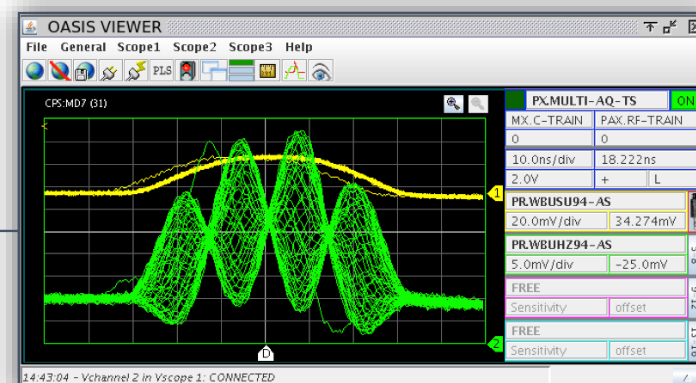
- Computational physics is somewhere between experimental and theoretical physics

Analytical studies
- Very reduced system
- Immediate global behavior of the system/scalings

$$-i\frac{\Omega}{\beta c} f_1 g_1\, e^{-i\Omega s/(\beta c)} = \left(-f_1 \frac{Q_y}{R}\frac{\partial g_1}{\partial \theta} - g_1 \frac{\omega_s}{\beta c}\frac{\partial f_1}{\partial \phi}\right) e^{-i\Omega s/(\beta c)}$$
$$+ \frac{e^2}{\beta^2 E C} f_0 \sqrt{\frac{2J_y R}{Q_y}}\, \sin\theta\, g_0' \frac{y^n}{n!}$$
$$\times \int dz' \sum_{k=-\infty}^{\infty} \rho^{(m)}(z')\, e^{-i\Omega(z/(\beta c) - kT_0)}\, W_{mn}(z - z' - kcT_0)$$



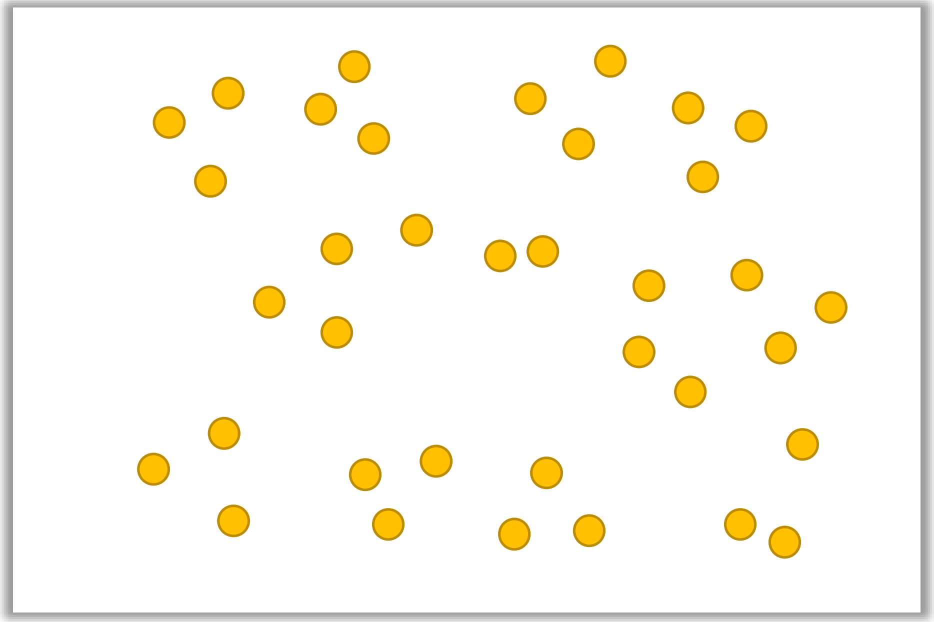
Numerical studies
- Intermediate system
- Full flexibility for scenarios
- Full diagnostics



Machine studies
- Full real system
- Reduced flexibility for scenarios
- Reduced diagnostics

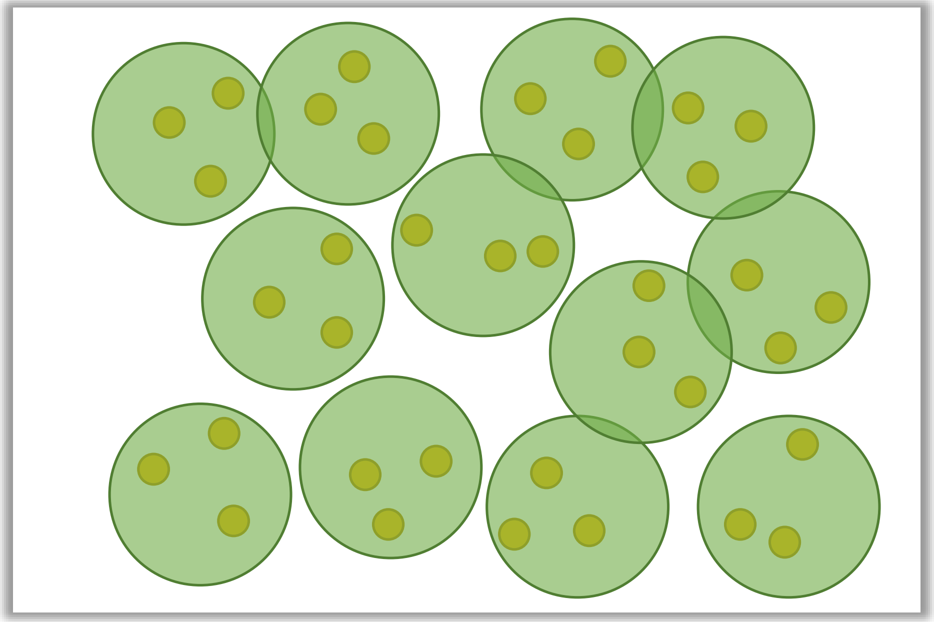
Macroparticle models

- Macroparticle models are used to study the dynamics of multi-particle systems, i.e. plasmas or beams
- Macroparticle models permit a seamless mapping of realistic systems into a computational environment – they are fairly easy to implement



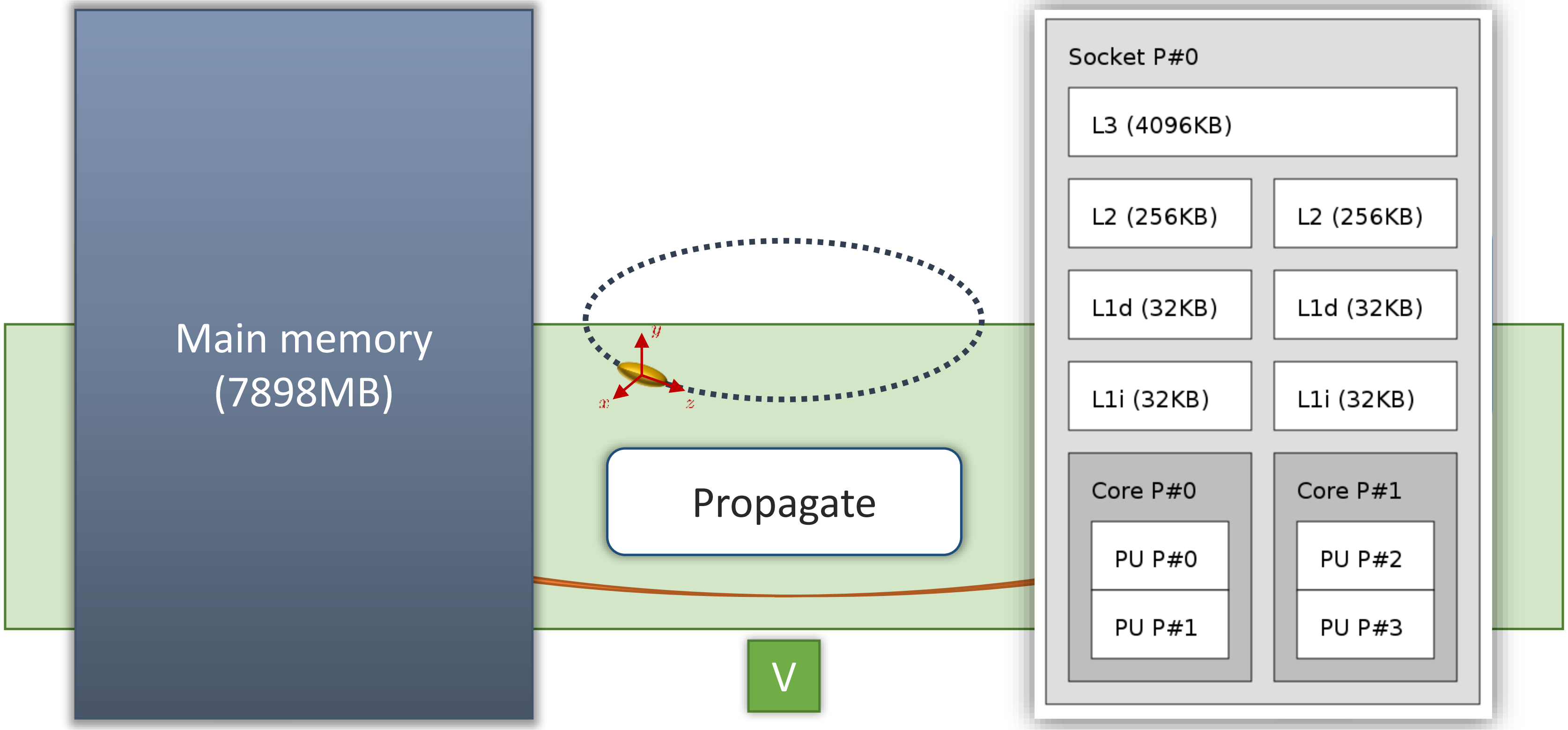
Macroparticle models

- Macroparticle models are used to study the dynamics of multi-particle systems, i.e. plasmas or beams
- Macroparticle models permit a seamless mapping of realistic systems into a computational environment – they are fairly easy to implement
- A macroparticle is a numerical representation of a clustered collection of physical particles → this increases the granularity of the numerical system (i.e. one must beware of numerical noise issues) but allows to treat large systems within the given limitations of computational resources



Accelerator-beam system – modelling

- Possible program layout



Storage

momenta

A function
knows ho
update th
coordinates and
momenta

Processing

Accelerator-beam system - modelling

- Possible program layout

By closing the loop, we introduce collective effects which can act as a feedback mechanism:
- New equilibrium solutions
- Damping or growth of the stationary solutions → instabilities

coordinates and momenta

update the coordinates and momenta

...

Numerical representation of the beam

Beam:

$$\begin{pmatrix} x_i \\ x'_i \end{pmatrix} \quad \begin{pmatrix} q_i \\ m_i \end{pmatrix}, \quad i = 1, \ldots, N$$

$$\begin{pmatrix} y_i \\ y'_i \end{pmatrix}$$

$$\begin{pmatrix} z_i \\ \delta_i \end{pmatrix}$$

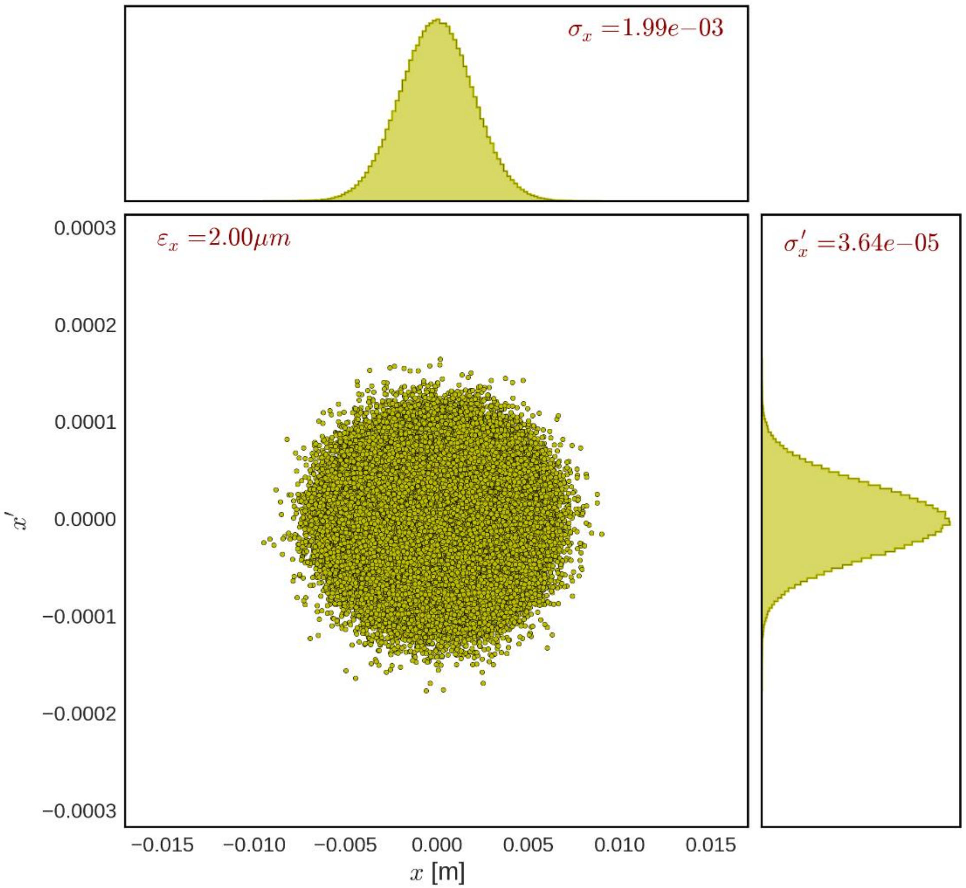
Macroparticlenumber

Can. conjugate coordinates and momenta

Memory (assume q, m are identical):

	array	array	array	array	array	array
count	**x**	**x'**	**y**	**y'**	**z**	**delta**
0	...	...	...	...	...	...
1	...	...	...	...	...	...
2	...	...	...	...	...	...
3	...	...	...	...	...	...
4	...	...	...	...	...	...
5	...	...	...	...	...	...
6	...	...	...	...	...	...
7	...	...	...	...	...	...
8	...	...	...	...	...	...
9	...	...	...	...	...	...

Numerical representation of the beam



count	x	x'	y	y'	z	de'
0	...	...	...	...	...	
1	...	...	...	...	...	
2	...	...	...	...		
3	...	...	...	...		

- Initial conditions of the beam/particles

Profile	Size	Matching
Gaussian	Emittance	Optics
Parabolic		
Flat		
…		

- We use random number generators to obtain random distributions of coordinates and momenta
- Example transverse Gaussian beam in the SPS with normalized emittance of 2 um (0.35 eVs longitudinal)

$$\varepsilon_\perp = \beta\gamma\sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$$
$$= \beta\gamma\sigma_x\sigma_{x'}$$
$$\varepsilon_\parallel = 4\pi\sigma_z\sigma_\delta\frac{p_0}{e}$$

Signpost

- We have identified with the beam and allocated a field of memory and filled this with the can. conjugate coordinates and momenta in accordance with our specifications for beam profile and emittance and machine optics.
- We are now ready to investigate how to implement the beam dynamics

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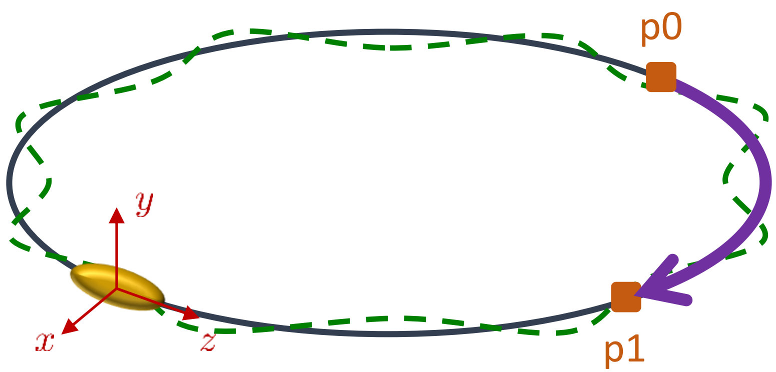
Numerical representation of the linear lattice



- Dipoles → orbit
- Quadrupoles → focusing
- Particle dynamics along the linear periodic lattice is described by Hill's equation

$$x'' + K(s)x = 0\,, \qquad K(s) = K(s+L)$$

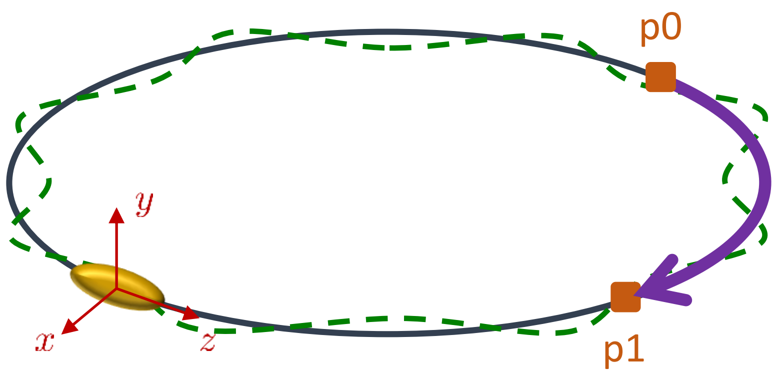
- Hill's equation can be solved to obtain the linear transfer map from one point to another along the ring:

$$\mathcal{M} = \begin{pmatrix} \sqrt{\beta_1} & 0 \\ -\frac{\alpha_1}{\sqrt{\beta_1}} & \frac{1}{\sqrt{\beta_1}} \end{pmatrix} \begin{pmatrix} \cos(\Delta\mu_{0\to1}) & \sin(\Delta\mu_{0\to1}) \\ -\sin(\Delta\mu_{0\to1}) & \cos(\Delta\mu_{0\to1}) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_0}} & 0 \\ \frac{\alpha_0}{\sqrt{\beta_0}} & \sqrt{\beta_0} \end{pmatrix}$$

$$Q_x = \oint \frac{\Delta\mu}{2\pi}$$

E. D. Courant and H. S. Snyder, *Theory of the Alternating-Gradient Synchrotron*, Annals of Physics 3 (1958)

Numerical representation of the linear lattice



1. The optics functions can be obtained from a Twiss file (e.g. MAD-X)
2. We can make the smooth approximation

$$\beta_x = \text{constant}$$

$$Q_x = \frac{C}{2\pi\beta_x}, \left(Q_s = \frac{\eta C}{2\pi\beta_z}\right)$$

$$\Delta\mu_x = Q_x \frac{L}{C}$$

L: Segment length

C: Ring circumference

$$\mathcal{M} = \begin{pmatrix} \sqrt{\beta_1} & 0 \\ -\frac{\alpha_1}{\sqrt{\beta_1}} & \frac{1}{\sqrt{\beta_1}} \end{pmatrix} \begin{pmatrix} \cos(\Delta\mu_{0\to1}) & \sin(\Delta\mu_{0\to1}) \\ -\sin(\Delta\mu_{0\to1}) & \cos(\Delta\mu_{0\to1}) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_0}} & 0 \\ \frac{\alpha_0}{\sqrt{\beta_0}} & \sqrt{\beta_0} \end{pmatrix}$$

$$\begin{pmatrix} x_i \\ x_i' \end{pmatrix}\bigg|_1 = \boxed{\mathcal{M}} \begin{pmatrix} x_i \\ x_i' \end{pmatrix}\bigg|_0$$

$$i = 1, \ldots, N$$

All matrix elements are constant

- The numerical implementation is then simply the matrix product applied to all macroparticles in all panes

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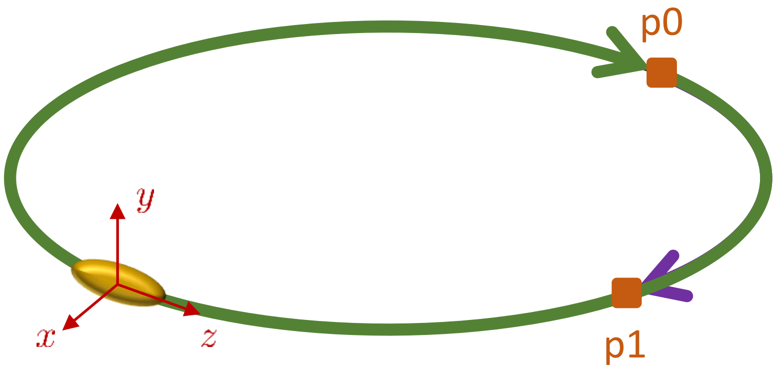
Numerical representation of the long. motion



- Particle dynamics in the longitudinal plane are described by the longitudinal equations of motion

$$z' = -\eta\,\delta$$

$$\delta' = \frac{e\,V_{\mathrm{RF}}}{m\gamma\beta^2 c^2\,C}\sin\left(\frac{2\pi h}{C}\,z\right)$$

- V_{RF}: RF voltage
- $h = \frac{\omega_{\mathrm{RF}}}{\omega_0}$: harmonic number
- ω_0: Revolution frequency
- C: circumference

$$z_{i,k+1/2} = z_{i,k} - \frac{\eta\,C}{2}\,\delta_{i,k}$$

$$\delta_{i,k+1} = \delta_{i,k} + \frac{e\,V_{\mathrm{RF}}}{m\gamma\beta^2 c^2}\sin\left(\frac{2\pi h}{C}\,z_{i,k+1/2}\right)$$

$$z_{i,k+1} = z_{i,k+1/2} - \frac{\eta\,C}{2}\,\delta_{i,k+1}$$

$$i = 1, \ldots, N$$

k: iteration/turn

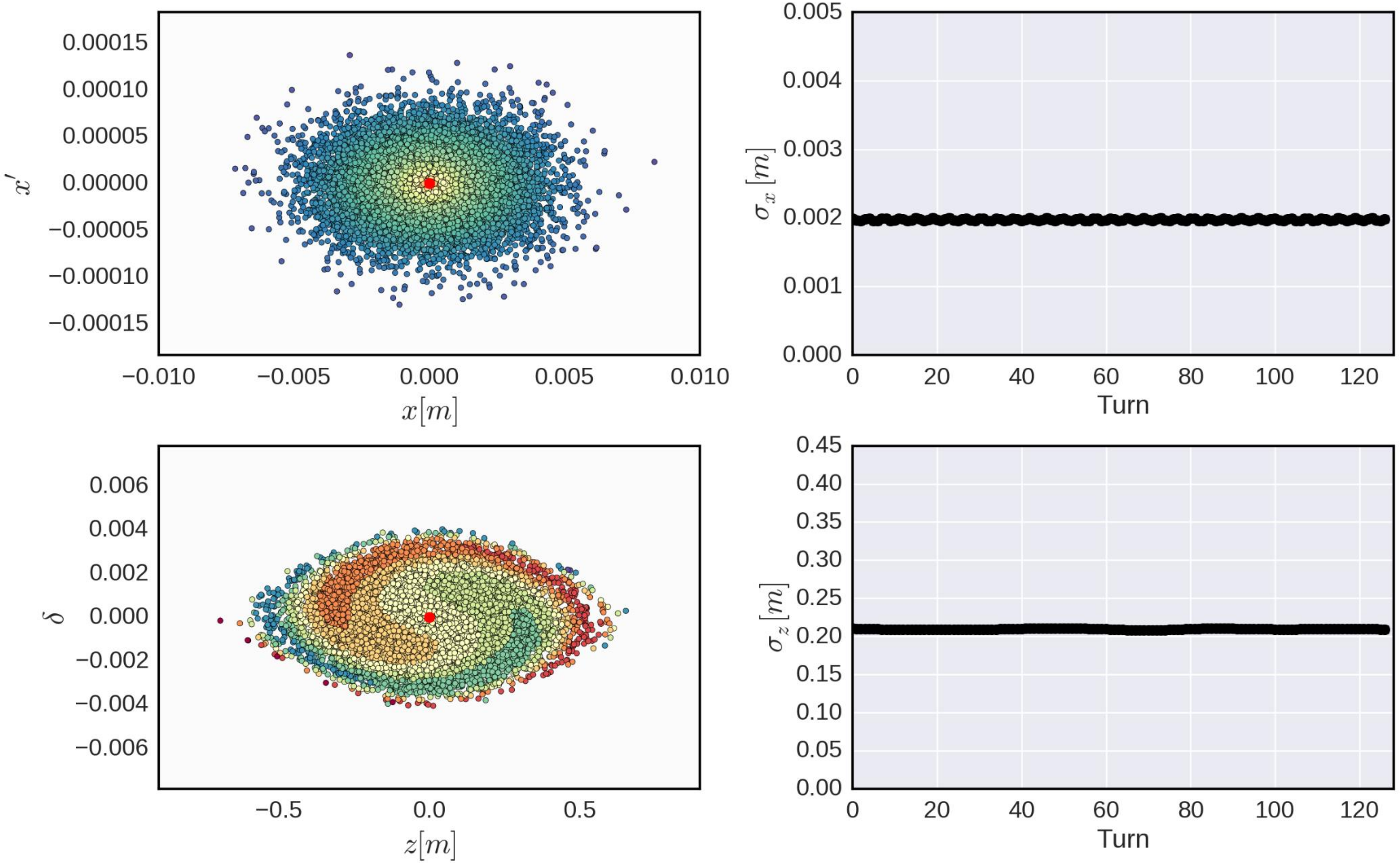
- These can be solved numerically via a symplectic integration scheme – iterative turn-by-turn advancement of the coordinates and momenta

W. Herr, *Tools for Non Linear Dynamics*, CAS Poland 2015

E. Forest, *Beam Dynamics: A New Attitude and Framwork*, 1998

Example: SPS – transverse and longitudinal

- We can now input the optics functions along with the RF parameters and observe the oscillations



