INTRABEAM SCATTERING

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Prologue

Intrabeam Scattering (IBS) is a multiple Coulomb scattering of charged particle beams (alternatively IBS is a diffusion process in all 3 transverse & longitudinal beam dimensions) ref. [1,3,7,8,15] & ref. [C,I,J]

IBS and *Touschek effect* are distinctive facets of Coulomb scattering event inside particle beams

- IBS in charged particle beams causes *small* changes of the colliding *particles momenta* by addition of *multiple* random *small-angle scattering* events, leading to:
  1. A *relaxation* to a thermal (energy) *equilibrium* via reallocation of the whole beam phase volume between the 3 transverse and longitudinal beam phase volumes (emittances).
  2. A continuous *diffusion growth* of the global beam phase volume *without equilibrium*, and reduction of the *beam lifetime* when the particles hit the *aperture*.
- *Touschek effect* is the particle *losses* due to *single* collision *events* at large scattering angles where only the energy transfer from transverse to longitudinal planes is examined (no particle redistribution done).
- IBS simulation consists to iteratively compute the particle *momentum variation* by *coulomb scattering* with the other particles of the beam and find the *growth rates* for the 3 degrees of freedom.
- *IBS* theory was later extended to include:
  - Amplitude & dispersion *derivatives* and lattice parameter *variations* around the *lattice*.
  - Horizontal-vertical betatron *linear coupling*.

Prologue

IBS in week focusing or smooth ring lattices can be related with scattering of *gas molecules* in a *closed box*, where the walls mimic the quadrupole focusing forces and the RF voltage keep the particles together. The *scattering* of the molecules leads to the *Maxwell-Boltzmann distribution* of the 3 velocity components (v_x, v_y, v_s) in which m is the molecule mass, T the temperature, k the Boltzmann's constant ($f\,d\boldsymbol{v}$ is normalized to unity):

$$f(\boldsymbol{v}_x, \boldsymbol{v}_y, \boldsymbol{v}_s) = \frac{1}{(2\pi kT/m)^{3/2}} e^{-m(\boldsymbol{v}_x{}^2+\boldsymbol{v}_y{}^2+\boldsymbol{v}_s{}^2)/(2kT)} \quad 1.1$$

The difference between *IBS* and *gaz molecule* scattering in a box is due to the ring orbit curvature:

- *Curvature* yields a *dispersion* so that a sudden change of *energy* will change the *betatron* amplitudes and initiate a *synchro-betatron* oscillation *coupling*.
- *Curvature* also leads to the *negative mass instability* i.e. if a particle accelerates above *transition* it becomes slower and behaves as a particle with negative mass and thus an *equilibrium* of particles *above transition energy* can't exist (transition energy $\gamma_t mc^2$ is got once $\gamma^2 = \gamma_t^2 = \frac{1}{\alpha_p} = \frac{dp/p}{dR/R}$ or $\frac{df/f}{dp/p} = \frac{1}{\gamma_t^2} - \frac{1}{\gamma_t^2} = 0$).
- *Above transition* the IBS effect is to increase the three bunch dimensions.
- *Below transition* an *equilibrium* particle distribution can *exists* (week focusing/smooth lattices).

The Intrabeam scattering effect

- Small angle **multiple Coulomb scattering** effect
  - Redistribution of beam momenta
  - Beam diffusion with impact on the beam quality (Brightness , luminosity, etc)

- **Different approaches** for the probability of scattering
  - Classical Rutherford cross section
  - Quantum approach
    - Relativistic "Golden Rule" for the 2-body scattering process

- **Several theoretical models** and their **approximations** developed over the years
  - Classical models of Piwinski (**P**) and Bjorken-Mtingwa (**BM**)
  - High energy approximations **Bane, CIMP, etc**
  - Integrals with analytic solutions

Courtesy F. Antoniou, Y. Papaphilippou, CERN

Lagrangian and Hamiltonian (briefly)

- We restrict to systems of N particles with $3N$ degrees of freedom described via Cartesian coordinates $\boldsymbol{r} = (\boldsymbol{r}_1 \cdots \boldsymbol{r}_N)$, $\boldsymbol{r}_i = (x, y, z)_i$, and $\boldsymbol{v} \equiv \dot{\boldsymbol{r}} = (\dot{\boldsymbol{r}}_1 \cdots \dot{\boldsymbol{r}}_N)$, $\dot{\boldsymbol{r}}_i = (\dot{x}, \dot{y}, \dot{z})_i$
- Assume the system exists in a ***conservative force field*** $\boldsymbol{F}^c(\boldsymbol{r})$ with *kinetic* energy $T(\boldsymbol{r}, \dot{\boldsymbol{r}})$ and *potential* $V(\boldsymbol{r})$ such as $\boldsymbol{F}^c(\boldsymbol{r}) = -\boldsymbol{\nabla}_r V(\boldsymbol{r}) \equiv -\partial V(\boldsymbol{r})/\partial \boldsymbol{r}$. The *Lagrangian* is defined as (ref. [A,B]):

$$L(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) \stackrel{\text{def}}{=} T(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) - V(\boldsymbol{r})$$

Lagrange's equations stem from the *variational principle*:

$$\delta I = \int_{t1}^{t2} L(\boldsymbol{r}, \dot{\boldsymbol{r}}, t)\, dt = 0 \quad \Rightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{r}}} - \frac{\partial L}{\partial \boldsymbol{r}} = 0$$

L is then recast in an *Hamiltonian* form H

$$H(\boldsymbol{r}, \boldsymbol{p}, t) \stackrel{\text{def}}{=} \dot{\boldsymbol{r}} \cdot \boldsymbol{p} - L(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) \qquad \boldsymbol{p} \stackrel{\text{def}}{=} \partial L/\partial \dot{\boldsymbol{r}}$$

$\boldsymbol{p}$: *conjugate momentum to* $\boldsymbol{r}$

e.g. $L(r, \dot{r}, \ddot{r}, t) = \ddot{r}^2 - 2f(t)r \Longrightarrow \frac{d^4 r}{dt^4} = f(t)$
each Lagrangian defines a theory (realistic?)

From which *Hamilton's equations* are derived:

$$\frac{d\boldsymbol{r}}{dt} = \frac{\partial H}{\partial \boldsymbol{p}} \qquad \frac{d\boldsymbol{p}}{dt} = -\frac{\partial H}{\partial \boldsymbol{r}}$$

$\dot{H} = 0$ if $H = H(\boldsymbol{r}, \boldsymbol{p}) \longrightarrow$ $H = T + V = E =$ constant energy

Lagrangian and Hamiltonian (briefly)

If the total force $\boldsymbol{F}$ acting on a system contains a ***conservative*** (*Hamiltonian*) part $\boldsymbol{F}^c(\boldsymbol{r})$ and a ***non-conservative*** (i.e. *non-strictly-Hamiltonian*) part $\boldsymbol{F}^{nc}(\boldsymbol{r},\dot{\boldsymbol{r}},t)$ representing *friction*, *inelastic processes*… ($\boldsymbol{F} = -\boldsymbol{\nabla}_r V(\boldsymbol{r}) + \boldsymbol{F}^{nc}$). The *Lagrangian* of the system is then written as:

$$\boxed{L(\boldsymbol{r},\dot{\boldsymbol{r}},t) \stackrel{\text{def}}{=} T(\boldsymbol{r},\dot{\boldsymbol{r}},t) - V(\boldsymbol{r})} \Rightarrow \boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{r}}} - \frac{\partial L}{\partial \boldsymbol{r}} = \boldsymbol{F}^{nc}}$$

since $\boldsymbol{F}^{nc} \neq -\frac{\partial \tilde{V}}{\partial \boldsymbol{r}} \equiv -\boldsymbol{\nabla}_{\boldsymbol{r}}\tilde{V}$

From $H(\boldsymbol{r},\boldsymbol{p},t) = \dot{\boldsymbol{r}}\cdot\boldsymbol{p} - L(\boldsymbol{r},\dot{\boldsymbol{r}},t)$ the *(non-Hamiltonian) equations* follow:

$$\left.\begin{array}{l} 1)\ \dfrac{\partial H}{\partial \boldsymbol{p}} = \dot{\boldsymbol{r}} - \dfrac{\partial L}{\partial \boldsymbol{p}} = \dot{\boldsymbol{r}} \\ 2)\ \dfrac{\partial H}{\partial \boldsymbol{r}} = -\dfrac{\partial L}{\partial \boldsymbol{r}} = \boldsymbol{F}^{nc} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\boldsymbol{r}}} = \boldsymbol{F}^{nc} - \dfrac{d\boldsymbol{p}}{dt} \end{array}\right]$$

$$\boxed{\frac{d\boldsymbol{r}}{dt} = \frac{\partial H}{\partial \boldsymbol{p}} \qquad \frac{d\boldsymbol{p}}{dt} = -\frac{\partial H}{\partial \boldsymbol{r}} + \boldsymbol{F}^{nc}} \quad 1.2$$

Liouville equation

- $\boldsymbol{\Gamma}$**−space**: *6N*−dim phase space coordinates, a single point (microstate) represents *N* particles labelled by *3N* positions $\boldsymbol{r}=(\boldsymbol{r}_1 \cdots \boldsymbol{r}_N)$ and momenta $\boldsymbol{p}=(\boldsymbol{p}_1 \cdots \boldsymbol{p}_N), \boldsymbol{r}_i=(x,y,z)_i$ and $\boldsymbol{p}_i=(p_x,p_y,p_z)_i$
- **Ensemble**: $\mathcal{N}$ copies of a specific microstate (N particles) each copy described by a different representative point in Γ−space ($\mathcal{N}\neq N$)
- $d\boldsymbol{\mathcal{N}}(\boldsymbol{r},\boldsymbol{p},\boldsymbol{t})$: number of microstates in the volume element $d\Gamma=\prod_{i=1}^{N} d\boldsymbol{r}_i d\boldsymbol{p}_i$ about any coordinate values $(\boldsymbol{r},\boldsymbol{p})$ at time t
- $\boldsymbol{\rho}(\boldsymbol{r},\boldsymbol{p},\boldsymbol{t})$: density of representative microstates ("coarse-graining" density $\rho(\boldsymbol{r},\boldsymbol{p},t)$ is obtained by disregarding variation of ρ below small resolution in Γ-space) ref. [1] & ref. [B-D]

Formal *density* definition

$$\rho(\boldsymbol{r},\boldsymbol{p},t)\boldsymbol{d\Gamma} = \lim_{\mathcal{N}\to\infty} \frac{d\mathcal{N}(\boldsymbol{r},\boldsymbol{p},t)}{\mathcal{N}}$$

Coarse-graining density

$$\rho(\boldsymbol{r},\boldsymbol{p},t)\Delta\boldsymbol{\Gamma} = \frac{\Delta\mathcal{N}(\boldsymbol{r},\boldsymbol{p},t)}{\mathcal{N}}$$

Liouville equation

- A microstate of N particles with coordinates $(\boldsymbol{r},\boldsymbol{p}) = (\boldsymbol{r}_i,\boldsymbol{p}_i)_{i=1\cdots N}$ at time t will be found at $t+\delta t$ with new coordinates $(\boldsymbol{r}'_i,\boldsymbol{p}'_i)_{i=1\cdots N} = (\boldsymbol{r}_i+\dot{\boldsymbol{r}}_i\,\delta t, \boldsymbol{p}_i+\dot{\boldsymbol{p}}_i\,\delta t+\mathcal{O}(\delta t^2))$
- The microstate density $\rho(\boldsymbol{r},\boldsymbol{p},t)$ at time t will become $\rho(\boldsymbol{r}',\boldsymbol{p}',t+\delta t)$ at $t+\delta t$
- The phase space volume $d\Gamma(t)$ at t will change into $d\Gamma(t+\delta t)$ at $t+\delta t$
- $d\mathcal{N}(\boldsymbol{r}',\boldsymbol{p}',t+\delta t) = d\mathcal{N}(\boldsymbol{r},\boldsymbol{p},t)$ because $(\boldsymbol{r}(t),\boldsymbol{p}(t))$ follow *Hamilton's* equations for (*conservative forces*) and thus *no trajectories cross* (do not escape the *6N-1 dim* surface $C(t)$ enclosing the microstates, $C(t)$ being itself a microstate !)

$$\begin{aligned}\Delta\mathcal{N}(\boldsymbol{r},\boldsymbol{p},t+\delta t)\\ = \Delta\mathcal{N}(\boldsymbol{r},\boldsymbol{p},t)\end{aligned} \Rightarrow \rho(\boldsymbol{r}',\boldsymbol{p}',t+\delta t)\int_{\text{in } C(t+\delta t)} d\Gamma(t+\delta t) = \rho(\boldsymbol{r},\boldsymbol{p},t)\int_{\text{in } C(t)} d\Gamma(t)$$

The relation between $d\Gamma' \stackrel{\text{def}}{=} d\Gamma(t+\delta t)$ with border $C' \stackrel{\text{def}}{=} C(t+\delta t)$ and $d\Gamma \stackrel{\text{def}}{=} d\Gamma(t)$, border $C \stackrel{\text{def}}{=} C(t)$ is

$$\int_{\text{in } C'} d\Gamma' = |J|\int_{\text{in } C} d\Gamma \qquad J = \frac{\partial(\boldsymbol{r}'_i,\boldsymbol{p}'_i)}{\partial(\boldsymbol{r}_i,\boldsymbol{p}_i)}\ (3N\times 3N\ \text{Jacobian}) \qquad \left.\begin{aligned}\boldsymbol{r}_i&=(x,y,z)_i\\ \boldsymbol{p}_i&=(p_x,p_y,p_z)_i\end{aligned}\right\} i=1\cdots N$$

Liouville equation

Using $(\boldsymbol{r}'_i, \boldsymbol{p}'_i) = (\boldsymbol{r}_i + \dot{\boldsymbol{r}}_i\, \delta t, \boldsymbol{p}_i + \dot{\boldsymbol{p}}_i\, \delta t)$ and the *Hamilton's equations* the determinant $|\det J|$ of the *Jacobian* matrix writes (1st order)

$$|\det J| = \begin{vmatrix} \frac{\partial r_1}{\partial r_1} + \frac{\partial \dot{r}_1}{\partial r_1}\delta t & \cdots & \frac{\partial p_N}{\partial r_1} + \frac{\partial \dot{p}_N}{\partial r_1}\delta t \\ \vdots & \ddots & \vdots \\ \frac{\partial r_1}{\partial p_N} + \frac{\partial \dot{r}_1}{\partial p_N}\delta t & \cdots & \frac{\partial p_{3N}}{\partial p_N} + \frac{\partial \dot{p}_{3N}}{\partial p_N}\delta t \end{vmatrix} = \begin{vmatrix} 1 + \frac{\partial \dot{r}_1}{\partial r_1}\delta t & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 + \frac{\partial \dot{p}_N}{\partial p_N}\delta t \end{vmatrix}$$

$$= 1 + \sum_{i=1}^{N} \left(\frac{\partial \dot{r}_i}{\partial r_i} + \frac{\partial \dot{p}_i}{\partial p_i} \right) \delta t + \mathcal{O}(\delta t^2) \quad \Rightarrow \quad \boxed{|\det J| = 1 + \mathcal{O}(\delta t^2)} \quad \Rightarrow$$

$$\boxed{\int_{\text{in } C(t+\delta t)} d\Gamma(t+\delta t) = \int_{\text{in } C(t)} d\Gamma(t)} \quad \Rightarrow$$

Liouville's theorem stems from the conservation of the phase space volume in Γ−space

Liouville equation

Liouville's theorem

The microstate density $\rho(\boldsymbol{r},\boldsymbol{p},t)$ in $\Gamma-$space behaves like an incompressible fluid

$\rho(\boldsymbol{r}',\ \boldsymbol{p}',\ t+\delta t) = \rho(\boldsymbol{r},\boldsymbol{p},\ t)$ $\Leftrightarrow$	$\dfrac{d\rho(\boldsymbol{r},\boldsymbol{p},t)}{dt} = 0$	1.3	Liouville's theorem

Equivalently ρ writes in differential form using the *Hamilton's equations* and *Poisson bracket*:

$\dfrac{d\rho}{dt} = \dfrac{\partial\rho}{\partial t} + \sum_{i=1}^{3N}\left(\dfrac{\partial\rho}{\partial r_i}\dot{r}_i + \dfrac{\partial\rho}{\partial p_i}\dot{p}_i\right) = 0$	$\dfrac{\partial\rho}{\partial t} + \dot{\boldsymbol{r}}\cdot\boldsymbol{\nabla}_r\rho + \dot{\boldsymbol{p}}\cdot\boldsymbol{\nabla}_p\rho = \boldsymbol{0}$	1.4	Liouville's formula
$\{\rho,H\} \stackrel{\text{def}}{=} \sum_{i=1}^{3N}\left(\dfrac{\partial\rho}{\partial r_i}\dfrac{\partial H}{\partial p_i} - \dfrac{\partial\rho}{\partial p_i}\dfrac{\partial H}{\partial r_i}\right)$	$\dfrac{\partial\rho}{\partial t} + \{\rho,H\} = 0$	1.5	

Liouville equation

Consider the (*non-strictly-Hamiltonian*) equations of motion for ***non-conservative forces*** $\boldsymbol{F}^{nc}$:

$$\dot{r}_i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H}{\partial r_i} + F_i^{nc} \Rightarrow \frac{\partial \dot{r}_i}{\partial r_i} + \frac{\partial \dot{p}_i}{\partial p_i} = \frac{\partial F_i^{nc}}{\partial p_i} \Rightarrow \boxed{|\det J| = 1 + \sum_{i=1}^{N} \frac{\partial F_i^{nc}}{\partial p_i} \delta t}$$

$$\Rightarrow \int_{\text{in } C(t+\delta t)} d\Gamma(t+\delta t) = \left(1 + \delta t \sum_{i=1}^{N} \frac{\partial F_i^{nc}}{\partial p_i}\right) \int_{\text{in } C(t)} d\Gamma(t)$$

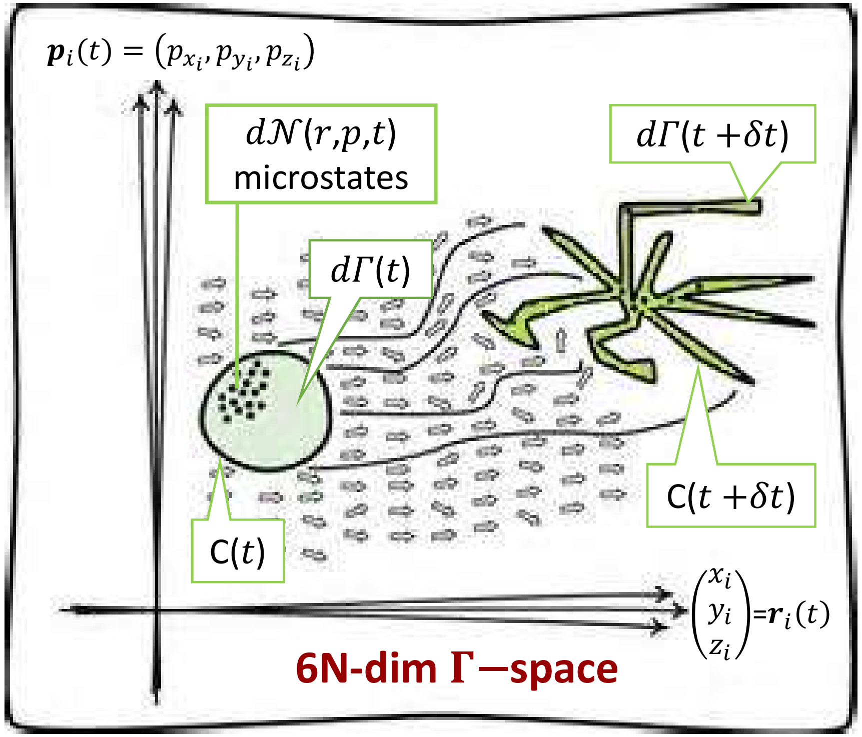
Liouville's theorem "violated" !?: incompressibility condition of $\rho(\boldsymbol{r}, \boldsymbol{p}, t)$ not satisfied i.e.

$$\rho(\boldsymbol{r}', \boldsymbol{p}', t+\delta t) = \left(1 + \delta t\, \boldsymbol{\nabla}_p \cdot \boldsymbol{F}^{nc}\right) \rho(\boldsymbol{r}, \boldsymbol{p}, t) \Rightarrow \frac{\rho(\boldsymbol{r}', \boldsymbol{p}', t+\delta t) - \rho(\boldsymbol{r}, \boldsymbol{p}, t)}{\delta t} = \boldsymbol{\nabla}_p \cdot \boldsymbol{F}^{nc}$$

Written in differential form this lead to the equivalent results:

$$\boxed{\frac{d\rho}{dt} = \boldsymbol{\nabla}_p \cdot \boldsymbol{F}^{nc}} \Leftrightarrow \boxed{\frac{\partial \rho}{\partial t} + \{\rho, H\} = \boldsymbol{\nabla}_p \cdot \boldsymbol{F}^{nc}} \Leftrightarrow \boxed{\frac{\partial \rho}{\partial t} + \dot{\boldsymbol{r}} \cdot \boldsymbol{\nabla}_r \rho + \dot{\boldsymbol{p}} \cdot \boldsymbol{\nabla}_p \rho = \boldsymbol{\nabla}_p \cdot \boldsymbol{F}^{nc}} \quad 1.6$$

Liouville equation



Microstate subset $d\mathcal{N}(r,p,t)$ inside the *6N*−dim volume $d\Gamma(t)$ of border $C(t)$ at t in Γ−space will occupy a distorted volume $d\Gamma(t+\delta t)$ of border $C(t+\delta t)$ at t+δt

Liouville (also called *collisionless Boltzmann*) *equation*

- Detailed account of the density $\rho(\boldsymbol{r}(\boldsymbol{t}), \boldsymbol{p}(\boldsymbol{t}), t)$ would require knowledge of $6N$ particle trajectories with initial conditions for all microstates of the sub-ensemble $d\mathcal{N}$ ($\sim 10^{23}$?!) in the (Γ−space) volume element $d\Gamma$.
- Practically it would be more suitable to place the phase trajectories of the *N* particles in the same *6*−dim phase space (μ−space): a single point represents one particle labelled by *3* positions $\boldsymbol{r}=(x, y, z)$ and 3 momenta $\boldsymbol{p}=(p_x, p_y, p_z)$.
- To reach this objective the 6*N*−dim *microstate* density $\rho(\boldsymbol{r}_1, \cdots \boldsymbol{r}_N, \boldsymbol{p}_1, \cdots \boldsymbol{p}_N, t)$ must be reduced a *6*−dim *particle* density $f_1(\boldsymbol{r}, \boldsymbol{p}, t)$ in (μ−space).
- This should be done via the *BBGKY hierarchy* framework to go from the *N-particles* (in Γ −space) to the *N-times 1-particle* (μ−space) description (ref. [1,2] & ref. [C]).

