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JÜLICH FORSCHUNGSZENTRUM

Introduction to Plasma Physics

CERN School on Plasma Wave Acceleration

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Outline

- Lecture 1: Introduction – Definitions and Concepts
- Lecture 2: Wave Propagation in Plasmas

Lecture 1: Introduction

What is a plasma?

Simple definition: a *quasi-neutral* gas of charged particles showing *collective behaviour*.

Quasi-neutrality: number densities of electrons, n_e, and ions, n_i, with charge state Z are *locally balanced*:

$$n_e \simeq Z n_i. \tag{1}$$

Collective behaviour: long range of Coulomb potential ($1/r$) leads to nonlocal influence of disturbances in equilibrium.

Macroscopic fields usually dominate over microscopic fluctuations, e.g.:

$$\rho = e(Zn_i - n_e) \Rightarrow \nabla.\mathbf{E} = \rho/\varepsilon_0$$

Where are plasmas found?

1. cosmos (99% of visible universe):
   - interstellar medium (ISM)
   - stars
   - jets
2. ionosphere:
   - $\leq$ 50 km = 10 Earth-radii
   - long-wave radio
3. Earth:
   - fusion devices
   - street lighting
   - plasma torches
   - discharges - lightning
   - plasma accelerators!

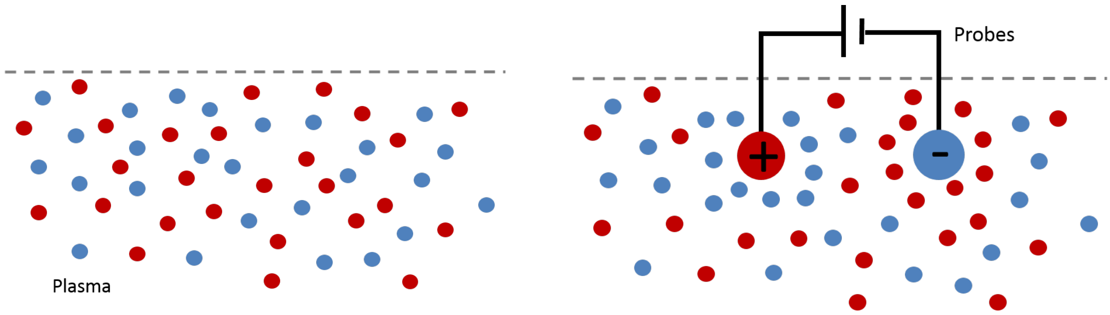
Plasma properties

Type	Electron density n_e (cm^{-3})	Temperature T_e (eV*)
Stars	10^{26}	2×10^3
Laser fusion	10^{25}	3×10^3
Magnetic fusion	10^{15}	10^3
Laser-produced	$10^{18} - 10^{24}$	$10^2 - 10^3$
Discharges	10^{12}	1-10
Ionosphere	10^6	0.1
ISM	1	10^{-2}

Table 1: Densities and temperatures of various plasma types

* 1eV $\equiv$ 11600K

Debye shielding



What is the potential $\phi(r)$ of an ion (or positively charged sphere) immersed in a plasma?

Debye shielding (2): ions vs electrons

For equal ion and electron temperatures ($T_e = T_i$), we have:

$$\frac{1}{2} m_e v_e^2 = \frac{1}{2} m_i v_i^2 = \frac{3}{2} k_B T_e \qquad (2)$$

Therefore,

$$\frac{v_i}{v_e} = \left(\frac{m_e}{m_i}\right)^{1/2} = \left(\frac{m_e}{Am_p}\right)^{1/2} = \frac{1}{43} \qquad \text{(hydrogen, Z=A=1)}$$

Ions are almost stationary on electron timescale!
To a good approximation, we can often write:

$$n_i \simeq n_0,$$

where the material (eg gas) number density, $n_0 = N_A \rho_m / A$.

Debye shielding (3)

In thermal equilibrium, the electron density follows a Boltzmann distribution*:

$$n_e = n_i \exp(e\phi/k_B T_e) \quad (3)$$

where n_i is the ion density and k_B is the Boltzmann constant.