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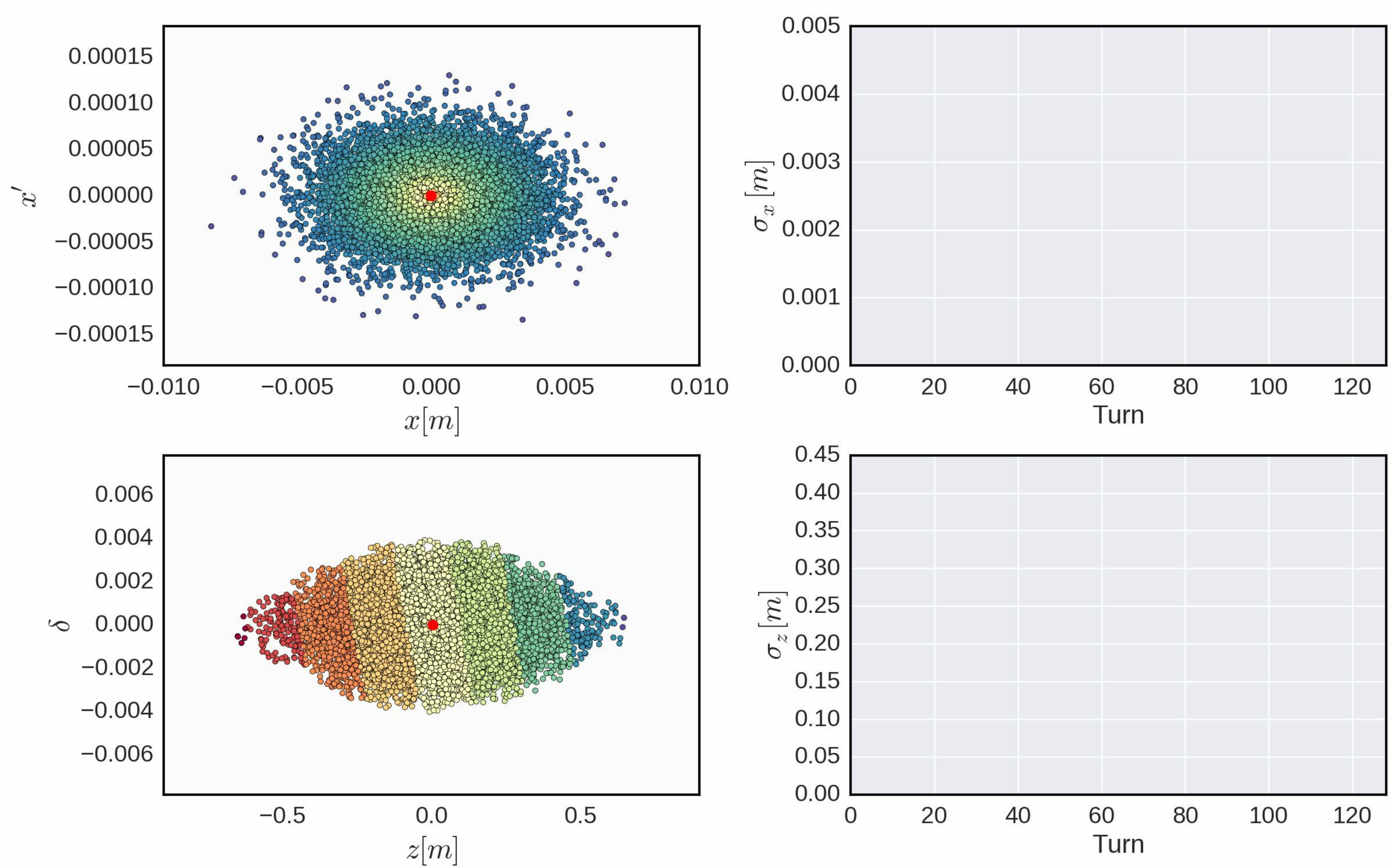
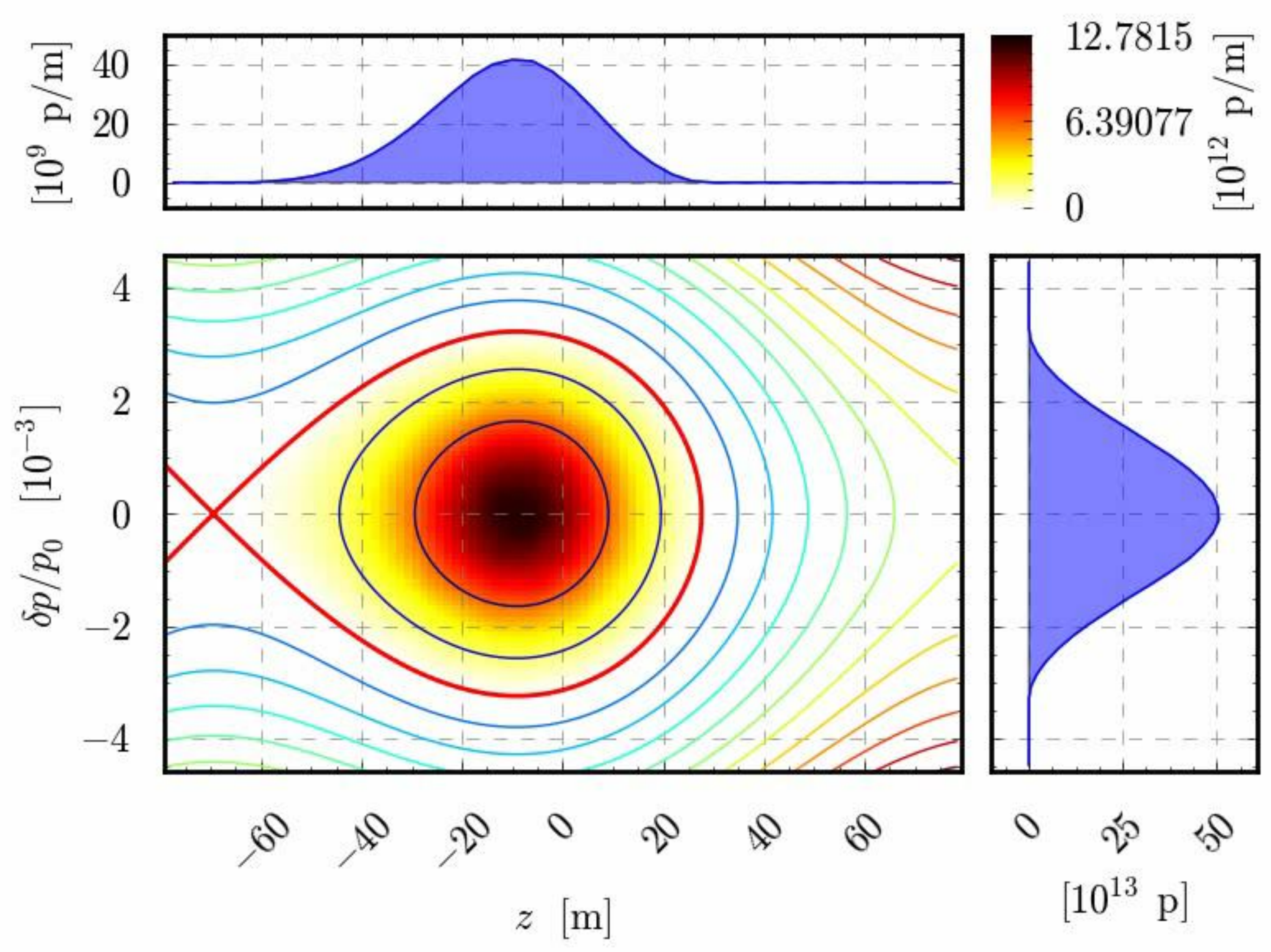
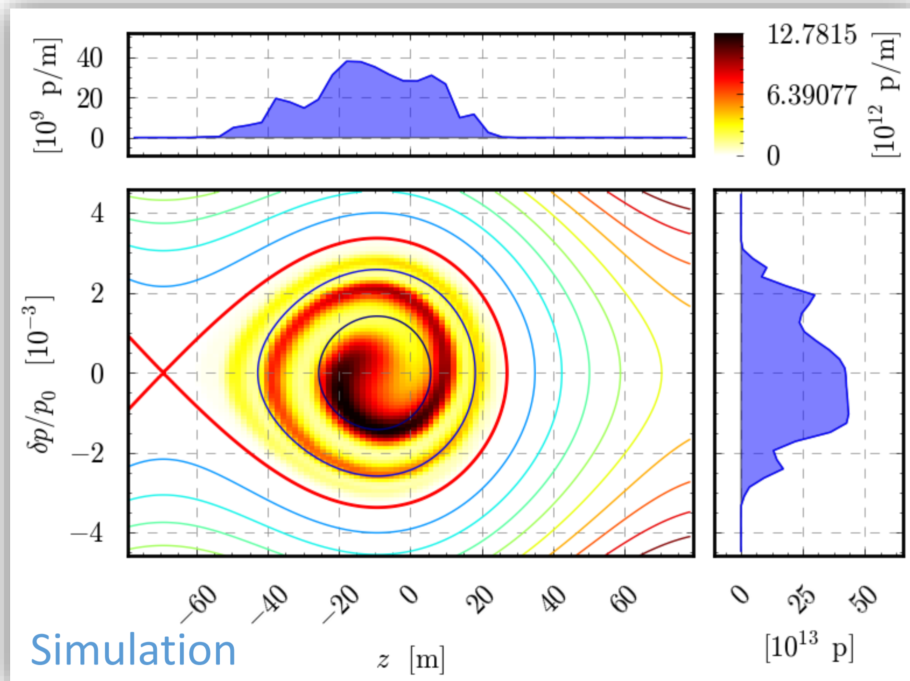
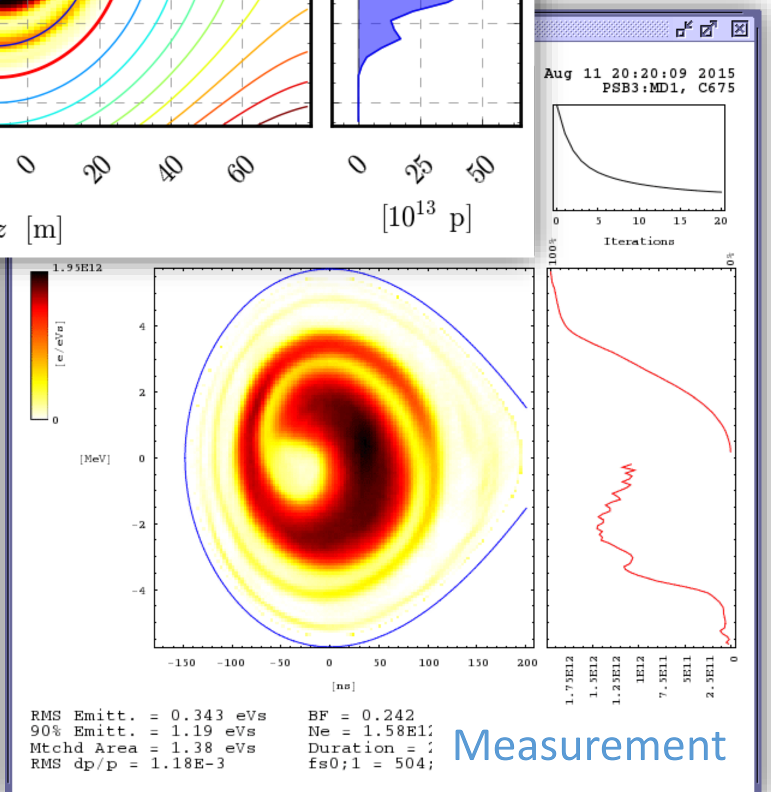


Example: PS Booster hollow bunches



Example: PS Booster hollow bunches





- Generation of ‘hollow’ bunches for mitigation of transverse space charge
- Modulation of the reference phase at 490 Hz with an amplitude of 18 deg (reconstructed from 3.4 mm amplitude and the waterfall plot) for 4 synchrotron periods
- Simulations are capable of reproducing as well as predicting

A. Oeftiger et al.

Signpost

- We have implemented a beam to initializing a field of memory with the beam's can. conjugate coordinates and momenta.
- We have implemented linear transverse tracking as a 2D matrix multiplication looped over the set of macroparticles in the beam.
- We have implemented longitudinal tracking as a second order symplectic integration scheme looped over the set of macroparticles in the beam.

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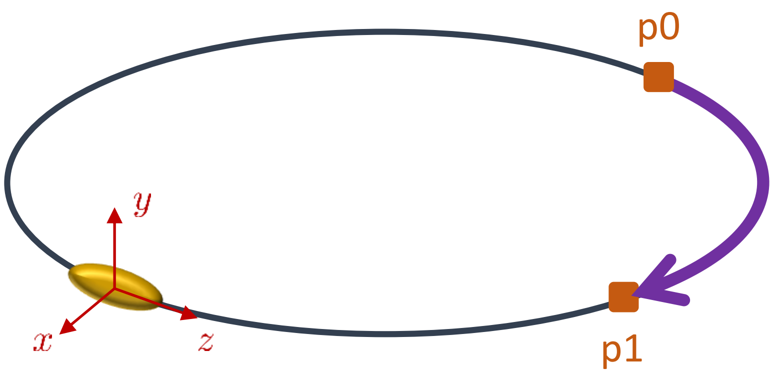
What else do we need?

- We have learned or we may know from operational experience that there are a set of crucial machine parameters to influence beam stability – among them chromaticity and amplitude detuning
- Chromaticity
  - Controlled with sextupoles – provides chromatic shift of bunch spectrum wrt. impedance
  - Changes interaction of beam with impedance
  - Damping or excitation of headtail modes
- Amplitude detuning
  - Controlled with octupoles – provides (incoherent) tune spread
  - Leads to absorption of coherent power into the incoherent spectrum → Landau damping
- Of course, to study intensity effects, these need to be included in our model – fortunately, this is pretty simple!

Chromaticity and amplitude detuning



$$\mathcal{M} = \begin{pmatrix} \sqrt{\beta_1} & 0 \\ -\frac{\alpha_1}{\sqrt{\beta_1}} & \frac{1}{\sqrt{\beta_1}} \end{pmatrix} \begin{pmatrix} \cos(\Delta\mu\ ) & \sin(\Delta\mu\ ) \\ -\sin(\Delta\mu\ ) & \cos(\Delta\mu\ ) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_0}} & 0 \\ \frac{\alpha_0}{\sqrt{\beta_0}} & \sqrt{\beta_0} \end{pmatrix}$$

$$\begin{pmatrix} x_i \\ x'_i \end{pmatrix}\Bigg|_1 = \boxed{\mathcal{M}} \begin{pmatrix} x_i \\ x'_i \end{pmatrix}\Bigg|_0$$

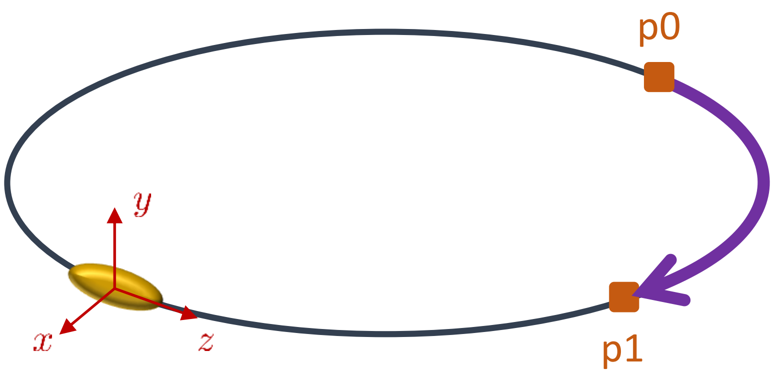
$$i = 1, \ldots, N$$

All matrix elements are constant

- The numerical implementation is then simply the matrix product applied to all macroparticles in all panes

Chromaticity and amplitude detuning

- Chromaticity or detuning with amplitude are implemented as a phase adjustment of each individual macroparticle



Chromaticity: coupling to longitudinal

Detuning with amplitude: continuous detuning

$$\Delta\mu_i \sim \Delta\mu_{0,\,i} + \boxed{\xi\,\delta_i} + \boxed{\alpha_{xx}\,J_{x,\,i} + \alpha_{xy}\,J_{y,\,i}}$$

$$\mathcal{M}_i = \begin{pmatrix} \sqrt{\beta_1} & 0 \\ -\frac{\alpha_1}{\sqrt{\beta_1}} & \frac{1}{\sqrt{\beta_1}} \end{pmatrix} \begin{pmatrix} \cos(\Delta\mu_i) & \sin(\Delta\mu_i) \\ -\sin(\Delta\mu_i) & \cos(\Delta\mu_i) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_0}} & 0 \\ \frac{\alpha_0}{\sqrt{\beta_0}} & \sqrt{\beta_0} \end{pmatrix}$$

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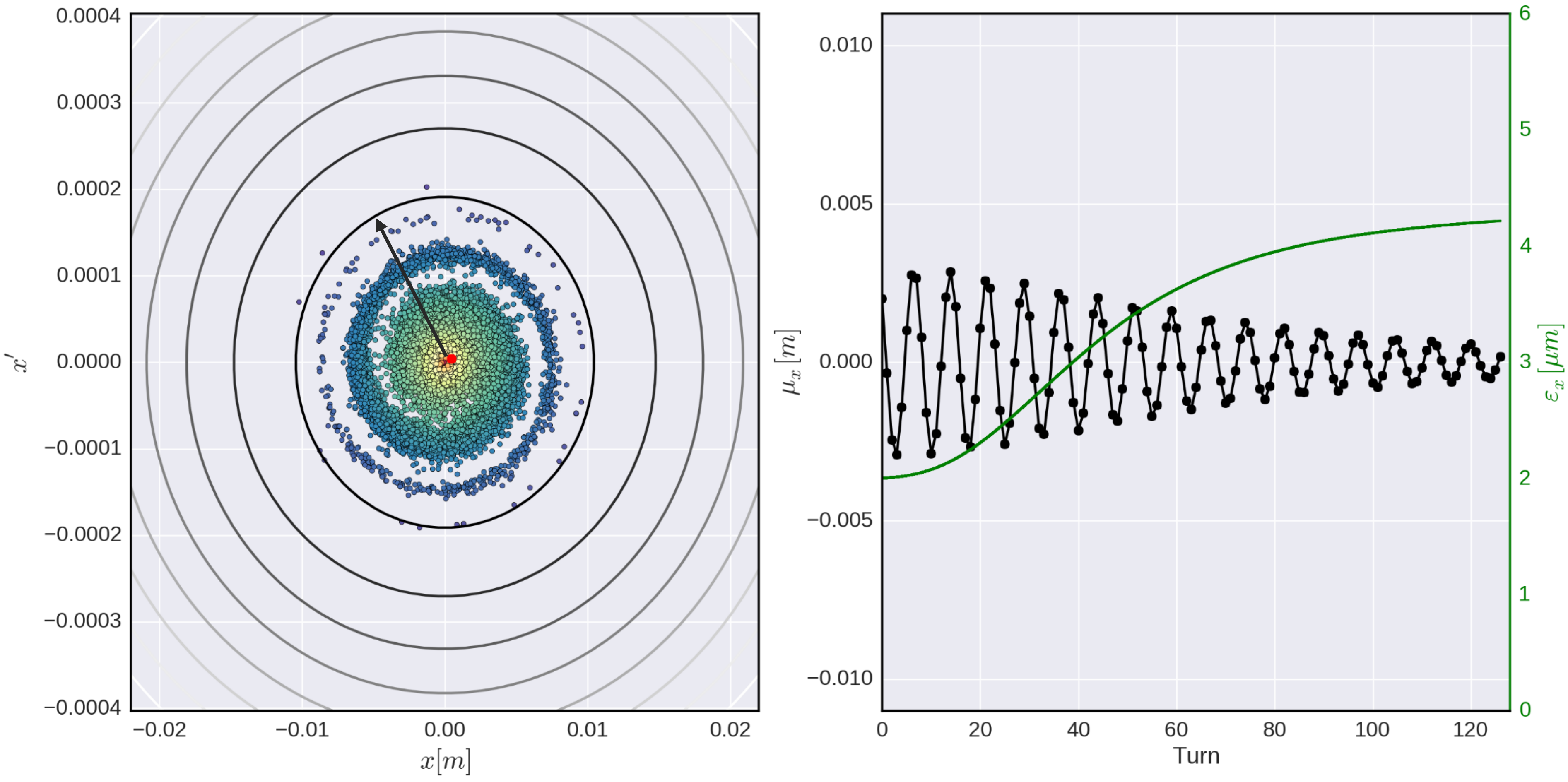
- The numerical implementation is then simply the matrix product applied to all macroparticles in all panes

All matrix elements are macroparticle dependent

Example: filamentation as result of detuning



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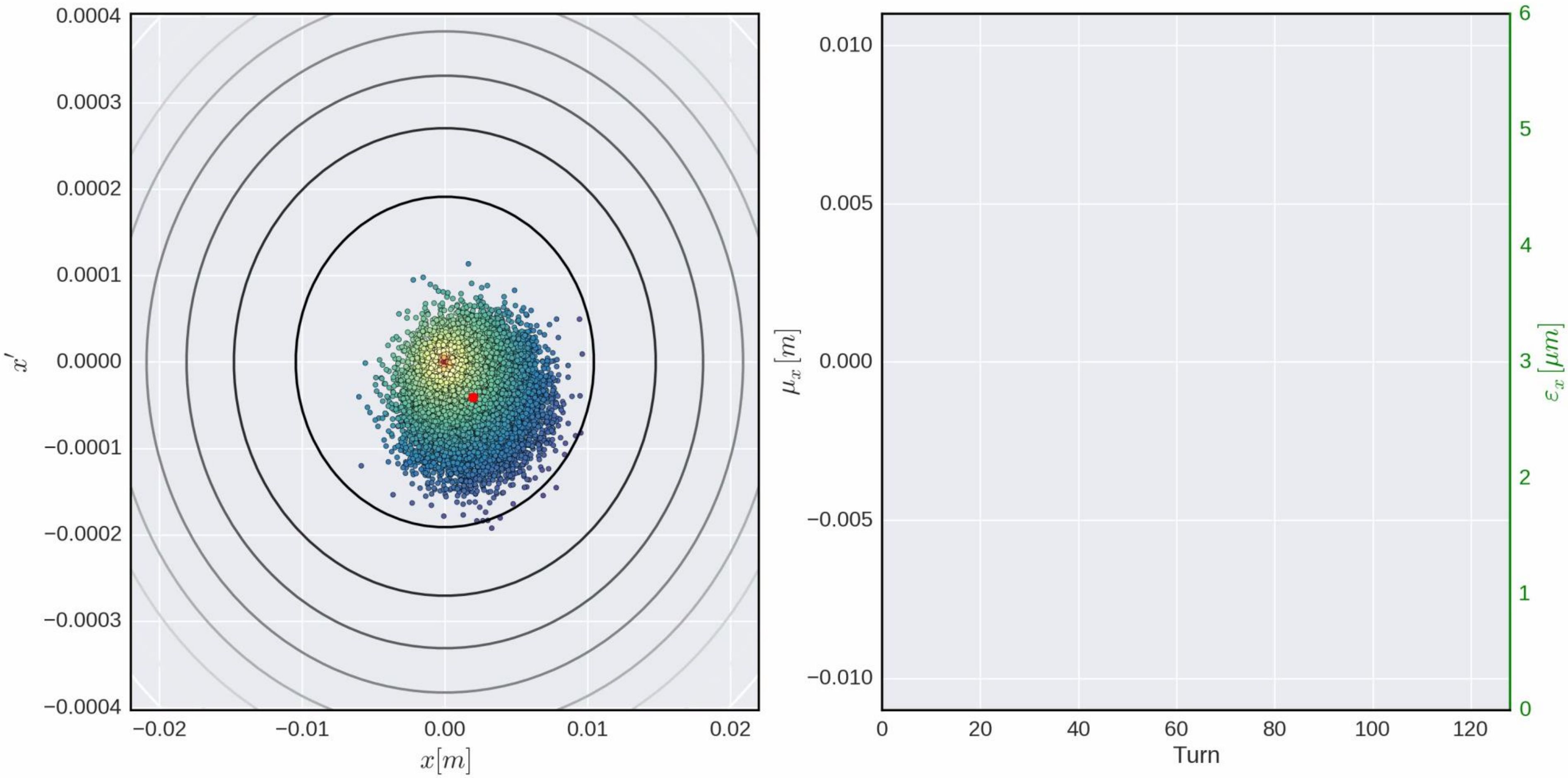

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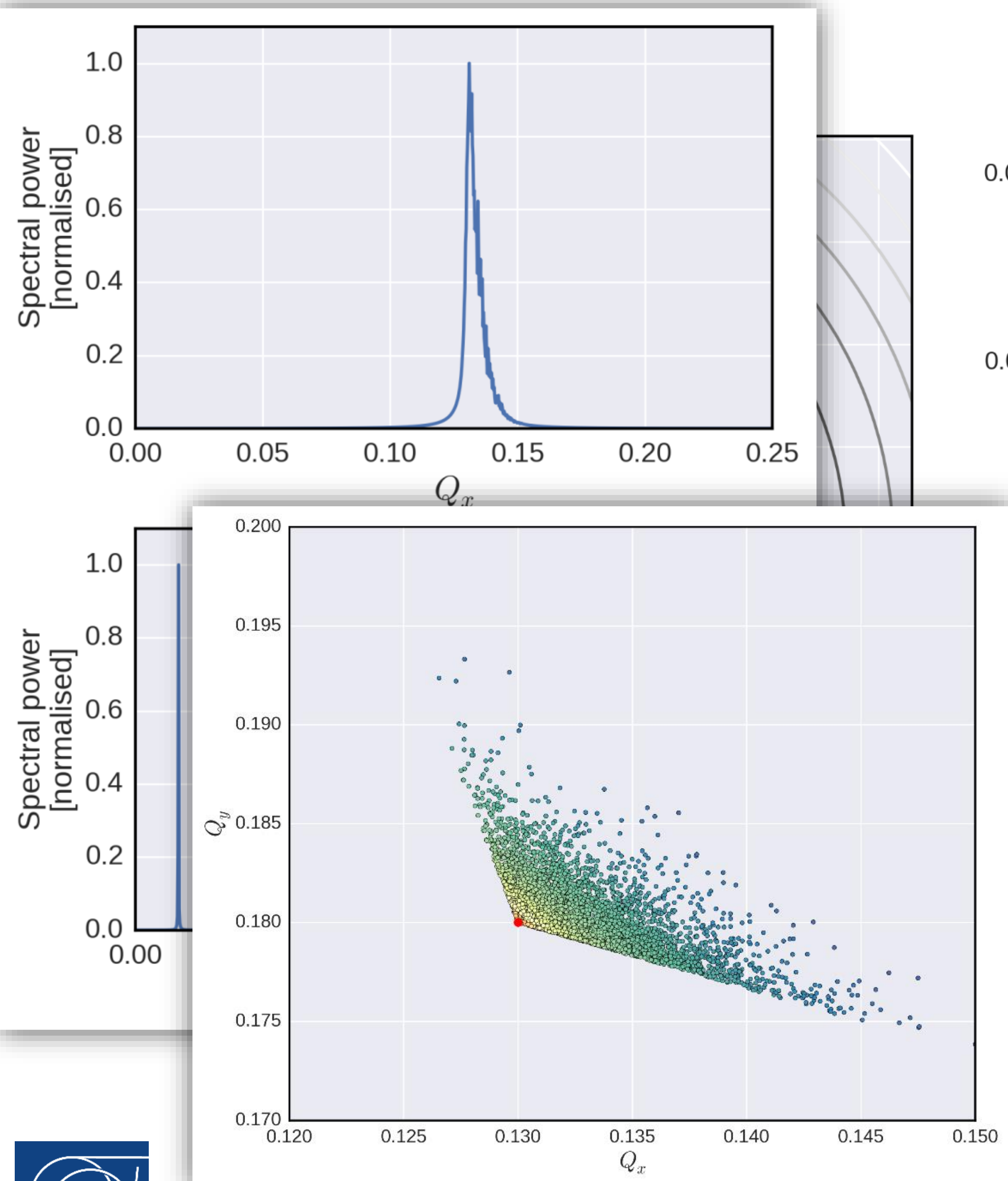

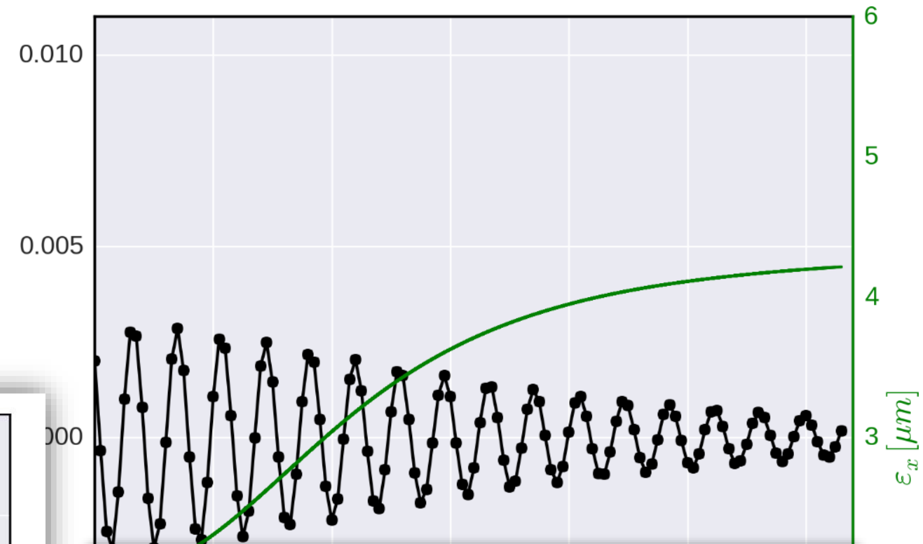

- Taking an FFT of the centroid motion (black curve) reveals the tune – interestingly there is a spread
- In the simulation we have access to the trajectory of each individual macropartice – we can equally perform an FFT of every macroparticle and plot the horizontal vs. vertical tune to obtain the tune footprint

Signpost

- We have added chromaticity and detuning with amplitude to our transverse tracking.
- We now have all the necessary single-particle dynamics implemented to serve as platform onto which we now will add collective effects interactions.