Liouville equation

- The full phase space density $\rho(\boldsymbol{r},\boldsymbol{p},t)$ contains too much information than needed to describe the equilibrium properties of particles (e.g. 1-particle density is enough to compute a gas pressure).
- The *N*-particle density $\rho(\boldsymbol{r}_1,\boldsymbol{p}_1,\cdots \boldsymbol{r}_N,\boldsymbol{p}_N,t)$ in $6N$–dim Γ–space is to be reduced to a *single particle density* $f_1(\boldsymbol{r},\boldsymbol{p},t)$ in 6–dim μ–space: the *state* of *each particle* being represented by a *single point*.
- $f_1(\boldsymbol{r},\boldsymbol{p},t)/N$ refers to the expectancy of finding any one of the *N* particles at time t with location $\boldsymbol{r}$ and momentum $\boldsymbol{p}$, computed from $\rho(\boldsymbol{r}_1,\boldsymbol{p}_1,\cdots \boldsymbol{r}_N,\boldsymbol{p}_N,t)$ by means of the formulae:

$$f_1(\boldsymbol{r},\boldsymbol{p},t) = \left\langle \sum_{i=1}^{N} \delta\,(\boldsymbol{r}-\boldsymbol{r}_i)\delta(\boldsymbol{p}-\boldsymbol{p}_i) \right\rangle \equiv \int d\Gamma \rho(\boldsymbol{r},\boldsymbol{p},t) \sum_{i=1}^{N} \delta\,(\boldsymbol{r}-\boldsymbol{r}_i)\delta(\boldsymbol{p}-\boldsymbol{p}_i)$$

with for any function $\mathcal{O}(\boldsymbol{r},\boldsymbol{p})$: $\langle \mathcal{O} \rangle = \int d\Gamma\, \rho(\boldsymbol{r},\boldsymbol{p},t)\mathcal{O}(\boldsymbol{r},\boldsymbol{p})$. Using the first pair of delta functions to compute one set of integrals we get, assuming a symmetric density when permuting particles:

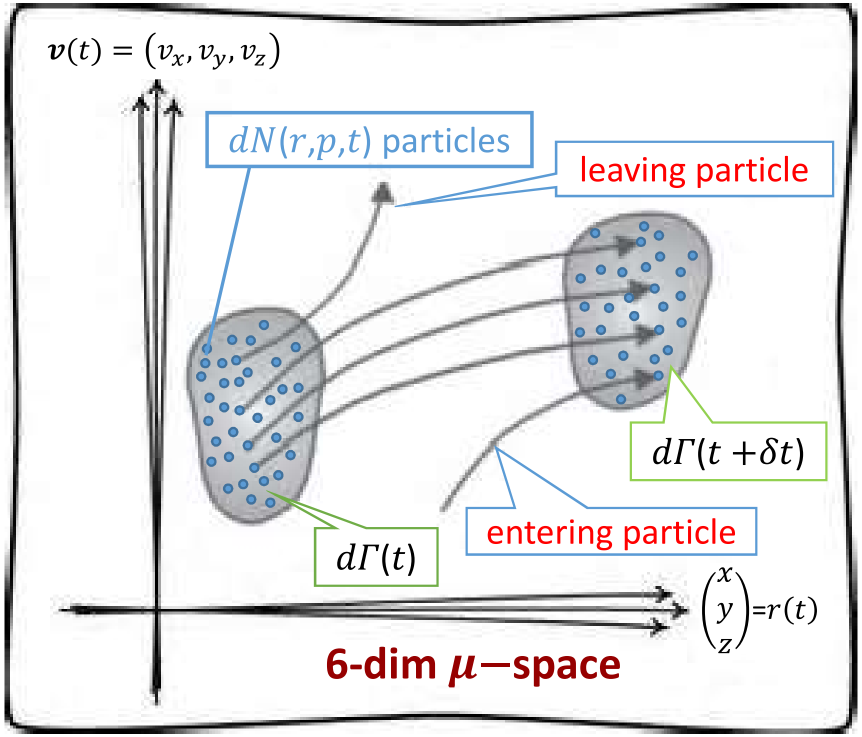
For many aims the reduced function f_1 governed by the BBGKY hierarchy (Bogoliubov, Born, Green, Kirkwood, Yvon) is all it is really needed to know about a N-particles system in the $6N$-dim Γ–space because it describes its density function in the 6-dim μ–space.

$$f_1(\boldsymbol{r},\boldsymbol{p},t) = N \int \prod_{i=2}^{N} d\boldsymbol{r}_i d\boldsymbol{p}_i\, \rho(\boldsymbol{r},\boldsymbol{p},\boldsymbol{r}_2,\boldsymbol{p}_2 \cdots, \boldsymbol{r}_N,\boldsymbol{p}_N,t)$$

e. g. N=2: $f_1(\boldsymbol{x}) = \iint d\boldsymbol{x}_1 d\boldsymbol{x}_2 \rho(\boldsymbol{x}_1,\boldsymbol{x}_2)\{\delta(\boldsymbol{x}-\boldsymbol{x}_1)+\delta(\boldsymbol{x}-\boldsymbol{x}_2)\} = \int d\boldsymbol{x}_2 \rho(\boldsymbol{x}_1{=}\boldsymbol{x},\boldsymbol{x}_2) + \int d\boldsymbol{x}_1 \rho(\boldsymbol{x}_1,\boldsymbol{x}_2{=}\boldsymbol{x}) = 2\int d\boldsymbol{x}_2 \rho(\boldsymbol{x},\boldsymbol{x}_2)$

f_1 is *normalized* to N and ρ to 1

Boltzmann collision equation



Particle subset $dN(r,p,t)$ inside μ−space at t+δt due to collisions in the time δt

Liouville formula needs then to be adapted to *Boltzmann collision equation* when considering *particle interactions*

- As a result of collisions during the time interval δt particles that were inside the volume $d\Gamma = d\boldsymbol{r}d\boldsymbol{p}$ in the 6-dim μ−space may be removed from it and particles outside $d\Gamma$ may end up inside it.
- The net *gain* or *loss* of particles as a result of *collisions* during δt inside $d\Gamma$ is denoted:

$$\left[\frac{\delta f_1(\boldsymbol{r}_1, \boldsymbol{p}_1, t)}{\delta t}\right]_{\mathrm{coll}} d\boldsymbol{r}d\boldsymbol{p}\delta t$$

where $(\delta f_1/\delta t)_{\mathrm{coll}}$ means the rate of change of f_1. Hence the *Liouville equation* turns into the *collision Boltzmann equation*

$$\frac{\partial \rho}{\partial t} + \dot{\boldsymbol{r}} \cdot \boldsymbol{\nabla}_r \rho + \dot{\boldsymbol{p}} \cdot \boldsymbol{\nabla}_p \rho = \left(\frac{\delta f_1}{\delta t}\right)_{\mathrm{coll}} \equiv \boldsymbol{\nabla}_p \cdot \boldsymbol{F}^{nc} = \frac{\partial}{\partial \boldsymbol{r}} \cdot \boldsymbol{F}^{nc} \quad (1.7)$$

non conservative force field

Boltzmann collision equation

Heuristic assumptions are made to «derive» the *Boltzmann collision* equation:

- f_1 does not vary visibly over the *distance* of *interparticle force range* and over the *time scale* of the *interaction*.
- Disregard *external force* effects on the *collision cross-section* size.
- Consider only *binary collisions*.
- "*Molecular chaos*" assumption: the interacting particle momenta (velocities), before collision, are assumed to be uncorrelated, i.e.
  - the joint probability $f_2(\boldsymbol{r}, \boldsymbol{p}_1, \boldsymbol{r}, \boldsymbol{p}_2, t)$ of having, at position $\boldsymbol{r}$ and time t, particles 1 & 2 of momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ is equal to $f_1(\boldsymbol{r}, \boldsymbol{p}_1, t) f_1(\boldsymbol{r}, \boldsymbol{p}_2, t)$ (supposing that collisions are local in space so that the 2 particles sit at the same point).

- Generally the joint probability density would be equal to $f_1(\boldsymbol{r}, \boldsymbol{p}_1, t) f_1(\boldsymbol{r}, \boldsymbol{p}_2, t)[1+\mathrm{K}_2(\boldsymbol{r}, \boldsymbol{p}_1, \boldsymbol{p}_2, t)]$ where $\mathrm{K}_2(\boldsymbol{r}, \boldsymbol{p}_1, \boldsymbol{p}_2, t)$ is a *correlation* function.
- To by-pass the *molecular chaos* approximation the alternative is to work with the equations of the BBGKY hierarchy (Bogoliubov, Born, Green, Kirkwood, Yvon) ref. [B,C,E,F].

Boltzmann collision equation

Let's start with an Hamiltonian $H(\boldsymbol{r},\boldsymbol{p})$ with no interacting collision potential between particle pairs (e.g. Coulomb scattering potential). This Hamiltonian will just contain:

- *Particle kinetic energy* (for non relativistic charged particles)
- *External potential* $\Phi(\boldsymbol{r})$ (e.g. electromagnetic field for charged particle beam)

$$H(\boldsymbol{r},\boldsymbol{p}) = \sum_{i=1}^{N}\left[\frac{p_i^2}{2m} + \Phi(\boldsymbol{r}_i)\right]$$

From Liouville's formula in terms of Poisson bracket and replacing the $6N$–dim density ρ in Γ–space by the 6–dim density f_1 in μ–space we get:

$$\{H, f_1\} = \frac{\partial H}{\partial \boldsymbol{r}_1}\frac{\partial f_1}{\partial \boldsymbol{p}_1} - \frac{\partial H}{\partial \boldsymbol{p}_1}\frac{\partial f_1}{\partial \boldsymbol{r}_1} = \frac{\partial \Phi}{\partial \boldsymbol{r}_1}\frac{\partial f_1}{\partial \boldsymbol{p}_1} - \frac{\boldsymbol{p}_1}{m}\frac{\partial f_1}{\partial \boldsymbol{r}_1}$$

$$\boxed{\frac{\partial f_1}{\partial t} + \{f_1, H\} = \boldsymbol{0}} \Rightarrow \frac{\partial f_1}{\partial t} - \frac{\partial \Phi}{\partial \boldsymbol{r}_1}\frac{\partial f_1}{\partial \boldsymbol{p}_1} + \frac{\boldsymbol{p}}{m}\frac{\partial f_1}{\partial \boldsymbol{r}_1} = 0 \qquad 1.8$$

collisionless Boltzmann equation

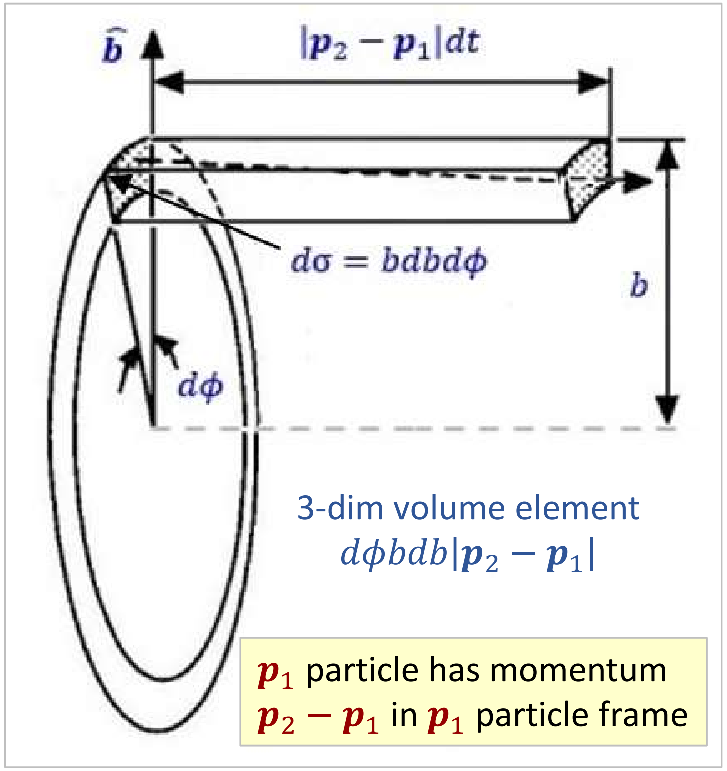
The external force $\boldsymbol{F} = m\boldsymbol{a}$ (e.g. in a plasma) includes the Lorentz force $\dot{\boldsymbol{p}} = e\,(\boldsymbol{E} + \dot{\boldsymbol{r}} \times \boldsymbol{B})$ due to externally applied fields.

Boltzmann collision equation

Collision terms:

The interaction result is characterized by the *net rate* at which *collisions increase* or *decrease* the particle number entering the *6-dim phase-space* slice $\partial \boldsymbol{r} \partial \boldsymbol{p}$ in time δt (named $\delta\mathcal{R}$) defined as: $\delta\mathcal{R}=\delta\mathcal{R}_{+}-\delta R_{-}$ where $\delta\mathcal{R}_{\pm}$ are the particle number *injected*/*ejected* in $\partial \boldsymbol{r} \partial \boldsymbol{p}$ by collisions in δt

- For $\delta\mathcal{R}_{-}$: particles are shared in 2 groups, the 1st of momenta in the interval $\partial \boldsymbol{p}$ about $\boldsymbol{p_1}$ and the 2nd of all other momenta denoted $\boldsymbol{p_2}$, the particles ejected from $\partial \boldsymbol{r} \partial \boldsymbol{p}$ are the number of collisions that the $\boldsymbol{p_1}$'s have with all other $\boldsymbol{p_2}$'s (not in 1st group) in δt. To compute $\delta\mathcal{R}_{-}$ all collisions between pairs of particles that eject one of them out of the interval $\partial \boldsymbol{p}$ about $\boldsymbol{p_1}$ are considered:
  - One particle is in $\partial \boldsymbol{r} \partial \boldsymbol{p}$ near $(\boldsymbol{r_1}, \boldsymbol{p_1})$ the other in $\partial \boldsymbol{r_2} \partial \boldsymbol{p_2}$ near $(\boldsymbol{r_2}, \boldsymbol{p_2})$
  - The $\boldsymbol{p_2}$'s in $\partial \boldsymbol{r_2}$ suffer a collision with the $\boldsymbol{p_1}$'s in $\partial \boldsymbol{r}$ in time δt.
- For $\delta\mathcal{R}_{+}$: consider all pair-particle collisions that send one particle into the momentum interval $\partial \boldsymbol{p}$ about $\boldsymbol{p_1}$ in time δt which is the inverse of the original collision $(\boldsymbol{p}'_1, \boldsymbol{p}'_2) \rightleftarrows (\boldsymbol{p_1}, \boldsymbol{p_2})$



Boltzmann collision equation

The number of particles *injected*/*ejected* into $\partial\boldsymbol{r}\partial\boldsymbol{p}$ by collisions in time δt are:

$$\delta\mathcal{R}_- = \int_{(\boldsymbol{r}_2,\boldsymbol{p}_2)} f_1(\boldsymbol{r},\boldsymbol{p}_1,t) f_1(\boldsymbol{r},\boldsymbol{p}_2,t)\, d\boldsymbol{r} d\boldsymbol{p}_1 d\boldsymbol{p}_2 \quad \delta\mathcal{R}_+ = \int_{(\boldsymbol{r}'_2,\boldsymbol{r}'_2)} f_1(\boldsymbol{r}',\boldsymbol{p}'_1,t) f_1(\boldsymbol{r}',\boldsymbol{p}'_2,t)\, d\boldsymbol{r}' d\boldsymbol{p}'_1 d\boldsymbol{p}'_2$$

All $\boldsymbol{p}_2$ particles shown (see fig. above) in the cylinder of height $|\boldsymbol{p}_2 - \boldsymbol{p}_1|\delta t$ and base area $bdbd\phi$ suffer a collision with the $\boldsymbol{p}_1$ particle in time δt $\Rightarrow$ $d\boldsymbol{r} = |\boldsymbol{p}_2 - \boldsymbol{p}_1|\delta t bdbd\phi$ (idem for $\boldsymbol{p}'_1$, $\boldsymbol{p}'_2$). Also since $d^3\boldsymbol{r}d^3\boldsymbol{p} = d^3\boldsymbol{r}'d^3\boldsymbol{p}'$

$$\Rightarrow \quad \delta\mathcal{R}_- = \left(\int f_1 d\boldsymbol{p}_2\, |\boldsymbol{p}_2 - \boldsymbol{p}_1| bdbd\phi\right) d^3\boldsymbol{r} d^3\boldsymbol{p}_1 \delta t \quad \delta\mathcal{R}_+ = \left(\int f_1 d\boldsymbol{p}_2\, |\boldsymbol{p}_2 - \boldsymbol{p}_1| bdbd\phi\right) d^3\boldsymbol{r} d^3\boldsymbol{p}_1 \delta t$$

$$\Rightarrow \quad \boxed{\delta\mathcal{R} = \int [f_1(\boldsymbol{r}',\boldsymbol{p}'_1,t) f_1(\boldsymbol{r}',\boldsymbol{p}'_2,t) - f_1(\boldsymbol{r},\boldsymbol{p}_1,t) f_1(\boldsymbol{r},\boldsymbol{p}_2,t)] |\boldsymbol{p}_2 - \boldsymbol{p}_1| d\boldsymbol{r} d\boldsymbol{p}_1 bdbd\phi}$$

From Liouville equation the net number of particles that enter the 6-dim phase element $d\boldsymbol{r}d\boldsymbol{p}$ keeping on a particle trajectory during δt is zero. Likewise the collisionless Boltzmann equation writes:

$$\boxed{\delta\mathcal{R}_{\text{Liouville}} \equiv d\boldsymbol{r}d\boldsymbol{p}\,\delta t \left[\frac{\partial f_1}{\partial t} - \frac{\partial\Phi}{\partial \boldsymbol{r}_1}\frac{\partial f_1}{\partial \boldsymbol{p}_1} + \frac{\boldsymbol{p}}{m}\frac{\partial f_1}{\partial \boldsymbol{r}_1}\right] = 0}$$

Boltzmann collision equation

Hence the above term $\delta\mathcal{R}$ can be cast into the form:

$$\frac{\delta\mathcal{R}}{d\boldsymbol{r}d\boldsymbol{p}_1\delta t}=\int[f_1(\boldsymbol{r},\boldsymbol{p}_1',t)f_1(\boldsymbol{r},\boldsymbol{p}_2',t)-f_1(\boldsymbol{r},\boldsymbol{p}_1,t)f_1(\boldsymbol{r},\boldsymbol{p}_2,t)]|\boldsymbol{p}_2-\boldsymbol{p}_1|d\boldsymbol{p}_2 bdbd\phi\equiv\left[\frac{\delta f_1(\boldsymbol{r},\boldsymbol{p}_1,t)}{\delta t}\right]_{\text{coll}}$$

- The quantity $bd\phi db\equiv d\sigma$ having dimensions of area can be written as $d\sigma=(d\sigma/d\Omega)d\Omega$ in which $|d\sigma/d\Omega|$ is the *differential cross-section* (see below).
- Replacing $|\boldsymbol{p}_2-\boldsymbol{p}_1|/m$ by the *velocity* $|\boldsymbol{v}_2-\boldsymbol{v}_1|$ (*non relativistic* particles) the collision term writes:

$$\left(\frac{\delta f_1}{\delta t}\right)_{\text{coll}}=\int d\boldsymbol{v}_2\int d\Omega\left|\frac{d\sigma}{d\Omega}\right||\boldsymbol{v}_2-\boldsymbol{v}_1|\,[f_1(\boldsymbol{r},\boldsymbol{p}_1',t)f_1(\boldsymbol{r},\boldsymbol{p}_2',t)-f_1(\boldsymbol{r},\boldsymbol{p}_1,t)f_1(\boldsymbol{r},\boldsymbol{p}_2,t)] \quad 1.9$$

Putting $(\delta f_1/\delta t)_{\text{coll}}$ in the collisionless Boltzmann equation yields the *Boltzmann collision equation*:

$$\left(\frac{\partial}{\partial t}-\frac{\partial\Phi}{\partial\boldsymbol{r}_1}\frac{\partial}{\partial\boldsymbol{p}_1}+\frac{\boldsymbol{p}_1}{m}\frac{\partial}{\partial\boldsymbol{r}_1}\right)f_1=\int d\boldsymbol{v}_2\int d\Omega\left|\frac{d\sigma}{d\Omega}\right||\boldsymbol{v}_2-\boldsymbol{v}_1|\,[f_1(\boldsymbol{r},\boldsymbol{p}_1',t)f_1(\boldsymbol{r},\boldsymbol{p}_2',t)-f_1(\boldsymbol{r},\boldsymbol{p}_1,t)f_1(\boldsymbol{r},\boldsymbol{p}_2,t)] \quad 1.10$$

Particle interactions modify the Liouvillian flow

Boltzmann collision equation

Kinematics of collisions:

- A cylindrical polar coordinates is taken to do the above integral: the *scattering* angle θ refers to the x-axis *parallel* to $\boldsymbol{p}_2 - \boldsymbol{p}_1$ (before x_1), the perpendicular plane is parametrized by the y-axis parallel to the *impact parameter* $\hat{\boldsymbol{b}}$ (unit vector) and by the angle ϕ, r_m is the *distance of closest approach*.

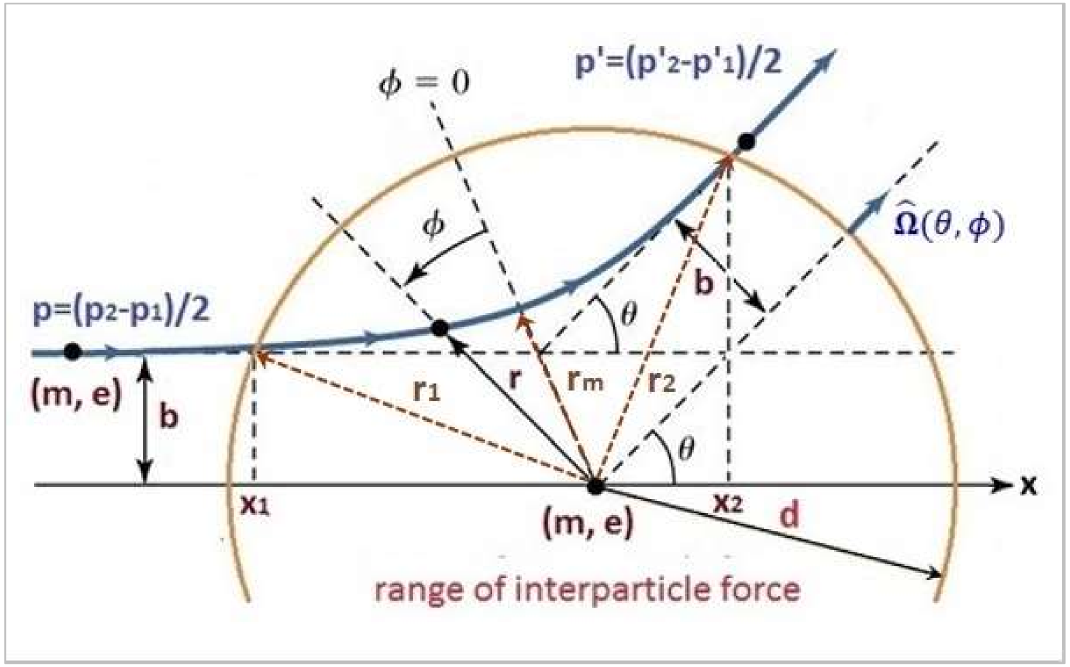
- *Non-relativistic* collision of 2 particles of mass m and momenta $\boldsymbol{p}_{1,2} = m\boldsymbol{v}_{1,2}$ seen from a frame in which one particle is at rest at $x = 0$.
- The out-going momenta $\boldsymbol{p}'_{1,2}$ are given from the conditions:
  1. Conserved momentum: $\boldsymbol{p}'_2 + \boldsymbol{p}'_1 = \boldsymbol{p}_2 + \boldsymbol{p}_1$
  2. Conserved energy:

     $|\boldsymbol{p}'_2|^2 + |\boldsymbol{p}'_1|^2 = |\boldsymbol{p}_2|^2 + |\boldsymbol{p}_1|^2 \Leftrightarrow$

     $\boldsymbol{p}'_2 - \boldsymbol{p}'_1 = |\boldsymbol{p}_2 - \boldsymbol{p}_1|\hat{\Omega}(\theta,\phi) \Rightarrow$

     $|\boldsymbol{p}'_2 - \boldsymbol{p}'_1| \equiv |\boldsymbol{p}_2 - \boldsymbol{p}_1|$ (constant modulus)

     where $\hat{\boldsymbol{\Omega}}(\theta,\phi)$ is a *solid angle* unit vector



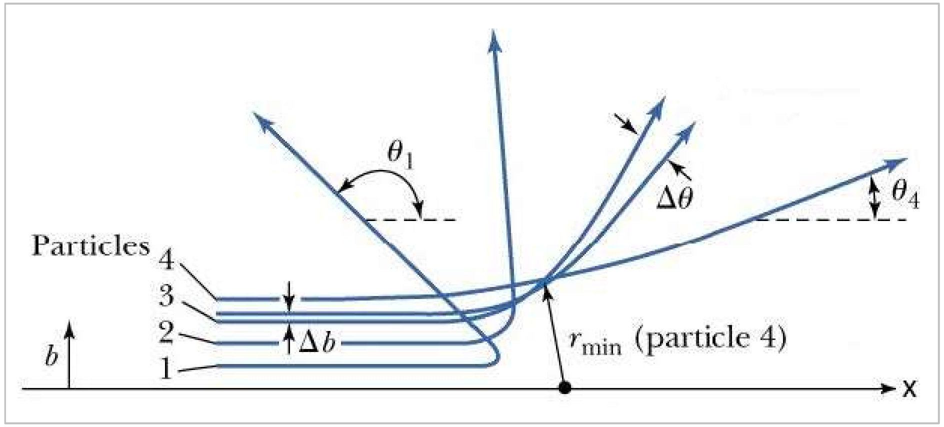
Boltzmann collision equation

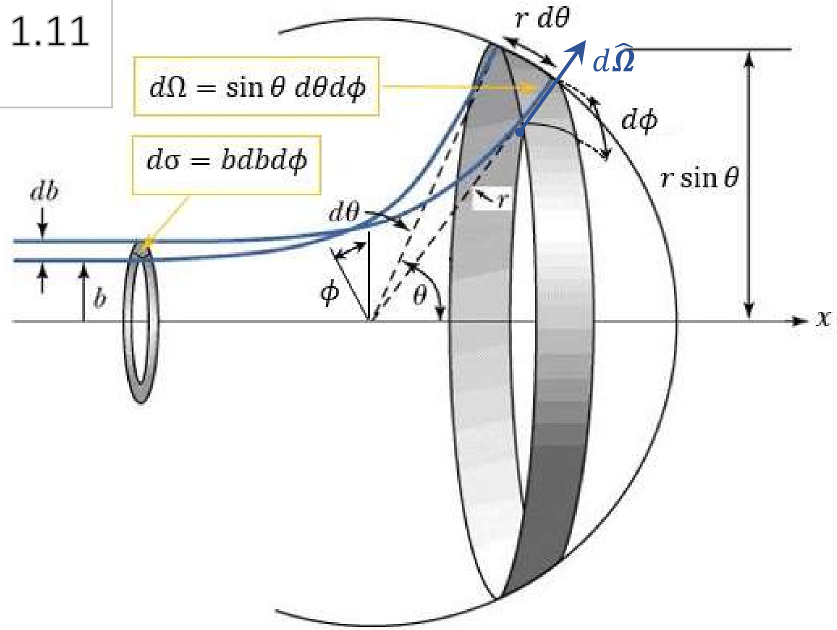
- Differential cross-section: $\boxed{|d\sigma/d\Omega| = |db/d\theta|}$ [m^2] (ref. [19] & ref. [B-D])
  - This is the number of particles *scattered* per *unit time*, unit *incident flux* and oriented *solid angle* $\widehat{\boldsymbol{\Omega}}(\theta,\phi)$ (the absolute value $|\cdots|$ comes because θ usually decreases when b increases)
  - Geometrically the next figures show a scattering process with $d\Omega = \sin\theta\, d\theta d\phi$ and $d\sigma = b\, dbd\phi$ where θ depends on the *interparticle force* law, the *relative momentum* $|\boldsymbol{p}_2 - \boldsymbol{p}_1|$ and *impact parameter* b