From Gauss' law (Poisson's equation):

$$\nabla^2\phi = -\frac{\rho}{\varepsilon_0} = -\frac{e}{\varepsilon_0}(n_i - n_e) \quad (4)$$

* See, eg: F. F. Chen, p. 9

Debye shielding (4)

Combining (4) with (3) in spherical geometry[a] and requiring $\phi \to 0$ at $r = \infty$, get solution:

Exercise

$$\phi_D = \frac{1}{4\pi\varepsilon_0}\frac{e^{-r/\lambda_D}}{r}. \quad (5)$$

with

Debye length

$$\lambda_D = \left(\frac{\varepsilon_0 k_B T_e}{e^2 n_e}\right)^{1/2} = 743\left(\frac{T_e}{\text{eV}}\right)^{1/2}\left(\frac{n_e}{\text{cm}^{-3}}\right)^{-1/2} \text{cm} \quad (6)$$

[a] $\nabla^2 \to \frac{1}{r^2}\frac{d}{dr}(r^2\frac{d\phi}{dr})$

Debye sphere

An *ideal* plasma has many particles per Debye sphere:

$$N_D \equiv n_e \frac{4\pi}{3} \lambda_D^3 \gg 1. \qquad (7)$$

$\Rightarrow$ Prerequisite for collective behaviour.

Alternatively, can define *plasma parameter*:

$$g \equiv \frac{1}{n_e \lambda_D^3}$$

Classical plasma theory based on assumption that $g \ll 1$, which also implies dominance of collective effects over collisions between particles.

Collisions in plasmas

At the other extreme, where $N_D \leq 1$, screening effects are reduced and collisions will dominate the particle dynamics. A good measure of this is the *electron-ion collision rate*, given by:

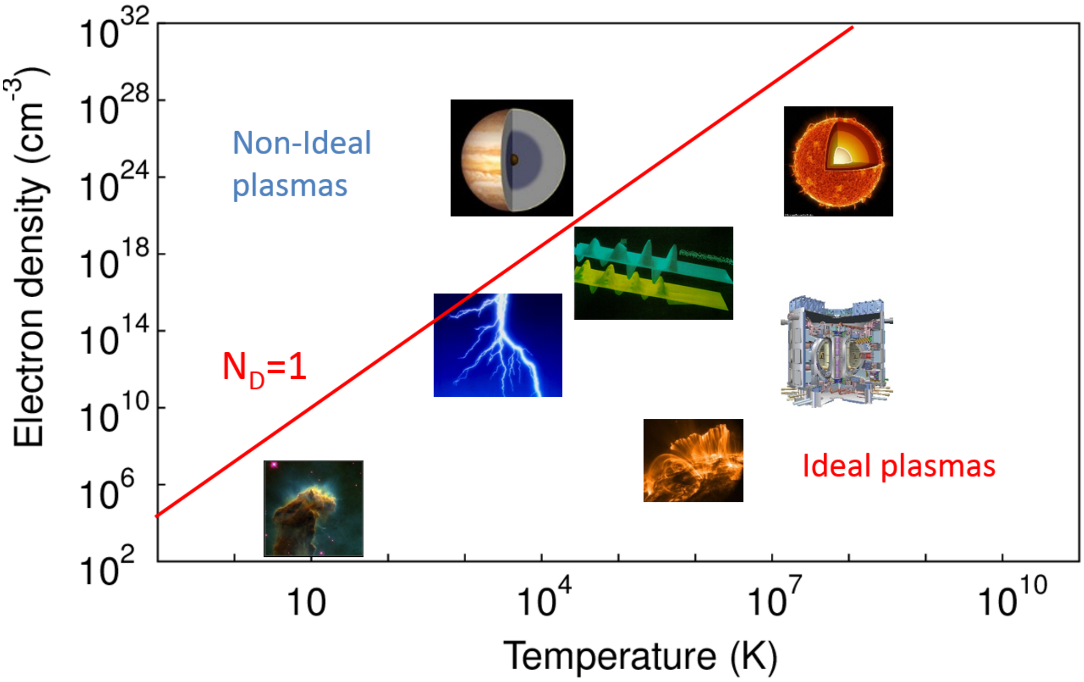
$$\nu_{ei} = \frac{\pi^{\frac{3}{2}} n_e Z e^4 \ln \Lambda}{2^{\frac{1}{2}} (4\pi\varepsilon_0)^2 m_e^2 v_{te}^3} \mathrm{s}^{-1}$$

$v_{te} \equiv \sqrt{k_B T_e / m_e}$ is the electron thermal velocity and $\ln \Lambda$ is a slowly varying term (Coulomb logarithm) $O(10 - 20)$.