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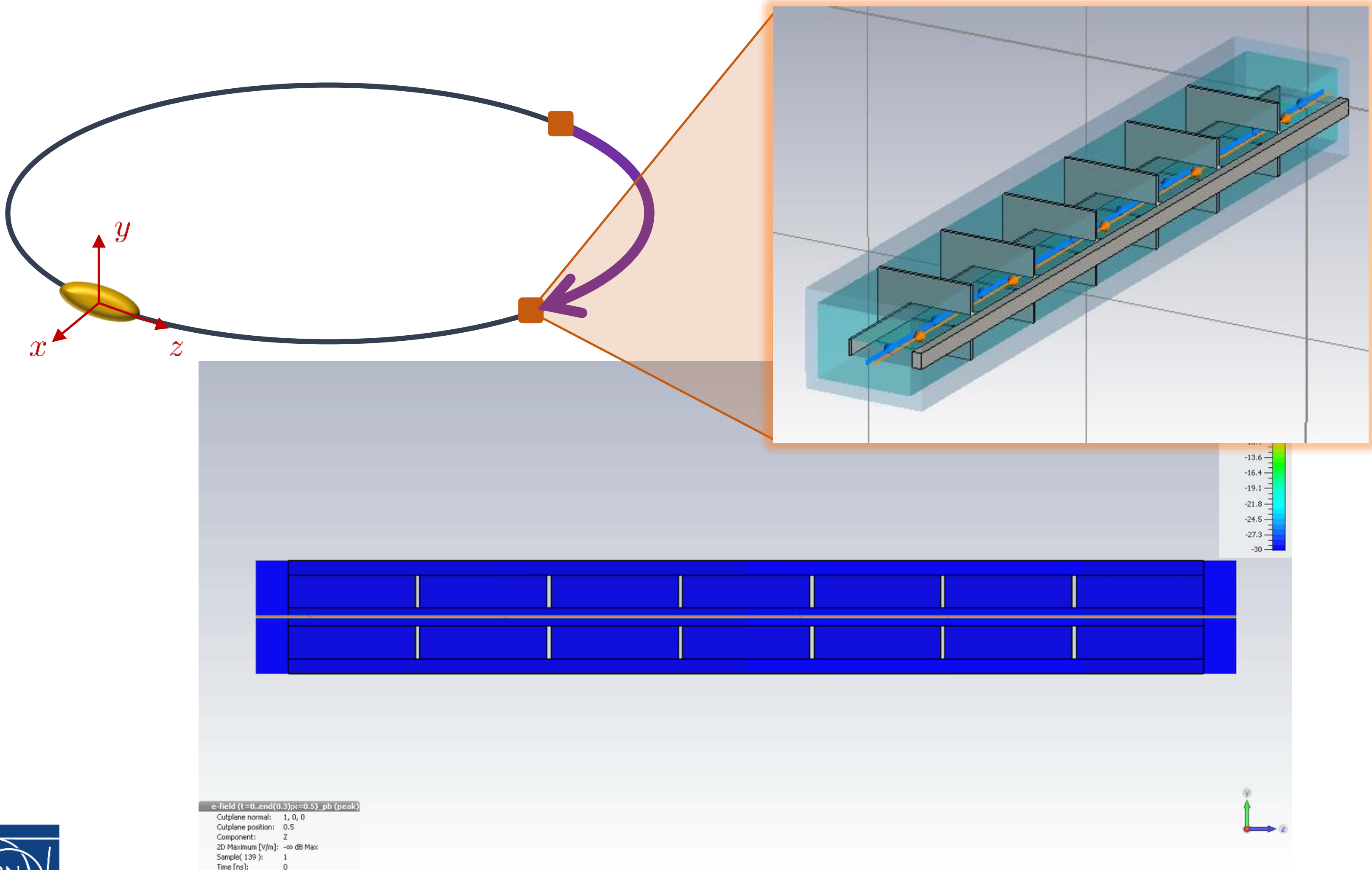
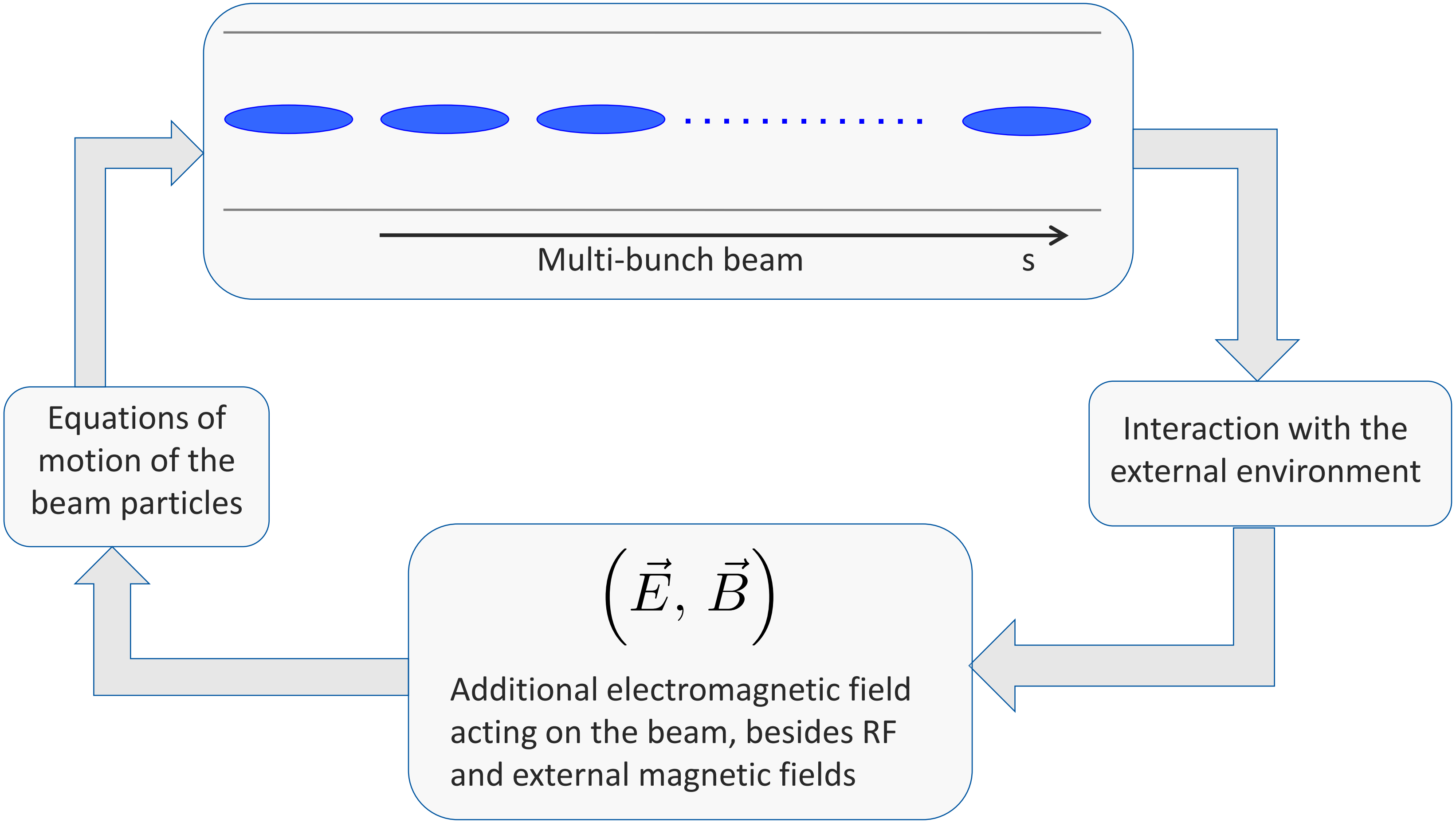
Accelerator beam system - wakefields

- Our first 'real' collective interaction from impedances



Introduction to the general problem

The CERN Accelerator School



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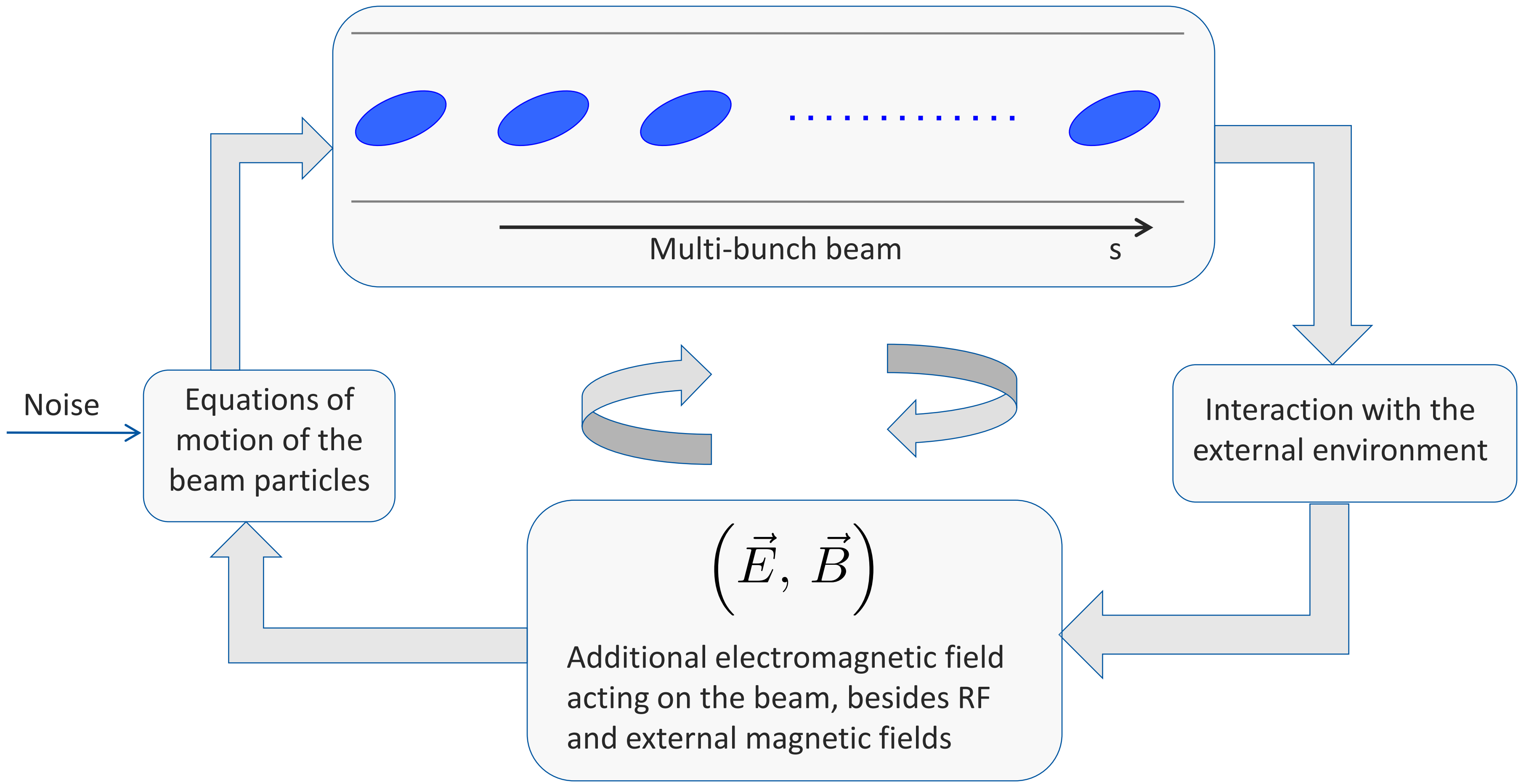
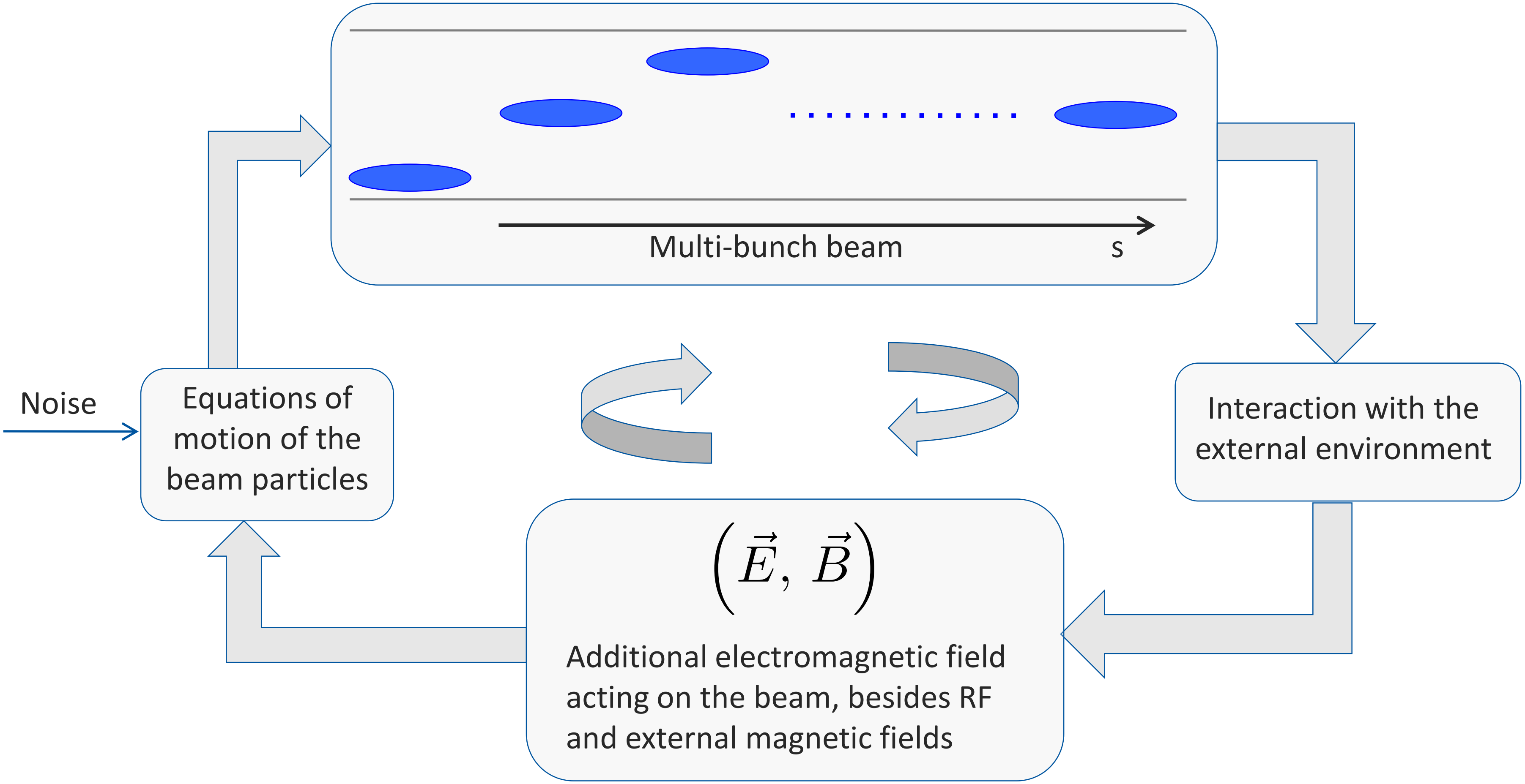


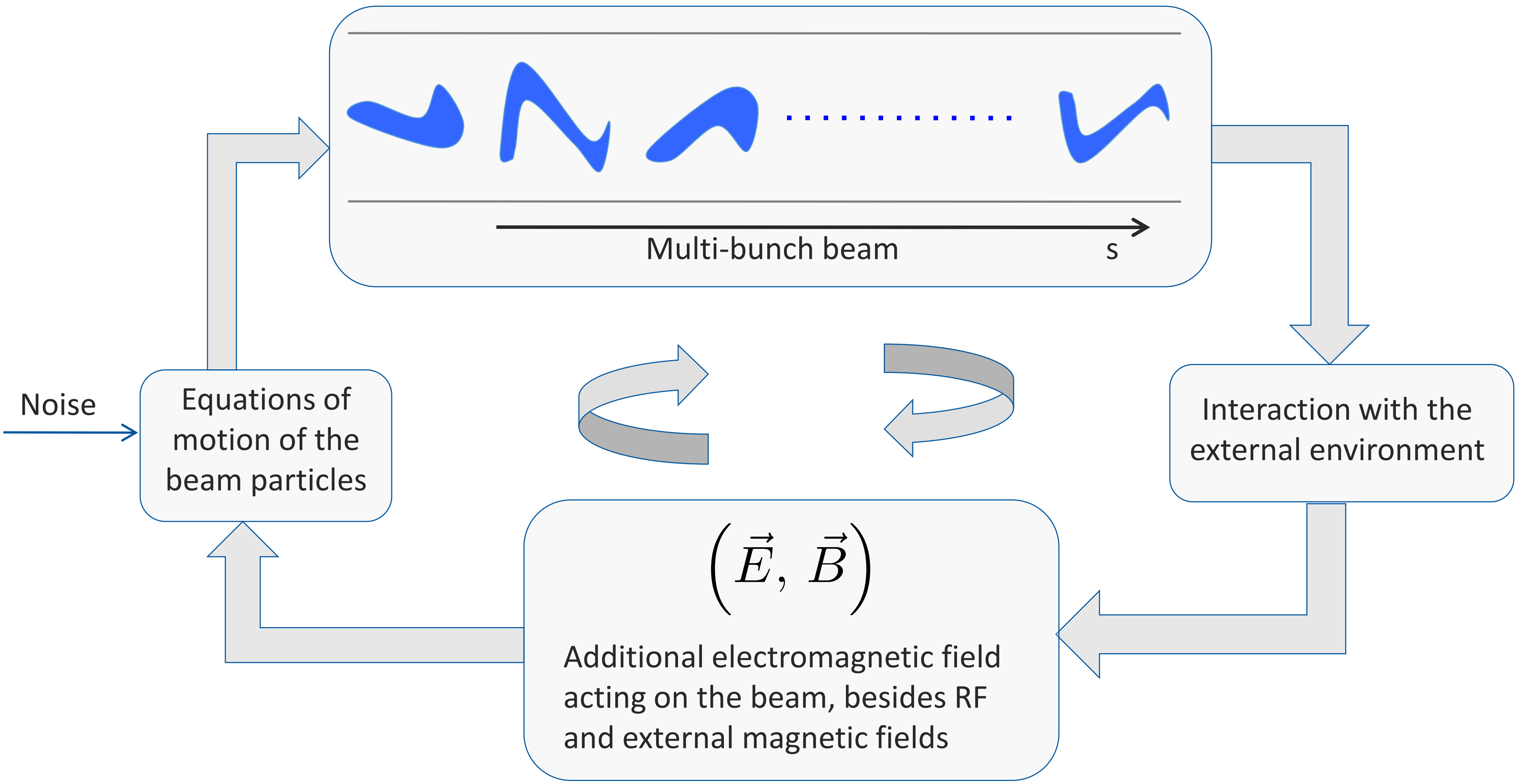
When the loop closes, either the beam will find a new stable equilibrium configuration …

Introduction to the general problem



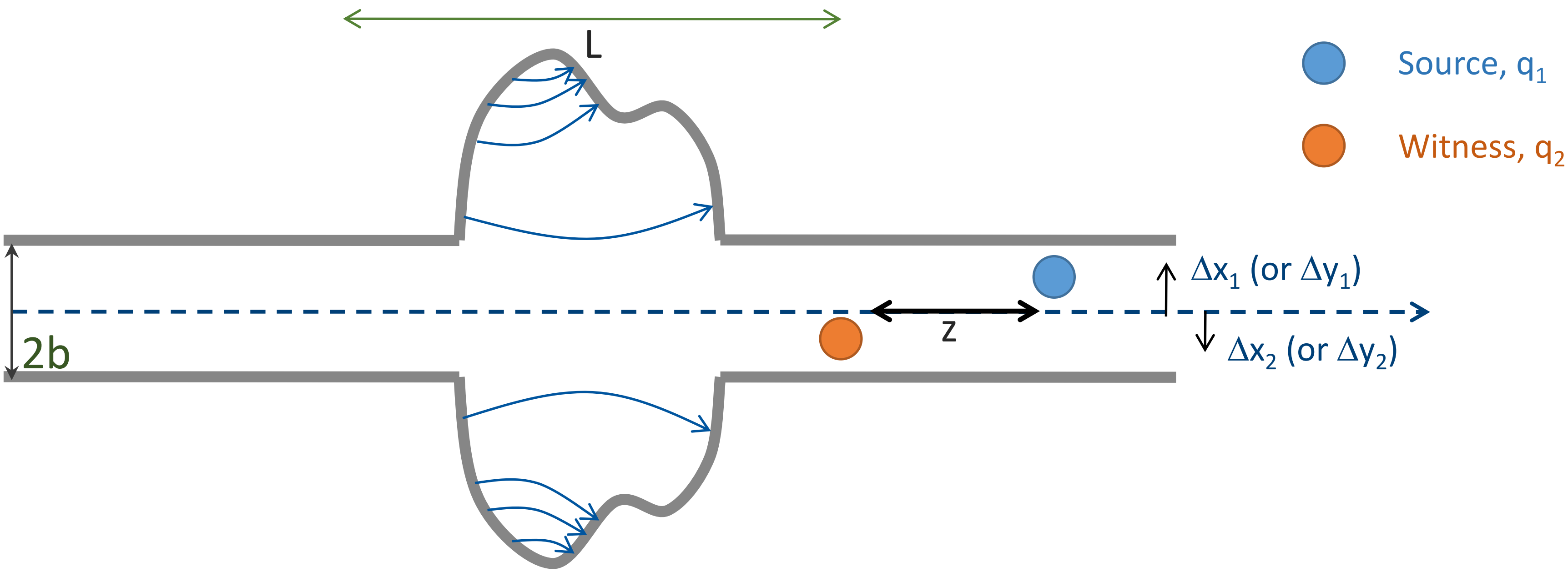
… or it might develop an instability along the bunch train …

Introduction to the general problem



… or also an instability affecting different bunches independently of each other

Reminder of wake functions



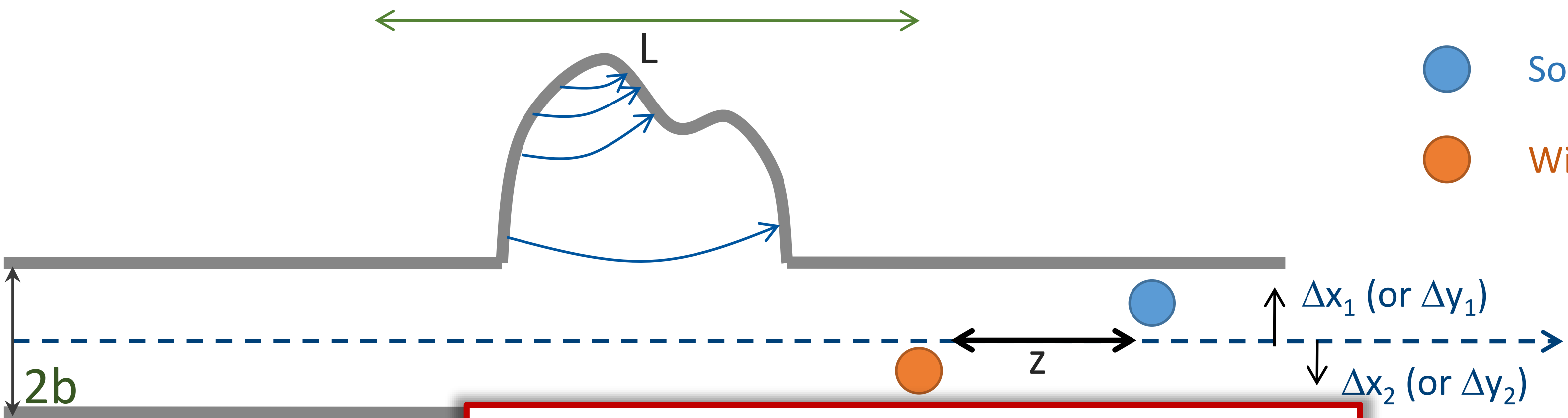
- Longitudinal wake fields

$$\int F_s(z,s)\,ds = -q_1 q_2 \left(W_{||}(z) + O(\Delta x_1) + O(\Delta x_2)\right)$$