- Rutherford scattering:

$$\left|\frac{d\sigma}{d\Omega}\right| = \left(\frac{me^2}{4\pi\epsilon_0|\boldsymbol{p}_2 - \boldsymbol{p}_1|^2}\right)^2 \frac{1}{\sin^4(\theta/2)} \tag{1.11}$$

  - Small θ yield large b ($\theta_{\min}=0 \rightarrow b_{\max}=\infty$)





Equilibrium particle density

- **Equilibrium**: At equilibrium the 1−particle density $f_1(\boldsymbol{r},\boldsymbol{p})$ has no explicit time dependence:

  $$\partial f_1/\partial t = 0 \longrightarrow \{H_1, f_1\} = 0 \longrightarrow f_1 = f_1(H_1) \text{ with } H_1(\boldsymbol{r},\boldsymbol{p}) = \boldsymbol{p}^2/2m + \Phi(\boldsymbol{r})$$

- **Maxwell-Boltzmann distribution**: Similarly at equilibrium the collision integral vanishes: (ref. [C,D,F])

$$f_1(\boldsymbol{r},\boldsymbol{p}_1) f_1(\boldsymbol{r},\boldsymbol{p}_2) = f_1(\boldsymbol{r},\boldsymbol{p}_1') f_1(\boldsymbol{r},\boldsymbol{p}_2') \qquad \ln f_1(\boldsymbol{r},\boldsymbol{p}_1) + \ln f_1(\boldsymbol{r},\boldsymbol{p}_2) = \ln f_1(\boldsymbol{r},\boldsymbol{p}_1') + \ln f_1(\boldsymbol{r},\boldsymbol{p}_2')$$

where the l.h.s. refers to momenta before collision the r.h.s. to the those after collision.
The equality is satisfied by any additive *invariant* quantities during the collision, e.g.

$$\ln f_1(\boldsymbol{r},\boldsymbol{p}) = -\beta[\boldsymbol{p}^2/2m + \Phi(\boldsymbol{r})] \Longrightarrow \boxed{f_1(\boldsymbol{r},\boldsymbol{p}) = \alpha e^{-\beta[\boldsymbol{p}^2/2m + \Phi(\boldsymbol{r})]}}$$

α and β are constants, from which the *Maxwell-Boltzmann velocity density* (for $\Phi(\boldsymbol{r}) = 0$) follows:

For a gaz of N particles in a box volume V for $\boldsymbol{p} = m\boldsymbol{v}$, $\boldsymbol{u}$ an overall *drift*, k the *Boltzmann* constant (the integral of f_1 over the ***3-dim*** box volume V is equal to N since $f_1 d\boldsymbol{p}$ must be normalized to N):

$$f_1(\boldsymbol{v}) = \frac{N}{V}\left(\frac{\beta m}{2\pi}\right)^{3/2} e^{-\beta m(\boldsymbol{v}-\boldsymbol{u})^2/2} \xLeftrightarrow{\beta = 1/kT} f_1(\boldsymbol{v}) = \frac{N}{V}\frac{1}{(2\pi kT/m)^{3/2}} e^{-m(\boldsymbol{v}-\boldsymbol{u})^2/(2kT)} \qquad 1.12$$

INTRABEAM SCATTERING

- Part 2: Intrabeam scattering
  - Core IBS model
  - IBS analytical model
  - Original Piwinski model
  - Bjorken-Mtingwa model
- Part 3: Applications
  - IBS & LHC (7 TeV)
  - IBS & ELENA (100 keV)
  - Epilogue
- Appendices: Feynman rules

The Intrabeam scattering effect

- Theoretical models calculate the **IBS growth rates**:

$$\frac{1}{T_i} \propto \frac{N}{\gamma \varepsilon_{xn} \varepsilon_{yn} \varepsilon_{sn}} f(optics, \gamma, \varepsilon_{xn}, \varepsilon_{yn}, \varepsilon_{sn})$$

- Complicated integrals averaged around the rings
  - Depend on optics and beam properties

- ✓ They have been well benchmarked for hadron machines
- ➢ For lepton machines the work is in progress
  - ➢ Need to benchmark the IBS effect in the presence of SR and QE
  - ➢ Studies and publications from: ATF(2001), CesrTA, SLS, SPEAR3
- ➢ Main drawbacks:
  - Gaussian beams assumed
  - Betatron coupling not trivial to be included
  - Impact on damping process (especially in strong IBS regimes)?
- ➢ Tracking codes **SIRE** (A. Vivoli) and **CMAD-IBStrack** (M. Pivi, T. Demma)
  - Based on the classical Rutherford cross section

Courtesy F. Antoniou, Y. Papaphilippou, CERN

Core IBS model

Continuation… from Part 1

Transverse & longitudinal beam growth rate estimate: A strategy in 7 steps

- In conformity with Piwinski's approach (refs. [3,5]), calculations of beam size *growth*/*decrease rates* caused by IBS effect are sketched out to give a sound idea of the process.
- The kinematics & dynamics of charged particle pair collisions is delineated over the following steps:

1. Transform the momenta of the colliding particles from the *LAB* to the centre of mass (*CM*) frame
2. Calculate the changes in momenta due to an *elastic collision*.
3. Transform of the momenta back to the *LAB* frame.
4. Relate the changes in *momenta* to changes in *transverse* & *longitudinal emittances*.
5. Average over the *scattering angle* distribution using the classical *Rutherford* cross-section.
6. Average over the *particle momentum* & *position* distributions in a bunch.
7. Calculate the *growth*/*fall rates* of mean *betatron* oscillation amplitudes & *momentum spread* in a bunch.

Strategy step 1-3: momenta kinematics

Core IBS model

According to Piwinski (ref. [3,5]) the relative longitudinal and transverse *momentum changes* after a two particles (labelled 1, 2) collision can be cast (after some hard-working task) into the form:

$$\delta \boldsymbol{p}_{1,2} = \boldsymbol{p}'_{1,2} - \boldsymbol{p}_{1,2} =$$

defining

$$\theta = \frac{p_{x_1} - p_{x_2}}{p} \equiv x'_1 - x'_2 \qquad \zeta = \frac{p_{z1} - p_{z2}}{p} \equiv z'_1 - z'_2$$

$$\gamma\xi = \frac{p_1 - p_2}{p} \qquad 2\alpha \equiv \alpha_1 + \alpha_2 = \sqrt{\theta^2 + \zeta^2}$$

$$\frac{\delta p_s}{p} \approx \frac{\delta p}{p} = \frac{\gamma}{2}\left[2\alpha \cos\bar{\phi} \sin\bar{\psi} + \xi(\cos\bar{\psi} - 1)\right]$$

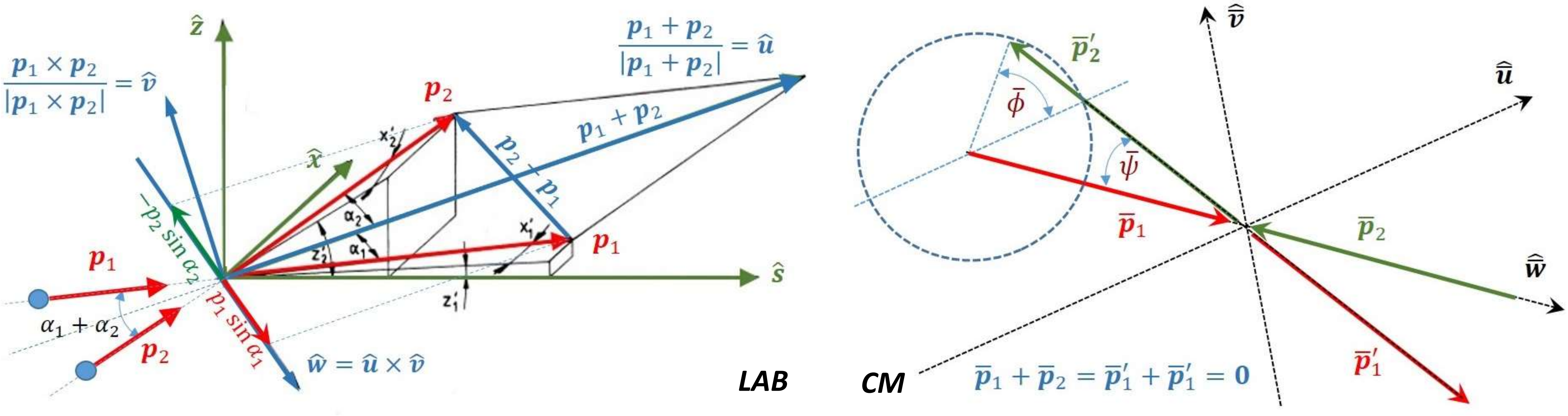
$$2\frac{\delta p_x}{p} = \left[\zeta\sqrt{1 + \frac{\xi^2}{4\alpha^2}} \sin\bar{\phi} - \frac{\xi\theta}{2\alpha}\cos\bar{\phi}\right]\sin\bar{\psi} + \theta(\cos\bar{\psi} - 1) \qquad 2.1$$

$$2\frac{\delta p_z}{p} = \left[\theta\sqrt{1 + \frac{\xi^2}{4\alpha^2}} \sin\bar{\phi} - \frac{\xi\zeta}{2\alpha}\cos\bar{\phi}\right]\sin\bar{\psi} + \zeta(\cos\bar{\psi} - 1)$$

- $\delta \boldsymbol{p}_{1,2}$ are the back momenta *Lorentz transform* from momenta in ad-hoc *CM* frame $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}})$-axes to the *LAB* frame $(\hat{\boldsymbol{s}}, \hat{\boldsymbol{x}}, \hat{\boldsymbol{z}})$-axes ($p_{1,2} = |\boldsymbol{p}_{1,2}|$, $\boldsymbol{p}$ is the mean particle momentum, $\hat{\boldsymbol{s}}$ =unit vector, γ the *Lorentz* factor, $\bar{\psi}$ & $\bar{\phi}$ the axial & azimuthal collision angles in *CM*, $2\alpha \equiv \alpha_1 + \alpha_2$ the angle between particle momenta in *LAB*) (ref. [K])
- $\boldsymbol{p}'_{1,2}$ are the rotated momenta after collision with angles $\bar{\psi}$ & $\bar{\phi}$ (expressed in *LAB* frame).
- $\boldsymbol{p}_{1,2}$ are the momenta before collision written as $\boldsymbol{p}_{1,2} = p_{s_{1,2}}(1, x'_{1,2}, z'_{1,2})$ via $(\hat{\boldsymbol{s}}, \hat{\boldsymbol{x}}, \hat{\boldsymbol{z}})$-coordinates in *LAB* frame and $\boldsymbol{p}_{1,2} = p_{s_{1,2}}(\cos\alpha_{1,2}, 0, \pm\sin\alpha_{1,2})$ via $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}})$-coordinates in *CM* frame (see next Fig.)

Strategy step 1-3: momenta kinematics

Core IBS model



- Particle momenta $\boldsymbol{p}_{1,2}$ *before* collision in ***LAB*** frames $(\hat{s}, \hat{x}, \hat{z})$
- Relation between initial $\boldsymbol{p}_{1,2}$ and final $\boldsymbol{p}'_{1,2}$ is quite complex
- The overlaid $(\hat{u}, \hat{v}, \hat{w})$ frame is aligned on ***CM*** particle motion

- Particle momenta *before* collision $(\overline{\boldsymbol{p}}_1, \overline{\boldsymbol{p}}_2)$ and *after* $(\overline{\boldsymbol{p}}'_1, \overline{\boldsymbol{p}}'_2)$ in the *CM* frame $(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{v}}, \widehat{\boldsymbol{w}})$ ($\widehat{\boldsymbol{u}}$ is the Lorentz-transformed longitudinal axis from *LAB* to *CM* frame)

The *change of particle momentum* after collision leads to a parallel change of the particle *invariants* (i.e. longitudinal & transverse *emittances*) which result supposing that transverse particle positions are not altered during the interaction time (assumed to be short enough).

Strategy step 4: emittance changes

Core IBS model

- The radial particle movement from the closed orbit is the sum of betatron & momentum deviation.
- The invariants are the beam emittances $\varepsilon_{x,z}$ & H (for *bunched beams*) in which $\alpha_{x,z}, \beta_{x,z}, \gamma_{x,z}$ are the Twiss parameters, with $\beta_{x,z}\gamma_{x,z}-\alpha^2_{x,z}=1$, $2\alpha_{x,z}=-\beta'_{x,z}$, Ω is the synchrotron frequency:

$x = x_\beta + D_x \Delta p/p$	$z = z_\beta$	$\varepsilon_x = \gamma_x x_\beta^2 + 2\alpha_x x_\beta x'_\beta + \beta_x x'^2_\beta$	2.2
$x' \equiv p_x/p = x'_\beta - D'_x \Delta p/p$	$z' \equiv p_z/p = z'_\beta$	$H = (\Delta p/p)^2 + \Omega^{-2}\left[\frac{d}{dt}(\Delta p/p)\right]^2$	

The change $\delta\varepsilon_{x,z}$ of $\varepsilon_{x,z}$ works out as (swap x with z for $\delta\varepsilon_z$):

$$\delta\varepsilon_x = \gamma_x(2x_\beta\delta x_\beta + \delta x_\beta^2) + 2\alpha_x\left(x'_\beta\delta x_\beta + x_\beta\delta x'_\beta + \delta x_\beta\delta x'_\beta\right) + \beta_x\left(2x'_\beta\delta x'_\beta + \delta x'^2_\beta\right) \quad 2.3$$

Assuming there is no vertical dispersion i.e. $D_z=D'_z=0$ and that $x_{1,2}$ & $z_{1,2}$ stay constant during the short collision time so that only $x'_{1,2}$ & $z'_{1,2}$ vary with the momentum change. Since $\delta(\Delta p/p)=\delta p/p$ as the mean momentum $p=|\boldsymbol{p}|$ is constant without acceleration, the variations δx_β, $\delta x'_\beta$, $\delta z'_\beta$ can be written in term of betatron amplitudes as follows:

(e.g. $\delta x=\delta x_\beta+D_x\Delta p/p=\delta x_\beta+D_x\delta(\Delta p/p)=\delta x_\beta+D_x\delta p/p\equiv 0 \Longrightarrow \delta x_\beta=-D_x\delta p/p$)

Strategy step 5: scattering angle averages

Core IBS model

$$\delta x_\beta = -D_x \delta p/p \quad \delta x'_\beta = \delta p_x/p - D'_x \delta p/p \quad \delta z'_\beta = \delta p_z/p \qquad 2.4$$

The changes $\delta\varepsilon_{x,z}$ & δH of $\varepsilon_{x,z}$ & H *after collision* can be rewritten (in which $\widetilde{D}_x = \alpha_x D_x + \beta_x D'_x$ and by disregarding the time variation of Ω during the collision) as:

$$\frac{\delta\varepsilon_x}{\beta_x} = -\frac{2}{\beta_x}\left[x_\beta(\gamma_x D_x + \alpha_x D'_x) + x'_\beta \widetilde{D}_x\right]\frac{\delta p}{p} + \frac{D_x^2 + \widetilde{D}_x^2}{\beta_x^2}\left[\frac{\delta p}{p}\right]^2 + 2\left[x'_\beta + \frac{\alpha_x}{\beta_x}x_\beta \frac{\delta p_x}{p}\right] + \left[\frac{\delta p_x}{p}\right]^2 - \frac{2\widetilde{D}_x}{\beta_x}\frac{\delta p}{p}\frac{\delta p_x}{p} \qquad 2.5$$

$$\frac{\delta\varepsilon_z}{\beta_z} = 2\left[z'_\beta + \frac{\alpha_z}{\beta_z}z_\beta \frac{\delta p_z}{p}\right] + \left[\frac{\delta p_z}{p}\right]^2 \qquad 2.6$$

$$\delta H = 2\frac{\Delta p}{p}\frac{\delta p}{p} + \left[\frac{\delta p}{p}\right]^2 \qquad 2.7$$

The phase space volume variation is got by averaging the change of the particle invariant over the collisions.

- For a scattering process, Piwinski introduced the derivatives $d\langle\varepsilon_{x_1,z_2}\rangle/d\bar{t}$, i.e. the mean emittance change of a 1st particle by averaging with all betatron angles (or momentum spread) of a 2nd particle.
- Further averages over positions, betatron angles (or momentum deviations) of the 1st particle must be done to get the total mean emittance change of all particles: i.e. integrate over the phase space with the probability density law P ($\bar{P}$) in the *LAB* & *CM* frames. In formula this writes as follows:

Strategy step 5:
scattering angle averages

Core IBS model

Holds the physics of the collisions

$$\frac{d}{d\bar{t}}\frac{\langle \varepsilon_{x_1} \rangle}{\beta_x} = \left\langle \int 2c\bar{\beta}\bar{P} \int_{\bar{\psi}_{\min}}^{\pi} d\,\bar{\psi} \int_0^{2\pi} d\bar{\phi} \left| \frac{d\bar{\sigma}}{d\bar{\Omega}} \right| \frac{\delta \varepsilon_{x_1}}{\beta_x} \sin\bar{\psi}\, d\bar{V} \right\rangle \quad 2.8$$

The small bracket $\langle \cdot \rangle$ denotes an average over all particles, the outer bracket means an average round the ring circumference, $|d\bar{\sigma}/d\bar{\Omega}|$ is the *Rutherford* differential cross-section for the scattering into a solid angle element $d\bar{\Omega}(\bar{\phi},\bar{\psi})$ in the *CM* frame. The *proper time* intervals in *CM* & *LAB* frames are $d\bar{t}$ & dt with $dt=\gamma d\bar{t}$, $2c\bar{\beta}$ is the relative velocity of two colliding particles with $\bar{\boldsymbol{v}}_1+\bar{\boldsymbol{v}}_2=0$ in *CM* frame. P is defined as a probability density product using 12 variables and can be expressed in *LAB* into the form (defining for short $\eta_{1,2} \stackrel{\text{def}}{=} \Delta p_{1,2}/p_{1,2}$) (ref. [3,15]):

$$P = P_{12\text{var}} \stackrel{\text{def}}{=} P_{\eta s}(\eta_1, s_1)\, P_{\eta s}(\eta_2, s_2)\, P_{x_\beta x'_\beta}(x_{\beta_1}, x'_{\beta_1}) P_{x_\beta x'_\beta}(x_{\beta_2}, x'_{\beta_2}) P_{z\,z'}(z_1, z'_1) P_{z\,z'}(z_2, z'_2) \quad 2.9$$

Among the 12 variables 3 are dependent since during the short collision time the 2 particle positions are assumed not to change i.e. $s_1=s_2=s \quad x_1=x_{\beta_1}+D_x\eta_1 \equiv x_2=x_{\beta_2}+D_x\eta_2 \quad z_1=z_{\beta_1} \equiv z_2=z_{\beta_2}$, thus:

$$P = P_{9\text{var}} \stackrel{\text{def}}{=} P_\eta(\eta_1)P_\eta(\eta_2)P_s(s_1)P_{x_\beta}(x_{\beta_1})P_{x'_\beta}(x'_{\beta_1})P_{x'_\beta}(x'_{\beta_2})P_z(z_1)P_{z'}(z'_1)P_{z'}(z'_2) \quad 2.10$$

Strategy step 5:
scattering angle averages

Core IBS model

The scattering angle distribution is now examined. The *Rutherford* differential cross-section Eq. 1.11 for *non-relativistic* Coulomb collisions in a *CM* frame (i.e. $\bar{\beta} \ll 1$) of 2 *ions* of charge Z and atomic mass A is:

$$\left|\frac{d\bar{\sigma}}{d\bar{\Omega}}\right| = \left(\frac{AmZ^2e^2}{4\pi\varepsilon_0|\bar{\boldsymbol{p}}_2 - \bar{\boldsymbol{p}}_1|^2}\right)^2 \frac{1}{\sin^4(\bar{\psi}/2)} = \left(\frac{Z^2 r_0 mc^2}{2\bar{T}}\right)^2 \frac{1}{\sin^4(\bar{\psi}/2)} = \left(\frac{Z^2}{A}\frac{r_0}{4\bar{\beta}^2}\right)^2 \frac{1}{\sin^4(\bar{\psi}/2)} \qquad 2.11$$

with $\bar{T} = |\bar{\boldsymbol{p}}_2 - \bar{\boldsymbol{p}}_1|^2/2Am = 2Am\bar{\beta}^2c^2$ is the ion *kinetic energy*, $2Am\bar{\beta}c$ is the *relative momentum* between the hitting ions, for which $\bar{\boldsymbol{p}}_1 + \bar{\boldsymbol{p}}_2 = 0$ in *CM*, $r_0 = e^2/4\pi\varepsilon_0 mc^2$ is the *classical proton radius*, $r_\mathrm{i} = r_0 Z^2/A$ is the *classical ion radius* (ref. [19] & ref. [B-D]).

To evaluate $\bar{\beta}$ the above expression $|\bar{\boldsymbol{p}}_2 - \bar{\boldsymbol{p}}_1| = 2m\bar{\beta}c$ in the *CM* frame must be Lorentz transformed back to the *LAB* frame to link $\bar{\beta}c$ with βc. All calculations done we find, providing $\bar{\beta} \ll 1$, $\bar{\gamma} \approx 1$ (ref. [6,7]):

$$\bar{\beta} \approx \frac{\beta\gamma}{2}\sqrt{\left(\frac{p_1 - p_2}{\gamma p}\right)^2 + (x_1' - x_2')^2 + (z_1' - z_2')^2} = \frac{\beta\gamma}{2}\sqrt{\xi^2 + \theta^2 + \zeta^2} \qquad 2.12$$

wherein βc is the average particle velocity in the *LAB* frame.