Can show that

$$\frac{\nu_{ei}}{\omega_p} \simeq \frac{Z \ln \Lambda}{10 N_D}; \quad \text{with} \quad \ln \Lambda \simeq 9 N_D / Z$$

Plasma classification

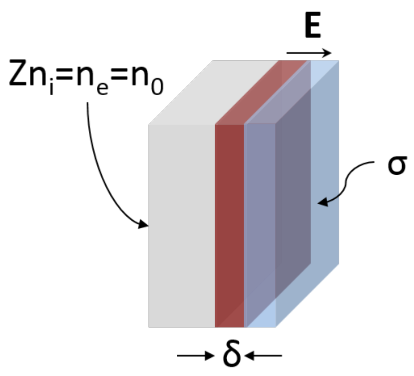


Model hierarchy

1. First principles N-body molecular dynamics
2. Phase-space methods – Vlasov-Boltzmann
3. 2-fluid equations
4. Magnetohydrodynamics (single, magnetised fluid)

- Time-scales: 10^{-15} – 10^{3} s
- Length-scales: 10^{-9} – 10 m
- Number of particles needed for first-principles modelling (1): 10^{21} (tokamak), 10^{20} (laser-heated solid)

Plasma oscillations: capacitor model



Consider electron layer displaced from plasma slab by length δ. This creates two 'capacitor' plates with surface charge $\sigma = \pm e n_e \delta$, resulting in an electric field:

$$\boldsymbol{E} = \frac{\sigma}{\varepsilon_0} = \frac{e n_e \delta}{\varepsilon_0}$$

Capacitor model (2)

The electron layer is accelerated back towards the slab by this restoring force according to:

$$m_e \frac{dv}{dt} = -m_e \frac{d^2\delta}{dt^2} = -eE = \frac{e^2 n_e \delta}{\varepsilon_0}$$

Or:

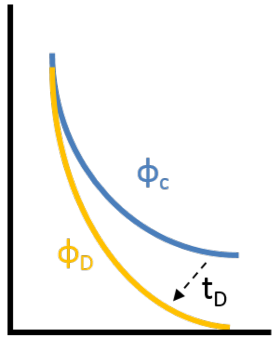
$$\frac{d^2\delta}{dt^2} + \omega_p^2 \delta = 0,$$

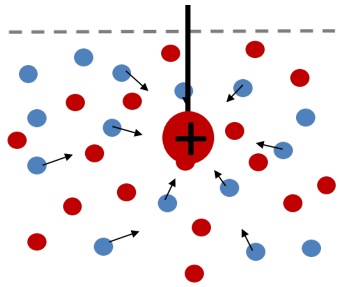
where

Electron plasma frequency

$$\omega_p \equiv \left(\frac{e^2 n_e}{\varepsilon_0 m_e}\right)^{1/2} \simeq 5.6 \times 10^4 \left(\frac{n_e}{\mathrm{cm}^{-3}}\right)^{1/2} \mathrm{s}^{-1}. \quad (8)$$

Response time to create Debye sheath





For a plasma with temperature T_e (and thermal velocity $v_{te} \equiv \sqrt{k_B T_e / m_e}$), one can also define a characteristic *reponse time* to recover quasi-neutrality:

$$t_D \simeq \frac{\lambda_D}{v_{te}} = \left(\frac{\varepsilon_0 k_B T_e}{e^2 n_e} \cdot \frac{m}{k_B T_e} \right)^{1/2} = \omega_p^{-1}.$$

External fields: underdense vs. overdense

If the plasma response time is shorter than the period of a external electromagnetic field (such as a laser), then this radiation will be *shielded out*.