Zeroth order with source and test centred usually dominant

Higher order terms
Usually negligible for small offsets

Reminder of wake functions



We have truncated to the first order, thus neglecting

⇒ First order coupling terms between x and y planes

⇒ All higher order terms in the wake expansion (including mixed higher order terms with products of the dipolar/quadrupolar offsets)

- Transverse wake fi

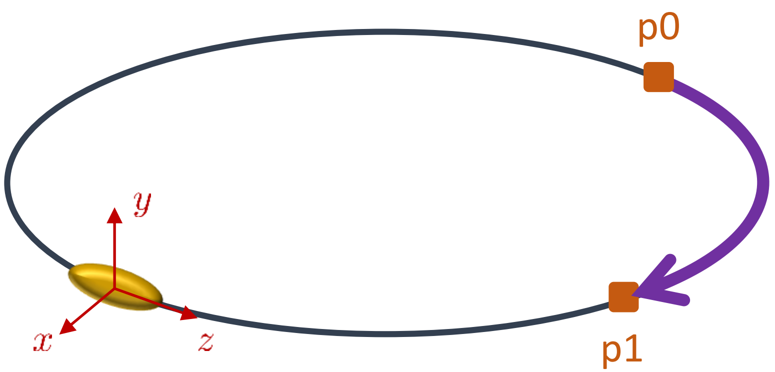
$$\int F_x(z,s)\,ds = \quad \ldots \quad z)\,\Delta x_2)$$

Zeroth orde... for asymmetric structures → Orbit offset

... depends on source particle → Orbit offset

...pole wakes – depends on witness particle → Detuning

Numerical implementation of wakefields



- We have learned how to track a macroparticle beam through a linear periodic lattice and how to include chromaticity and detuning with amplitude

Chromaticity: coupling to longitudinal

Detuning with amplitude: continuous detuning

$$\Delta\mu_i \sim \Delta\mu_{0,\,i} + \boxed{\xi\,\delta_i} + \boxed{\alpha_{xx}\,J_{x,\,i} + \alpha_{xy}\,J_{y,\,i}}$$

$$\mathcal{M}_i = \begin{pmatrix} \sqrt{\beta_1} & 0 \\ -\frac{\alpha_1}{\sqrt{\beta_1}} & \frac{1}{\sqrt{\beta_1}} \end{pmatrix} \begin{pmatrix} \cos(\Delta\mu_i) & \sin(\Delta\mu_i) \\ -\sin(\Delta\mu_i) & \cos(\Delta\mu_i) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_0}} & 0 \\ \frac{\alpha_0}{\sqrt{\beta_0}} & \sqrt{\beta_0} \end{pmatrix}$$

$$\begin{pmatrix} x_i \\ x_i' \end{pmatrix}\Bigg|_1 = \mathcal{M}_i \begin{pmatrix} x_i \\ x_i' \end{pmatrix}\Bigg|_0$$

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Numerical implementation of wakefields

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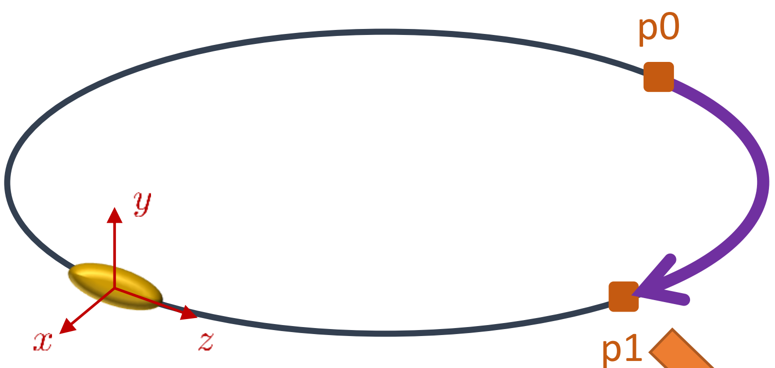

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Now, we want to add a wake field interaction at p1 that applies a kick, i.e. updates the momenta of each macroparticle

Numerical implementation of wakefields

The CERN Accelerator School

- To be numerically more efficient, the beam is longitudinally sliced into a set of slices
- Provided the slices are thin enough to sample the wake fields, the wakes can be assumed constant within a single slice
- The kick on to the set of macroparticles in slice 'i' generated by the set of macroparticles in slice 'j' via the wake fields now becomes:

$$\Delta x'[i] = -\frac{e^2}{m\gamma\beta^2c^2} \times \sum_{j=0}^{\text{n\_slices}} \begin{cases} N[j] \cdot W_{Cx}[i-j] \\ N[j]\langle x\rangle[j] \cdot W_{Dx}[i-j] \\ N[j] \cdot W_{Qx}[i-j]\,\Delta x[i] \end{cases}$$

- The wake functions are obtained externally from electromagnetic codes such as ACE3P, CST, GdfidL, HFSS…
- In the tracking code, the wake fields at p1 need to update the macroparticle momenta (i.e. they provide a kick)
- The kick on to a macroparticle 'i' generated by all macroparticles 'j' via the wake fields is:

$$\Delta x'_i = -\frac{e^2}{m\gamma\beta^2c^2} \times \sum_{j=0}^{\text{n\_macroparticles}} \begin{cases} W_{Cx}(z_i - z_j) \\ \Delta x_j \cdot W_{Dx}(z_i - z_j) \\ W_{Qx}(z_i - z_j)\,\Delta x_i \end{cases}$$

Numerical implementation of wakefields

The CERN Accelerator School

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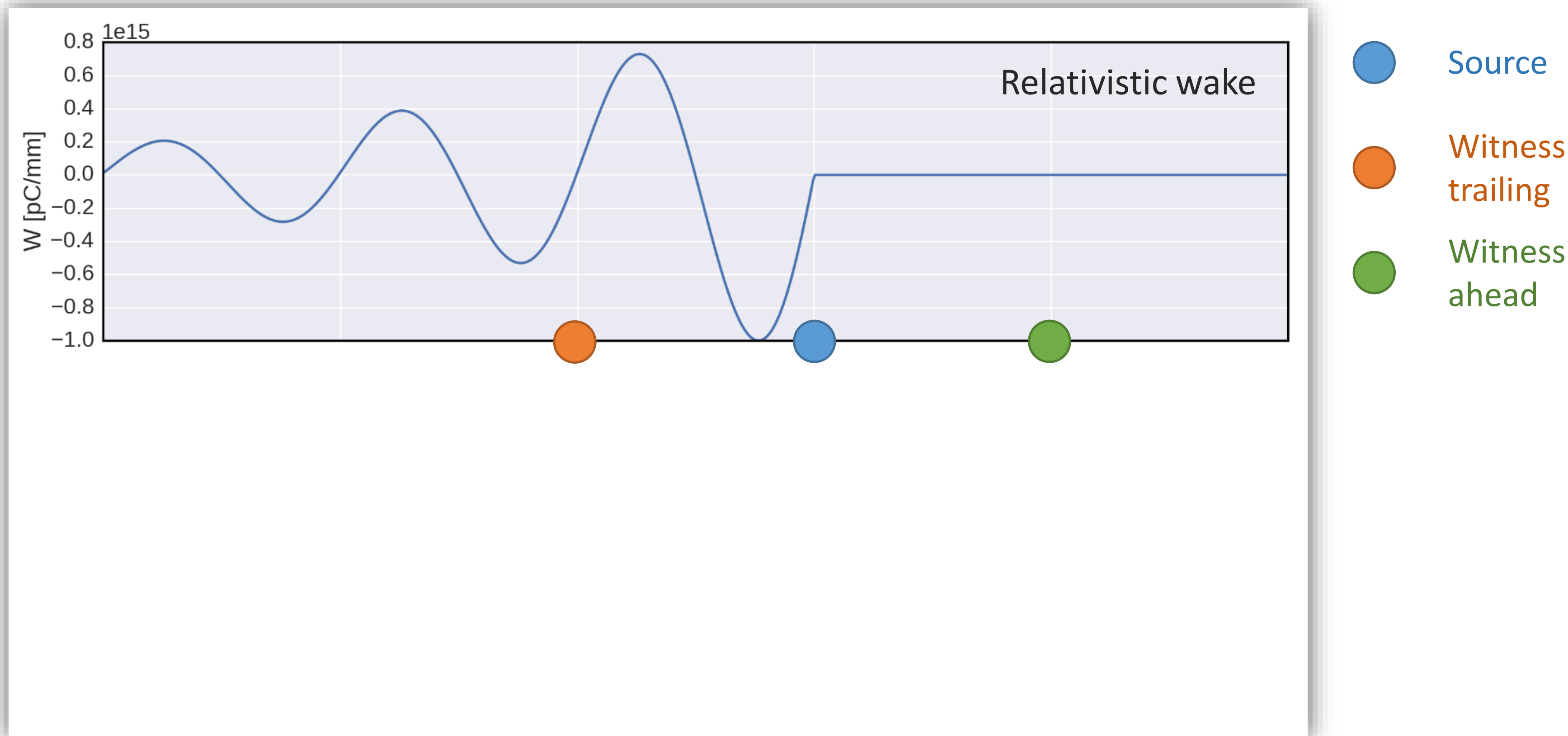
- N[i]: number of macroparticles in slice 'i' → can be pre-computed and stored in memory
- W[i]: wake function pre-computed and stored in memory for all differences i-j

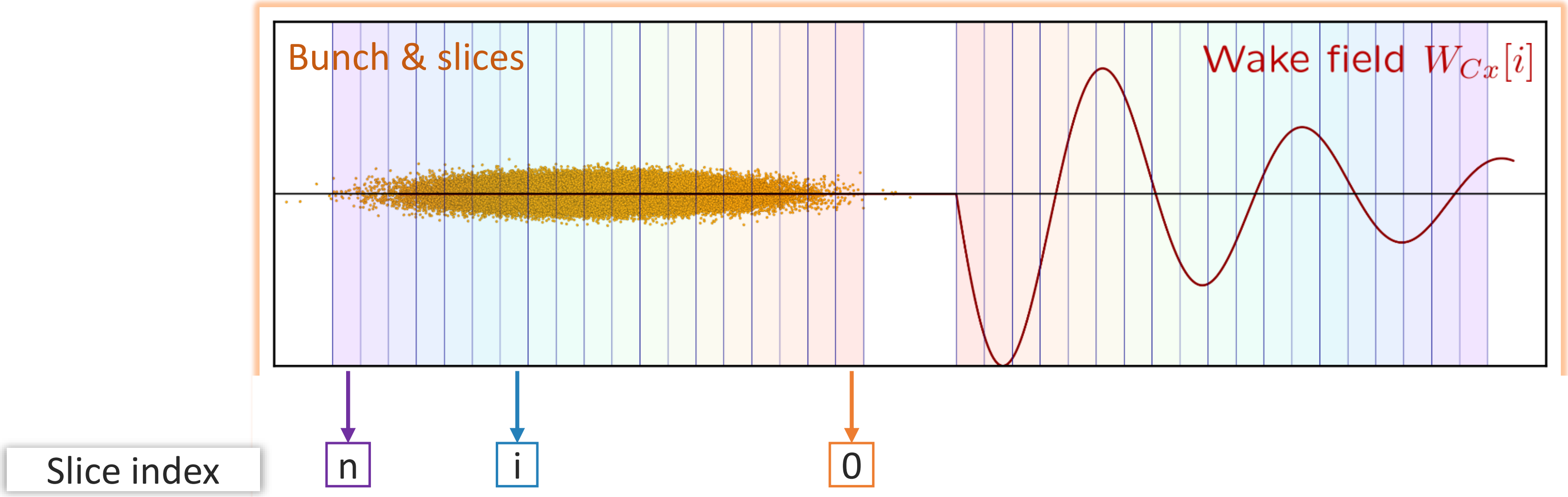
Count	0	1	2	3	4	5	6
N[i]	…	…	…	…	…	…	…
W[i]	…	…	…	…	…	…	…

Relativistic vs. non-relativistic wakes

- Relativistic wakes only affect trailing particles following the source particle
- Finite values range for negative distances, i.e. (-L, 0) or "tail – head"
  - L: bunch length
- Nonrelativistic wakes can also affect particles ahead of the source particle
- Finite values extend from (-L, L) or "tail –head" & "head – tail"
  - L: bunch length

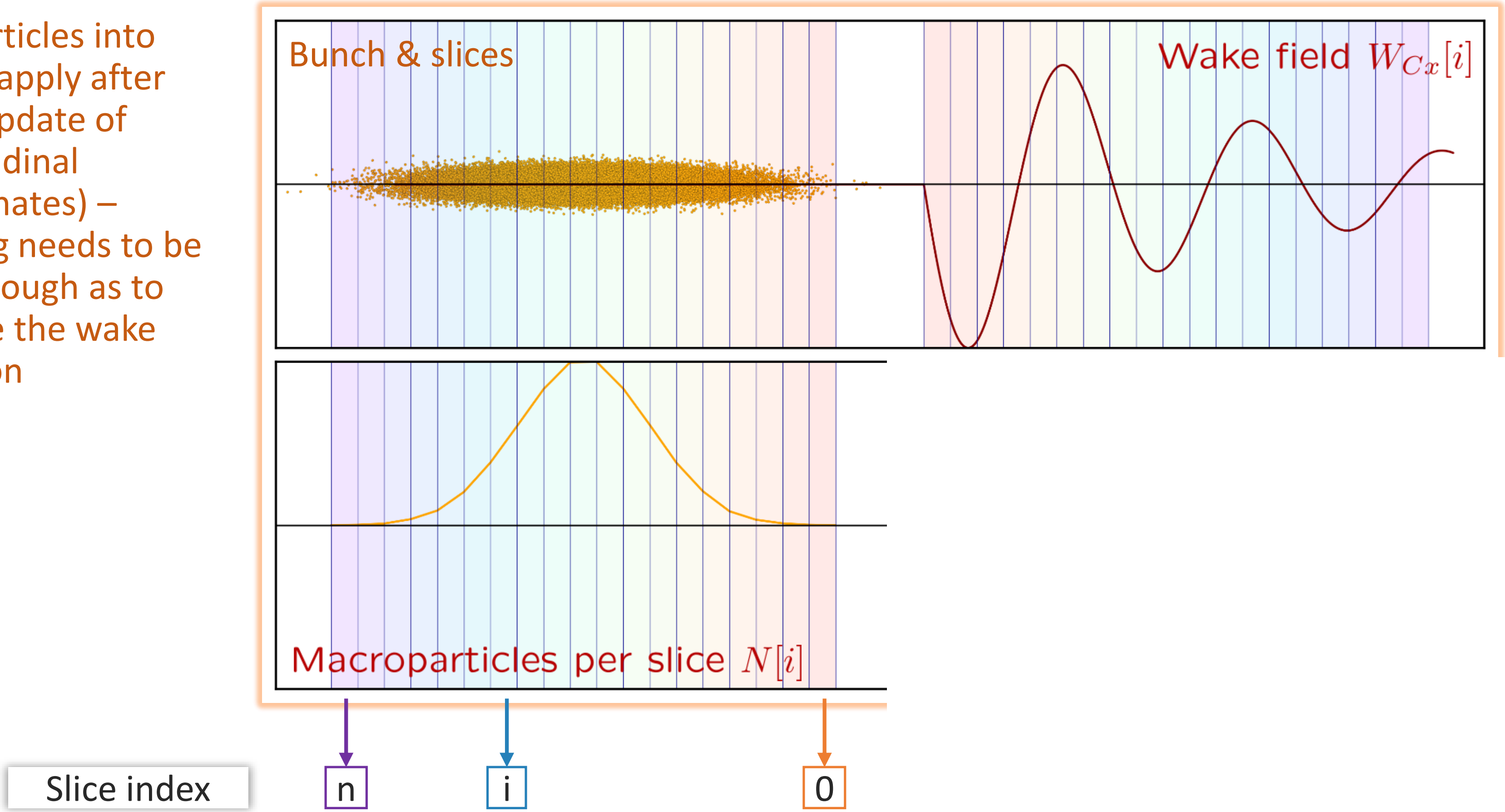


Numerical implementation of wakefields



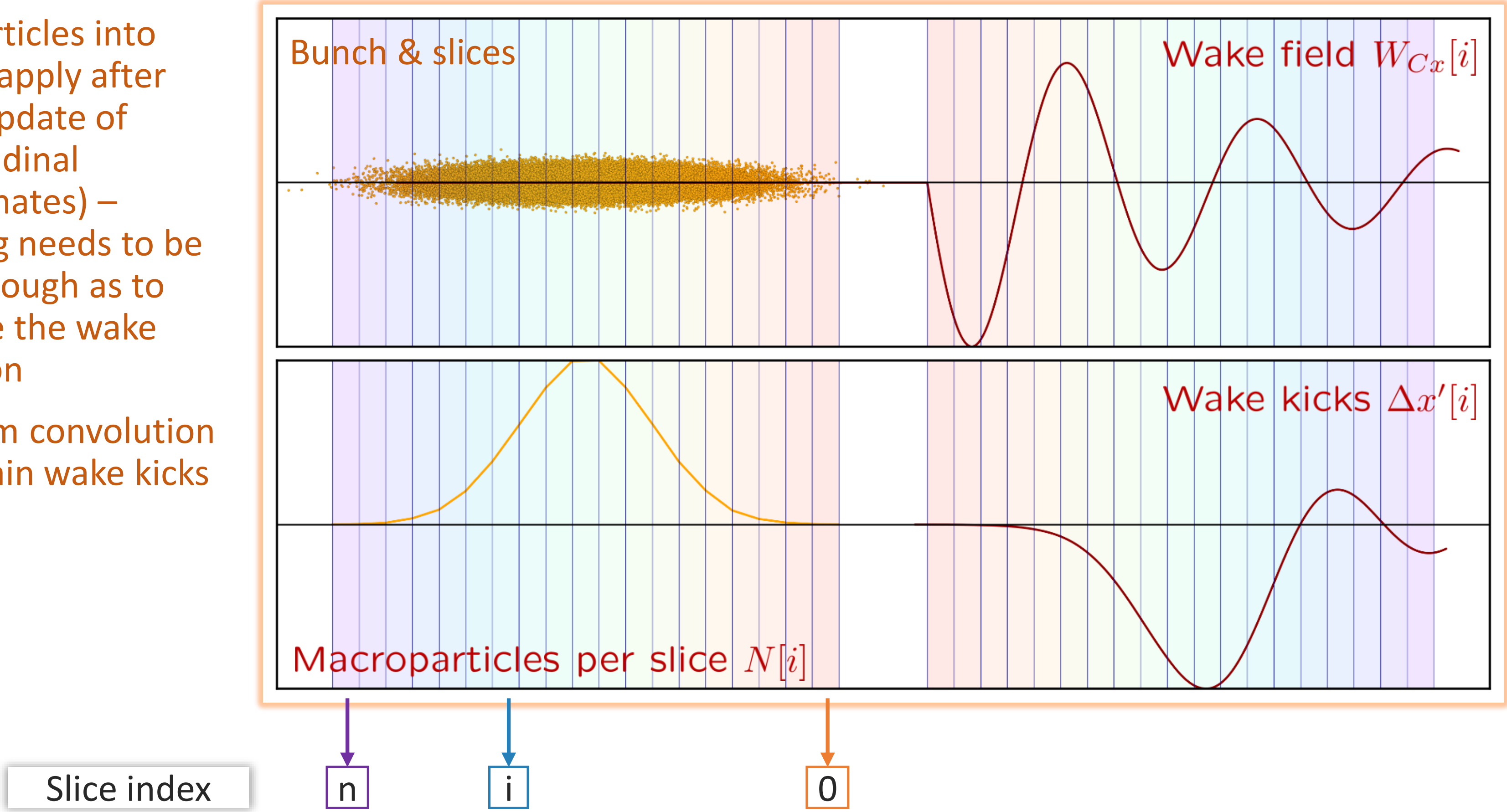
Numerical implementation of wakefields

1. Bin particles into slices (apply after each update of longitudinal coordinates) – binning needs to be fine enough as to sample the wake function



Numerical implementation of wakefields

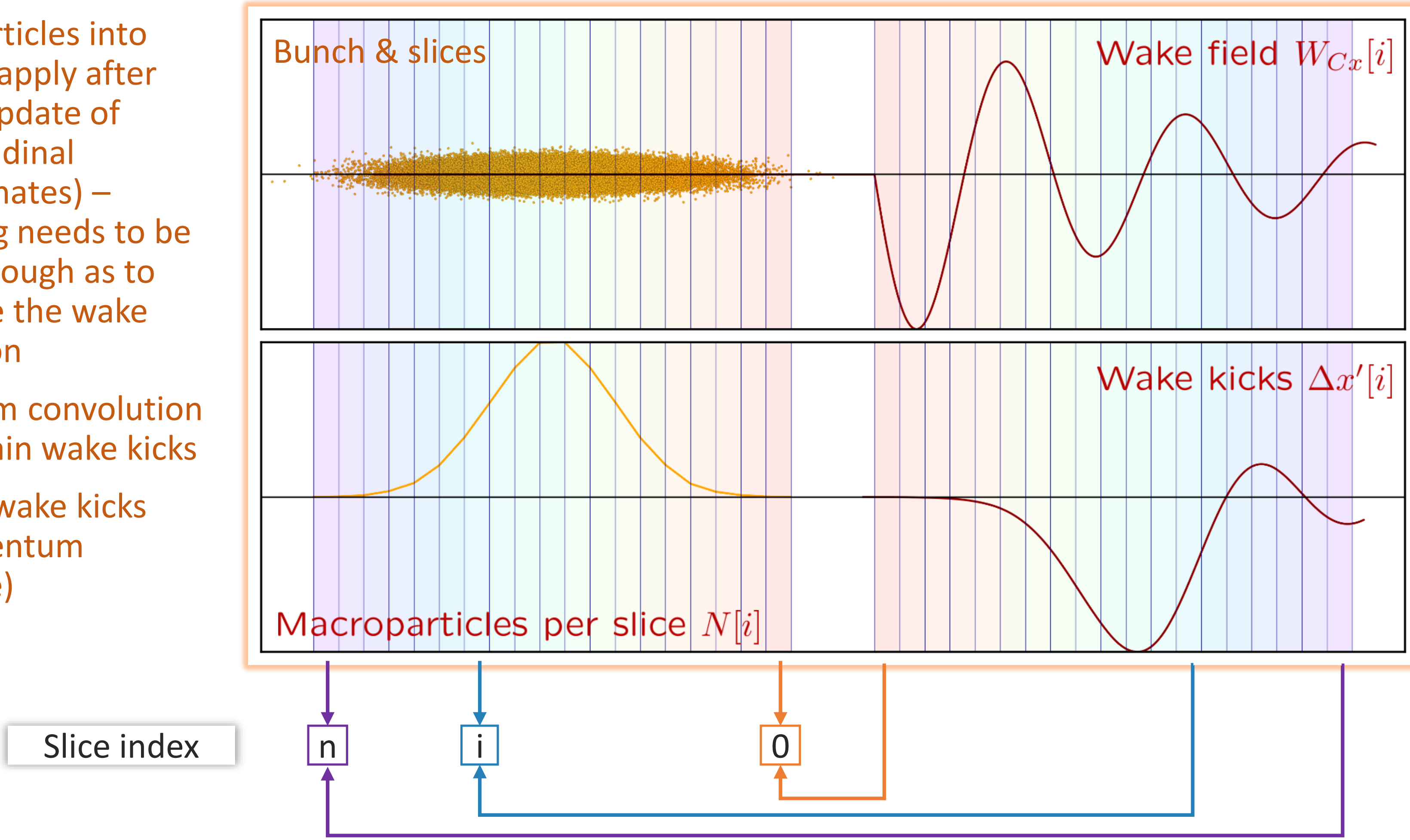
1. Bin particles into slices (apply after each update of longitudinal coordinates) – binning needs to be fine enough as to sample the wake function
2. Perform convolution to obtain wake kicks



$$\Delta x'[i] = -\frac{e^2}{m\gamma\beta^2 c^2} \sum_{j=0}^{i} N[j] \cdot W_{Cx}[i-j]$$

Numerical implementation of wakefields

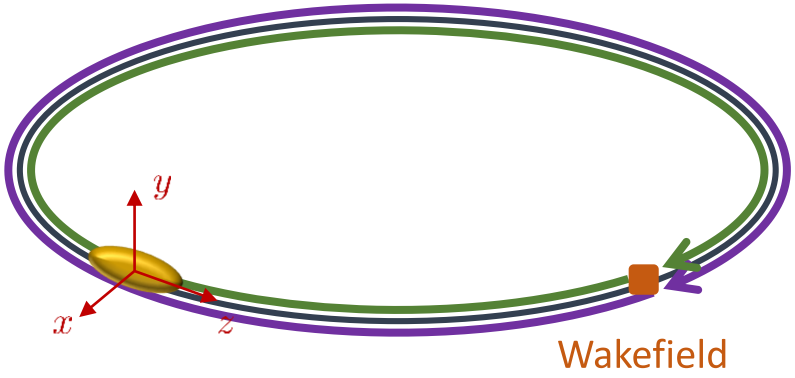
1. Bin particles into slices (apply after each update of longitudinal coordinates) – binning needs to be fine enough as to sample the wake function
2. Perform convolution to obtain wake kicks
3. Apply wake kicks (momentum update)



$$\Delta x'[i] = -\frac{e^2}{m\gamma\beta^2c^2}\sum_{j=0}^{i} N[j]\cdot W_{Cx}[i-j]\,, \qquad x'[i] \to x'[i] + \Delta x'[i]\,, \qquad i = 1,\ldots,\text{n\_slices}$$

Summary – where are we?