Strategy step 5: scattering angle averages

Core IBS model

The integration required to work out Eq. 2.8 can be done as follows, where the integral I_{x_1} aims to integrate the mean time-derivative of $\langle\varepsilon_{x_1}\rangle/\beta_x$. To do this replace $\delta p/p$ & $\delta p_x/p$ (Eq. 2.5) with their expression in terms of $\gamma,\,\xi,\,\theta,\,\zeta,\,\bar{\phi},\,\bar{\psi}$ (Eq. 2.1). Then, integrate $\delta\varepsilon_{x_1}/\beta_x$ over $\bar{\psi}$ & $\bar{\phi}$ (e.g. *Mathematica*) with Eq.2.11 and expand the scattering integrals to first order in $\bar{\psi}_{\min}$ yields:

$$I_{x_1} \equiv \iint_{\bar{\Omega}} d\bar{\Omega}\left|\frac{d\bar{\sigma}}{d\bar{\Omega}}\right|\frac{\delta\varepsilon_{x_1}}{\beta_x} = \int_{\bar{\psi}_{\min}}^{\pi} d\,\bar{\psi}\int_0^{2\pi} d\bar{\phi}\left|\frac{d\bar{\sigma}}{d\bar{\Omega}}\right|\frac{\delta\varepsilon_{x_1}}{\beta_x}\sin\bar{\psi} =$$
$$-\frac{\pi r_i^2}{8\bar{\beta}^4}\left\{\xi^2+\zeta^2-2\theta^2+\frac{{D_x}^2+{\widetilde{D}_x}^2}{\beta_x^2}\gamma^2(\zeta^2+\theta^2-2\xi^2)+\frac{6\widetilde{D}_x}{\beta_x}\gamma\xi\theta\right\}+\frac{\pi r_i^2}{4\bar{\beta}^4}\left\{\frac{4x_{\beta_1}}{\beta_x}\left(\gamma_x D_x\gamma\xi+\alpha_x(D'_x\gamma\xi-\theta)\right)\right.$$
$$\left.+\,4x'_{\beta_1}\left(\frac{\widetilde{D}_x\gamma\xi}{\beta_x}-\theta\right)+\xi^2+\zeta^2+\frac{{D_x}^2+{\widetilde{D}_x}^2}{\beta_x^2}\gamma^2(\zeta^2+\theta^2)+\frac{2\widetilde{D}_x}{\beta_x}\gamma\xi\theta\right\}\ln\left[\frac{2}{\bar{\psi}_{\min}}\right] \qquad 2.13$$

The smallest angle $\bar{\psi}_{\min}$ is defined by the maximum *impact parameter* $\bar{b}_{\max}$ as (r_i is the ion radius):

$$\tan\left(\frac{\bar{\psi}_{\min}}{2}\right)\approx\frac{r_i}{2\bar{\beta}^2\bar{b}_{\max}} \qquad r_i=\frac{r_0 Z^2}{A} \qquad 2.14$$

To get tractable results it was assumed that $\bar{\psi}_{\min}\ll 1$, thus $2\bar{\beta}^2\bar{b}_{\max}/r_i\gg 1$

Strategy step 5: scattering angle averages

Core IBS model

The two brackets (Eq.2.13) have similar tiny values as the angles ξ, θ and $\zeta \ll 1$; but the first bracket is negligible compared to the second one since it is multiplied by the *Coulomb logarithm* (with usual values between 10 and 20). Hence, after rearranging the integral I_{x_1}, it follows, with $d\bar{\Omega} = \sin\bar{\psi}\, d\bar{\psi}\, d\bar{\phi}$:

$$I_{x_1} = \int_{\bar{\psi}_{\min}}^{\pi} d\,\bar{\psi} \int_0^{2\pi} d\bar{\phi} \left|\frac{d\bar{\sigma}}{d\bar{\Omega}}\right| \frac{\delta\varepsilon_{x_1}}{\beta_x} \sin\bar{\psi} = \frac{\pi {r_i}^2}{4\bar{\beta}^4} \times$$
$$\left\{\frac{4x_{\beta_1}}{\beta_x}\left[\gamma_x D_x \gamma\xi + \alpha_x (D'_x\gamma\xi - \theta)\right] + 4x'_{\beta_1}\left[\frac{\tilde{D}_x\gamma\xi}{\beta_x} - \theta\right] + \xi^2 + \zeta^2 + \frac{{D_x}^2 + {\tilde{D}_x}^2}{\beta_x^2}\gamma^2[\zeta^2 + \theta^2] + \frac{2\tilde{D}_x}{\beta_x}\gamma\xi\theta\right\} \ln\left[\frac{2\bar{\beta}^2\bar{b}_{\max}}{r_i}\right] \quad 2.15$$

$$\bar{C}_{\log} \stackrel{\text{def}}{=} \ln\left[\frac{2\bar{\beta}^2\bar{b}_{\max}}{r_i}\right] = \ln\left[\frac{2}{\bar{\psi}_{\min}}\right] \quad 2.16$$

$\bar{C}_{\log}$ or $C_{\log}$ are the *Coulomb logarithms* in *CM* or *LAB* frames. The log dependence makes the *Coulomb log* slowly changing over a big range of the elements concerned in its definition (ref. [8,10] & ref. [G]).

The integrals I_{z_1} and I_{s_1} for the vertical and longitudinal momenta can be worked out too (zero vertical dispersion is supposed). Then $(I_{x_1}, I_{z_1}, I_{s_1})$ will give the transverse and longitudinal scattering integrals (δH is now changed in $\delta H \approx 2\eta\delta p_s/p + (\delta p_s/p)^2$ since $\delta p \approx p_s$, with $\eta \stackrel{\text{def}}{=} \Delta p/p$):

Strategy step 5: scattering angle averages

Core IBS model

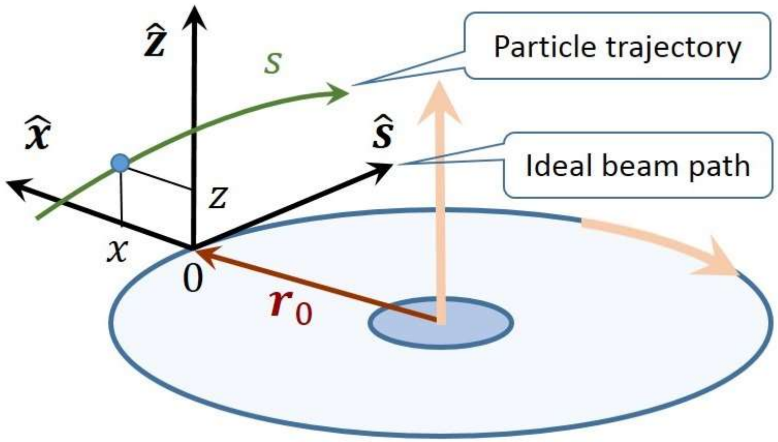
$$\begin{pmatrix} I_{s_1} \\ I_{x_1} \\ I_{z_1} \end{pmatrix} \equiv \int_{\bar{\psi}_{\min}}^{\pi} d\,\bar{\psi} \int_0^{2\pi} d\bar{\phi} \sin\bar{\psi} \left|\frac{d\bar{\sigma}}{d\bar{\Omega}}\right| \begin{pmatrix} \delta H_1/\gamma^2 \\ \delta\varepsilon_{x_1}/\beta_x \\ \delta\varepsilon_{z_1}/\beta_z \end{pmatrix} = \frac{\pi {r_i}^2}{4\bar{\beta}^4} \ln\left[\frac{2\bar{\beta}^2 \bar{b}_{\max}}{r_i}\right]$$
$$\times \begin{Bmatrix} -\dfrac{4\eta_1}{\gamma}\xi + \theta^2 + \zeta^2 \\ \dfrac{4x_{\beta_1}}{\beta_x}[\gamma_x D_x \gamma\xi + \alpha_x(D'_x\gamma\xi - \theta)] + 4x'_{\beta_1}\left[\dfrac{\widetilde{D}_x\gamma\xi}{\beta_x} - \theta\right] + \xi^2 + \zeta^2 + \dfrac{{D_x}^2 + {\widetilde{D}_x}^2}{\beta_x^2}\gamma^2[\zeta^2 + \theta^2] + \dfrac{2\widetilde{D}_x}{\beta_x}\gamma\xi\theta \\ -\dfrac{4\alpha_z z_1}{\beta_z}\zeta - 4z'_1\zeta + \xi^2 + \theta^2 \end{Bmatrix} \qquad 2.17$$

The computation of the mean change of the invariants ε_{x_1,z_1} & H of all particles due to the multiple particle collisions requires to average the above three integrals of the two colliding particles over the 12 variables, reduced to 9 as ($s_{1,2}, x_{1,2}, z_{1,2}$) are dependent (cf. Eq. 2.10) via the probability P ($\bar{P}$).

Strategy step 5:
scattering angle averages

Core IBS model

In the *CM* frame all derivatives d/ds are reduced by γ because of the *Lorentz* contraction along s (e.g. $\bar{P}=P/\gamma$, $\bar{\sigma}_{x'_\beta}=\sigma_{x'_\beta}/\gamma$), the transverse sizes & relative momentum spread stay unchanged (e.g. $\bar{\sigma}_{x_\beta}=\sigma_{x_\beta}$, $\bar{\sigma}_\eta=\sigma_\eta$, $\bar{b}_{\max}=b_{\max}$) and the bunch length turns into $\bar{\sigma}_s=\gamma\sigma_s$.



Accelerator & storage ring moving coordinates

$\boldsymbol{r}(s)=\boldsymbol{r}_0(s)+d\boldsymbol{r}(s\quad d\boldsymbol{r}(s)=x(s)\hat{\boldsymbol{x}}+z(s)\hat{\boldsymbol{z}}$

The *relative velocity* between 2 scattering particles in the *CM* frame is $2c\bar{\beta}$. Let's call $\bar{P}_{\text{scat}}$ the *likelihood* (or plausibility) for a collision per unit time and solid angle $d\bar{\Omega}$ in the *CM* frame. Suppose the probability P is specified in *LAB* frame; hence $\bar{P}=P/\gamma$ plus an "underlying" time gap $d\bar{t}=dt/\gamma$ induce two factors γ for P_{scat}. So, through the *Rutherford* cross-section formula (Eq. 2.11) the scattering likelihood per unit time in a storage ring converts from $\bar{P}_{\text{scat}}$ to P_{scat} by way of:

$$\bar{P}_{\text{scat}}=2c\bar{\beta}\bar{P}\left|\frac{d\bar{\sigma}}{d\bar{\Omega}}\right|d\bar{\Omega}\qquad P_{\text{scat}}=\frac{2c\bar{\beta}P}{\gamma^2}\left|\frac{d\bar{\sigma}}{d\bar{\Omega}}\right|\sin\bar{\psi}\,d\bar{\psi}\,d\bar{\phi}\qquad 2.18$$

Strategy step 6: particle beam averages

Core IBS model

Changing the 9 variables of the distribution P into new ones $\eta, s, \xi, x_\beta, x'_\beta, \theta, z, z', \zeta$ gives a new $\mathcal{P}$ via:

$$x_{\beta_{1,2}} = x_\beta \mp D_x\gamma\xi/2 \quad x'_{\beta_{1,2}} = x'_\beta \pm (\theta - D'_x\gamma\xi)/2$$
$$z'_{1,2} = z' \pm \zeta/2 \quad \eta_{1,2} = \eta \pm \gamma\xi/2 \tag{2.19}$$
$$x_{1,2} = x \qquad z_{1,2} = z \quad s_{1,2} = s$$

and thus:

$$P\left(\eta_1, \eta_2, s_1, x_{\beta_1}, x'_{\beta_1}, x'_{\beta_2}, z_1, z'_1, z'_2\right) \mapsto \mathcal{P}\left(\eta, \xi, s, x_\beta, x'_\beta, \theta, z, z', \zeta\right) \tag{2.20}$$

The Jacobian of the transformation is $|\det J| = \gamma$. The relation between the new and initial phase volume elements is related to the transformation of multiple integrals by:

$$\int_V P dV = \int_{\mathcal{V}} |\det J|\, \mathcal{P} d\mathcal{V} \tag{2.21}$$

$$dV = d\eta_1 d\eta_2 ds_1 dx_{\beta_1} dx'_{\beta_1} dx'_{\beta_2} dz_1 dz'_1, dz'_2$$
$$d\mathcal{V} = d\eta d\xi ds dx_\beta dx'_\beta d\theta dz dz' d\zeta$$

The mean invariant change Eq. 2.8 can be rewritten as follows via Eq. 2.17, swapping the variables $\eta_1, x_{\beta_1}, x'_{\beta_1}, z_1, z'_1, s_1$ with $\eta, x_\beta, x'_\beta, z, z', s$ (Eq. 2.19). This yields the formal result (integrals over ξ, θ, ζ are from $-\infty$ to ∞):

$$\frac{d}{dt}\begin{bmatrix} \langle H\rangle/\gamma^2 \\ \langle \varepsilon_x\rangle/\beta_x \\ \langle \varepsilon_z\rangle/\beta_z \end{bmatrix} = \left\langle \int_{\mathcal{V}} \frac{2c\bar{\beta}\mathcal{P}}{\gamma^2} \begin{bmatrix} I_s \\ I_x \\ I_z \end{bmatrix} d\mathcal{V} \right\rangle \tag{2.22}$$

No hypothesis regarding $\mathcal{P}$ were made up to here

Strategy step 6: particle beam averages

Core IBS model

By construction $\mathcal{P}$ is symmetrical as regards to ξ, θ, ζ. Hence, the integrals over $\{-\infty, \infty\}$ vanish for the linear terms in ξ, θ, ζ of the integrand. So, just keep the factors $\xi^2, \theta^2, \xi^2, \zeta^2$ and Eq. 2.22 reduces to:

$$\frac{d}{dt}\begin{bmatrix}\langle H\rangle/\gamma^2\\ \langle\varepsilon_x\rangle/\beta_x\\ \langle\varepsilon_z\rangle/\beta_z\end{bmatrix} = \left\langle \frac{\pi c r_i^2}{2}\int_{\mathcal{V}} \frac{d\mathcal{V}}{\bar{\beta}^3\gamma}\mathcal{P}(\eta, s, \xi, x_\beta, x'_\beta, \theta, z, z', \zeta)\ln\left[\frac{2\bar{\beta}^2\bar{b}_{\max}}{r_i}\right] \times \begin{Bmatrix}\theta^2+\zeta^2-2\xi^2\\ \xi^2+\zeta^2-2\theta^2+\dfrac{{D_x}^2+{\widetilde{D}_x}^2}{\beta_x^2}\gamma^2(\zeta^2+\theta^2)-\dfrac{2\gamma_x {D_x}^2}{\beta_x}\gamma^2\xi^2-\dfrac{2D'_x}{\beta_x}(\alpha_x D_x+\widetilde{D}_x)\gamma^2\xi^2\\ \xi^2+\theta^2-2\zeta^2\end{Bmatrix}\right\rangle \qquad 2.23$$

- Eq. 2.23 for the average change of the invariants $\varepsilon_{x,z}$ & H makes no a priory assumption about the particle density *distribution* $\mathcal{P}$ in the bunch.
- To formulate IBS analytical models it is frequently assumed that the betatron amplitudes, angles, momentum deviations and synchrotron coordinates are *Gaussian* distributed for *bunched* beams since 'Gaussian integration' is rather easy to make.

Strategy step 6:
particle beam averages

IBS analytical model

Let us describe Gaussian *distributions* $P_{x_\beta x'_\beta}$ & $P_{z_\beta z'_\beta}$ in terms of the primary variables $\eta_1, \eta_2, s_1, \ldots z'_2$ in *LAB* frame (with $z_{\beta_{1,2}} \equiv z_{1,2}$ & $z'_{\beta_{1,2}} \equiv z'_{1,2}$ assuming $D_z = D'_z = 0$) for the betatron amplitudes & angles and $P_{\eta s}$ for momentum and bunch length deviations (*bunched* beams) (ref. [9]):

$$P_{x_\beta x'_\beta} = \frac{\sqrt{1+\alpha_x^2}}{2\pi\sigma_{x_\beta}\sigma_{x'_\beta}} \exp\left[-Q(x_\beta, x'_\beta)\right] \quad Q(x_\beta, x'_\beta) = \frac{1+\alpha_x^2}{2}\left(\frac{x_\beta^2}{\sigma_{x_\beta}^2} + \frac{2x_\beta x'_\beta \alpha_x}{\sigma_{x_\beta}\sigma_{x'_\beta}\sqrt{1+\alpha_x^2}} + \frac{{x'_\beta}^2}{\sigma_{x'_\beta}^2}\right) \tag{2.24}$$

$$P_{\eta s} = P_\eta P_s = \frac{1}{2\pi\sigma_\eta\sigma_s} \exp\left[-\frac{\eta^2}{2\sigma_\eta^2} - \frac{(s-s_0)^2}{2\sigma_s^2}\right]$$

The same in vertical $P_{z_\beta z'_\beta}$. Here σ_{x_β}, $\sigma_{x'_\beta}$, σ_η are rms values of the related variables, σ_s the rms bunch length, $\Delta s = s - s_0$ the synchrotron coordinate (position relative to the synchronous particle).

$Q =$ constant is a tilted ellipse with correlation coefficient $\rho_x = \alpha_x/\sqrt{1+\alpha_x^2}$. The density distribution P must be well-matched to the *Courant-Snyder invariant* $\varepsilon_x = \gamma_x x_\beta^2 + 2\alpha_x x_\beta x'_\beta + \beta_x x'^2_\beta$ (related to the phase space area used by the beam, i.e. $\varepsilon_x = \text{area}/\pi$).

Strategy step 6: particle beam averages

IBS analytical model

Consider a Gaussian beam with a rms value σ_{x_β}. For a phase area covering a fraction F of this beam the emittance at F [%] of particles in phase space is (the function $F(\varepsilon_x)$ being the *cumulative probability*):

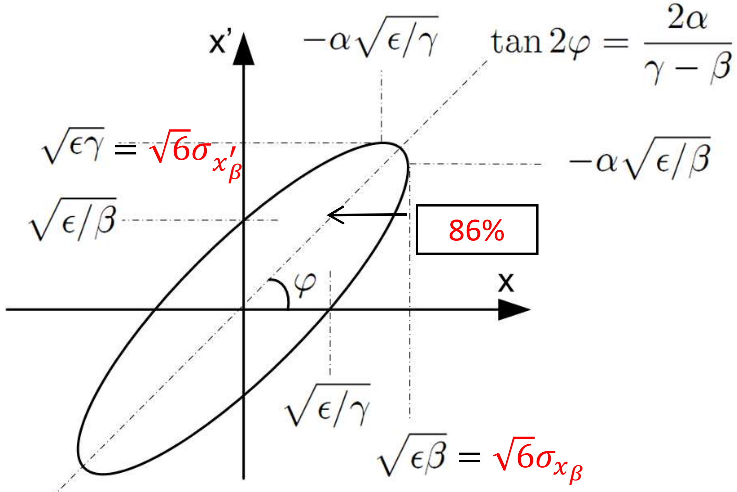
$$\varepsilon_x = -\left(2\sigma_{x_\beta}^2/\beta_x\right)\ln[1-F]$$

E.g. $\varepsilon_x = (1,4,6)\sigma_{x_\beta}^2/\beta_x$ picking $F=(39,86,95)\%$. Unlike F, "*projected emittances*", whose beam sizes cover a beam fraction F^{proj} projected onto the betatron amplitude axis, e.g. $\varepsilon_x^{\text{proj}} = (1,4,6)\sigma_{x_\beta}^2/\beta_x$ (the same as before) picking $F^{\text{proj}} = (68,95,99)\%$. Also, $P_{\varepsilon_x}(\varepsilon_x)$ writes:

$$P_{\varepsilon_x}(\varepsilon_x) \stackrel{\text{def}}{=} \mathrm{d}F(\varepsilon_x)/\mathrm{d}\varepsilon_x \quad \longrightarrow \quad P_{\varepsilon_x} = \frac{\beta_x}{2\sigma_{x_\beta}^2}\exp\left[-\frac{\beta_x\varepsilon_x}{2\sigma_{x_\beta}^2}\right]$$

$P_{x_\beta x'_\beta}$ can be also rephrased (so $P_{z_\beta z'_\beta}$):

$$P_{x_\beta x'_\beta} = \frac{\beta_x}{2\sigma_{x_\beta}^2}\exp\left[-\frac{\beta_x}{2\sigma_{x_\beta}^2}\left(\gamma_x x_\beta^2 + 2\alpha_x x_\beta x'_\beta + \beta_x x_\beta'^2\right)\right]$$



Phase space elliptical contour enclosing 86% of the beam

Strategy step 6: particle beam averages

IBS analytical model

- The next step is to convert the *LAB* frame distribution P, stated in 9 variables $\eta_{1,2}, s_1, x_{\beta_1}, x'_{\beta_{1,2}}, z_1, z'_{1,2}$, into $\mathcal{P}$, expressed in terms of the 9 variables $\eta, \xi, s, x_\beta, x'_\beta, \theta, z, z', \zeta$ (cf. Eq. 2.20 and e.g. Eq. 2.24
- In turn the density distribution $\mathcal{P}$ is integrated over the 6 variables $\eta,\ s,\ x_\beta,\ x'_\beta,\ z,\ z'$ yielding $\mathcal{P}(\xi, \theta, \zeta)$ in terms of the 3 left over variables ξ, θ, ζ.

To simplify we neglect the derivatives of the dispersion and betatron functions (D_z=0 early premise):

$$D'_{x,z} = \beta'_{x,z} = -2\alpha_{x,z} = 0 \quad \Longrightarrow \quad \widetilde{D}_{x,z} = \alpha_{x,z} D_{x,z} + \beta_{x,z} D'_{x,z} = 0 \quad \gamma_{x,z} = \beta_{x,z}^{-1}$$

Using the change of variables Eq. 2.19, $x'_{\beta_{1,2}}$ cuts to $x'_\beta \pm \theta/2$ as $D'_x = 0$ and Eq. 2.24 rewrites like:

$$\mathcal{P}_{x_\beta x'_\beta}\left(x_\beta \mp \frac{D_x \gamma \xi}{2}, x'_\beta + \frac{\theta}{2}\right) \quad \mathcal{P}_{zz'}\left(z, z' \pm \frac{\zeta}{2}\right) \quad \mathcal{P}_\eta(\eta \pm \frac{\gamma\xi}{2}) \quad \mathcal{P}_s^2(s) \qquad 2.25$$

- Considering N_b particles in a bunch; after integrating (with *Mathematica*) the 6 distributions $\mathcal{P}$'s over $\eta, s, x_\beta, x'_\beta, z, z'$ we get (the 3 lasting integrals over ξ, θ, ζ will be solved later):

Strategy step 6: particle beam averages

IBS analytical model

$$\mathcal{P}(\xi,\theta,\zeta)=N_b\prod_{u=\eta,\,s,\,x_\beta}^{x'_\beta,\,z,\,z'}\int_{-\infty}^{\infty}\mathcal{P}_u\left(u\pm\frac{\gamma\lambda_u}{2}\right)\mathcal{P}_u\left(u\mp\frac{\gamma\lambda_u}{2}\right)du=$$
$$=N_b\frac{\exp\left[-\frac{\gamma^2\xi^2}{4}\left(\frac{1}{\sigma_\eta^2}+\frac{D_x^2}{\sigma_{x_\beta}^2}\right)-\frac{\theta^2}{4\sigma_{x'_\beta}^2}-\frac{\zeta^2}{4\sigma_{z'}^2}\right]}{64\pi^3\sigma_{x_\beta}\sigma_{x'_\beta}\sigma_z\sigma_{z'}\sigma_\eta\sigma_s} \quad 2.26$$

In which λ_u stands for any ξ, 0, $D_x\xi$, θ, or ζ. Now $\mathcal{P}$ reduces to a function of ξ, θ, ζ.

For example let's compute the 2 terms $\mathcal{P}_\eta$ in Eq. 2.26 (using Eq. 2.24 for $P_\eta(\eta)$ and remembering that 2 particle momenta are involved in each interaction). The result of Gaussian integration is:

$$P_\eta(\eta_1,\eta_2)=P_\eta(\eta_1)P_\eta(\eta_2)\longmapsto\mathcal{P}_\eta\left(u\pm\frac{\gamma\xi}{2}\right)\mathcal{P}_\eta\left(u\mp\frac{\gamma\xi}{2}\right)\quad\int_{-\infty}^{\infty}\mathcal{P}_\eta\left(u\pm\frac{\gamma\xi}{2}\right)\mathcal{P}_\eta\left(u\mp\frac{\gamma\xi}{2}\right)d\eta=\frac{1}{2\sqrt{\pi}\sigma_\eta}\exp\left[-\frac{\gamma^2\xi^2}{4\sigma_\eta^2}\right]$$

Strategy step 6: particle beam averages

IBS analytical model

At that point we introduce $\mathcal{P}(\xi,\theta,\zeta)$ into the mean invariant change $\varepsilon_{x,z}$ & H Eq. 2.23, wherein all the variables are expressed in *LAB* frame except $\bar{\beta}$ (since $\bar{b}_{\max}=b_{\max}$). The Lorentz factor $\bar{\beta}$ is so converted back to *LAB* frame with Eq. 2.12: $2\bar{\beta} \approx \beta\gamma\sqrt{\xi^2+\theta^2+\zeta^2}$, yielding (with $q \overset{\text{def}}{=} \beta\gamma\sqrt{2b_{\max}/r_i}$):

$$\frac{d}{dt}\begin{bmatrix}\langle H\rangle/\gamma^2\\ \langle\varepsilon_x\rangle/\beta_x\\ \langle\varepsilon_z\rangle/\beta_z\end{bmatrix} = 4\mathcal{A}\left\langle \iiint_{-\infty}^{\infty}\frac{d\xi d\theta d\zeta}{(\xi^2+\theta^2+\zeta^2)^{3/2}}\exp\left[-\frac{\gamma^2\xi^2}{4}\left(\frac{1}{\sigma_\eta^2}+\frac{D_x^2}{\sigma_{x_\beta}^2}\right)-\frac{\theta^2}{4\sigma_{x'_\beta}^2}-\frac{\zeta^2}{4\sigma_{z'}^2}\right]\right.$$
$$\left.\times\begin{Bmatrix}\theta^2+\zeta^2-2\xi^2\\ \xi^2+\zeta^2-2\theta^2+\dfrac{D_x^2}{\beta_x^2}\gamma^2(\zeta^2+\theta^2-2\xi^2)\\ \xi^2+\theta^2-2\zeta^2\end{Bmatrix}\ln\left[\frac{q^2}{4}(\xi^2+\theta^2+\zeta^2)\right]\right\rangle \qquad 2.27$$

with for bunched beams:

$$\mathcal{A}=\frac{cr_i^2N_b}{64\pi^2\beta^3\gamma^4\sigma_{x_\beta}\sigma_{x'_\beta}\sigma_z\sigma_{z'}\sigma_\eta\sigma_s}=\frac{cr_i^2N_b}{64\beta^3\gamma^4\varepsilon_x\varepsilon_z\varepsilon_s}\qquad \varepsilon_s=\sigma_\eta\sigma_s\ [m] \qquad 2.28$$

Also $\varepsilon_s=\pi p\sigma_s\sigma_\eta(\beta c)^{-1}$ [eVs] p is the momentum [eV/c]

The integrals over ξ, θ, ζ must still be solved to work out the mean change of the invariants.