- We are now ready to track a full turn including the interaction with wake fields



$$\left.\begin{pmatrix} x_i \\ x_i' \end{pmatrix}\right|_{k+1} = \mathcal{M}_i \left.\begin{pmatrix} x_i \\ x_i' \end{pmatrix}\right|_{k}$$

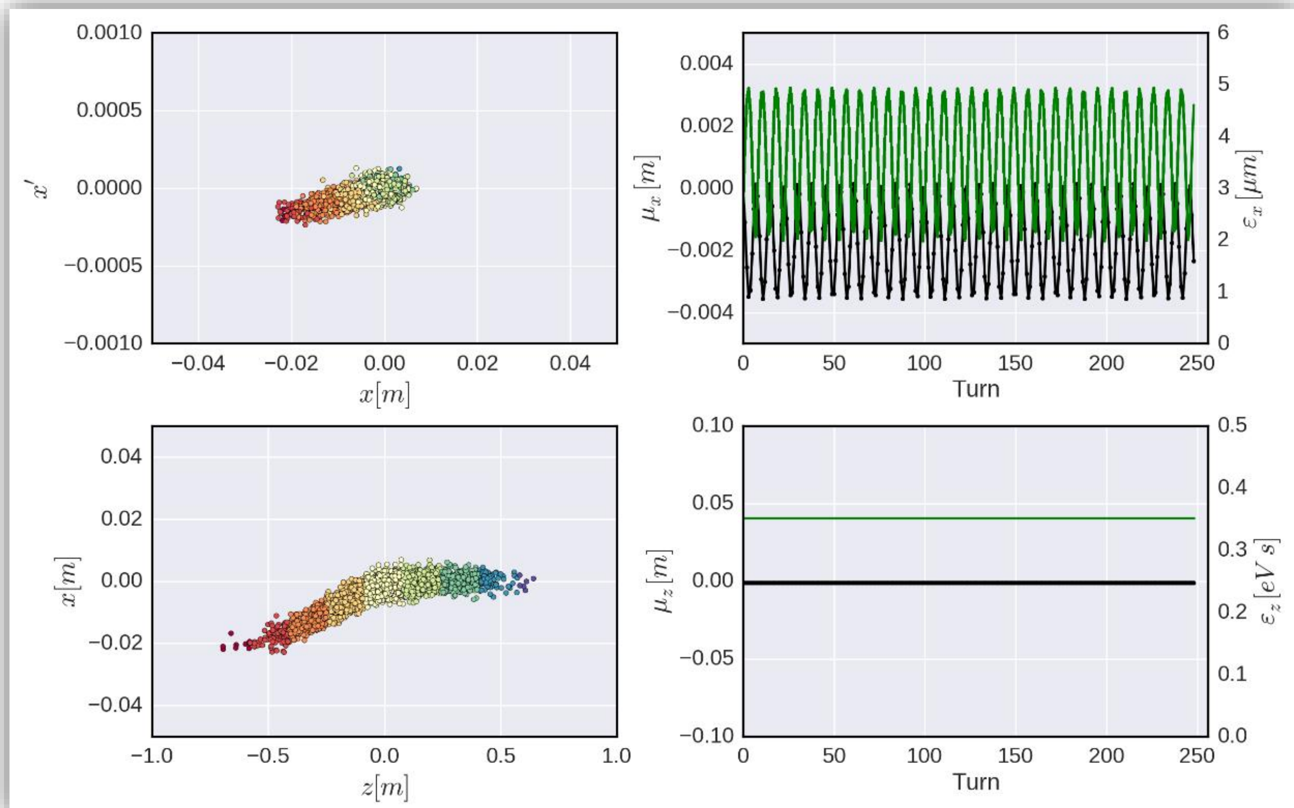
1. Initialise a macroparticle distribution with a given emittance
2. Update transverse coordinates and momenta according to the linear periodic transfer map – adjust the individual phase advance according to chromaticity and detuning with amplitude
3. Update the longitudinal coordinates and momenta according to the leap-frog integration scheme
4. Update momenta only (apply kicks) according to wake field generated kicks
5. Repeat turn-by-turn…

Examples – constant wakes

$$\Delta x'[i] = -\frac{e^2}{m\gamma\beta^2c^2}\sum_{j=0}^{i} N[j]\cdot W_{Cx}[i-j]$$

Dipolar term → orbit kick

Slice dependent change of closed orbit (if line density does not change)



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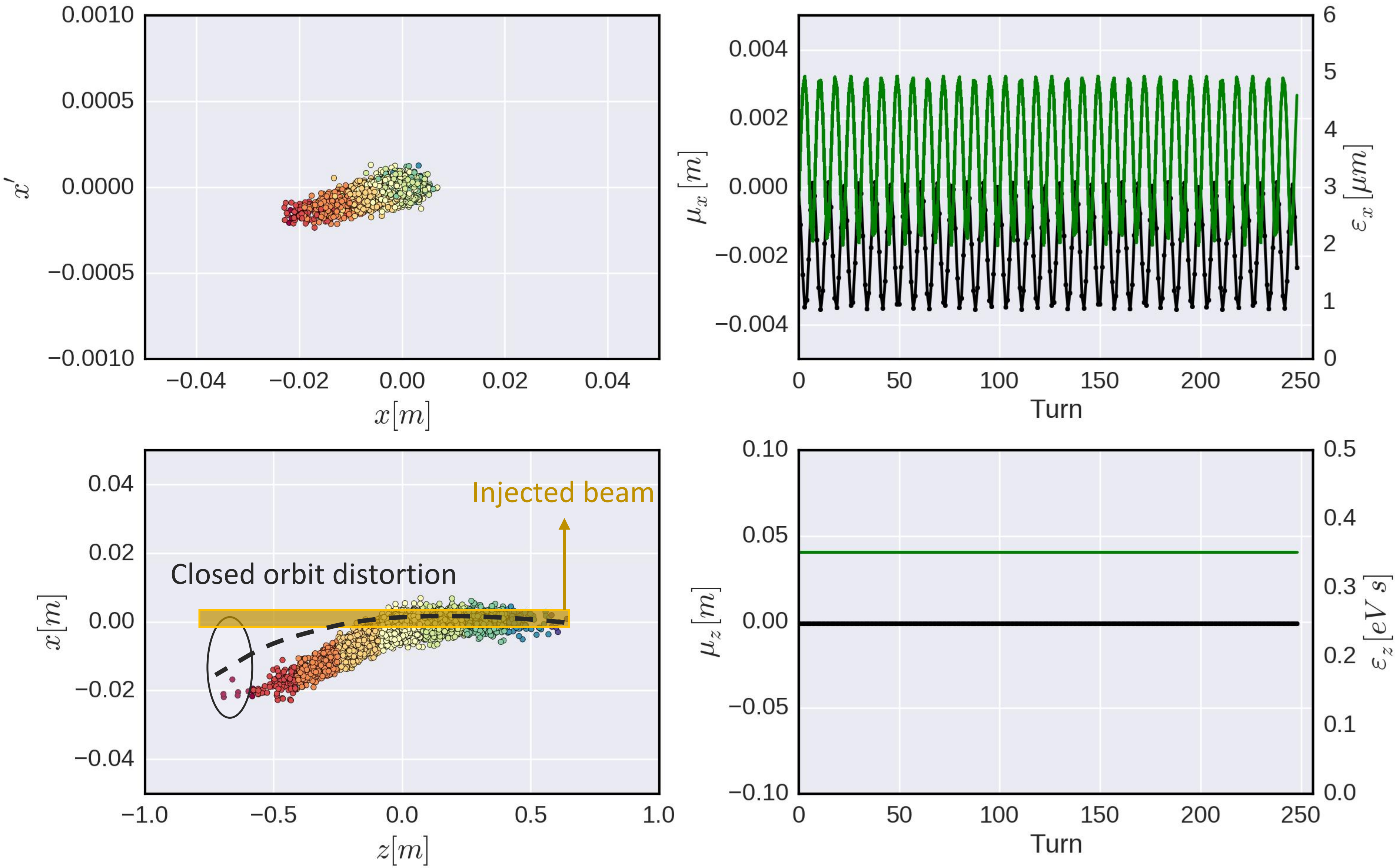

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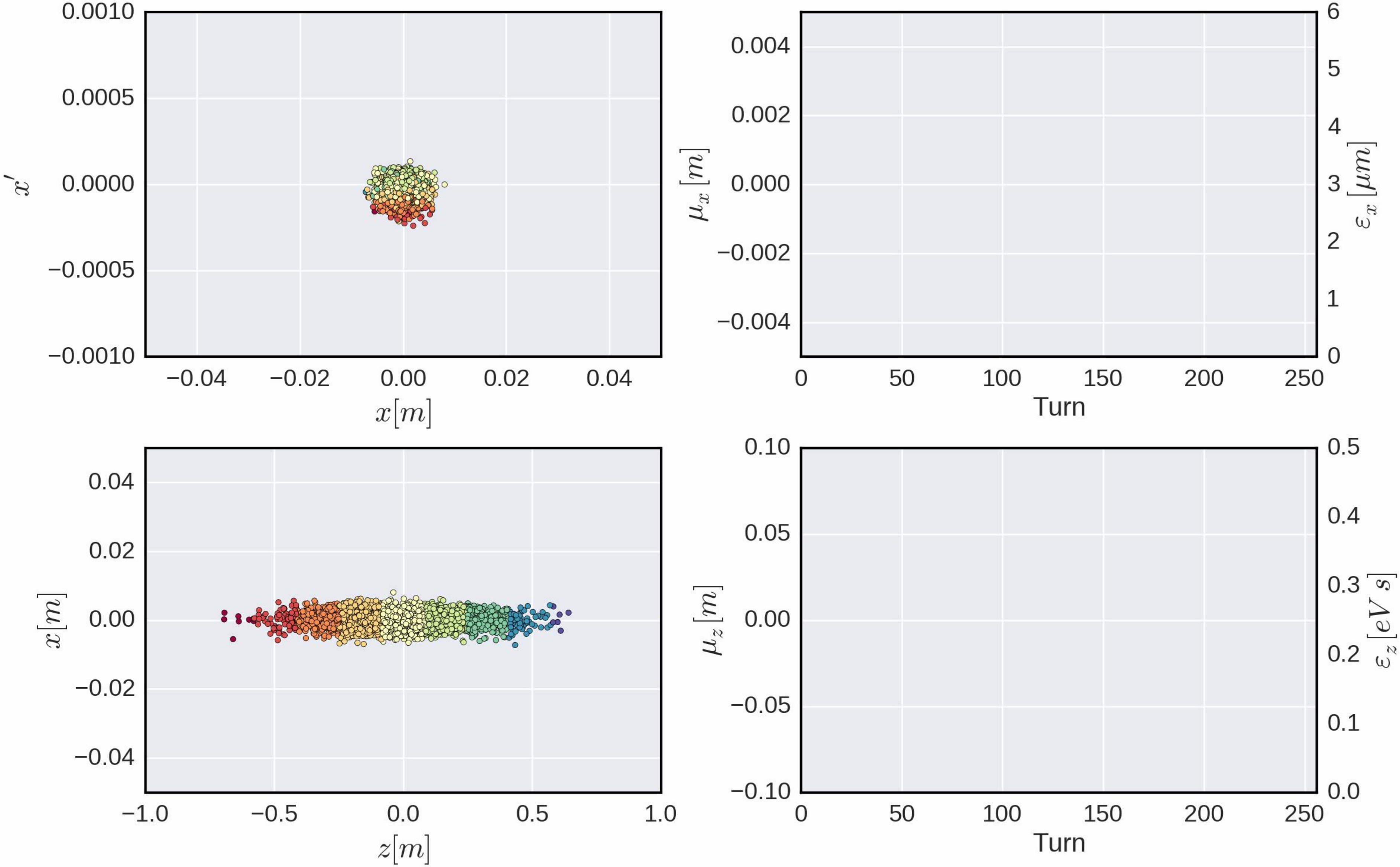

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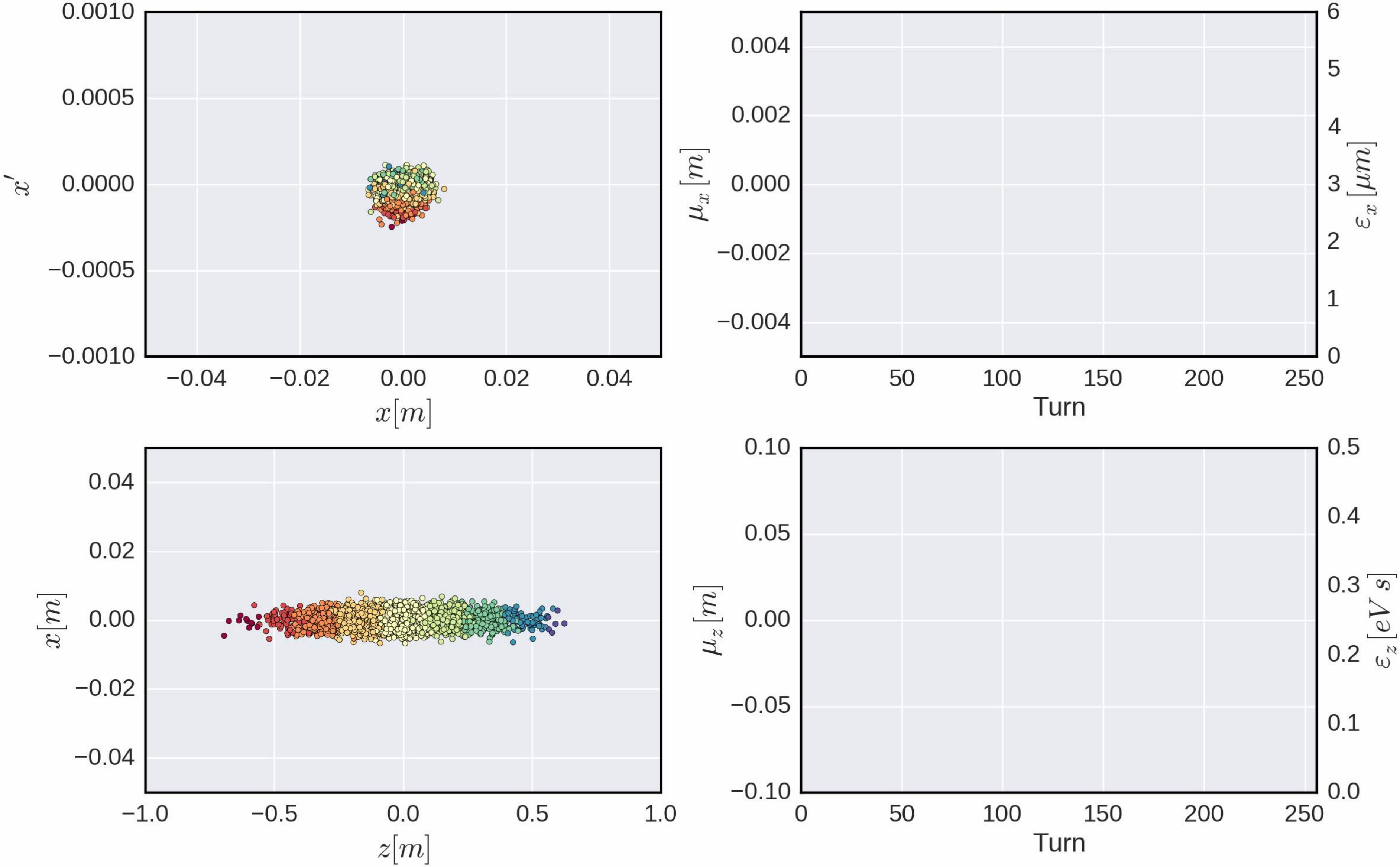

Examples – dipole wakes

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Offset dependent orbit kick
→ kicks can accumulate

- Without synchrotron motion:
  kicks accumulate turn after turn – the beam is unstable → beam break-up in linacs

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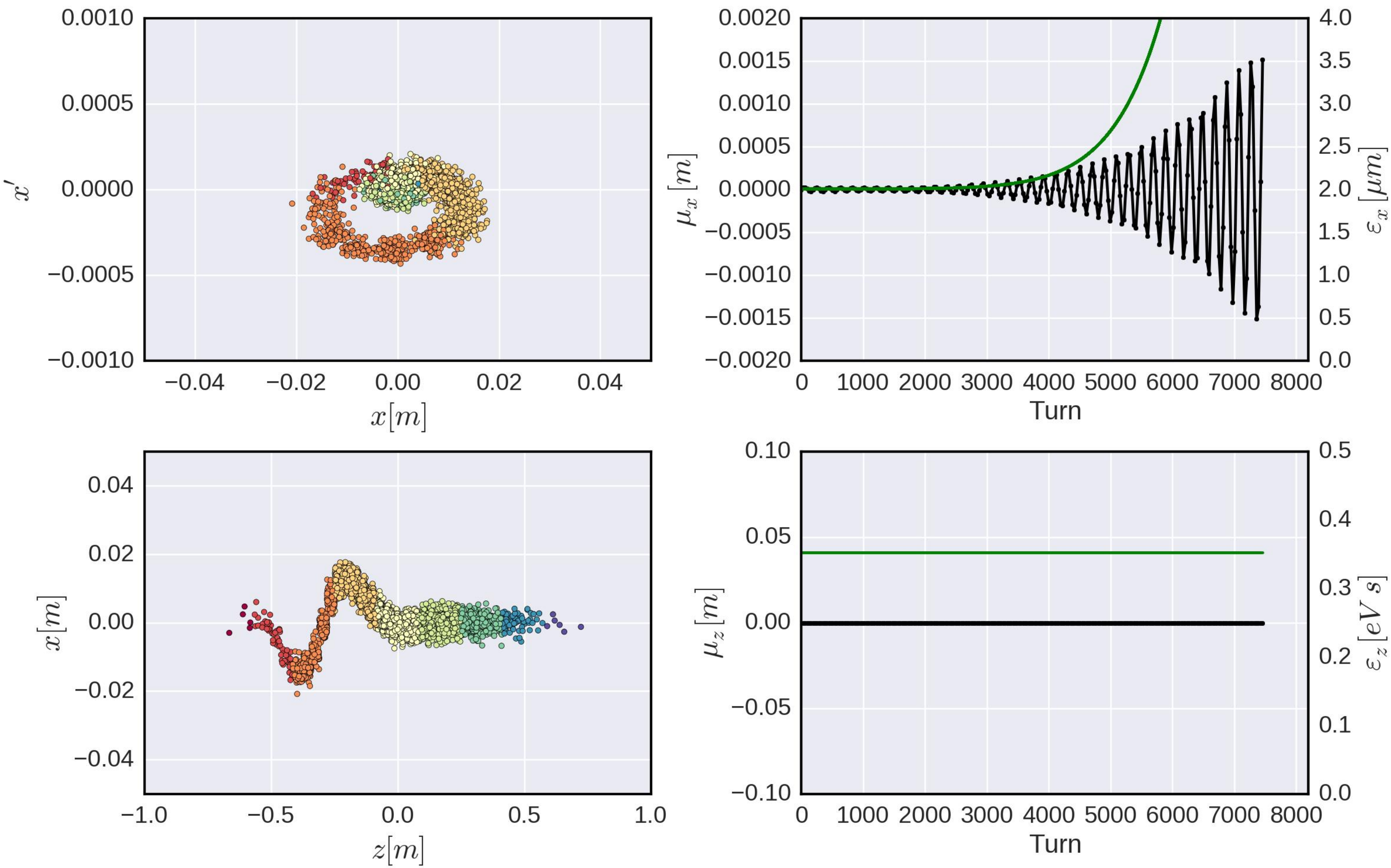
With synchrotron motion we can get into a feedback loop

- Without synchrotron motion:
  kicks accumulate turn after turn – the beam is unstable → beam break-up in linacs
- With synchrotron motion:
  - Chromaticity = 0
    - Synchrotron sidebands are well separated → beam is stable
    - Synchrotron sidebands couple → (transverse) mode coupling instability
  - Chromaticity ≠ 0
    - Headtail modes → beam is unstable (can be very weak and often damped by non-linearities)

Dipole wakes – beam break-up



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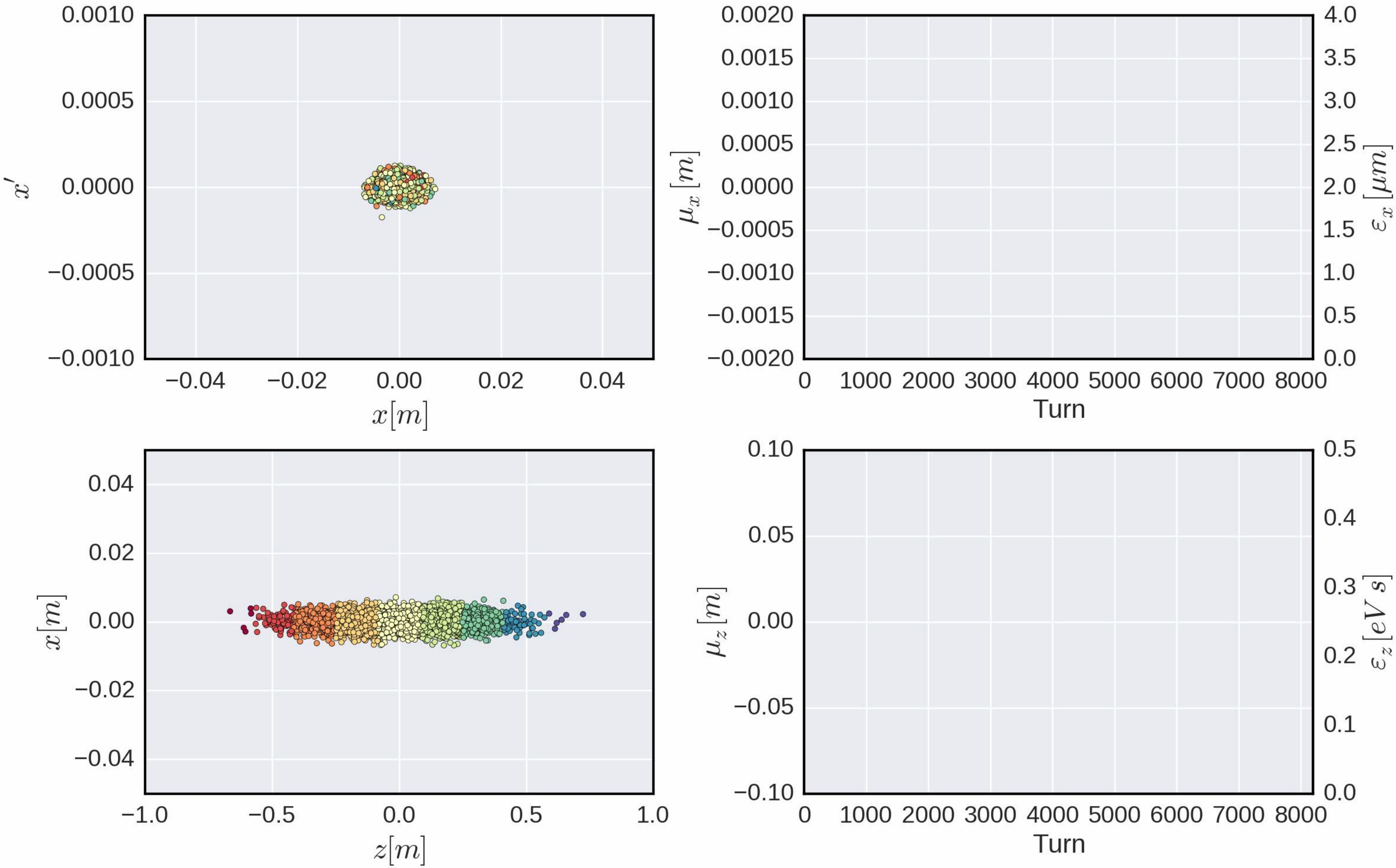
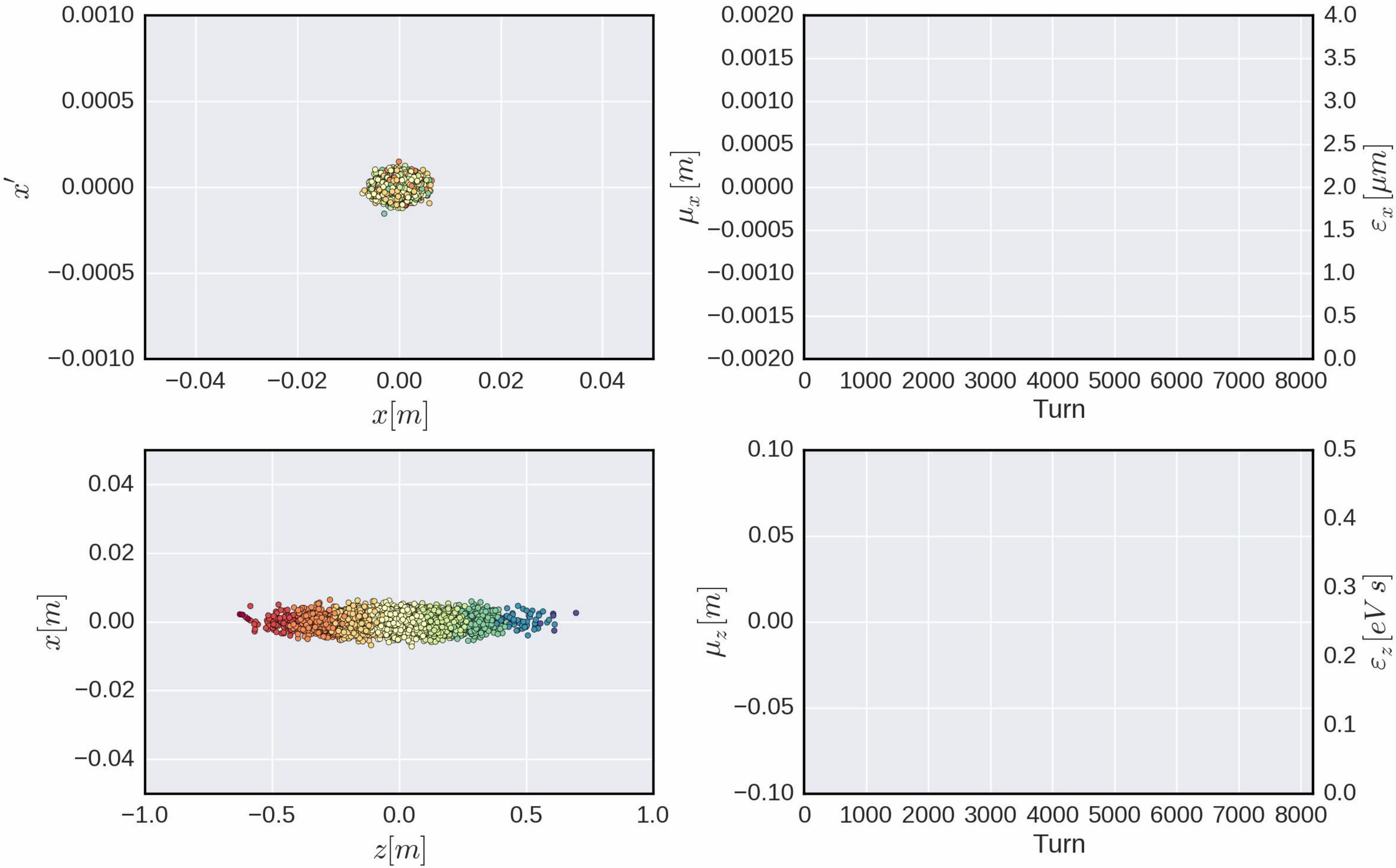


Dipole wakes – TMCI below threshold



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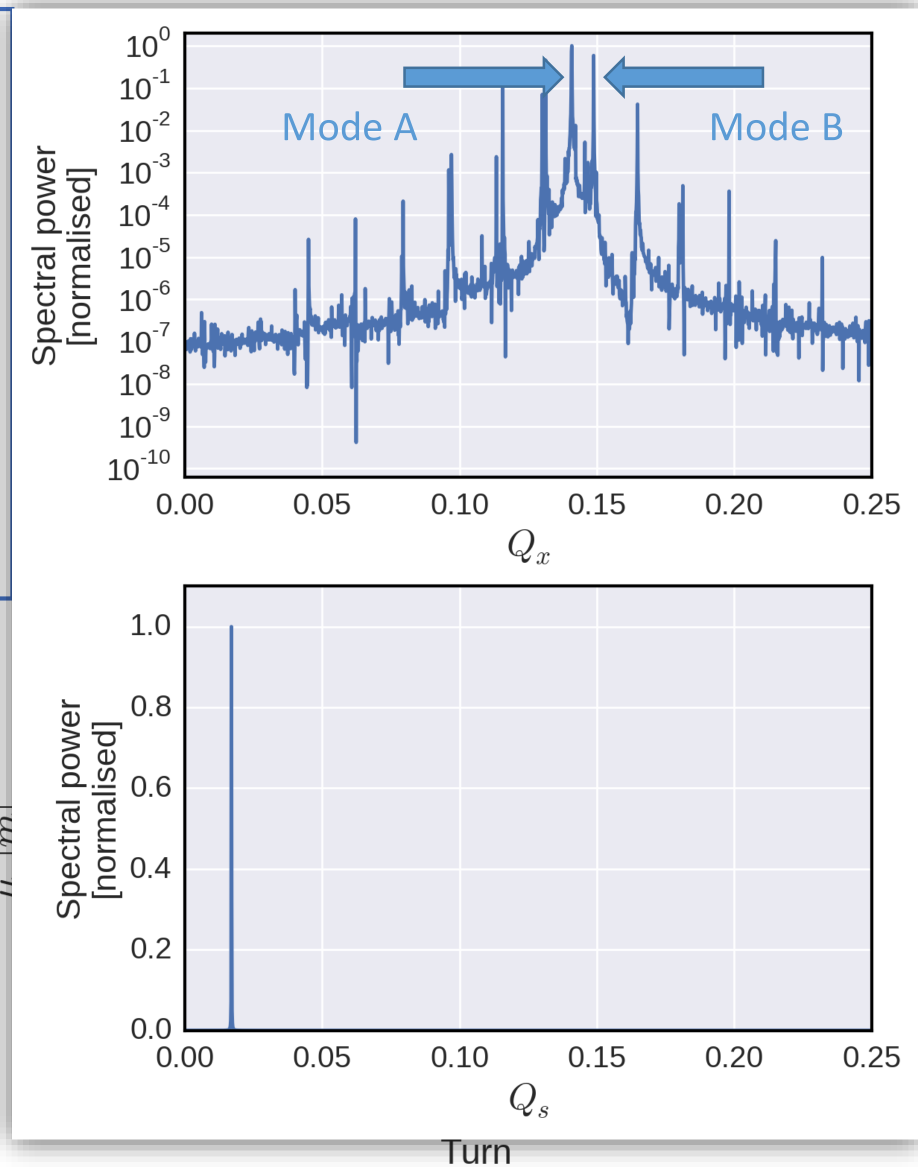
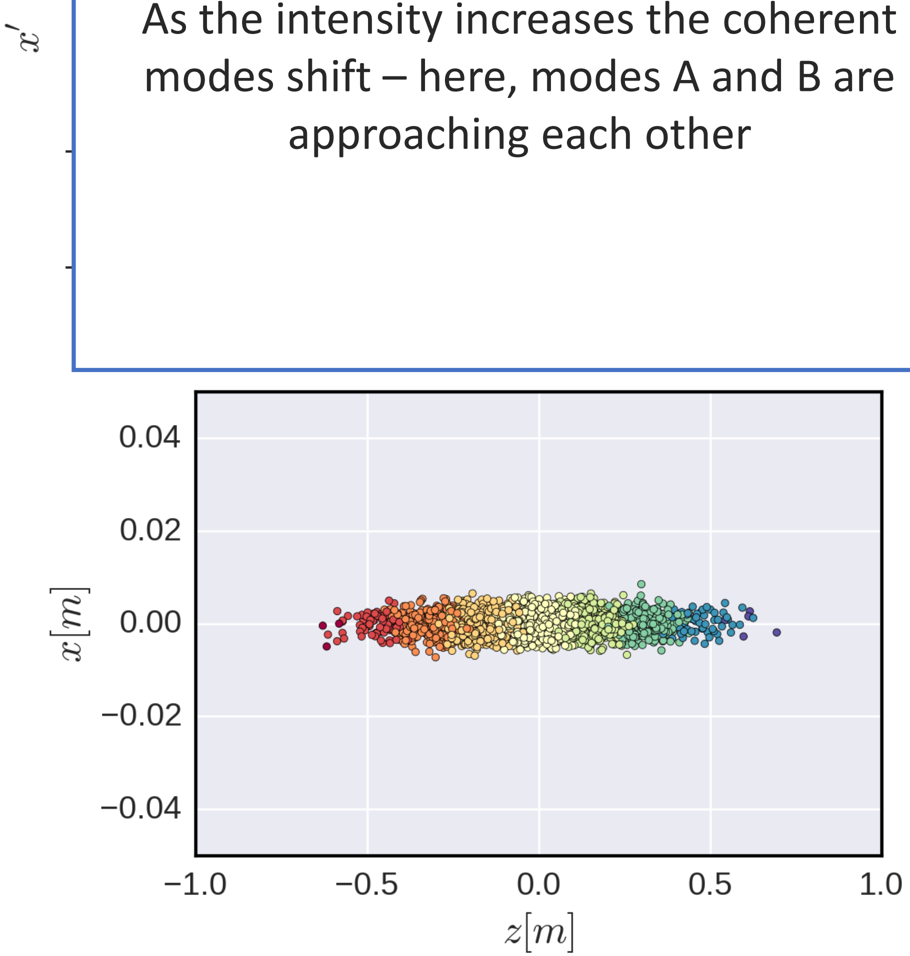
As the intensity increases the coherent modes shift – here, modes A and B are approaching each other





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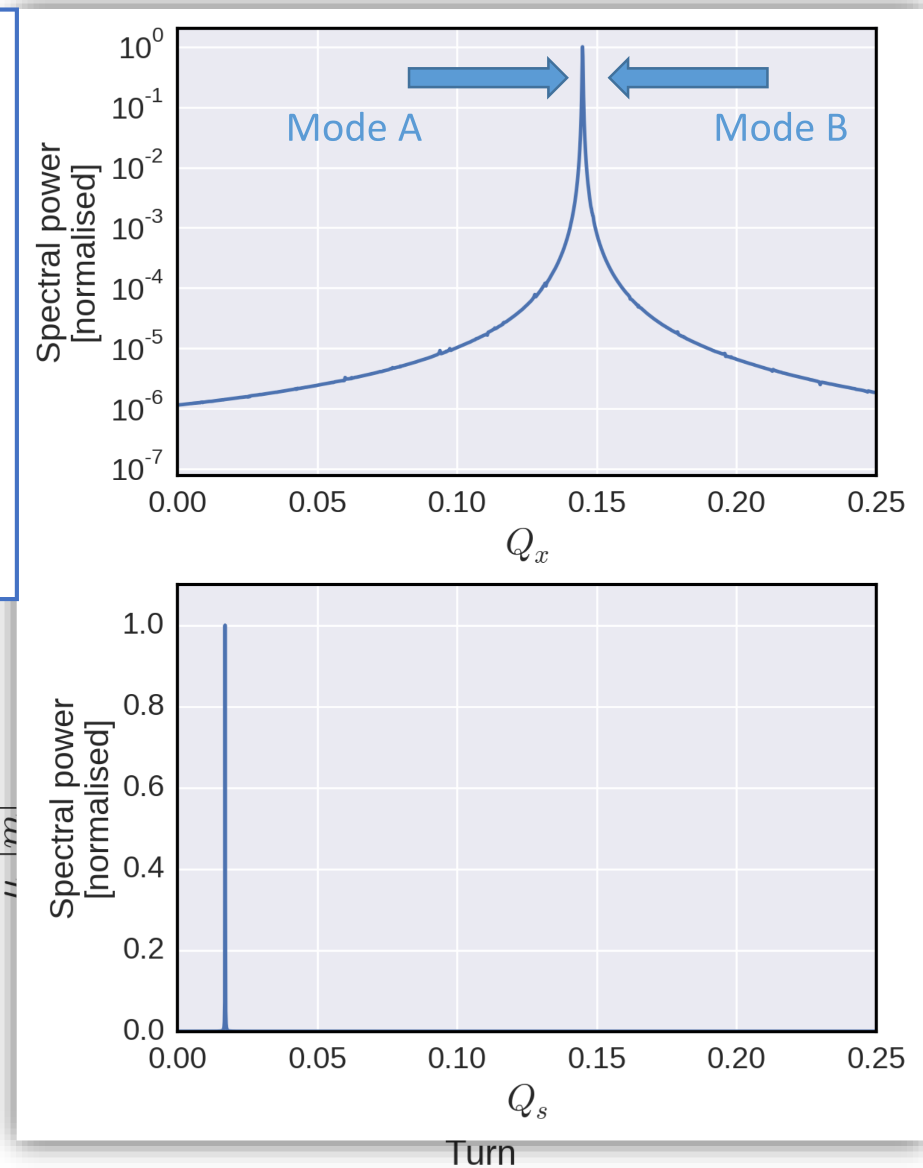
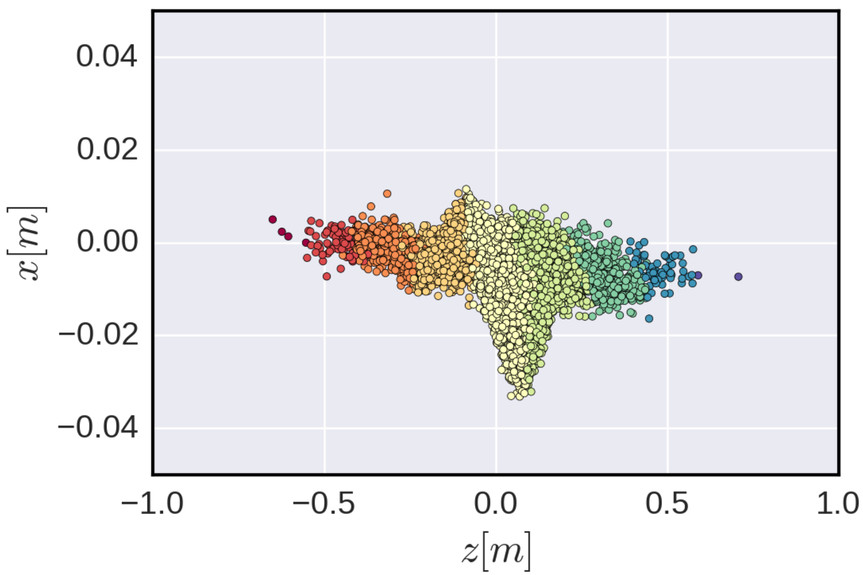
When the two modes merge a fast coherent instability arises – the transverse mode coupling instability (TMCI) which often is a hard intensity limit in many machines





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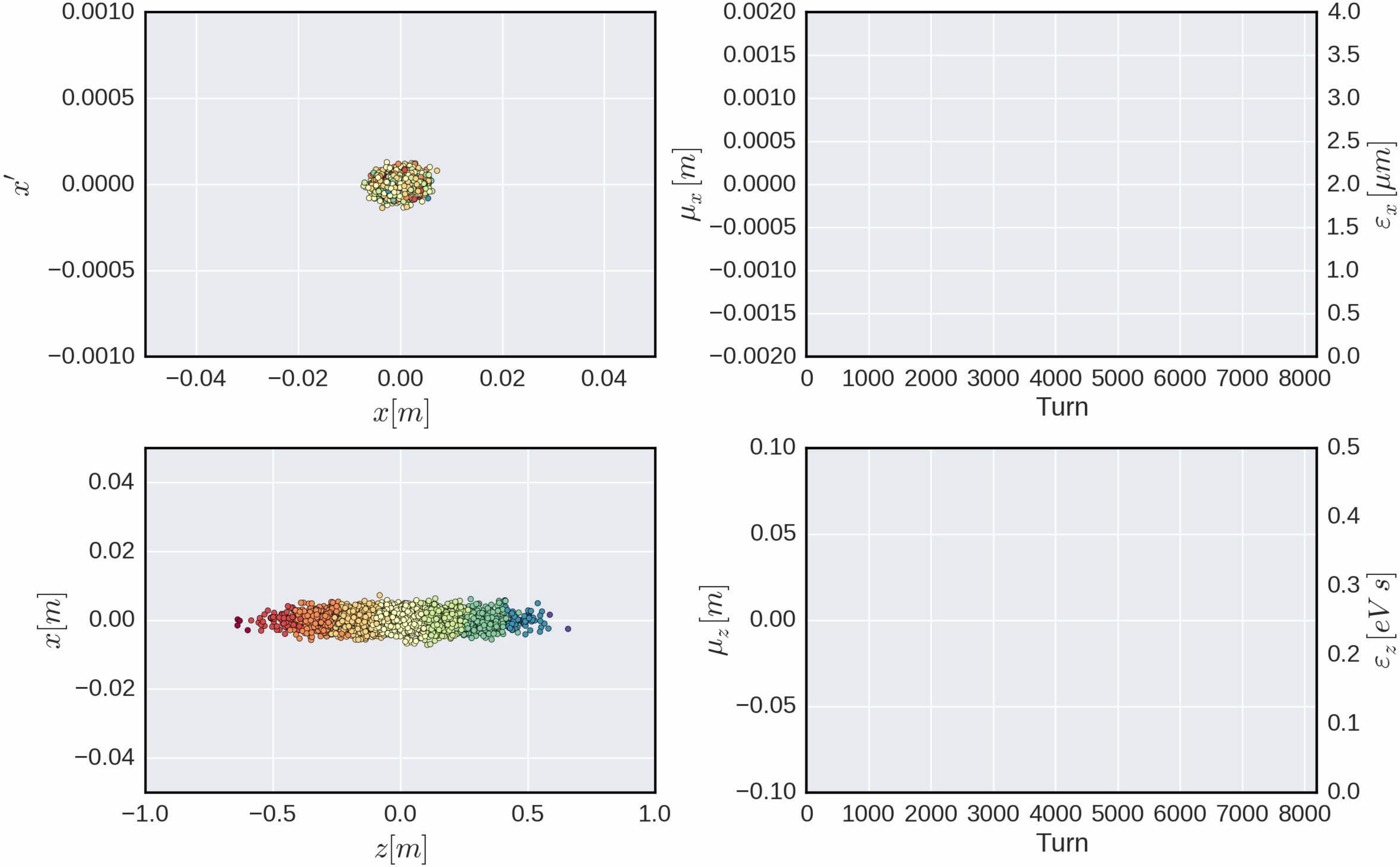
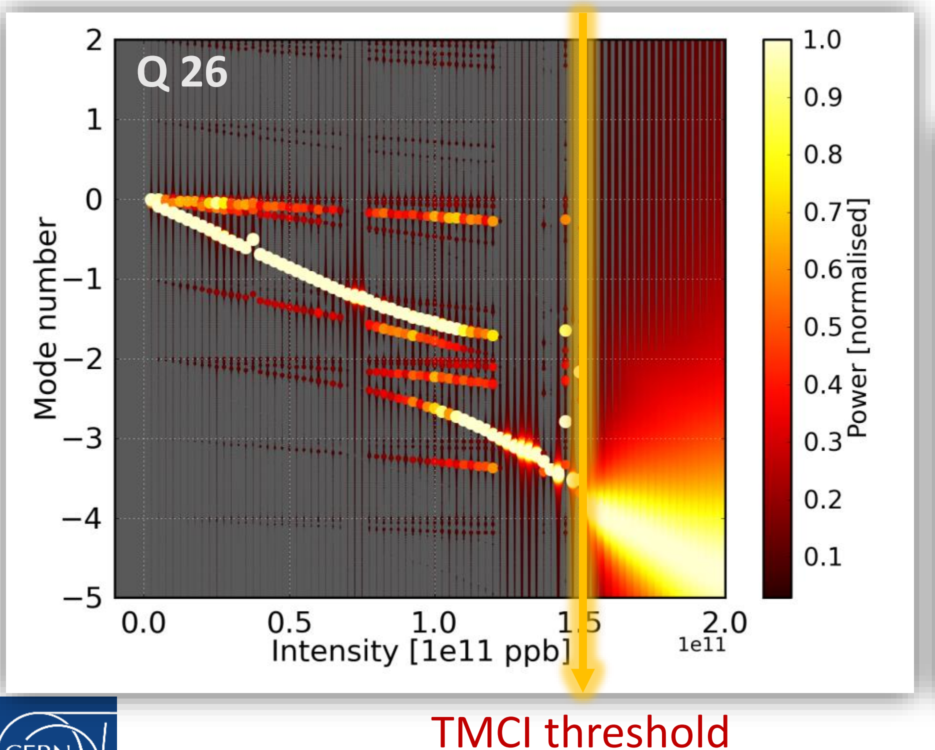
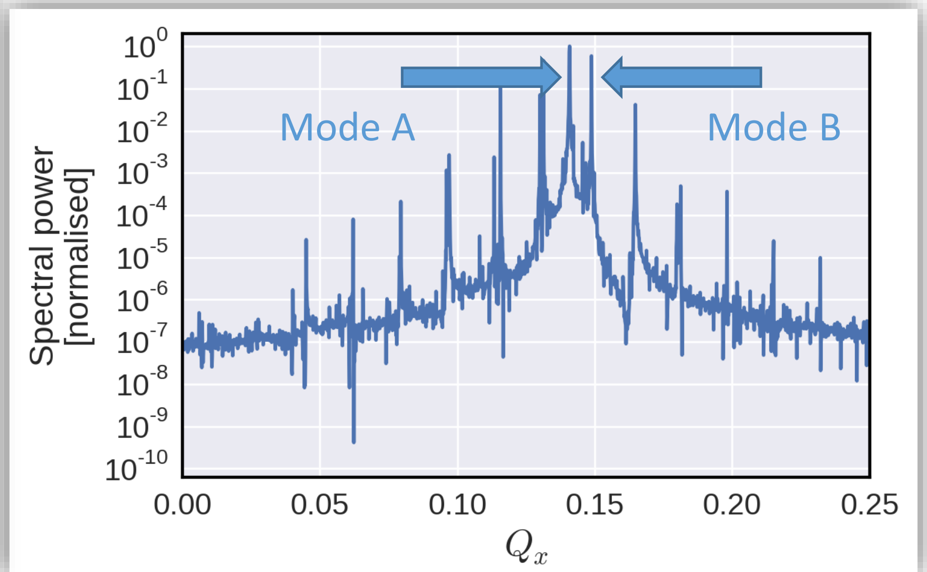


Raising the TMCI threshold – SPS Q20 optics

- In simulations we have the possibility to perform scans of variables, e.g. we can run 100 simulations in parallel changing the beam intensity
- We can then perform a spectral analysis of each simulation…
- … and stack all obtained plot behind one another to obtain…
- … the typical visualization plots of TMCI





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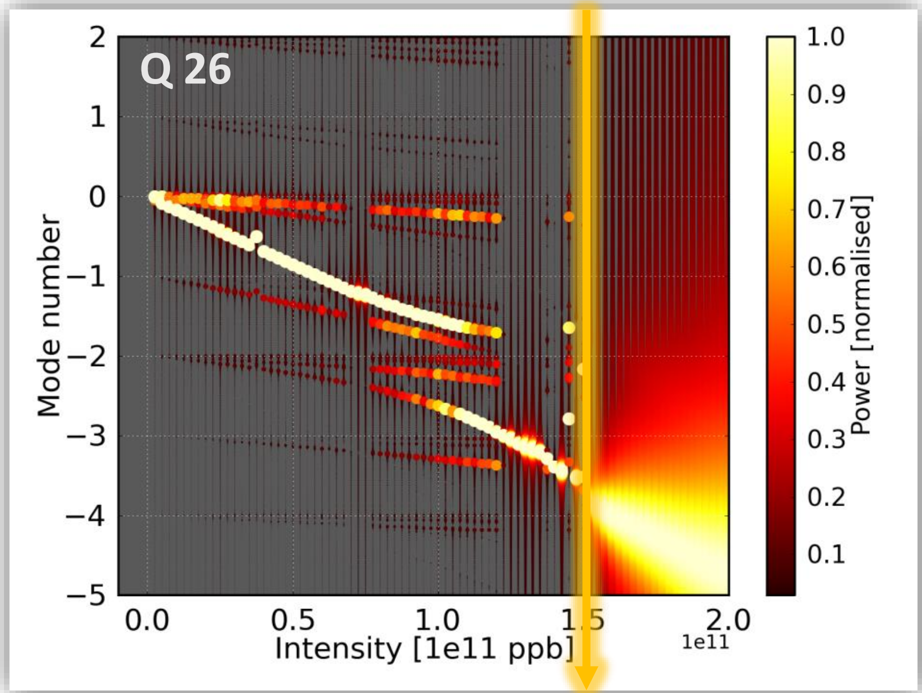

TMCI threshold

The mode number is given as

$$m = \frac{Q_x - Q_{x0}}{Q_s}$$

The modes are separated by the synchrotron tune.