Strategy step 7: growth rates calculation

Original Piwinski model

In his initial model (1974) Piwinski (ref. [3]) developed formulae for the IBS growth rates $1/\tau_{\eta,x,z}$ as the change in the betatron oscillation amplitudes $\sigma_{x_\beta z_\beta}$ (equal to the square root of emittances $\varepsilon_{x,z}$) and momentum spread σ_η per unit time caused by scattering events (with $\langle H\rangle \approx \langle\eta\rangle^2 = \sigma_\eta^2$):

$$\frac{1}{\tau_\eta} = \frac{1}{\sigma_\eta}\frac{d\sigma_\eta}{dt} \equiv \frac{1}{2\langle H\rangle}\frac{d\langle H\rangle}{dt} \qquad \frac{1}{\tau_{x,z}} = \frac{1}{\sigma_{x_\beta,z_\beta}}\frac{d\sigma_{x_\beta,z_\beta}}{dt} \equiv \frac{1}{2\langle\varepsilon_{x,z}\rangle}\frac{d\langle\varepsilon_{x,z}\rangle}{dt} \equiv \frac{1}{\langle\varepsilon_{x,z}\rangle^{1/2}}\frac{d\langle\varepsilon_{x,z}\rangle^{1/2}}{dt} \tag{2.29}$$

To this end, besides cancelling $\alpha_{x,z}$, $D'_{x,z}$ and D_z, he makes use of the *smooth focusing approximation*, in which only the mean values of the lattice functions are considered:

$$\langle\beta_x\rangle \approx \frac{\langle R\rangle}{Q_x} \quad \langle D_x\rangle \approx \frac{\langle\beta_x\rangle}{Q_x} = \frac{R}{Q_x^2} \implies \frac{\langle D_x\rangle}{R} \approx \frac{1}{Q_x^2} \quad \Rightarrow \quad \frac{1}{\gamma_t^2} \stackrel{\text{def}}{=} \alpha_p \approx \frac{\langle D_x\rangle}{R} \approx \frac{1}{Q_x^2}$$

where a bracket means averaging, and ρ, $\langle R\rangle$, Q_x, α_p , γ_t, η_t are the ring curvature and mean radius, the *betatron tune*, *momentum compaction factor*, *transition energy* and *slip factor*. Also:

$$\sigma_{x_\beta,z_\beta} = \sqrt{\beta_{x,z}\varepsilon_{x,z}} \quad \sigma_{x'_\beta,z'_\beta} = \sqrt{\frac{\varepsilon_{x,z}}{\beta_{x,z}}} \quad \alpha_p \stackrel{\text{def}}{=} \frac{1}{2\pi\langle R\rangle}\oint\frac{D_x(s)}{\rho(s)}ds = \left\langle\frac{D_x(s)}{\rho(s)}\right\rangle \quad \eta_t \stackrel{\text{def}}{=} \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$$

Strategy step 7: growth rates calculation

Original Piwinski model

Notice that the form of the 1st column in Eq. 2.27 ($\gamma^{-2}\mathrm{d}\langle H\rangle/\mathrm{dt}$ & $\beta_{x,z}^{-1}\mathrm{d}\langle\varepsilon_{x,z}\rangle/\mathrm{dt}$) does not fit Eq. 2.29 ($[2\langle H\rangle]^{-1}\mathrm{d}\langle H\rangle/\mathrm{dt}$ & $\langle\varepsilon_{x,z}\rangle^{-1/2}\mathrm{d}\langle\varepsilon_{x,z}\rangle^{1/2}/\mathrm{dt}$) for $1/\tau_{\eta,x,z}$. So, new quantities a, b, c, d (cf. next slide) and q (see 2.27), are added to the 1st column of Eq. 2.27 for good match with the IBS growth rates $1/\tau_{\eta,x,z}$. To this end we do a double change of variables to convert Eq. 2.27 to coordinates $\xi,\theta,\zeta\mapsto 2(u,v,w)/q$ and next to spherical coordinates $(u,v,w)\mapsto\sqrt{r}(\sin\mu\cos\nu,\sin\mu\sin\nu,\cos\mu)$. After some work we get:

$$\begin{pmatrix}1/\tau_\eta\\1/\tau_x\\1/\tau_z\end{pmatrix}=\left\langle\frac{q^2}{2c^2}\begin{bmatrix}(1-d^2)\\a^2\\b^2\end{bmatrix}\frac{d}{dt}\begin{bmatrix}\langle H\rangle/\gamma^2\\\langle\varepsilon_x\rangle/\beta_x\\\langle\varepsilon_z\rangle/\beta_z\end{bmatrix}\right\rangle=\frac{\mathcal{A}}{c^2}\left\langle\int_0^\infty dr\int_0^\pi d\mu\int_0^{2\pi}d\nu\right.$$
$$\left.\times\sin[\mu]\exp[-rD(\mu,\nu)]\ln[r]\begin{Bmatrix}(1-d^2)g_1(\mu,\nu)\\a^2g_2(\mu,\nu)+d^2g_1(\mu,\nu)\\b^2g_3(\mu,\nu)\end{Bmatrix}\right\rangle \quad 2.30$$

$$D(\mu,\nu)=\frac{1}{c^2}\left(b^2\cos^2[\mu]+\sin^2[\mu](\cos^2[\nu]+a^2\sin^2[\nu])\right)$$
$$g_1(\mu,\nu)=1-3\sin^2[\mu]\cos^2[\nu]\quad g_2(\mu,\nu)=1-3\sin^2[\mu]\sin^2[\nu]\quad g_3(\mu,\nu)=1-3\cos^2[\mu] \quad 2.31$$

Strategy step 7: growth rates calculation

Original Piwinski model

The functions $D, g_{1,2,3}$ were introduced for convenience (keeping in mind that $z = z_\beta, z' = z'_\beta$ Eq. 2.2):

$$\frac{1}{\sigma_h^2} = \frac{1}{\sigma_\eta^2} + \frac{D_x^2}{\sigma_{x_\beta}^2} \to \sigma_h = \frac{\sigma_\eta \sigma_{x_\beta}}{\sigma_x} \quad a = \frac{\sigma_h}{\gamma \sigma'_{x_\beta}} = \frac{\sigma_h}{\gamma}\sqrt{\frac{\beta_x}{\varepsilon_x}} = \frac{\beta_x \sigma_\eta}{\gamma \sigma_x} \quad b = \frac{\sigma_h}{\gamma \sigma'_{z_\beta}} = \frac{\sigma_h}{\gamma}\sqrt{\frac{\beta_z}{\varepsilon_z}} = \frac{\beta_z \sigma_\eta}{\gamma \sigma_z}$$

$$c = \frac{q\sigma_h}{\gamma} = \sigma_h \sqrt{\frac{2\beta^2 b_{\max}}{r_i}} \quad q = \gamma \exp\left[\frac{C_{\log}}{2}\right] \quad d^2 = 1 - \frac{\sigma_h^2}{\sigma_\eta^2} \to d = \frac{D_x \sigma_\eta}{\sigma_x} \qquad 2.32$$

$C_{\log}$ is the Coulomb log factor (Eq. 2.16)

The aim is to write Eq. 2.30 in a reduced form. To this end a *scattering function* $f(a,b,c)$ is introduced instead of the functions $g_{1,2,3}$, in which $\rho = r/c^2$ replaces r and $D_0(\mu,\nu)$ swaps with $D(\mu,\nu)$ (Eq. 2.31):

$$f(a,b,c) = 2\int_0^\pi d\mu \int_0^{2\pi} d\nu \sin[\mu](1 - 3\cos^2[\mu]) \int_0^\infty d\rho \log[c^2\rho] \exp[-\rho D_0(\mu,\nu)]$$
$$D_0(\mu,\nu) = \sin^2[\mu](a^2\cos^2[\nu] + b^2\sin^2[\nu]) + \cos^2[\mu] \qquad 2.33$$

f is integrable over the variable ρ. So, solving it by *Mathematica* reduces f to the double integral:

Strategy step 7: IBS rise times

Original Piwinski model

$$f(a,b,c) = 2\int_0^{\pi} d\mu \int_0^{2\pi} d\nu \sin[\mu](1-3\cos^2[\mu]) \frac{2\log[c] - C_{\text{Euler}} - \log[D_0(\mu,\nu)]}{D_0(\mu,\nu)}$$

where $C_{\text{Euler}} \approx 0.5772$ is Euler's constant.

In line with the Evans & Zotter approach (ref. [4]), f is first converted by a change of variables μ, ν to $x = \cos\mu$, $z = 2\nu$ using the periodicity of $\sin^2$ and $\cos^2$ with π and the symmetry about $\pi/2$, allowing to replace the limit π of μ by $\pi/2$ and 2π of ν by $\pi/2$ (providing one multiplies the integral by an additional factor 8). Thus, after tricky working f can be shrunk to the single integral:

$$f(a,b,c) = 8\pi \int_0^1 \left(2\ln\left[\frac{\tilde{C}}{2}\left\{\frac{1}{\sqrt{P(x)}} + \frac{1}{\sqrt{Q(x)}}\right\}\right] - C_{\text{Euler}} \right) \frac{1-3x^2}{\sqrt{P(x)Q(x)}} dx \qquad 2.34$$

with

$$P(x) = a^2 + (1-a^2)x^2$$
$$Q(x) = b^2 + (1-b^2)x^2$$

$$a = \frac{\sigma_h}{\gamma\sigma_{x'_\beta}} \quad b = \frac{\sigma_h}{\gamma\sigma_{z'_\beta}} \quad c = \frac{q\sigma_h}{\gamma} = \sigma_h\left(\frac{2\beta^2 b_{\max}}{r_i}\right)^{1/2} \quad \tilde{C} = \log[c^2] - C_{\text{Euler}}$$

Strategy step 7: IBS rise times

Original Piwinski model

Eq. 2.34 is now the "*new scattering function*" $f(a,b,c)$. It needs numerical integration but for a few cases (see ref. [4] for a clear and detailed derivation, and ref. [3,9,13,17] too).

After some more work the IBS growth rates for bunched beams Eq. 2.30 can be rewritten into the dense form below, that agrees with ref. [I], Eqs. 13.42-13.53, assuming none vertical dispersion function $D_z=0$:

$$\begin{pmatrix} \dfrac{1}{\tau_\eta} \\ \dfrac{1}{\tau_x} \\ \dfrac{1}{\tau_z} \end{pmatrix} = \mathcal{A} \left\langle \begin{pmatrix} \dfrac{\sigma_{x_\beta}^2}{\sigma_x^2} f(a,b,c) \\ f\left(\dfrac{1}{a},\dfrac{b}{a},\dfrac{c}{a}\right) + \dfrac{D_x^2\sigma_\eta^2}{\sigma_{x_\beta}^2} f(a,b,c) \\ f\left(\dfrac{1}{b},\dfrac{a}{b},\dfrac{c}{b}\right) \end{pmatrix} \right\rangle \qquad 2.35$$

in which, together with Eq. 2.32

$$\frac{\sigma_{x_\beta}^2}{\sigma_x^2} = 1 - \frac{D_x^2\sigma_\eta^2}{\sigma_x^2} = 1 - d^2 = \frac{\sigma_h^2}{\sigma_\eta^2}$$

Original Piwinski model

Invariants

- *Above transition energy* the particle property is often identify by a *negative mass* comportment.
- Association with a *gas* in a *closed box* is not valid and the overall *oscillation energy* can *increase*.
- The *beam behaviour* can be described via a global *invariant* which can be cast into a form close to the sum of the mean invariant change $\langle \varepsilon_{x,z} \rangle$ & $\langle H \rangle$ over the collisions for all particles, i.e. multiplying $\langle H \rangle / \gamma^2$ by $1 - \gamma^2 D_x^2 / \beta_x^2$) in the summation yields a non invariant quantity because D_x / β_x varies.
- *Smooth focusing* approx. for the *tune*, *momentum compaction factor* and *transition energy* yields:

$$\langle H \rangle \left(\frac{1}{\gamma^2} - \frac{D_x^2}{\beta_x^2} \right) + \frac{\langle \varepsilon_x \rangle}{\beta_x} + \frac{\langle \varepsilon_z \rangle}{\beta_z} \neq \text{constant} \Rightarrow \langle H \rangle \left(\frac{1}{\gamma^2} - \frac{1}{\gamma_t^2} \right) + \frac{\langle \varepsilon_x \rangle}{\beta_x} + \frac{\langle \varepsilon_z \rangle}{\beta_z} = \text{constant} \quad 2.36$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\langle H \rangle \left(\frac{1}{\gamma^2} - \frac{1}{\gamma_t^2} \right) + \frac{\langle \varepsilon_x \rangle}{\beta_x} + \frac{\langle \varepsilon_z \rangle}{\beta_z} \right] = 0 \quad 2.37$$

$$\eta_t = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$$ (*slip factor*)

- *Below transition* ($\eta_t < 0$) the sum of the 3 (positive) invariants is bounded, and thus the 3 oscillation energies. So the "emittances" are redistributed in all 3 phase planes, holding the whole phase space invariant. The distribution P is stable: equilibrium exists (like gaz molecules in a closed box).
- *Above transition* ($\eta_t \geq 0$) the overall oscillation energy can increase as $\eta_t > 0$: no equilibrium can exists.

Bjorken-Mtingwa model

Beam phase space density and emittance ε_x, ε_z, ε_s

In line with Piwinski (ref. [3]), *Gaussian* beam phase-space densities are chosen, since they can be put in exponential *canonical distributions* for momentum product separability (refs. [20],[B]). So, from ref. [8]:

$$P(\boldsymbol{r},\boldsymbol{p}) = \frac{N_b}{\Gamma} e^{-S(\boldsymbol{r},\boldsymbol{p})} \quad \Gamma = \int \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{p}\, e^{-S(\boldsymbol{r},\boldsymbol{p})} \quad S(\boldsymbol{r},\boldsymbol{p}) = S^{(\mathrm{x})} + S^{(\mathrm{z})} + S^{(\mathrm{s})} \tag{2.38}$$

Γ is the 6-dim *phase-space volume*, N_b the *particle* number *per bunch*, $\boldsymbol{r}=(x,z,\eta)$ & $\boldsymbol{p}=(p_x,p_z,p_s)$ the positions & momenta of the particles in the bunch, $\sigma_{x_\beta}, \sigma_{z_\beta}, \sigma_s, \sigma_\eta$ the rms bunch *width*, *height*, *length*, *momentum spread*, $\varepsilon_{x,z,s}$ the rms transverse & longitudinal emittances. The beam *Gaussian distribution* $S(\boldsymbol{r},\boldsymbol{p})$ is (s_0 is the synchronous particle position):

$$S^{(\mathrm{r})} \overset{r=x,z}{=} \frac{\beta_r}{2\sigma_{r_\beta}^2}\left(\gamma_r r_\beta^2 + 2\alpha_r r_\beta r'_\beta + \beta_r r'^2_\beta\right) \quad S^{(\mathrm{s})} = \frac{\eta^2}{2\sigma_\eta^2} + \frac{(s-s_0)^2}{2\sigma_s^2} \quad \varepsilon_r = \frac{\sigma_{r_\beta}^2}{\beta_r} \quad \varepsilon_s = \sigma_\eta \sigma_s$$
$$r' = \frac{\Delta p_r}{p} \quad \eta = \frac{\Delta p_s}{p} \quad \sigma_\eta = \frac{\sigma_p}{p} \quad r_\beta = r - D_r\eta \quad r'_\beta = r' - D'_r\eta \tag{2.39}$$

Bjorken-Mtingwa model

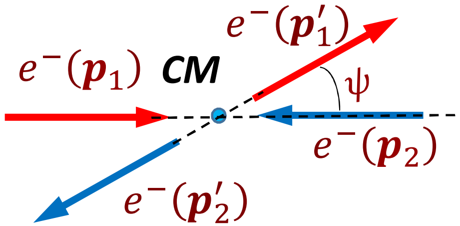
Two-body scattering in the CM frame

Bjorken & Mtingwa approach of IBS theory is based on the *S-matrix*, a time-evolution operator that relates the transition from an initial quantum state $|i\rangle$ to a final state $|f\rangle$ of a physical system facing to a collisional event. The matrix elements of S are the inner products $\langle f|S|i\rangle$, with characteristics:

- The squared modulus $|\langle f|S|i\rangle|^2$ yields the *probability* $\mathcal{P}$ for a *transition* from an initial to a final state.
- S is linked to an amplitude $\mathcal{M}$ stating the physical process: $\langle f|S|i\rangle=(2\pi)^4\delta^4(p_{1f}+p_{2f}-p_{1f}-p_{1f})\mathcal{M}$

In a 2–body scattering process particles 1 & 2 with energy-momentum 4-vectors $p_{1,2}\overset{\text{def}}{=}p^{\mu}_{1,2}$ interact each other to give after collision two 4-momenta $p'_{1,2}\overset{\text{def}}{=}p'^{\mu}_{1,2}$ (i.e. $\boldsymbol{p}_1+\boldsymbol{p}_2\rightarrow\boldsymbol{p}'_1+\boldsymbol{p}'_2$) whose transition rate is, expressed in the Heaviside-Lorentz (*HL*) units $\hbar=c=1$, (cf. ref. [8,11] and ref. [N,O]):

Eq. [2.40] stems from the electromagnetic scattering process of a spin-½ electron of mass m off a free pointlike and structureless spin+½ proton of mass M, called "Dirac proton", (in analogy with Eq. (7.42) and next ones in ref[O]).



$$\frac{d\mathcal{P}}{dt}=\frac{1}{2}\int d\boldsymbol{r}\frac{d\boldsymbol{p}_1}{\gamma_1}\frac{d\boldsymbol{p}_2}{\gamma_2}P(\boldsymbol{r},\boldsymbol{p}_1)P(\boldsymbol{r},\boldsymbol{p}_2)|\mathcal{M}|^2\frac{d\boldsymbol{p}'_1}{\gamma'_1}\frac{d\boldsymbol{p}'_2}{\gamma'_2}\frac{\delta^4(p'_1+p'_2-p_1-p_2)}{(2\pi)^2} \qquad 2.40$$

where $\mathcal{M}$ is the scattering *amplitude* to be computed and $\gamma_{1,2}=E_{1,2}/M$.

Bjorken-Mtingwa model

“real-life”≠“toy model“: particles & bosons have spin-0 & mass $m_i \geq 0$ & are their own antiparticles

Two-body scattering in the CM frame (for toy theory)

The metric is $4\mathrm{-momentum}^2 = \mathrm{energy}^2 - 3\mathrm{-momentum}^2$. E.g. in *HL* units with $c=1$ we get:

$$r \stackrel{\text{def}}{=} r^\mu \equiv (t,\boldsymbol{r}) = (t,x,z,s) \quad p \stackrel{\text{def}}{=} p^\mu \equiv (E,\boldsymbol{p}) = (E,p_x,p_z,p_s) \quad r_\mu = g_{\mu\nu} r^\mu = (t,-x,-z,-s) \quad p_\mu = g_{\mu\nu} p^\mu = (E,-p_x,-p_z,-p_s)$$

$$p_1 \cdot p_2 \stackrel{\text{def}}{=} p_1^\mu p_{2\mu} = E_1 E_2 - \boldsymbol{p}_1 \cdot \boldsymbol{p}_2 \quad r \cdot p \stackrel{\text{def}}{=} r^\mu p_\mu = tE - \boldsymbol{r} \cdot \boldsymbol{p} \quad \text{with} \quad g_{11}=1, g_{22}=g_{33}=g_{44}=-1,\ g_{\mu\neq\nu}=0$$

- For ease the *amplitude* $|\mathcal{M}|^2$ is computed for a Coulomb scattering among 2 electrons (not $e^- + p^+$!) via the exchange of a virtual photon γ with 4-momentum $q \stackrel{\text{def}}{=} q^\mu$, using the *Feynman diagram & rules*.
- To lessen the calculations in “*real-life*” collisions $e^- + e^- \rightarrow e^- + e^-$ with e^- of spin-$\frac{1}{2}$ and massless photon of spin 1, we use instead a “*toy model*“ which considers structureless particles and spinless bosons.
  - The *coupling constant* g_{E} in quantum electrodynamic *(QED)* specifies the interaction strength between electrons and photons; g_{E} is associated to the *fine structure constant* α_{E} by: $g_{\mathrm{E}} = \sqrt{4\pi\alpha_{\mathrm{E}}}$. In *HL* units ($\epsilon_0 = \hbar = c = 1$) $\alpha_{\mathrm{E}} = e^2/4\pi$ (with $e \approx 0.303 \approx \sqrt{4\pi/137.036}$, no charge unit), hence $g_{\mathrm{E}} = e$. In *SI* units $\alpha_{\mathrm{E}} = e^2/4\pi\epsilon_0\hbar c \approx 1/137$, thus $g_{\mathrm{E}} = e/\sqrt{\epsilon_0 \hbar c} = 0.303$.

    cf. K A Tomilin, Eur.J.Phys Vol 20 Nb 5 Sept 1999
  - A boson *propagator* $f(q)$ is associated to the wavy line in the Feynman diagram and represents the momentum transfer from one e^- to the other e^- through the virtual photon γ cf. Appendix 1-2.

Bjorken-Mtingwa model

Two-body scattering in the CM frame (for toy theory)

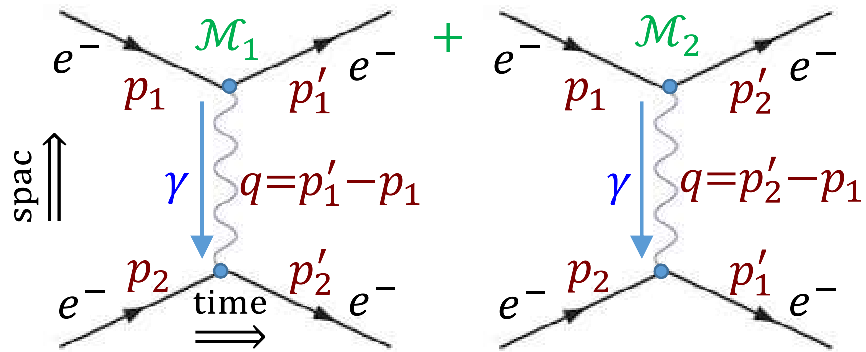
The *Feynman rules* for spin-0 *toy model* allow to calculate more easily the propagator $f(q)$ and scattering amplitude $\mathcal{M}$ for *elastic collisions* (4-momenta are *conserved*) cf. Appendix 1-2-3.