Raising the TMCI threshold – SPS Q20 optics

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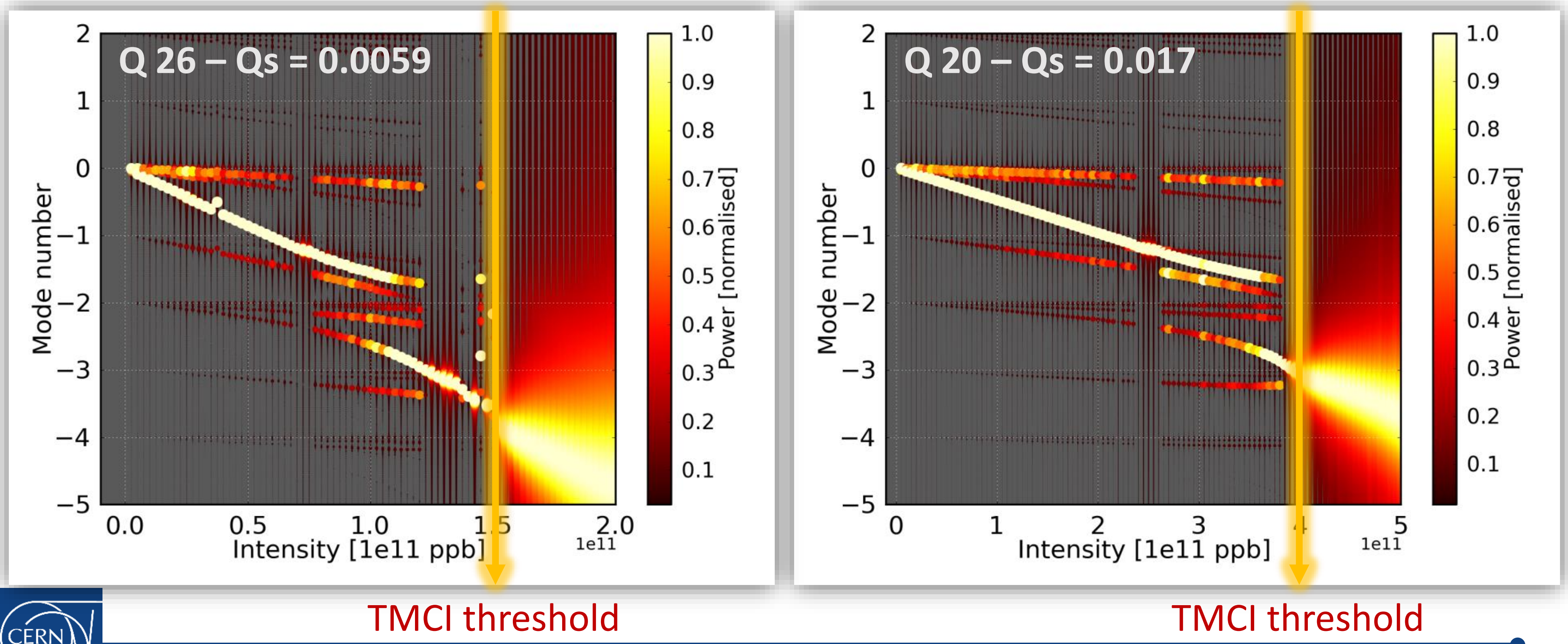
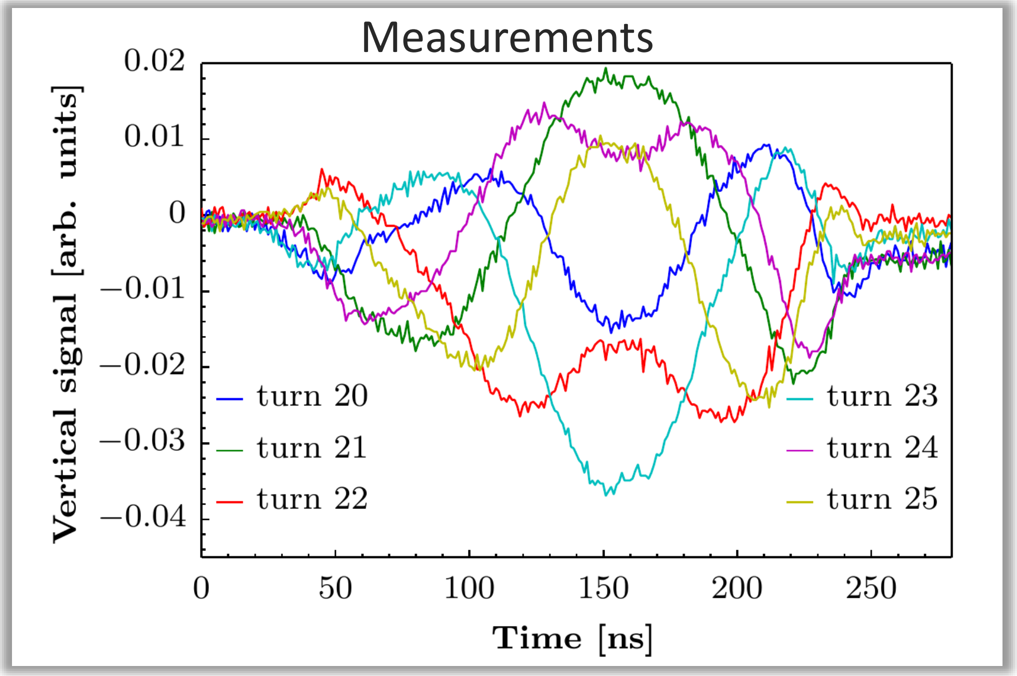


Example non-relativistic wakes



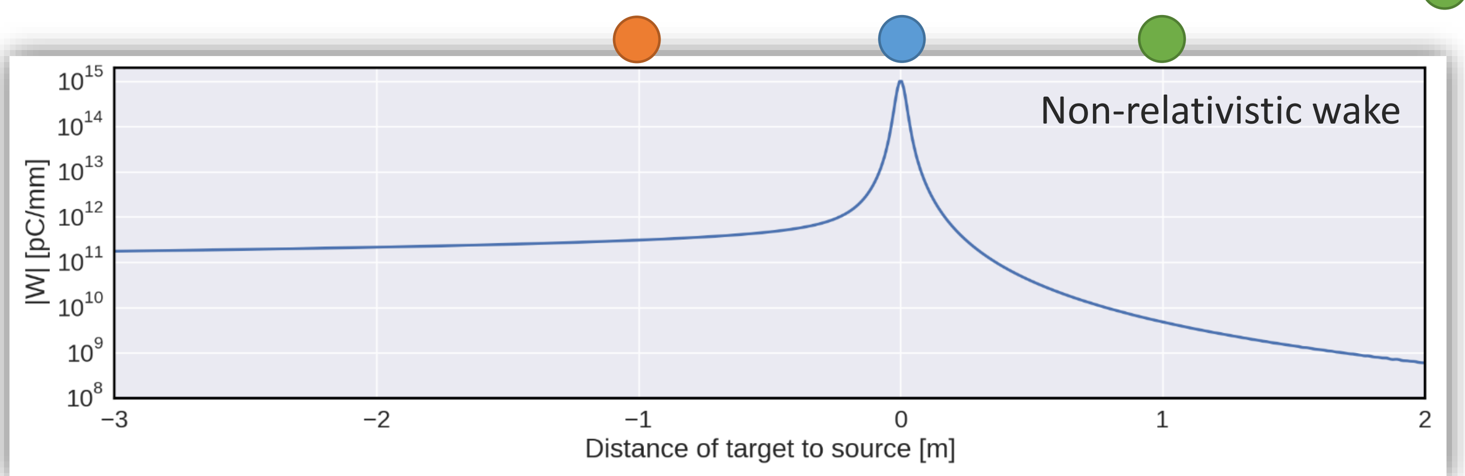
- PS injection oscillations show intra-bunch modulations
- These can only be reproduced when adding non-relativistic wakes caused by indirect space charge fields

A. Huschauer et al

Source

Witness trailing

Witness ahead



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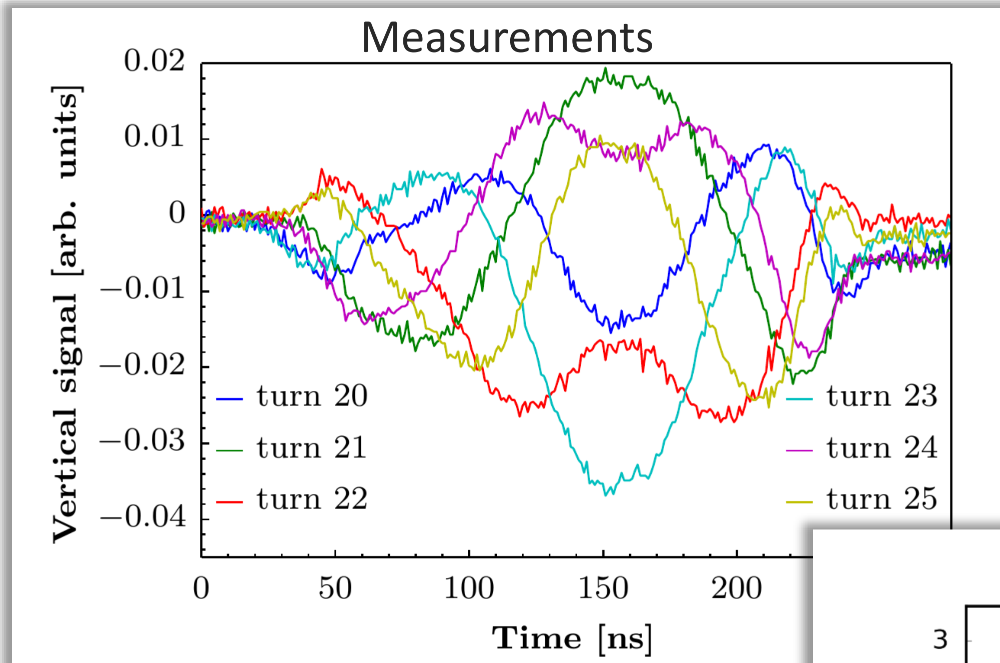

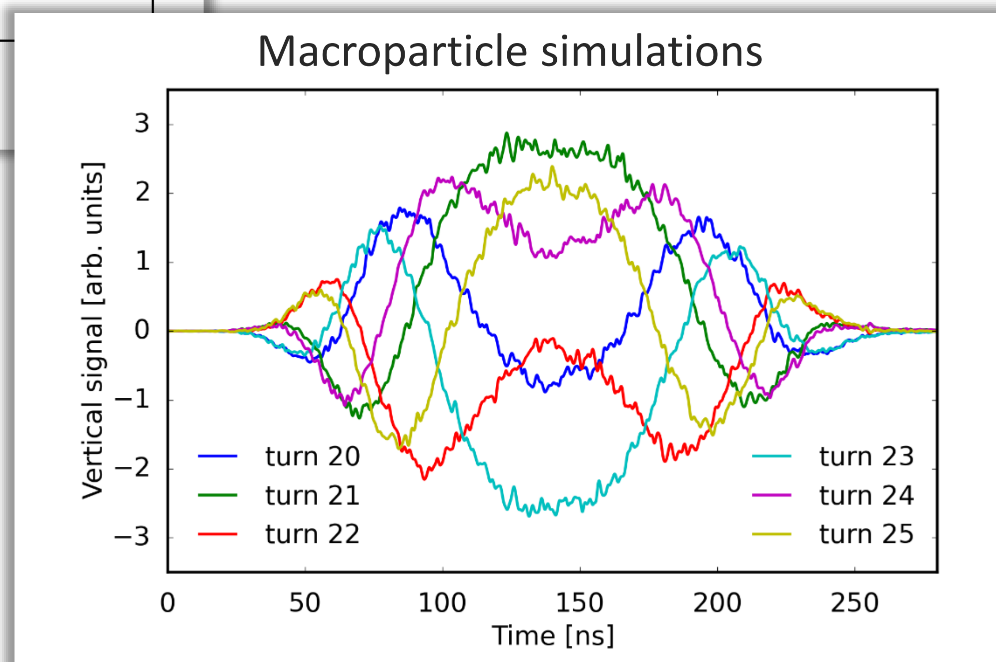

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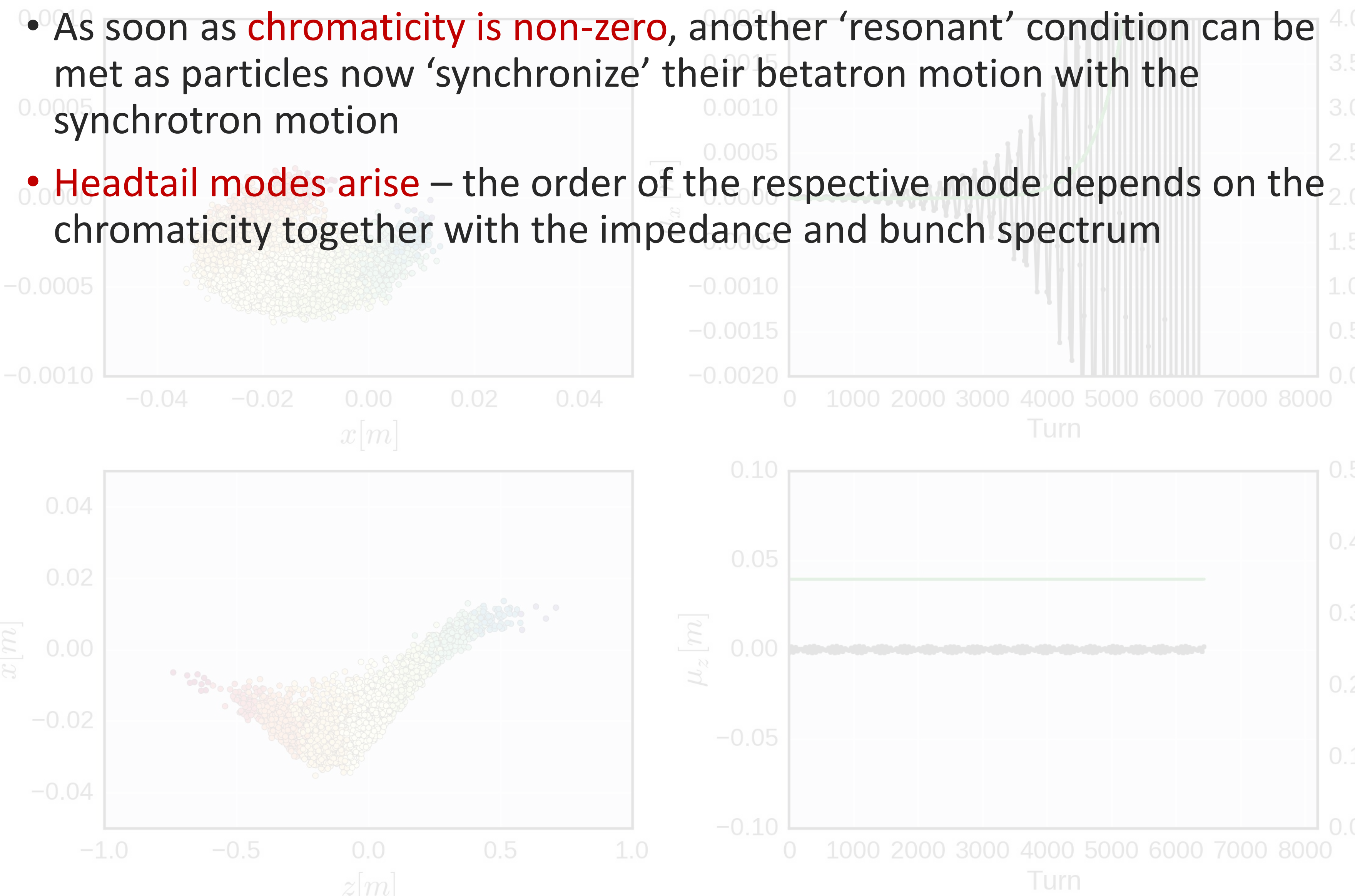
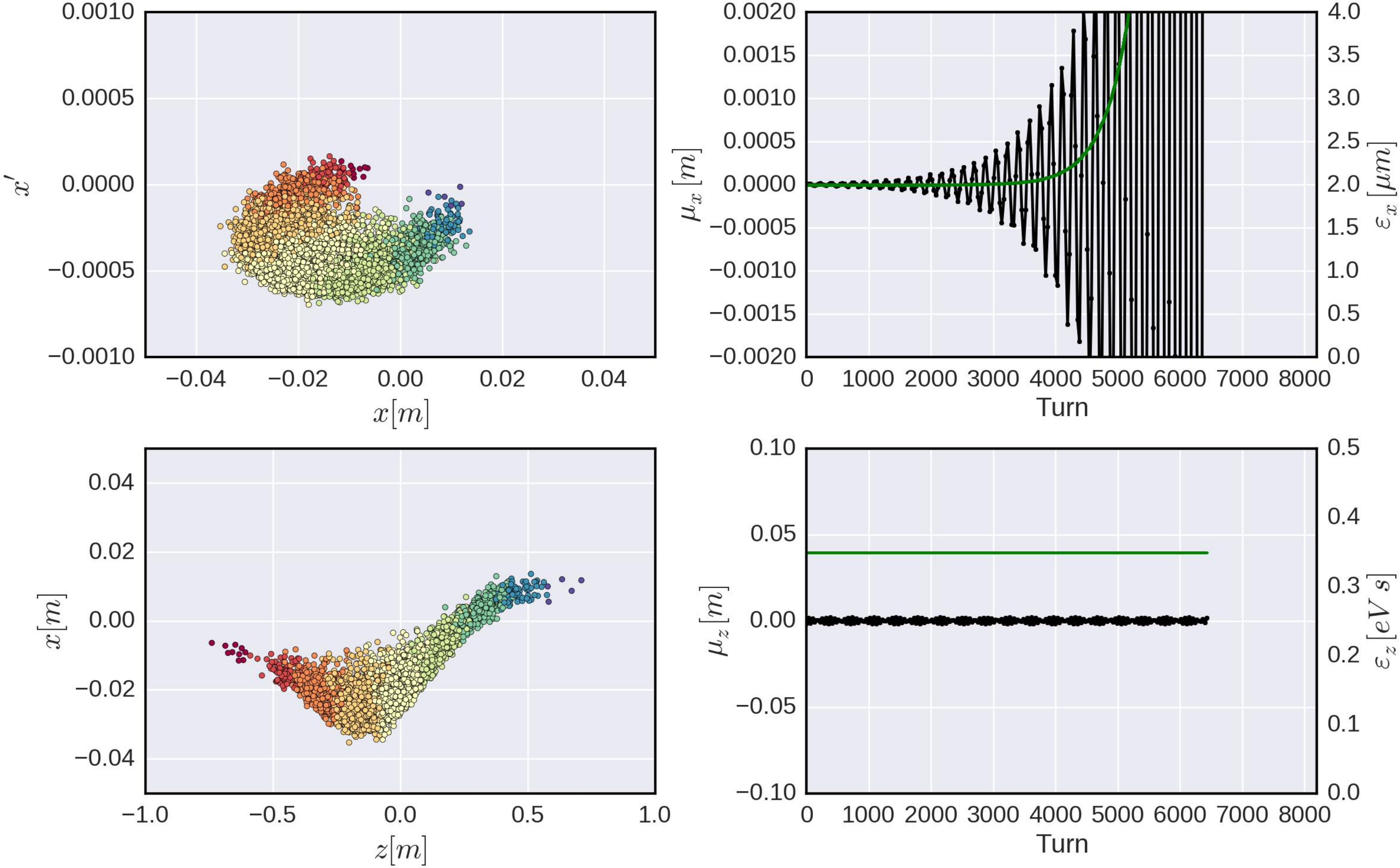
A. Huschauer et al

Dipole wakes – headtail modes

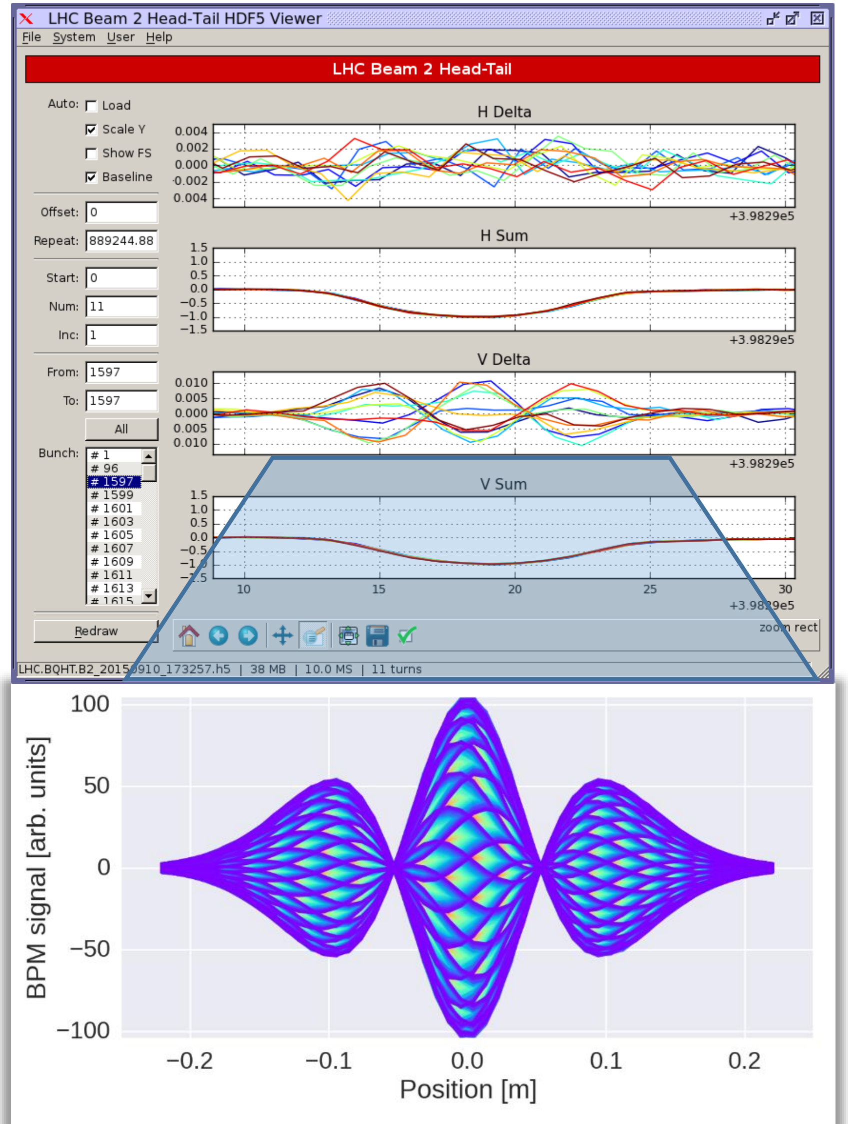
- As soon as chromaticity is non-zero, another ‘resonant’ condition can be met as particles now ‘synchronize’ their betatron motion with the synchrotron motion
- Headtail modes arise – the order of the respective mode depends on the chromaticity together with the impedance and bunch spectrum

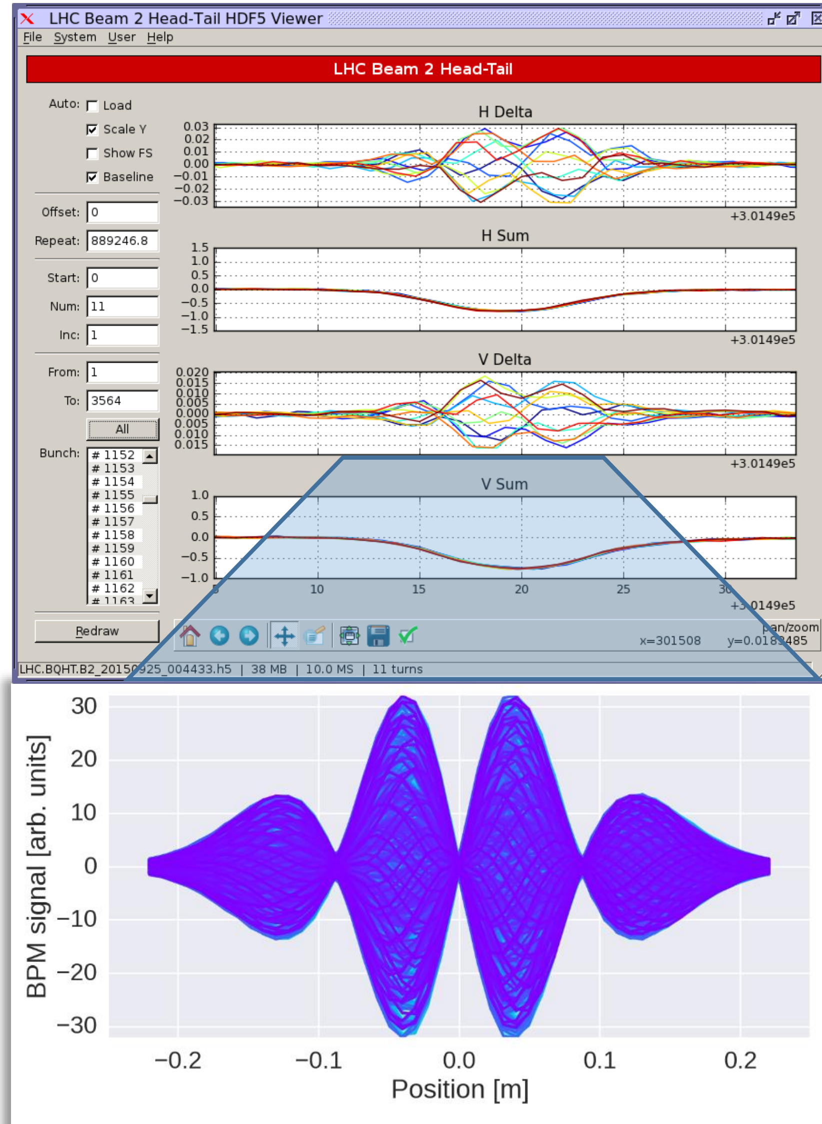


Dipole wakes – headtail modes



Example: Headtail modes in the LHC





End of part I

- Numerical methods allow us
  - to study conditions not realizable in a machine
  - to disentangle effects
  - to use unprecedented analysis tools
- Macroparticle models closely resemble real systems and are relatively easy to implement
- We have learned how to model and implement macroparticle simulations to study intensity effects in circular accelerators

Backup

Wakefields – rough formalism

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}K(s)x^2 + \sum_k \frac{e^2}{m\gamma\beta^2c^2\,C} \iiint \rho(x_s, z_s)\, w(x, x_s, z - z_s - kC)\, dx_s\, dz_s\, dx$$

Wakefields – rough formalism

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}K(s)x^2 + \sum_k \frac{e^2}{m\gamma\beta^2c^2\,C} \iiint \rho(x_s, z_s)\, w(x, x_s, z - z_s - kC)\, dx_s\, dz_s\, dx$$

$$= \ldots + \sum_k \frac{e^2}{m\gamma\beta^2c^2\,C} \iiint \rho(x_s, z_s) \sum_{mn} x^n x_s^m\, W_{mn}(z - z_s - kC)\, dx_s\, dz_s\, dx$$

$$= \ldots + \sum_k \frac{e^2}{m\gamma\beta^2c^2\,C} \sum_{mn} \int x^n \int \lambda_m(z_s)\, W_{mn}(z - z_s - kC)\, dz_s\, dx$$

$$\lambda_m(z_s) = \int \rho(x_s, z_s) x_s^m\, dx_s$$

- Expansion

Wakefields – rough formalism

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$$\lambda_m(z_s) = \int \rho(x_s, z_s)x_s^m\,dx_s$$

$$H = \frac{1}{2}p_x^2 + C + \boxed{A\,x} + \boxed{\frac{1}{2}B\,x^2} + \dots, \qquad \text{with } \frac{dq}{ds} = \frac{\partial H(p,q)}{\partial p}, \qquad \frac{dp}{ds} = -\frac{\partial H(p,q)}{\partial q}$$

Dipole term (n=1) → change of orbit

Quadrupole term (n=2) → change of tune

- Expansion – up to second order:

n	m	type
0	0, 1	
1	0	

Constant transverse wake (n=0, m=0)

Dipole transverse wake (n=0, m=1)

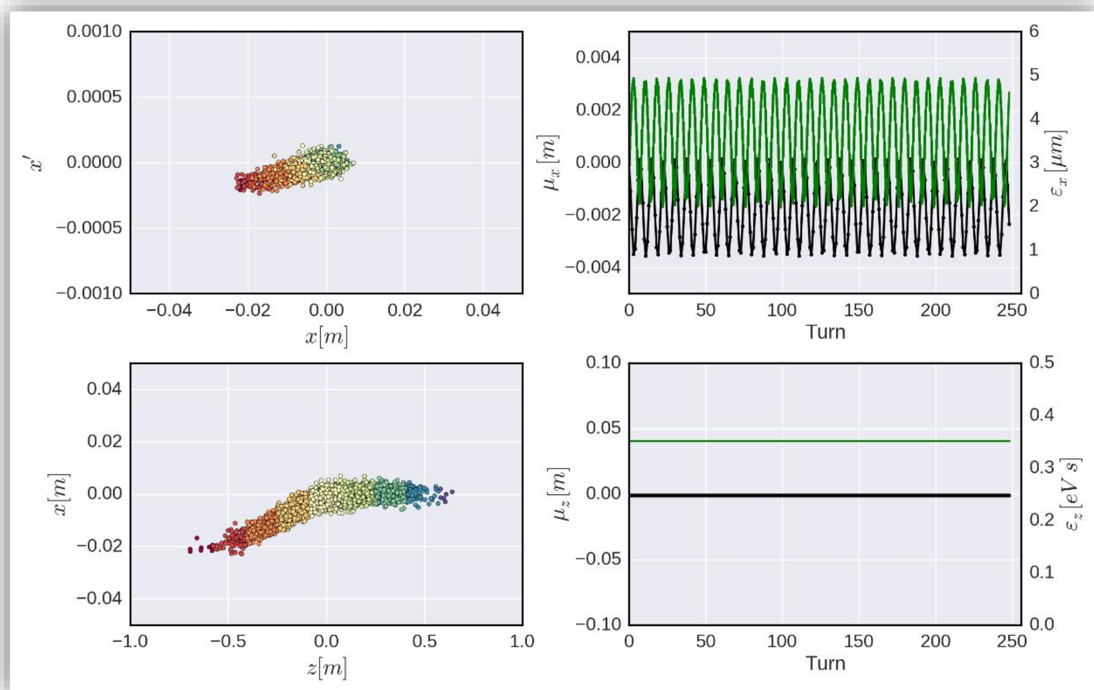
Quadrupole transverse wake (n=1, m=0)