- With rules 2-3 we write one coupling constant of $-\mathrm{i}g_{\mathrm{E}}$ for each vertex (whose product is $-g_{\mathrm{E}}^2$) and one propagator $f(q) = \mathrm{i}/q^2$ for the single internal line. The overall product is $-g_{\mathrm{E}}^2\mathrm{i}/q^2$ (with $q \stackrel{\text{def}}{=} q^\mu$).
- Then, with rules 4-5 we multiply this product by the δ-functions and integrate over q the 1st δ of the diagram with $\mathrm{d}^4q/(2\pi)^4$, and insert $q \mapsto p_1' - p_1$ (Eq. 2.42) in the 2nd δ–function gives for $m_\gamma = 0$:

which conserved the 4-momenta at the top & bottom vertex

- With Rules 6 the last δ-function is removed and the result is multiply by i. So the left Feynman diagram $\mathcal{M}_1$ follows:

$$-\frac{\mathrm{i}g_{\mathrm{E}}^2}{q^2 - m_\gamma^2}(2\pi)^4\delta^4(p_1 - p_1' + q)(2\pi)^4\delta^4(p_2 - p_2' - q)$$



The 2 Feynman diagrams contribute to the particles scattering process

$$\mathcal{M}_1 \stackrel{\text{def}}{=} g_{\mathrm{E}}^2 f(q) = -\mathrm{i}g_{\mathrm{E}}^2 \int \frac{1}{q^2}(2\pi)^4\delta^4(p_1 - p_1' + q)(2\pi)^4\delta^4(p_2 - p_2' - [p_1' - p_1])\frac{\mathrm{d}^4q}{(2\pi)^4}\mathrm{i} = \frac{g_{\mathrm{E}}^2}{(p_1' - p_1)^2} \qquad 2.41$$

Bjorken-Mtingwa model

Two-body scattering in the CM frame (for toy theory)

Eq. 2.41 holds because

$$\int \frac{\mathrm{d}^4 q}{q^2}\,\delta^4(q-[p_1'-p_1]) = (p_1'-p_1)^{-2}$$

Finally, since $g_{\mathrm{E}}=e$ in *HL* units Eq. [2.41] writes:

$$\mathcal{M}_1 = \frac{g_{\mathrm{E}}^2}{(p_1'-p_1)^2} \equiv \frac{g_{\mathrm{E}}^2}{4\boldsymbol{p}^2\sin^2[\bar{\psi}/2]} \qquad 2.42$$

with $\quad q \stackrel{\text{def}}{=} q_\mu=(p_1'-p_1)^\mu \qquad q_\mu^2=q^\mu q_\mu=(p_1'-p_1)^2$

To see the link of $\mathcal{M}_1$ with a collisional process let's rewrites $q_\mu^2=(p_1'-p_1)^2$ (*HL* units):

$$q_\mu^2 = (p_1'-p_1)^2 = (p_1'^2+p_1^2-2p_1\cdot p_1') = E_1^2+E_1'^2-\boldsymbol{p}_1^2-\boldsymbol{p}_1'^2-2(E_1E_1'-\boldsymbol{p}_1\cdot\boldsymbol{p}_2') =$$

$$(E_1-E_1')^2-(\boldsymbol{p}_1^2+\boldsymbol{p}_1'^2-2|\boldsymbol{p}_1||\boldsymbol{p}_1'|\cos\bar{\psi}) = -2\boldsymbol{p}^2(1-\cos\bar{\psi}) = -4\boldsymbol{p}^2\sin^2[\bar{\psi}/2]$$

Elastic collisions: $\quad E_1=E_1'=E_2=E_2' \quad |\boldsymbol{p}_1|=|\boldsymbol{p}_1'|=|\boldsymbol{p}_2|=|\boldsymbol{p}_2'| \quad \boldsymbol{p}_1\cdot\boldsymbol{p}_1'=\boldsymbol{p}^2\cos\bar{\psi} \quad \boldsymbol{p}_1\cdot\boldsymbol{p}_2'=-\boldsymbol{p}^2\cos\bar{\psi}$

$\boldsymbol{p} \stackrel{\text{def}}{=} \boldsymbol{p}_1$ is the incident momentum particle 1 and $\bar{\psi}$ is the *CM* frame scattering angle between $\boldsymbol{p}_1$ & $\boldsymbol{p}_1'$ after collision ($\pi+\bar{\psi}$ is the scattering angle between $\boldsymbol{p}_1$ & $\boldsymbol{p}_2'$, so: $-2\boldsymbol{p}^2(1-\cos[\pi+\bar{\psi}])$=-$4\boldsymbol{p}^2\cos^2[\bar{\psi}/2]$).

for the $\mathcal{M}_2$ right Feynman diagram

Bjorken-Mtingwa model

Two-body scattering in the CM frame (for toy theory)

- The scattering amplitude $\mathcal{M}_2$ for the right Feynman diagram above is derived by exchanging p_1' with p_2' in Eq. 2.42 giving:

$$\mathcal{M}_2 \equiv \frac{g_E^2}{(p_2' - p_1)^2} = \frac{g_E^2}{4\boldsymbol{p}^2 \cos^2[\bar{\psi}/2]}$$

- Thus, the full amplitude $\mathcal{M}=\mathcal{M}_1+\mathcal{M}_2$ for the process $e^-+e^-\to e^-+e^-$ is, with $g_E^2=4\pi\alpha_E=e^2$, *HL* units:

$$\mathcal{M} = \frac{g_E^2}{4\boldsymbol{p}^2}\left(\frac{1}{\sin^2[\bar{\psi}/2]} + \frac{1}{\cos^2[\bar{\psi}/2]}\right) = -\frac{e^2}{\boldsymbol{p}^2 \sin^2\bar{\psi}} \quad \Longrightarrow \quad |\mathcal{M}|^2 = \left(\frac{e^2}{\boldsymbol{p}^2 \sin^2\bar{\psi}}\right)^2 \qquad 2.43$$

- For two-body scattering in the *CM* frame with all 4 particle masses even, the *Rutherford differential cross section* is given by Eq. A.1, derived using the "Fermi's Golden rule", cf. ref. [M,P,Q] see Appendix 3.

- At that stage, $d\mathcal{P}/dt$ Eq. 2.40 can be rewritten introducing $|\mathcal{M}|^2$ and the beam distribution $P(\boldsymbol{r},\boldsymbol{p})$ Eq. 2.39 into it, yielding ($|\boldsymbol{p}|=|\boldsymbol{p}_1|=|\boldsymbol{p}_2|$):

$$\frac{d\mathcal{P}}{dt} = \frac{N_b}{2\Gamma^2}\int d\boldsymbol{r}\,\frac{d\boldsymbol{p}_1}{\gamma_1}\frac{d\boldsymbol{p}_2}{\gamma_2}\,e^{-S(\boldsymbol{r},\boldsymbol{p}_1)-S(\boldsymbol{r},\boldsymbol{p}_2)}\left(\frac{e^2}{|\boldsymbol{p}|^2\sin^2\bar{\psi}}\right)^2\frac{d\boldsymbol{p}_1'}{\gamma_1'}\frac{d\boldsymbol{p}_2'}{\gamma_2'}\,\frac{\delta^4(p_1'+p_2'-p_1-p_2)}{(2\pi)^2} \qquad 2.44$$

Bjorken-Mtingwa model

How to find the growth times τ_{xzs} from the above results?

Final steps of IBS theory providing quantifiable growth rates

Bjorken & Mtingwa took on a vertical dispersion function $D_z=0$ to develop their equation (3.4) in ref. [8]. Its proof needs arduous work. The *IBS growth rates* τ_u^{-1} below (with $u=x,z,s$) for *bunched* beams (Eq. 2.29) including the non-zero vertical dispersion $D_z\neq 0$ refs. [14,21] are derived from B & M Eq. (3.4):

$$\frac{1}{\tau_u}=\frac{1}{\tau_{x,z,\eta}}=\frac{1}{\sigma_{x,z,\eta}}\frac{d\sigma_{x,z,\eta}}{dt}\stackrel{\text{def}}{=}\left\{\langle\varepsilon_{x,z}\rangle^{-1/2}\frac{d\langle\varepsilon_{x,z}\rangle^{1/2}}{dt},\frac{1}{\sigma_\eta}\frac{d\sigma_\eta}{dt}\right\}\equiv\left\{\frac{1}{2\langle\varepsilon_{x,z}\rangle}\frac{d\langle\varepsilon_{x,z}\rangle}{dt},\frac{1}{2\sigma_\eta^2}\frac{d\sigma_\eta^2}{dt}\right\}$$

$$\frac{1}{\tau_u}=\frac{c r_i^2 N_b C_{\log}}{16\pi\beta^3\gamma^4\varepsilon_x\varepsilon_z\sigma_s\sigma_\eta}\left\langle\int_0^\infty\frac{d\lambda}{\sqrt{\det[L+\lambda I]}}\{\mathrm{Tr}[L_u]\,\mathrm{Tr}[(L+\lambda I)^{-1}]-3\,\mathrm{Tr}[L_u(L+\lambda I)^{-1}]\}\right\rangle \tag{2.45}$$

where the bracket $\langle\cdot\rangle$ denotes an average around the ring circumference, and with $L=L_x+L_z+L_s$:

$$L_x=\frac{\beta_x}{\varepsilon_x}\begin{pmatrix}1 & -\gamma\phi_x & 0\\ -\gamma\phi_x & \gamma^2 H_x/\beta_x & 0\\ 0 & 0 & 0\end{pmatrix}\quad L_z=\frac{\beta_z}{\varepsilon_z}\begin{pmatrix}1 & 0 & 0\\ 0 & \gamma^2 H_z/\beta_z & -\gamma\phi_z\\ 0 & -\gamma\phi_z & 1\end{pmatrix}\quad L_s=\frac{\gamma^2}{\sigma_\eta^2}\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix} \tag{2.46}$$

Bjorken-Mtingwa model

Final steps of IBS theory providing quantifiable growth rates

with:

$$\emptyset_{x,z}=\frac{D_{x,z}\alpha_{x,z}+D'_{x,z}\beta_{x,z}}{\beta_{x,z}} \quad H_{x,z}=\frac{D^2_{x,z}+\beta^2_{x,z}\emptyset^2_{x,z}}{\beta_{x,z}} \quad \Delta_x=\frac{\gamma^2 H_x}{\beta_x} \quad \Delta_z=\frac{\gamma^2 H_z}{\beta_z} \quad \Delta_s=\frac{\gamma^2}{\sigma^2_\eta}$$

Here the emittance $\langle\varepsilon_{x,z}\rangle$ is the *projected r.m.s. emittances* on the betatron amplitude (x, z)-axes, also written $\left\langle\varepsilon^{\mathrm{proj}}_{x,z}\right\rangle \stackrel{\mathrm{def}}{=} \sigma^2_{x_\beta,z_\beta}/\beta_{x,z}$ whose beam profile covers 68% of the beam. This is not the *r.m.s. phase plane emittance* whose phase ellipse encloses only 39% beam fraction!

N_b is the number of particles per bunch, c is the speed of light, r_i the classical ion radius, β, γ the Lorentz factors & $\alpha_u, \beta_u, D_u, D'_u$ the optics parameters. The longitudinal emittance ε_s is defined either by the product $\varepsilon_s=\sigma_s\sigma_\eta$ [m] or the momentum p such that $\varepsilon_s=\pi p\sigma_s\sigma_\eta\beta^{-1}c^{-1}$ [eVs] (*bunched* beam) and σ_s, σ_η are the bunch length and momentum spread. $C_{\log}$ in Eq. 2.45 is the Coulomb log factor.

After the bracket expansion in Eq. 2.45 the growth rates are simplified throughout right approximations (ref. [8]) and some work in the next form, cf. ref. [14] too:

As well high energy IBS approximations to Bjorken-Mtingwa theory were made by Bane & Mtingwa: refs. [12,16]

Bjorken-Mtingwa model

Final steps of IBS theory providing quantifiable growth rates

$$\frac{1}{\tau_u} = \frac{N_b c r_0^2 C_{\log}}{\gamma 8\pi\beta^3\gamma^3\varepsilon_x\varepsilon_y\sigma_s\sigma_\eta}\frac{Z^4}{A^2}\left\langle \Delta_u \int_0^\infty d\lambda \frac{(a_u\lambda + b_u)\sqrt{\lambda}}{(\lambda^3 + a\lambda^2 + b\lambda + c)^{3/2}} \right\rangle \quad 2.47$$

The 9 coefficients $a, b, c, a_{x,} b_x, a_{z,} b_z, a_{s,} b_s$ (not reproduced here) depend on the optics parameters of the storage ring lattice (cf. ref. [21]).

For illustration, the following figure displays the evolution of the *Coulomb log* for the ELENA 100 keV low-energy antiproton decelerator ring calculated with Eq. 2.48.

$$C_{\log} = \min\left[\frac{\ln r_{\max}}{\ln r_{\min}}\right] \quad r_{\max} = \min[\sigma_x, \lambda_D] \quad r_{\min} = \max\left[r_{\min}^C, r_{\min}^{QM}\right] \quad 2.48$$

In $C_{\log}$ the impact parameter $r_{\min}$ is the larger of the *classical* distance of closest approach $r_{\min}^C$ and the *quantum* diffraction limit from the nuclear radius $r_{\min}^{QM}$, and $r_{\max}$ is the smaller of the mean rms *beam size* σ_x=$\sqrt{\langle\beta_x\rangle\varepsilon_x}$ and the Debye length λ_D. All these variables are explicitly defined as follows:

Bjorken-Mtingwa model

Final steps of IBS theory providing quantifiable growth rates

$$\lambda_D = \frac{7.434}{Z}\sqrt{\frac{2E_\perp}{\rho}} \quad \rho = \frac{N_b \times 10^{-6}}{\sqrt{64\pi^3\langle\beta_x\rangle\varepsilon_x\langle\beta_y\rangle\varepsilon_y\sigma_z^2}} \quad E_\perp = \frac{(\gamma^2-1)E_0}{2}\frac{\varepsilon_x}{\langle\beta_x\rangle}$$

$$r_{\min}^{C} = \frac{1.44\times 10^{-9}Z^2}{2E_\perp} \quad r_{\min}^{QM} = \frac{1.973\times 10^{-13}}{\sqrt{8E_\perp E_0}}$$

(see ref. [10])

in which ρ is the particle volume density [m^{-3}] and $E_\perp$ is the transverse beam kinetic energy [eV] in the centre-of-mass frame.

Coulomb logarithm (ELENA ring)

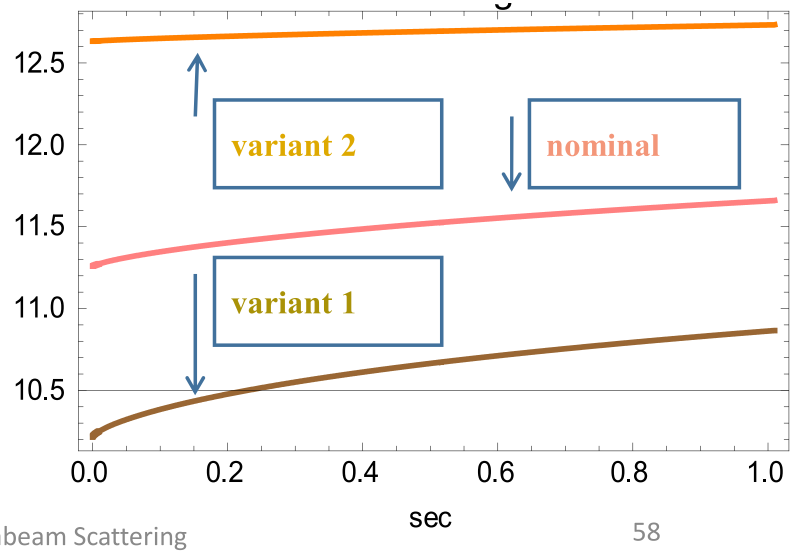


Fig. caption: Evolution of the calculated Coulomb logarithm during 1 s on a 100 keV plateau for the nominal ELENA beam and the first two variants (see table slide 73).

INTRABEAM SCATTERING

- Part 3: Applications
  - IBS & LHC (7 TeV)
  - IBS & ELENA (100 keV)
  - Epilogue
- Appendices: Feynman rules

Continuation… from Part 2

IBS Calculations

The IBS growth rates in one turn (or one time step)

Horizontal, vertical and longitudinal **equilibrium states** and **damping times** due to SR damping

$$\frac{1}{T_i} = \langle f_i \rangle$$

Complicate integrals averaged around the ring

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}(\varepsilon_x - \varepsilon_{x0}) + \frac{2\varepsilon_x}{T_x(\varepsilon_x, \varepsilon_y, \sigma_p)}$$

$$\frac{d\varepsilon_y}{dt} = -\frac{2}{\tau_y}(\varepsilon_y - \varepsilon_{y0}) + \frac{2\varepsilon_y}{T_y(\varepsilon_x, \varepsilon_y, \sigma_p)}$$

$$\frac{d\sigma_p}{dt} = -\frac{1}{\tau_p}(\sigma_p - \sigma_{p0}) + \frac{\sigma_p}{T_p(\varepsilon_x, \varepsilon_y, \sigma_p)}$$

If = 0 → **Steady State emittances**

If ≠0

- Steady state exists if we are below transition or in the presence of SR damping
- dt should be much smaller than the IBS growth times
- Good scanning of optics is important in order not to skip large IBS kick points

Courtesy F. Antoniou, Y. Papaphilippou, CERN

IBS & LHC (7 TeV)

LHC and SLHC beam parameter with improved variants

$\mathcal{L} = \dfrac{f_{rev} n_b N_b \gamma}{2 r_p \beta^*} \lvert \Delta Q_{bb} \rvert$	LHC Luminosity with nominal beam intensity		SLHC Luminosity	
	Case 1 Initial IR triplet	Case 2 IR phase 1 triplet: $\beta^* = 0.30$ m reduced emittance	Case 3 Ultimate N_b: $\beta^* = 0.25$ m reduced emittance	Case 4 >Ultimate N_b: $\beta^* = 0.15$ m reduced emittance
$\mathrm{N_b}\ (10^{11})$	1.15	1.15	1.70	2.36
$\varepsilon^n_{H,V} = \varepsilon^n = \gamma\varepsilon$ rms μm	3.75	2.54	2.65	2.60
β^* m	0.55	0.30	0.25	0.15
$\sigma^*_{H,V} = \sigma^*$ μm	16.58	10.11	9.40	7.21
σ_{BL} mm	75.50	75.50	75.50	75.50
$\sigma_{\Delta p/p}\ (10^{-4})$	1.13	1.13	1.13	1.13
ε_L rms eVs	0.62	0.62	0.62	0.62
Crossing angle θ μrad	285	337	355	454
ΔQ_{bb} head-on**	1.00	1.09	1.43	1.37
$\mathcal{L}$uminosity (10^{34}) cm^{-2}s^{-1}	1.00	2.00	4.65	10.29

** ΔQ_{bb} normalized to the value of the nominal beam

- 1st case: nominal beam and LHC parameters at top energy give the nominal luminosity of 10^{34}cm^{-2}s^{-1}
- 2nd case: new optics will rise the crossing angle to 337 μrad and the luminosity to 2×10^{34}cm^{-2}s^{-1}
- 3rd case: will raise the head-on beam-beam tune shift to 1.43 and the luminosity to 4.65×10^{34}cm^{-2}s^{-1}
- 4th case: with an intensity of 2.36×10^{11} protons/bunch a top luminosity of $\sim 10^{35}$cm^{-2}s^{-1} can be got.

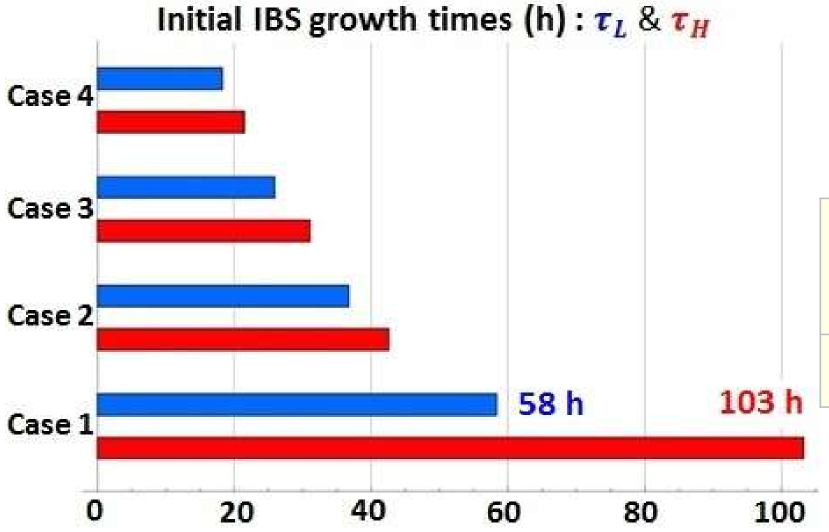
IBS & LHC (7 TeV)

IBS effects in the SLHC

IBS (**Bjorken-Mtingwa** model) and *synchrotron radiation* calculation to estimate the *LHC* & *SLHC* beam emittances evolution during *7 TeV physics coasts* are done for the *4 nominal* & *reduced emittance beam* cases

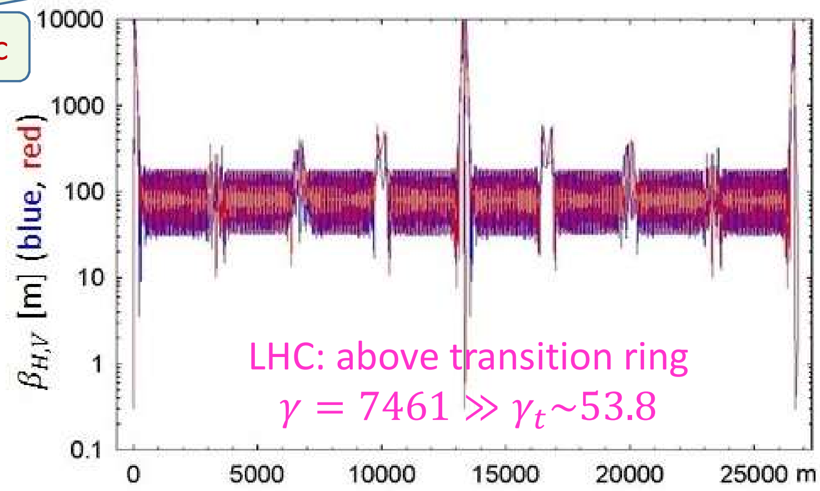
○ IBS growth rates:	$$\frac{1}{\tau_{L,H,V}} = \frac{N_b c r_0^2 C_{\log}}{8\pi\beta^3\gamma^4\varepsilon_H\varepsilon_V\sigma_{BL}\sigma_{\Delta p/p}}\langle H_{L,H,V}\rangle$$	3.1
○ Longitudinal emittance:	$$\varepsilon_L = \pi p \sigma_{BL}\sigma_{\Delta p/p}(\beta c)^{-1} \quad [\text{eVs}]$$	

p is the momentum in eV/c



		$\Delta\varepsilon_L/\varepsilon_L$	$\Delta\varepsilon_H/\varepsilon_H$	$\Delta\varepsilon_V/\varepsilon_V$
1st case	Initial IR triplet	16%	9%	-0.0001%
2nd case	IR phase 1 triplet ($\beta^* = 0.30$ m) reduced emittance	24%	21%	-0.001%
3rd case	Ultimate N_b ($\beta^* = 0.25$ m) reduced emittance	32%	27%	-0.001%
4th case	>Ultimate N_b ($\beta^* = 0.15$ m) reduced emittance	44%	37%	-0.001%

IBS emittance growth after a 10 hours beam coast



SLHC betatron functions for $\beta^* = 0.30$ m (IP1 and IP5)

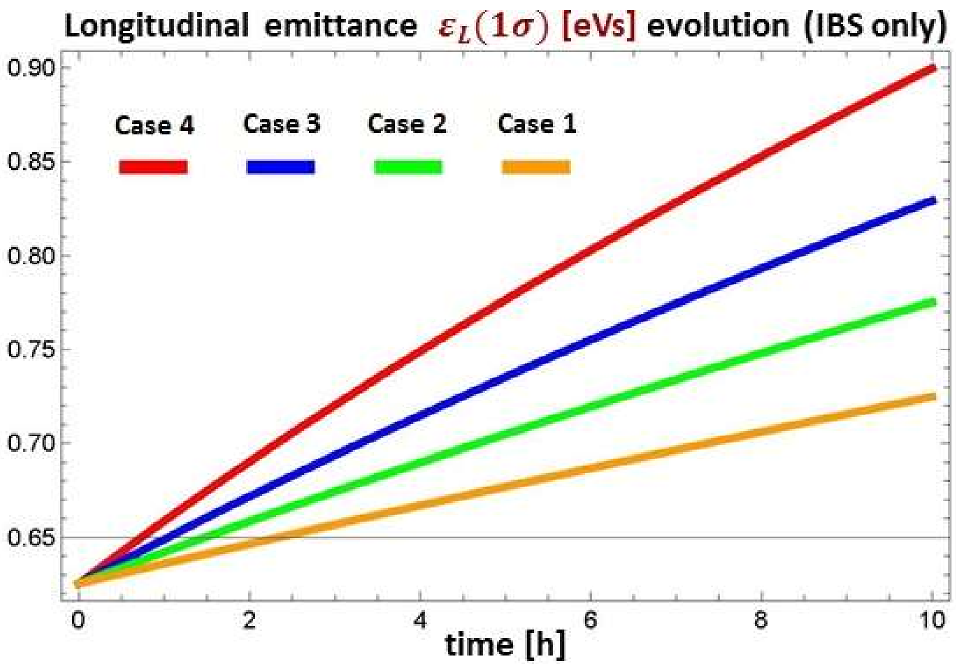
IBS & LHC (7 TeV)

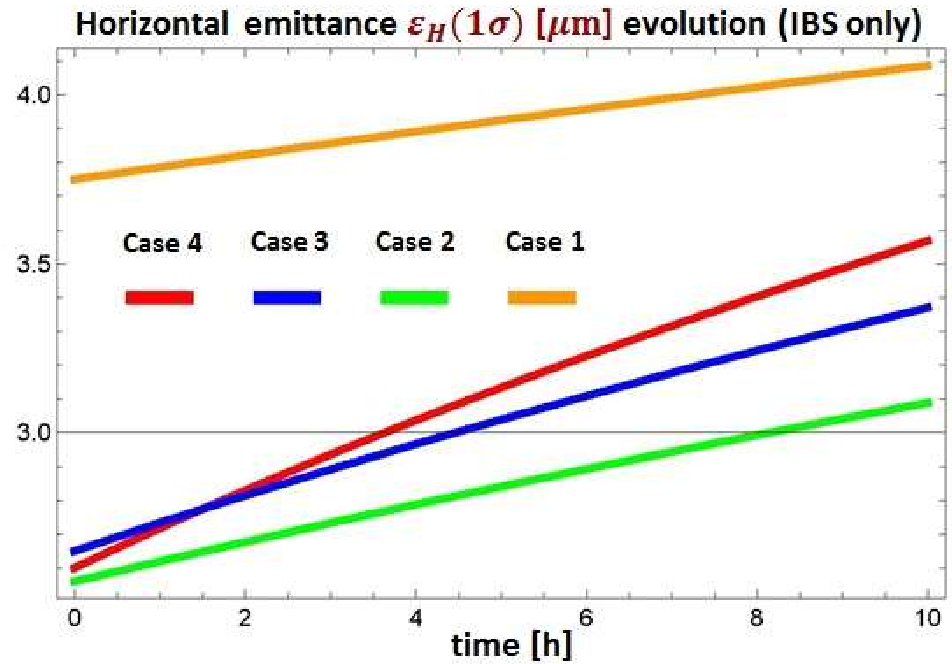
IBS effects in the SLHC

- A constant beam intensity for the duration of the beam storage period is assumed in the computations.
- The next 2 figures show the evolution of the *longitudinal* & *horizontal emittances* over a *10 hours beam coast*.
- IBS growth-rates $\tau_{L,H,V}^{-1}$ were calculated iteratively by step Δt of 5 minutes updating the emittances at each iteration i:

$$\varepsilon_{L,H,V}(i+1) = \varepsilon_{L,H,V}(i)e^{\Delta t/\tau_{L,H,V}(i)} \quad (3.2)$$

$$i = i+1 \quad \tau_{L,H,V}^{-1}(i+1) = d\ln\varepsilon_{L,H,V}(i)/dt \quad (3.3)$$

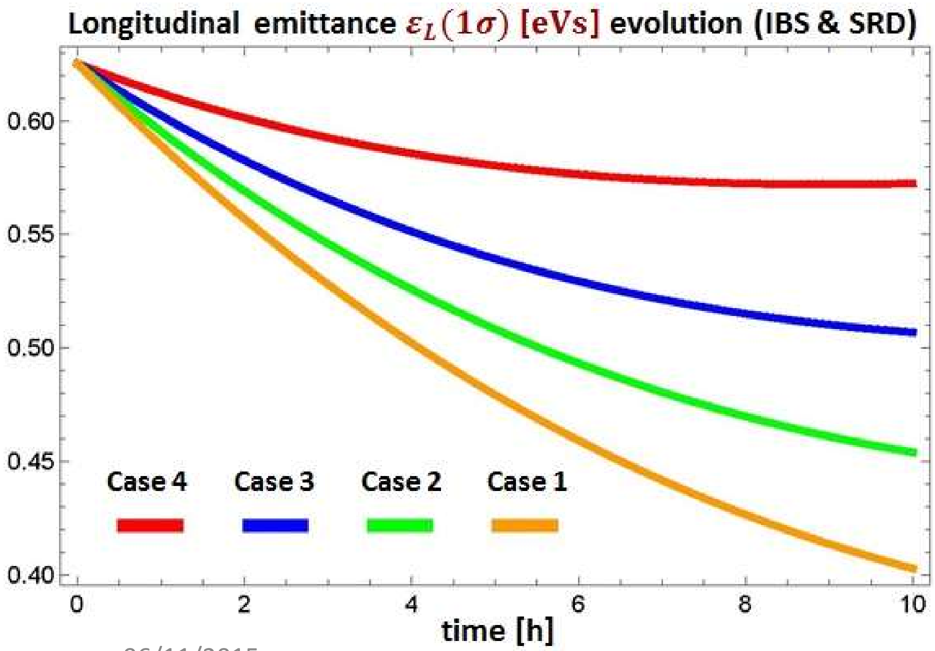


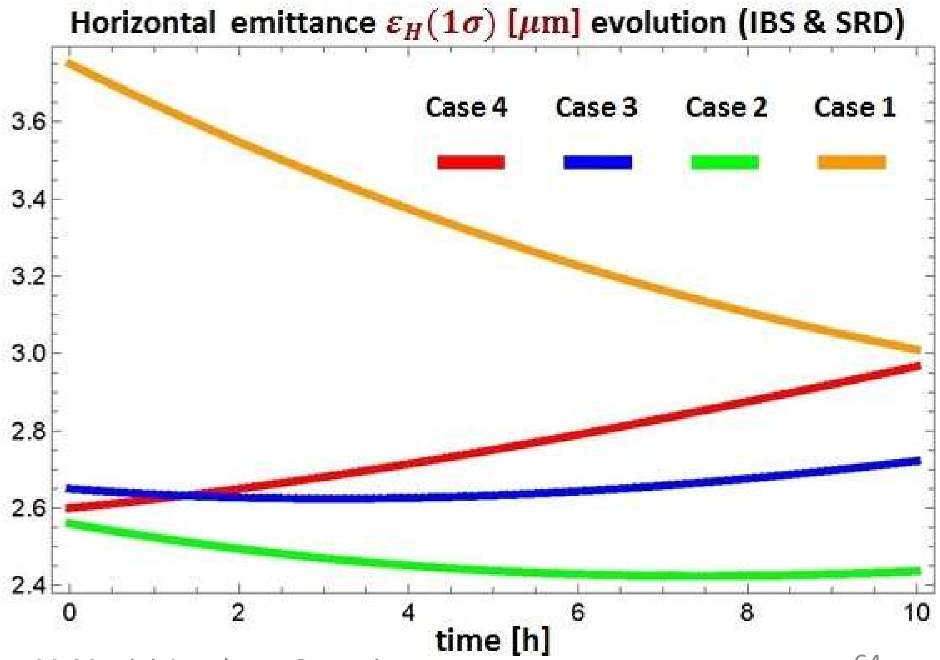


IBS & LHC (7 TeV)

IBS & synchrotron radiation damping effects in the SLHC

- The synchrotron radiation turns into a visible effect for the LHC/SLHC proton beams at 7 TeV collision energy. *Emittances shrink* with *damping* times of: **12.9 h** in the *longitudinal* and **26.0** h in the 2 *transverse* planes.
- Synchrotron radiation damping (*SRD*) is modelled substituting in the previous formula $\tau_{L,H,V}(i)$ by $\left(\tau^{-1}_{L,H,V}(i) - \tau^{-1}_{\mathrm{srd}_{L,H,V}}\right)^{-1}$
- The next 3 figures show the evolution of the *longitudinal* & *transverse emittances* over a *10 hours beam coast*.
- *SRD* dominates the *IBS growth* in the *longitudinal* & *vertical* planes for the 4 cases, in *horizontal* the emittance damps over the all coast only for *case 1* while, for *cases 2-4* it grows at some point in time during the coast.

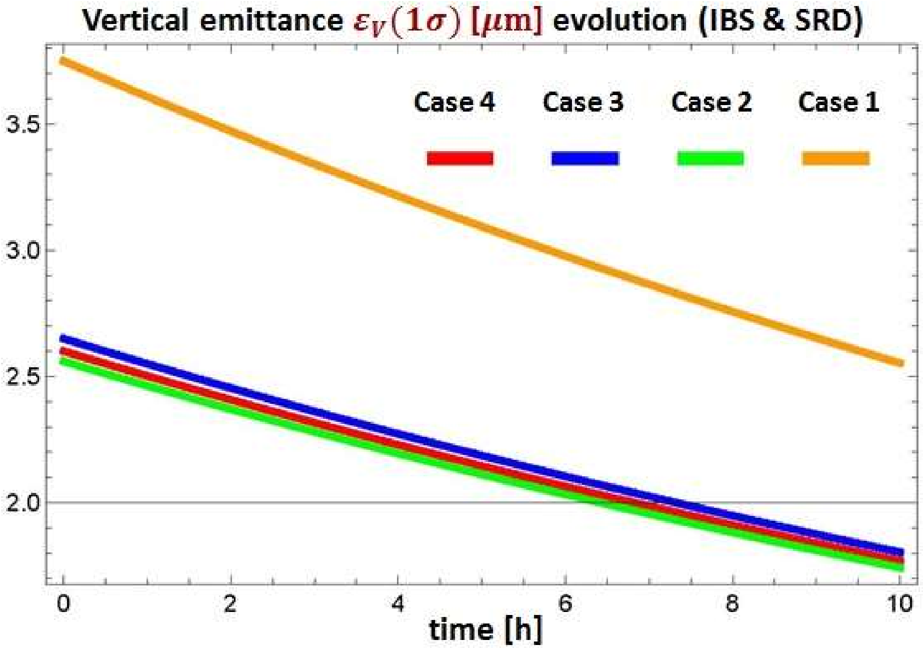




IBS & LHC (7 TeV)

IBS & synchrotron radiation damping effects in the SLHC

Table: *Emittance changes* after a 10 hours beam coast resulting from the effects of IBS and synchrotron radiation damping



		$\Delta\varepsilon_L/\varepsilon_L$	$\Delta\varepsilon_H/\varepsilon_H$	$\Delta\varepsilon_V/\varepsilon_V$
1st case	Initial IR triplet	-36%	-20%	-32%
2nd case	IR phase 1 triplet ($\beta^* = 0.30$ m) reduced emittance	-27%	-5%	-32%
3rd case	Ultimate N_b ($\beta^* = 0.25$ m) reduced emittance	-19%	3%	-32%
4th case	>Ultimate N_b ($\beta^* = 0.15$ m) reduced emittance	-8%	14%	-32%

IBS emittance changes after a 10 hours beam coast

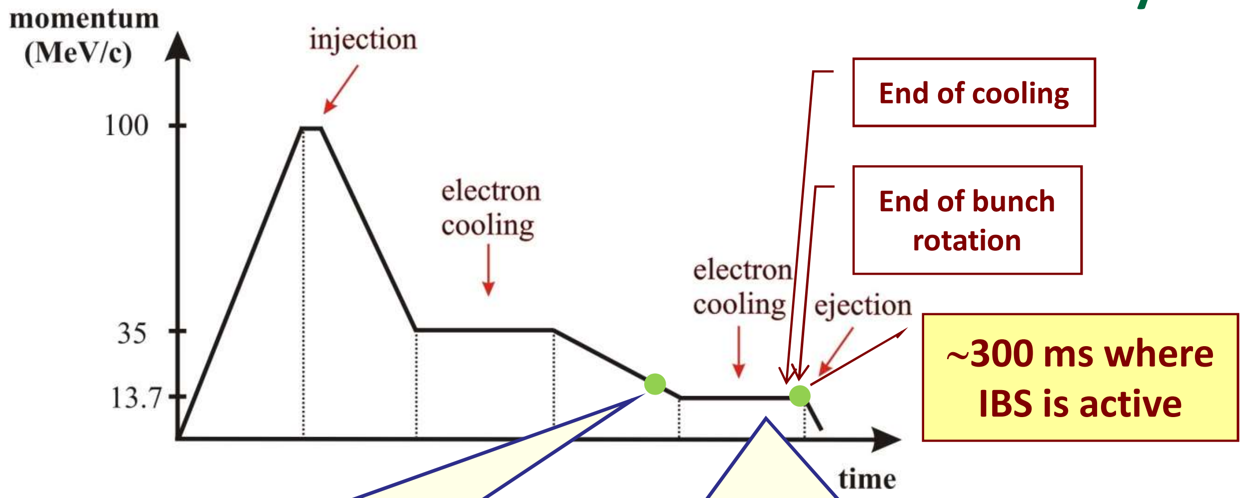
Conclusion

- **Longitudinal** & **vertical**: cases 1-2-3-4: *emittances* of all the *luminosity* scenarios are kept within target specifications.
- **Horizontal**: *emittances* stay in requirements cases 1-2: (*nominal* 10^{34} & *first IR upgrade* 2×10^{34} cm^{-2}s^{-1} luminosities, case 3: $\sim$*3% blow-up* expected (*ultimate* intensity $N_b = 2.36\times10^{11}$) & case 4: $\sim$*14%* ($\sim10^{35}$ cm^{-2}s^{-1}*peak* luminosity).

Globally for most scenarios the evolution of *emittances* during the 10 hours coast is kept inside the design values

IBS & ELENA (100 keV)

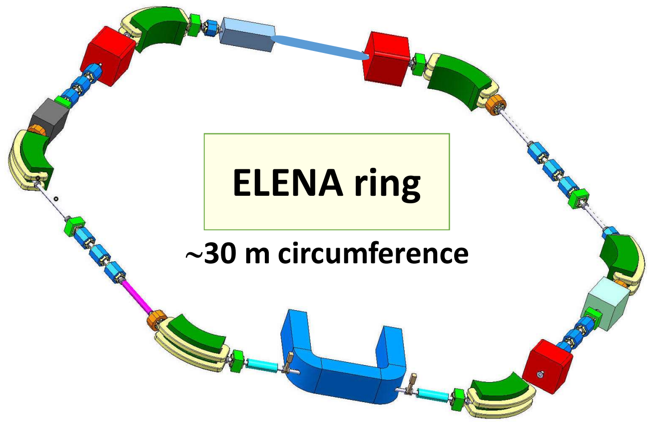
ELENA deceleration cycle



Momentum	~ 13.7 MeV/c
Beam intensity	2.5 10^7 (1 bunch)
Physical $\varepsilon_{H,V}$ (95%)	5 mm.mrad
$\Delta p/p$ (95%)	3 10^{-4}
Bunch length (95%)	10.1 m (circumf/3)

Momentum (energy)	13.7 MeV/c (100 keV)
Bunch intensity	6.25 10^6 (4 bunches)
Physical $\varepsilon_{H,V}$ (95%)	4 mm.mrad
$\Delta p/p$ (95%)	3 10^{-4}
Bunch length (95%)	1.3 m

ELENA (Extra Low Energy Antiproton) is a compact ring for *cooling* and more *deceleration* of **5.3 MeV** *antiprotons* sent by the Antiproton Decelerator to give dense beams at **100 keV** energies cf. ref. [22,23]



ELENA: below transition ring
$\gamma = 1.0001 < \gamma_t \sim 1.9$

- 1st plateau: 4 bunches injection at 100 MeV/c from AD followed by beam cooling.
- 2nd plateau: Deceleration down to 35 MeV/c and cooling again.
- 3rd plateau: Last deceleration down to 13.7 MeV/c, beam cooled down to emittances needed for ELENA experiments.

IBS & ELENA (100 keV)

Nominal beam parameter and variant study

Ejection momentum/energy	13.7MeV/c	100 keV
Injected/ejected beam intensity	3 10^7	2.5 10^7
Number of extracted bunches	4	
Extracted bunch intensity	6.25 10^6	

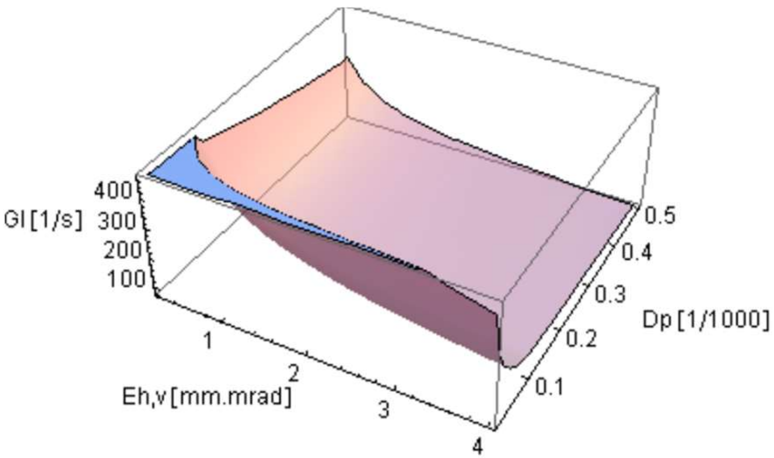
$\varepsilon_{H,V}^{rms} = 1\,\mu m, \sigma_{\Delta p/p} = 0.325\; m\;(75\; ns), \sigma_{\Delta p/p} = 0.075\,‰\;(7.510^{-5})$ $\varepsilon_{H,V}^{rms} = \pi p \sigma_L \sigma_{\Delta p/p} (\beta c)^{-1}$

	σ_{BL} m	$BL^{95\%}$ m	$\sigma_{\Delta p/p}$ ‰	$\Delta p/p^{95\%}$ ‰	$\varepsilon_L{}^{rms}$ eVs	$\varepsilon_L{}^{95\%}$ eVs	$\varepsilon_{H,V}^{rms}$ μm	$\varepsilon_{H,V}^{95\%}$ μm
Nominal beam	0.325	1.3	0.075	0.3	2.4 10^{-4}	9.6 10^{-4}	1.0	4.0
Variant 1	0.325	1.3	0.025	0.1	0.8 10^{-4}	3.2 10^{-4}	0.5	2.0
Variant 2	0.325	1.3	0.125	0.5	4.0 10^{-4}	16 10^{-4}	2.5	10.0

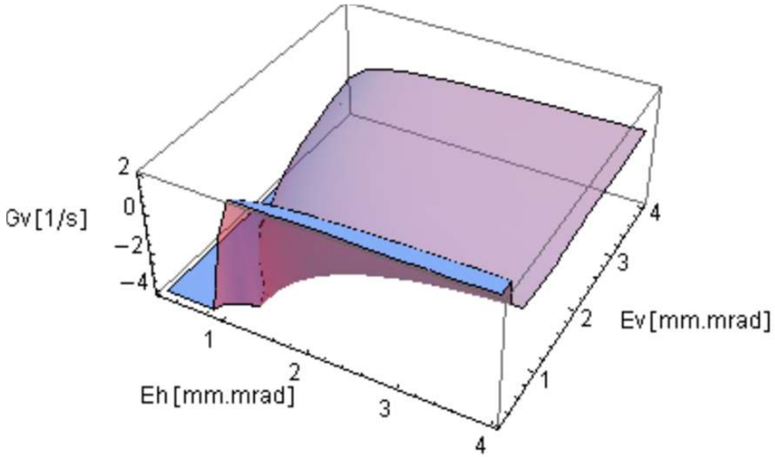
Initial nominal beam emittances with variants on the 100 keV plateau

IBS & ELENA (100 keV)

Longitudinal IBS



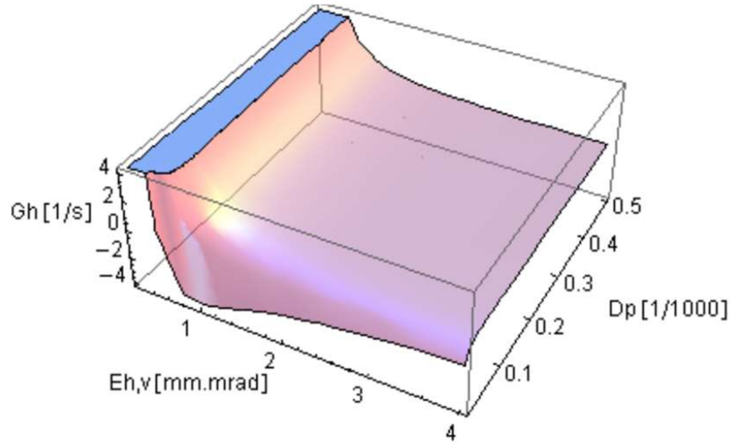
Growth-rate $1/\tau_L$ vs ($\sigma_{\Delta p/p}$, ε_H) for $\varepsilon_H = \varepsilon_V$ & σ_{BL}= 0.325 m



Growth-rate $1/\tau_V$ vs (ε_H, ε_V) for $\sigma_{\Delta p/p}$ =0.075 ‰ & σ_{BL}= 0.325 m

Vertical IBS

Horizontal IBS



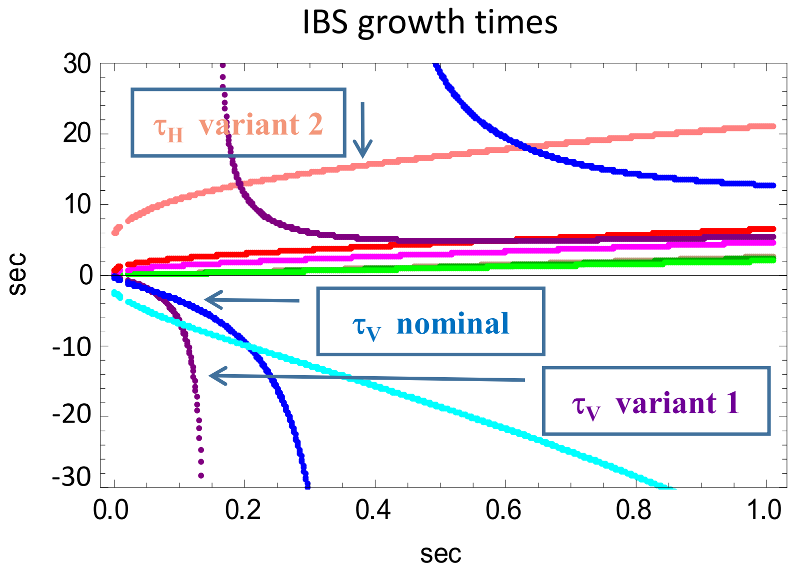
Growth-rate $1/\tau_H$ vs (ε_H, ε_V) for $\sigma_{\Delta p/p}$ =0.075 ‰ & σ_{BL}=0.325 m

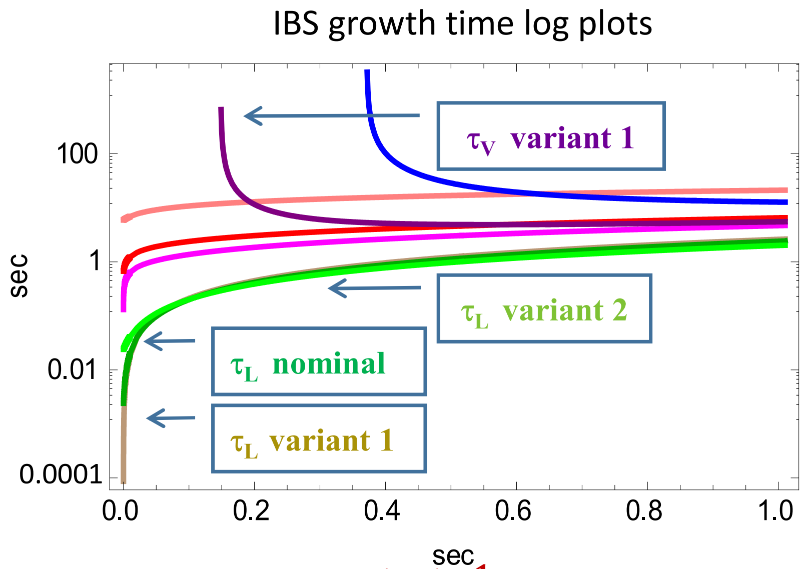
Bjorken-Mtingwa

IBS calculation model

IBS & ELENA (100 keV)