Examples – constant wakes

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}K(s)x^2 + \frac{e^2}{m\gamma\beta^2c^2\,C}\,x\sum_{j=0}^{n\_slices-1}\lambda(z_j)\,W_{01}(z-z_j)\,\Delta z_j$$

Dipolar term → orbit kick

Slice dependent change of closed orbit
(if line density does not change)



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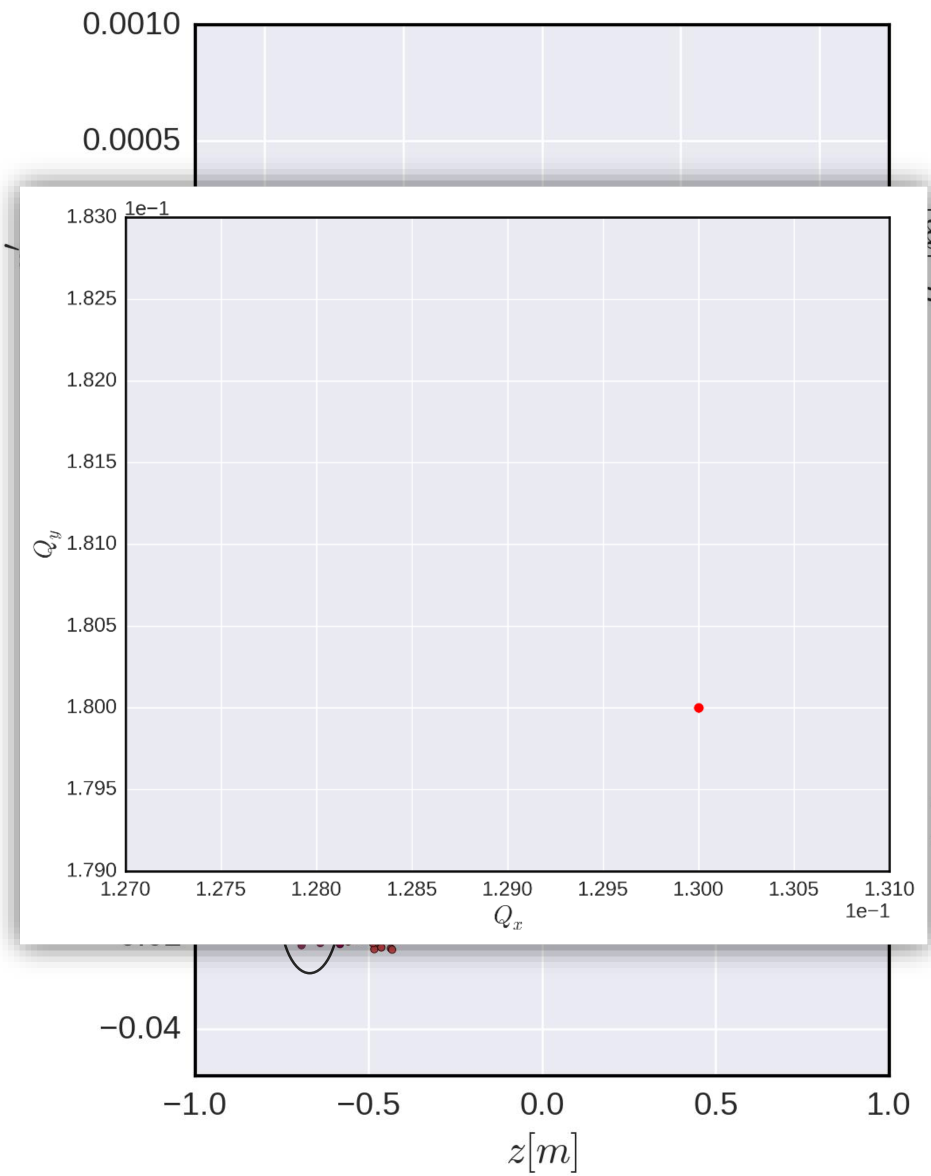
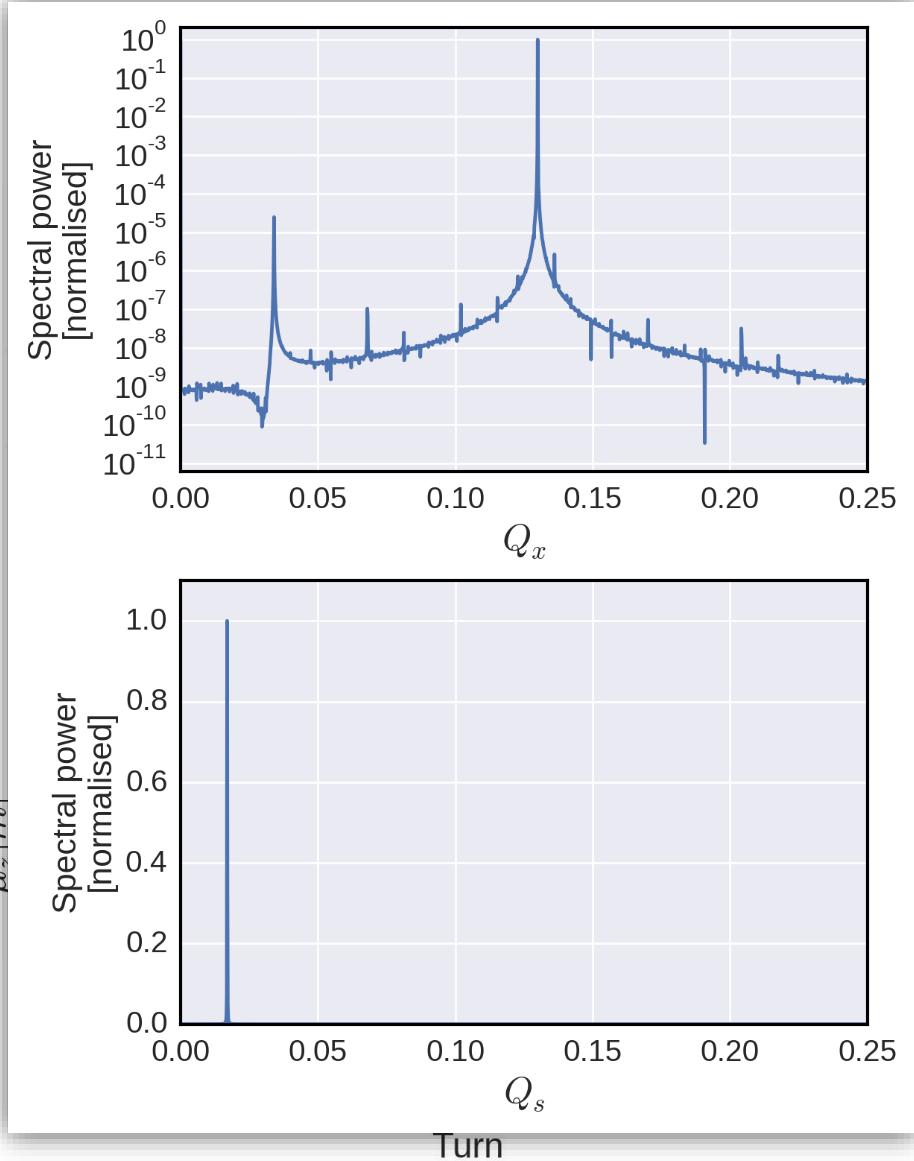


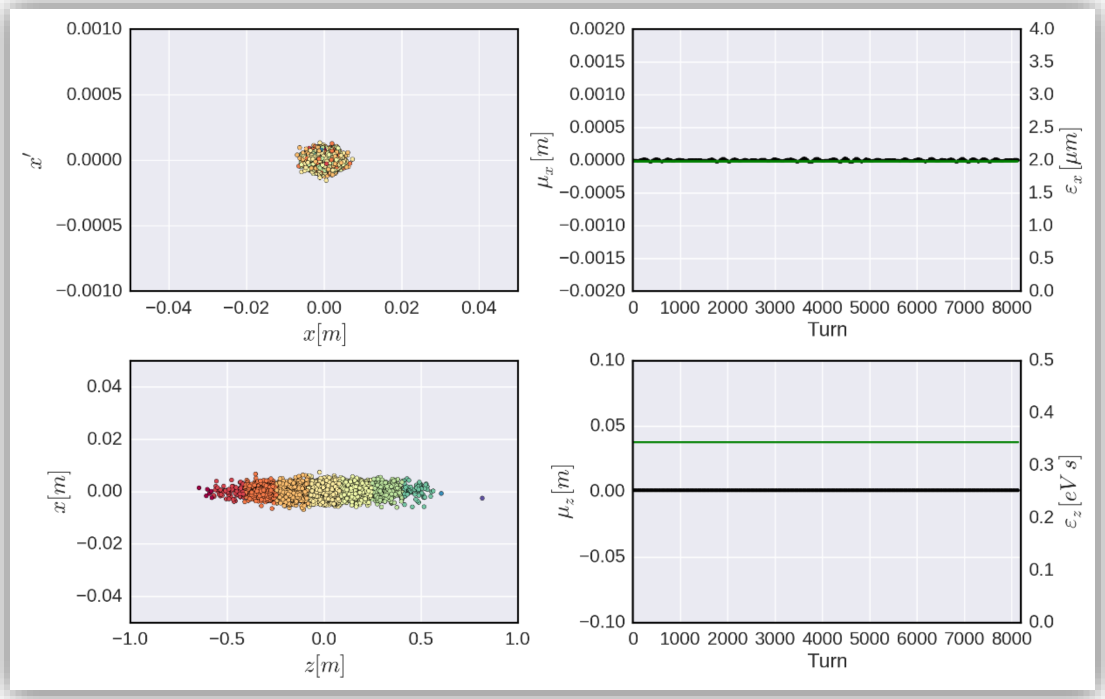


Examples – quadrupole wakes

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}K(s)x^2 + \frac{e^2}{m\gamma\beta^2c^2\,C}\,x^2 \sum_{j=0}^{\mathrm{n\_slices}-1} \lambda(z_j)\,W_{02}(z - z_j)\,\Delta z_j$$

Quadrupole term → tune kick

Slice dependent change of tune (if line density does not change)



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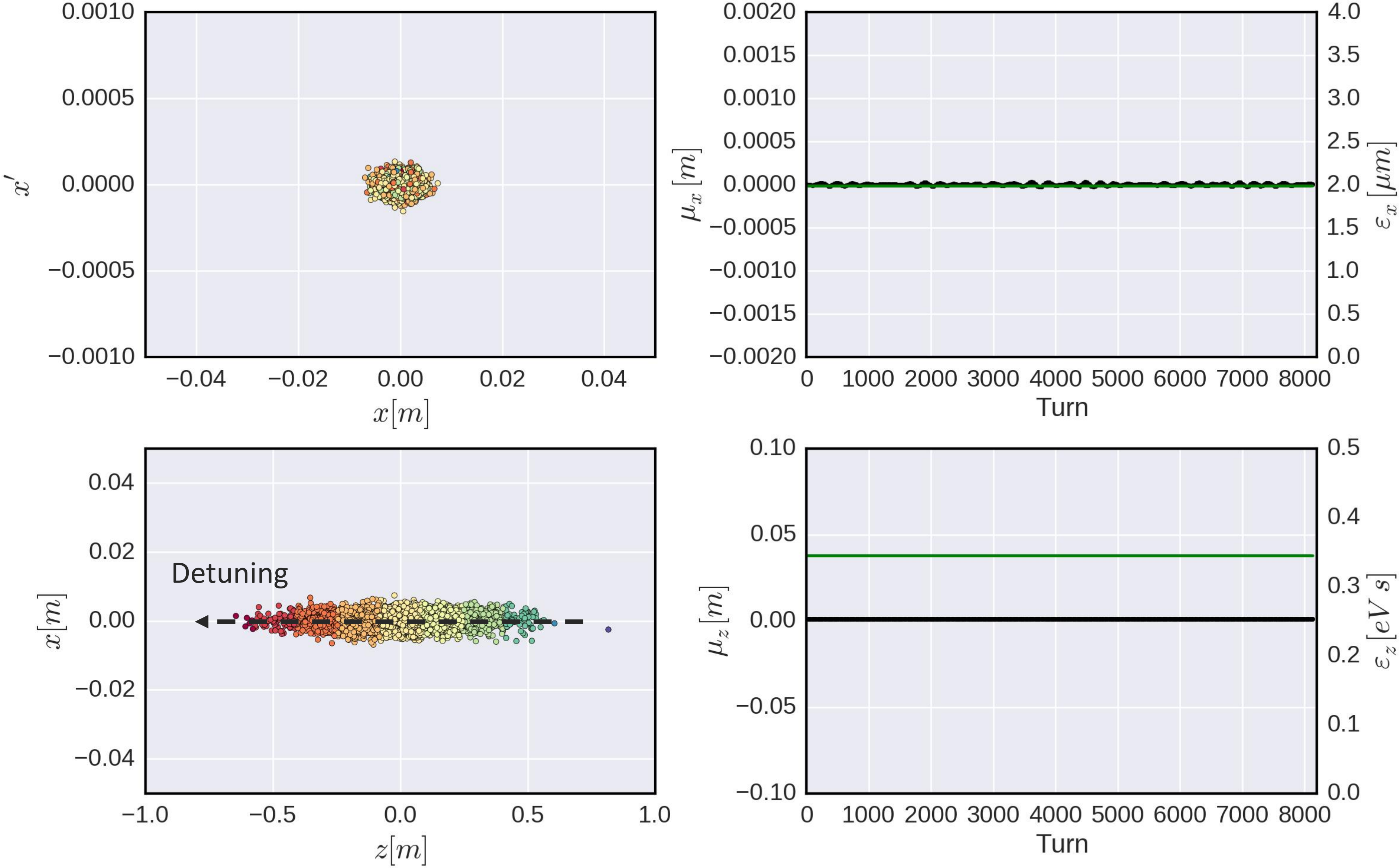

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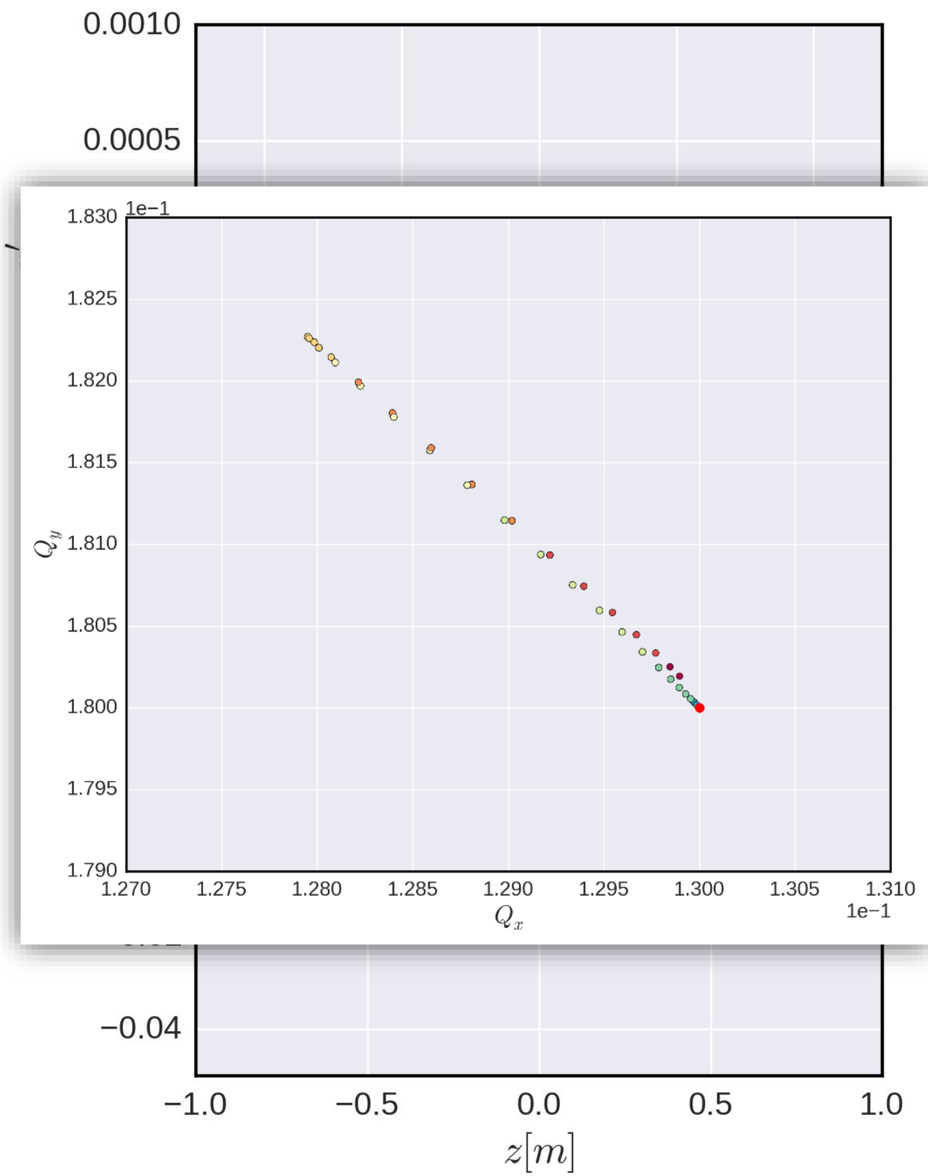

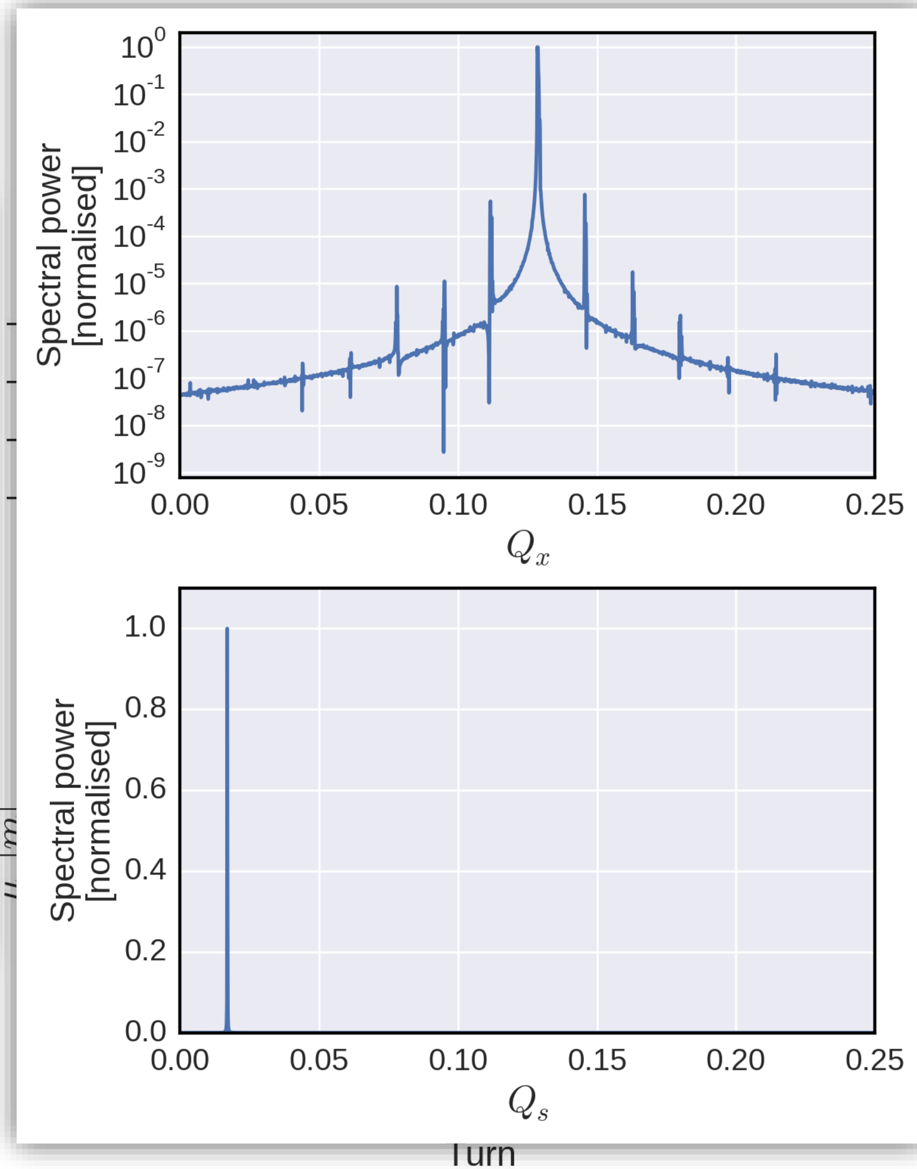

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Dipolar term → orbit kick

Offset dependent orbit kick
→ kicks can accumulate

- Without synchrotron motion:
  kicks accumulate turn after turn – the beam is unstable → beam break-up in linacs

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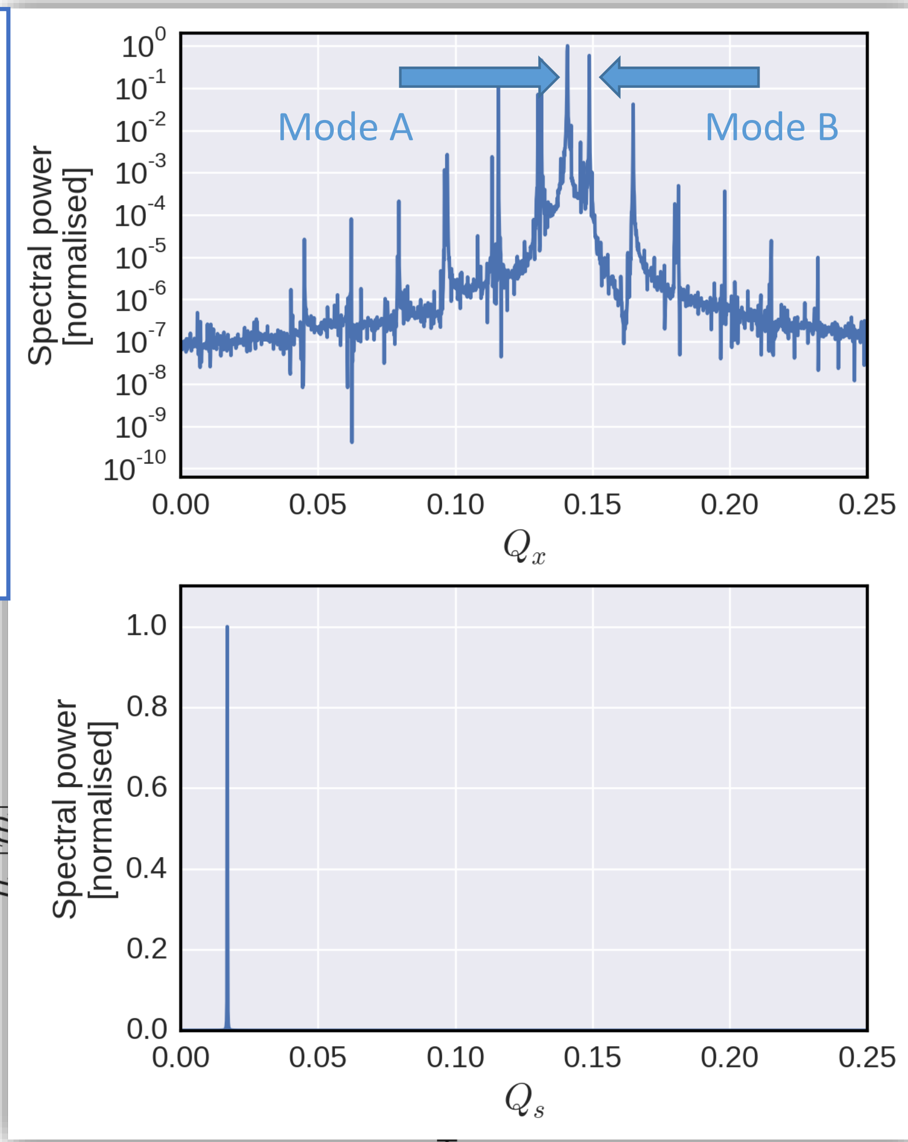
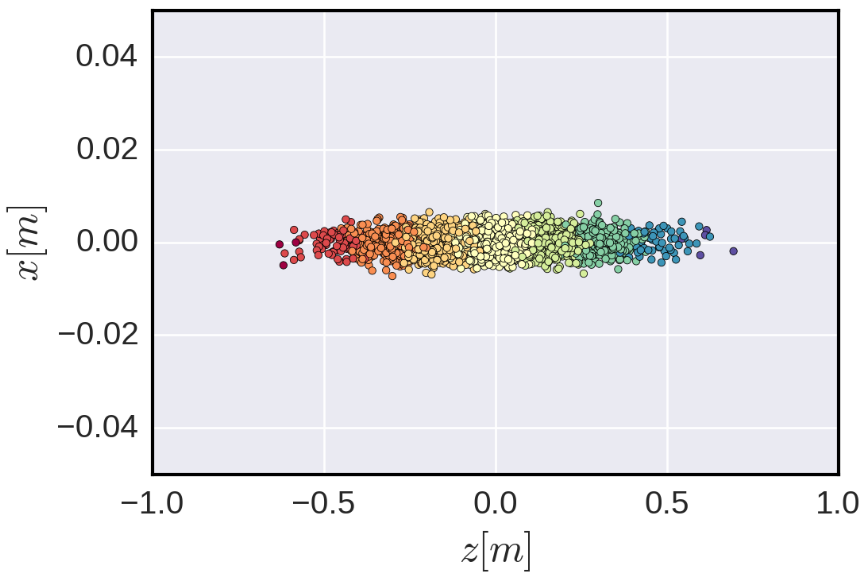
Dipole wakes – TMCI below threshold

As the intensity increases the coherent modes shift – here, modes A and B are approaching each other





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