IBS growth times evolution



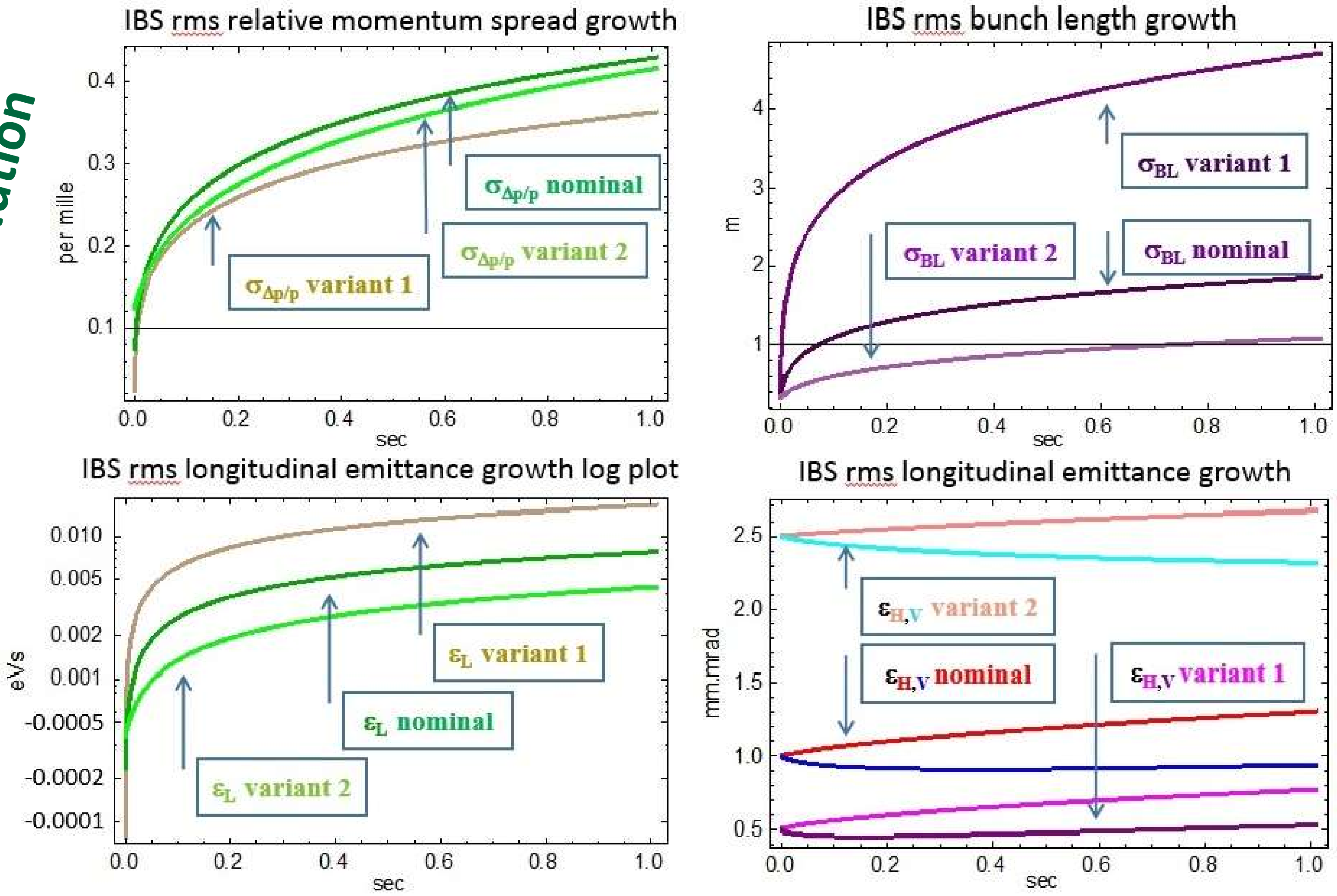


IBS growth-times $\tau_{L,H,V}$ evolution $\left(\varepsilon_L = \pi p \sigma_{BL} \sigma_{\Delta p/p} (\beta c)^{-1}\right)$

ELENA initial rms beam emittances and IBS growth times at 100 keV ejection								
	σ_{BL} m	$\sigma_{\Delta p/p}$ ‰	ε_L eVs	ε_H µm	ε_V µm	τ_L ms	τ_H s	τ_V s
Nominal beam	0.325	0.075	$2.4\ 10^{-4}$	1.0	1.0	2.40	0.67	-0.27
Variant 1	0.325	0.025	$0.8\ 10^{-4}$	0.5	0.5	0.09	0.13	-0.04
Variant 2	0.325	0.125	$4.0\ 10^{-4}$	2.5	2.5	24.0	5.92	-2.44

IBS & ELENA (100 keV)

IBS beam parameter evolution



IBS & ELENA (100 keV)

Comments on variant performance & study extra variants

Assuming one or several bunches circulate for ~1 s on the 100 keV *plateau*: the above plots show that *none* of the *3 scenarios* are fully *satisfactory* because the *bunch length* and *momentum spread* will suffer too much *blow-up* due to *IBS*:

Nominal: bunch length and momentum spread growth after 1 s on the 100 keV plateau is **Big !**

σ_{BL}(1s) =1.9 m , $\sigma_{\Delta p/p}$(1s) =0.4 ‰ (95% bunch length=7.4 m instead of 1.3 m !)

Variant 1: bunch length and momentum spread increases after 1 s on the 100 keV plateau is **Huge !**

σ_{BL}(1s) =4.7 m, $\sigma_{\Delta p/p}$(1s) =0.4 ‰ (95% bunch length=18.8 m !)

Variant 2: bunch length and momentum spread increases after 1 s on the 100 keV plateau is still too **Large !**

σ_{BL}(1s) =1.1 m, $\sigma_{\Delta p/p}$(1s) =0.4 ‰ (95% bunch length=4.3 m !)

	σ_{BL} m	$BL^{95\%}$ m	$\sigma_{\Delta p/p}$ ‰	$\Delta p/p^{95\%}$ ‰	$\varepsilon_L{}^{rms}$ eVs	$\varepsilon_L{}^{95\%}$ eVs	$\varepsilon_{H,V}^{rms}$ μm	$\varepsilon_{H,V}^{95\%}$ μm
variant 3	**0.325**	1.3	**0.250**	1	**$8\ 10^{-4}$**	$32\ 10^{-4}$	**1.0**	4.0
variant 4	**0.325**	1.3	**0.375**	1.5	**$12\ 10^{-4}$**	$48\ 10^{-4}$	**1.0**	4.0
variant 5	**0.325**	1.3	**0.500**	2	**$16\ 10^{-4}$**	$60\ 10^{-4}$	**1.0**	4.0

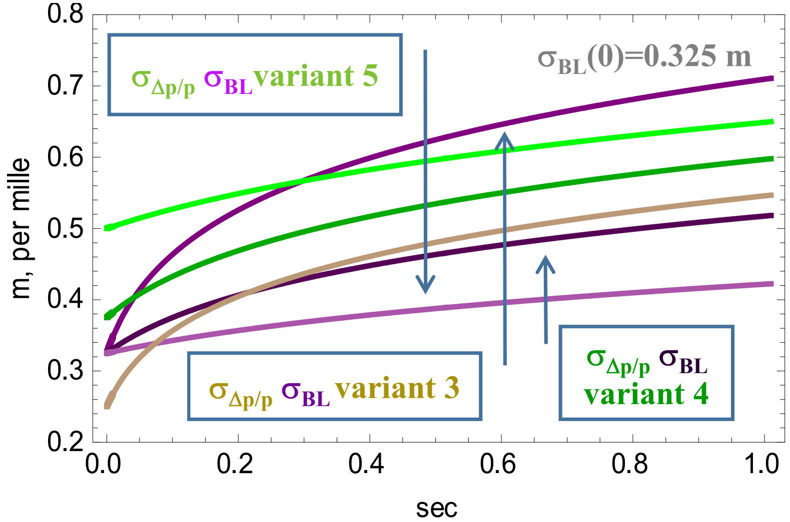
Three more variant scenarios with higher relative momentum spreads

IBS & ELENA (100 keV)

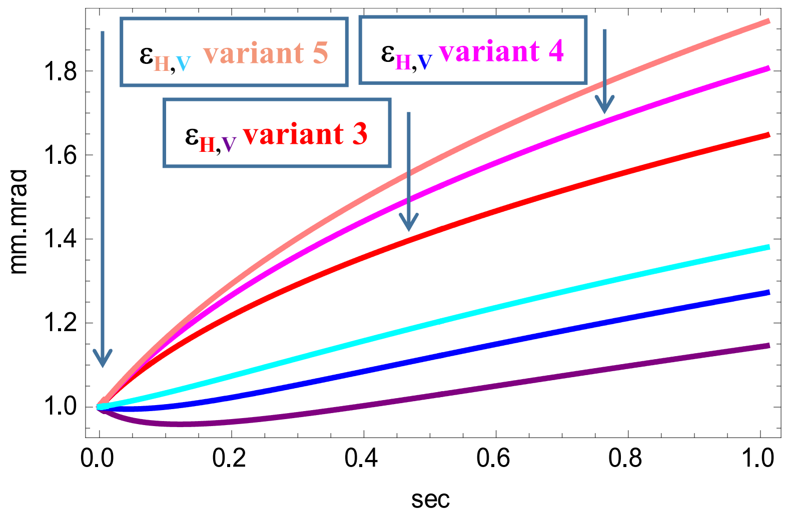
Additional IBS variant beam study

Plots of the beam parameter evolution for the three new variant scenarios

IBS rms bunch length and momentum spread growth



IBS rms physical transverse emittance growth



Evolution of the momentum spread and bunch length (left) and transverse emittances (right)

Summary of the IBS variant beam performance

The table shows that among the *3 new scenarios* investigated the *variant 5* is the *best* because the **bunch length** and **momentum spread** will suffer only **30% blow-up due to IBS after 1s** on the 100 keV plateau (**13% blow-up after 0.3s**)

Nominal: the bunch length and momentum spread growth after 1 [s] on the 100 keV plateau is **Big !**

σ_{BL}(1s) =1.9 m , $\sigma_{\Delta p/p}$(1s) =0.4 ‰ (95% bunch length=7.4 m instead of 1.3 m at t=0 !)

Variant 5: the bunch length and momentum spread growth after 1 [s] looks **Fine**

σ_{BL}(1s) =0.4 m, $\sigma_{\Delta p/p}$(1s) =0.6 ‰ (95% bunch length=1.7m !)

	$\sigma_{BL}(t)/\sigma_{BL}(0)$		$\sigma_{\Delta p/p}(t)/\sigma_{\Delta p/p}(0)$		$\varepsilon_L(t)/\varepsilon_L(0)$		$\varepsilon_H(t)/\varepsilon_H(0)$		$\varepsilon_V(t)/\varepsilon_V(0)$	
Growth factor at t=	1 s	0.3 s	1 s	0.3 s	1 s	0.3 s	1s	0.3 s	1 s	0.3 s
Nominal beam	**5.7**	**4.4**	**5.7**	**4.4**	**32.5**	**19.0**	**1.31**	**1.13**	**0.94**	**0.91**
variant 1	**14.5**	**11.3**	**14.5**	**11.3**	**205.0**	**125.3**	**1.25**	**1.54**	**1.05**	**0.92**
variant 2	**3.3**	**2.4**	**3.3**	**2.4**	**11.0**	**5.9**	**1.07**	**1.03**	**0.93**	**0.96**
variant 3	**2.19**	**1.75**	**2.19**	**1.75**	**4.78**	**3.04**	**1.65**	**1.29**	**1.15**	**0.98**
variant 4	**1.59**	**1.32**	**1.59**	**1.32**	**2.54**	**1.75**	**1.81**	**1.36**	**1.27**	**1.05**
variant 5	**1.30**	**1.13**	**1.30**	**1.13**	**1.69**	**1.29**	**1.92**	**1.40**	**1.38**	**1.12**

IBS beam growth factor: beam parameter at time t over the initial one at t=0 along the 100 keV plateau

Epilogue

- Exchange of energies between *horizontal & vertical* β-oscillations *& synchrotron* oscillations due to IBS was first studied by Piwinski (1974) for weak-focussing storage rings ref. [3].
- The derivatives of the amplitude function & dispersion β_x' & D_x' were implemented into a CERN code by Piwinski & Sacherer (1977) and used for *rise-time* calculations in diverse proton storage rings ref. [4].
- Likewise strong-focussing IBS *rise-times* were afterward derived by Bjorken-Mtingwa (1983) using a quantum electrodynamic theory approach, giving a new, broad and smart description of IBS theory ref. [8,11].
- Next IBS theory was extended by Piwinski (1990) to include a *linear coupling* (skew quads or solenoids) between *horizontal & vertical* β-oscillations (mixing the derivatives of vertical β_z'-function & dispersion D_z' in his theory).
- Between 2005 & 2012 the vertical lattice functions β_z' and D_z' were incorporated in the Bjorken-Mtingwa theory by Zimmermann ref. [14]. *Mathematica Notebooks* were written accordingly by diverse persons.
- Besides, Bane (2002) & Kubo, Mtingwa, Wolski (2005) adapted the Piwinski IBS theory to get growth times at high energies comparable to those of Bjorken-Mtingwa: yielding the *Completely Integrated Modified Piwinski* (*CIMP*) ref. [12,13]. Also, Mtingwa (2008) developed a fast computation estimate of the emittance growth rates for flat e^+ & e^- beams at high energy ref. [16], (e.g. aimed at damping rings and synchrotron light sources).
- The IBS growth times with *linear coupling* was applied to the generalized emittances specified by way of the β-oscillation eigenvectors (e.g. as calculated by *MADX*). The process was fully implemented into a *Mathematica Notebook* in 2012 ref. [18] and used for *ELENA* antiproton deceleration studies at 100 keV energy.

INTRABEAM SCATTERING

- Appendices: Feynman rules

Appendix 1: Feynman diagrams for QED

Case: 2-body scattering in CM frame

- **Feynman diagrams**: symbolic & qualitative description of elementary particle interactions (also show graphically the approximations of the *S-matrix* elements got by perturbative series expansion).
- **Particles**: are lines with arrows in space-time, time flows from left to right (or bottom to top), space direction is at right angles to the time direction (antiparticles travel backwards in time).
- **Arrows**: show the charge flux relative to time, where wavy lines represent *virtual particles* are bosons that mediate the interaction between the particles, and which are created (emitted) and annihilated soon after (e.g. photons). *Virtual particles* do not have mass of real particles: $m^2 \neq E^2 - \boldsymbol{q}^2$ ($m=0$ for γ).
- **Loops**: are closed patterns of virtual particles (in diagrams with high-order terms of the perturbative *S-matrix*'s expansion power series).

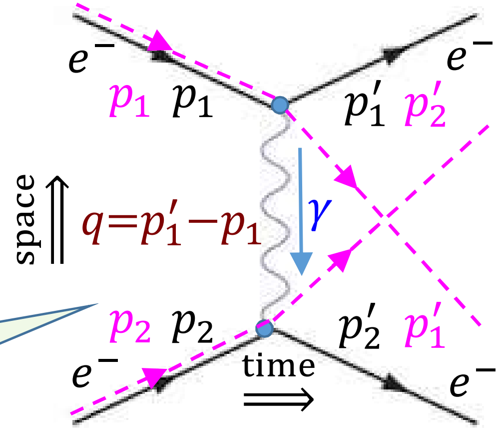


Fig. caption: Feynman diagram for electron–electron (e^-) scattering; the left-hand side of the diagram shows the initial state, the right-hand side the final one. The wavy line linking the 2 vertices belongs to neither the initial nor the final state, it illustrates "how the interaction occurs". The intermediate photon γ is virtual. Dashed lines show the diagram for exchange e^-e^- scattering.

Actually there are 2 Feynman diagrams as the 2 emerging e^- are undifferentiated, but the 2 incident e^- stay the same. So the 2 diagrams for direct and exchange e^-e^- scattering mirror the full process (cf. Eq. 2.41-2.43)

Appendix 2: Feynman rules for spinless particles & bosons

Case: 2-electrons elastic scattering in CM frame

Basic rules for a **toy model** used for the easiest diagrams with a single internal momentum (no loop which denotes perturbation terms)

1. **Label**: draw a line for each inward/outward external particle momenta $p_{1,2} \stackrel{\text{def}}{=} p_{1,2}^{\mu}$ & $p'_{1,2} \stackrel{\text{def}}{=} p'^{\mu}_{1,2}$ and the internal momentum $q \stackrel{\text{def}}{=} q^{\mu}$, with $q = p'_1 - p_1$ (i.e. momentum transfer carried by an exchanged boson).
2. **Vertex**: for each one give a factor $-\mathrm{i}g_{\mathrm{E}} (\mathrm{i} = \sqrt{-1})$, their products is $-g_{\mathrm{E}}^2$, g_{E} is the *coupling constant*.
3. **Propagator**: give to the single internal wavy line a factor $f(q) = \mathrm{i}/q^2$ for boson with spin-0 and zero mass (mimicked a photon γ). $f(q)$ acts for the momentum propagation among the 2 electrons in the interaction time, via a virtual photon. The global product is $(-\mathrm{i}g_{\mathrm{E}})^2 f(q) = -\mathrm{i}g_{\mathrm{E}}^2/q^2$.
4. **4-momenta conservation**: write a δ-function at each vertex (put a $+/-$ sign on the $p_{1,2}$, $p'_{1,2}$, q if the arrow points in/out a vertex). The above diagram gives: $(2\pi)^4 \delta^4(p_1 - p'_1 + q)$ & $(2\pi)^4 \delta^4(p_2 - p'_2 - q)$.
5. **Momenta integration**: multiply the δ-functions together. Fix $q \mapsto p'_1 - p_1$ in the 2nd δ-function and integrate the 1st δ-function over the internal 4-momentum q with $\mathrm{d}^4 q/(2\pi)^4$.
6. **Cancel**: the left over δ-function is cut off, the result is multiply by i, the product is $\mathcal{M}$.

ref. L,M,O,P,R

Note: the 4-energy-momentum formula $p_{1,2}^2 = E_{1,2}^2 - \boldsymbol{p}_{1,2}^2 \equiv m_{1,2}^2$ is valid for real particles but is violated for the *transitional states* bosons, called virtual particles, i.e. $q^2 = E^2 - \boldsymbol{q}^2 \neq m^2$ ($m = 0$ for physical photons). This is by virtue of the Heisenberg uncertainty principle $\Delta E \Delta t \approx \hbar$, as long as the virtual particle of energy E last only for a tiny time $\Delta t \lesssim \hbar/E$. So, the calculations of scattering processes are based on real and virtual particles to yield true results.

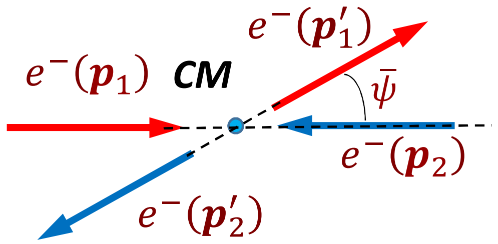
Appendix 3: Feynman rules for QED

Case: 2-electrons elastic scattering in CM frame (ref. L,M,N,O,P,Q,R,S)

- For two-body scattering in the *CM* frame with all 4 particle masses even, the *differential cross section* can be cast as Eq. A.1 (obtained via the Fermi's Golden rule, ref. [M,O,Q]).

$$\left(\frac{d\bar{\sigma}}{d\bar{\Omega}}\right)_{\mathrm{CM}} = \frac{|\mathcal{M}|^2}{64\pi^2(E_1+E_2)^2} \qquad \text{A.1}$$

- Fig. caption: kinematics of electron-electron scattering. The QED process amplitude $\mathcal{M}=\mathcal{M}_1$-$\mathcal{M}_2$ (not +) writes as follows, with $g_{\mathrm{E}}^4=(4\pi\alpha_{\mathrm{E}})^2=e^4$ in *HL* units. The 1st & 3rd terms in the brace are the amplitudes $\mathcal{M}_1$ & $\mathcal{M}_2$ of the single Feynman diagrams (cf. Eq. 2.41), the 2nd (mid) term gives the coupling strength:



$$|\mathcal{M}|^2=\frac{e^4}{4}32\left\{\frac{(p_1\cdot p_2)^2+(p_1\cdot p_2')^2+2m^2(m^2-p_1\cdot p_1')}{[(p_1-p_2')^2]^2} + 2\frac{(p_1\cdot p_2)^2-2m^2 p_1\cdot p_2}{(p_1-p_1')^2(p_1-p_2')^2} + \frac{(p_1\cdot p_2)^2+(p_1\cdot p_1')^2+2m^2(m^2-p_1\cdot p_2')}{[(p_1-p_2')^2]^2}\right\} \qquad \text{A.2}$$

- As $\boldsymbol{p}_1=-\boldsymbol{p}_2$ & $\boldsymbol{p}_1'=\boldsymbol{p}_2'$ for elastic collision of 2 electrons (of mass m), the next expressions hold:

$$E_1=E_1'=E_2=E_2'\stackrel{\mathrm{def}}{=}E \quad |\boldsymbol{p}_1|=|\boldsymbol{p}_1'|=|\boldsymbol{p}_2|=|\boldsymbol{p}_2'|\stackrel{\mathrm{def}}{=}|\boldsymbol{p}| \quad \text{with} \quad \boldsymbol{p}^2=E^2-m^2 \quad \text{and for ultrarelativistic limit } E\gg m\text{:} \quad \boldsymbol{p}^2\approx E^2$$

$$p_1\cdot p_2=p_1'\cdot p_2'=E^2+\boldsymbol{p}^2\approx 2E^2 \quad p_1\cdot p_{1,2}'=E^2\mp\boldsymbol{p}^2\cos\bar{\psi}\approx E^2(1\mp\cos\bar{\psi}) \quad \left(p_1-p_{1,2}'\right)^2=-2\boldsymbol{p}^2(1\mp\cos\bar{\psi})\approx -4E^2\begin{cases}\sin^2(\bar{\psi}/2)\\ \cos^2(\bar{\psi}/2)\end{cases} \qquad \text{A.3}$$

Appendix 3: Feynman rules for QED

Case: 2-electrons elastic scattering in CM frame (ref. L,M,N,O,P,Q,R,S)

○ The 4-momenta scalar products and square differences Eq. A.2 are changed with those of Eq. A.3 giving:

$$|\mathcal{M}|^2 = \frac{2e^4}{\boldsymbol{p}^4}\left\{\frac{(E^2+\boldsymbol{p}^2)^2+(E^2+\boldsymbol{p}^2\cos\bar{\psi})^2-2m^2\boldsymbol{p}^2(1-\cos\bar{\psi})}{(1-\cos\bar{\psi})^2}+2\,\frac{(E^2+\boldsymbol{p}^2)^2-2m^2(E^2+\boldsymbol{p}^2)}{\sin^2\bar{\psi}}+\frac{(E^2+\boldsymbol{p}^2)^2+(E^2-\boldsymbol{p}^2\cos\bar{\psi})^2-2m^2\boldsymbol{p}^2(1+\bar{\psi})}{(1+\cos\bar{\psi})^2}\right\} \qquad \text{A.4}$$

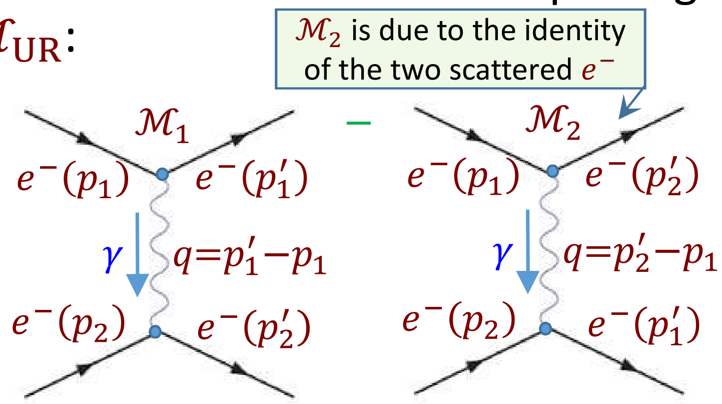
Ultra relativistic limit: $|\mathcal{M}_{\mathrm{UR}}|^2 \approx 4e^4\left\{\frac{1}{\cos^2[\bar{\psi}/2]}+\frac{1}{\sin^2[\bar{\psi}/2]}+1\right\}$

○ The *differential cross section* for unpolarised initial states & ultrarelativistic limit $E \gg m$ follows placing Eq. A.4 $|\mathcal{M}_{\mathrm{UR}}|^2$ in Eq. A.2 yielding the Möller scattering formula for $\mathcal{M}_{\mathrm{UR}}$:

$$\left[\left(\frac{d\bar{\sigma}}{d\bar{\Omega}}\right)_{\mathrm{CM}}\right]_{\mathrm{UR}} = \frac{|\mathcal{M}_{\mathrm{UR}}|^2}{64\pi^2(2E)^2} = \frac{e^4}{64\pi^2E^2}\left\{\frac{1}{\cos^2[\bar{\psi}/2]}+\frac{1}{\sin^2[\bar{\psi}/2]}+1\right\} \equiv \frac{e^4}{64\pi^2E^2}\frac{(3+\cos^2\bar{\psi})^2}{\sin^4\bar{\psi}} \qquad \text{A.5}$$

Compare the QED Eq. A.5 with the toy model Eq. 2.43

$$|\mathcal{M}|^2 = \frac{e^4}{\boldsymbol{p}^4\sin^4\bar{\psi}} \qquad \left(\frac{d\bar{\sigma}}{d\bar{\Omega}}\right) = \frac{|\mathcal{M}|^2}{64\pi^2E^2} \overset{E^2\approx\boldsymbol{p}^2}{=} \frac{e^4}{64\pi^2E^2}\frac{1}{\sin^4\bar{\psi}}$$



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