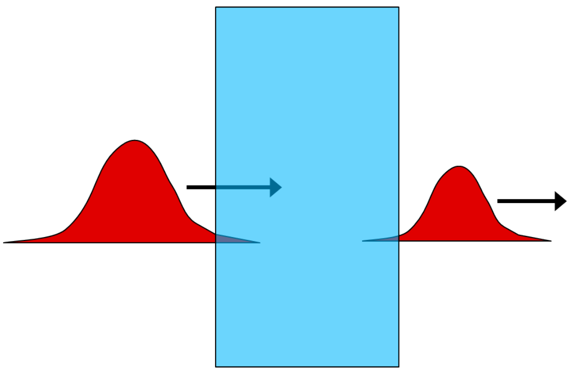


Figure 1: Underdense, $\omega > \omega_p$: plasma acts as nonlinear refractive medium

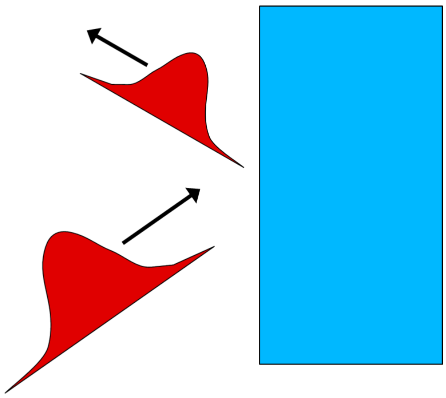


Figure 2: Overdense, $\omega < \omega_p$: plasma acts like mirror

The critical density

To make this more quantitative, consider ratio:

$$\frac{\omega_p^2}{\omega^2} = \frac{e^2 n_e}{\varepsilon_0 m_e} \cdot \frac{\lambda^2}{4\pi^2 c^2}.$$

Setting this to unity defines the wavelength for which $n_e = n_c$, or the

Critical density

$$n_c \simeq 10^{21} \lambda_\mu^{-2} \text{ cm}^{-3} \tag{9}$$

above which radiation with wavelengths $\lambda > \lambda_\mu$ will be reflected. cf: radio waves from ionosphere.

Plasma creation: field ionization

At the Bohr radius

$$a_B = \frac{\hbar^2}{me^2} = 5.3 \times 10^{-9} \text{ cm},$$

the electric field strength is:

$$\begin{aligned} E_a &= \frac{e}{4\pi\varepsilon_0 a_B^2} \\ &\simeq 5.1 \times 10^9 \text{ Vm}^{-1}. \end{aligned} \tag{10}$$

This leads to the *atomic intensity*:

$$\begin{aligned} I_a &= \frac{\varepsilon_0 c E_a^2}{2} \\ &\simeq 3.51 \times 10^{16} \text{ Wcm}^{-2}. \end{aligned} \tag{11}$$

A laser intensity of $I_L > I_a$ will *guarantee ionization* for any target material, though in fact this can occur well below this threshold value (eg: $\sim 10^{14}$ Wcm^{-2} for hydrogen) via *multiphoton* effects .

Ionized gases: when is plasma response important?

Simultaneous field ionization of many atoms produces a plasma with electron density n_e, temperature $T_e \sim 1 - 10$ eV. *Collective effects* important if

$$\omega_p \tau_{\text{interaction}} > 1$$

Example (Gas jet)

$\tau_{\text{int}} = 100$ fs, $n_e = 10^{17}$ cm^{-3} $\rightarrow \omega_p \tau_{\text{int}} = 1.8$
Typical gas jets: $P \sim 1$bar; $n_e = 10^{18} - 10^{19}$ cm^{-3}
Recall that from Eq.9, critical density for glass laser $n_c(1\mu) = 10^{21}$ cm^{-3}. Gas-jet plasmas are therefore *underdense*, since $\omega^2/\omega_p^2 = n_e/n_c \ll 1$.

Exploit plasma effects for: short-wavelength radiation; nonlinear refractive properties; high electric/magnetic fields; *particle acceleration*!

Relativistic field strengths

Classical equation of motion for an electron exposed to a linearly polarized laser field $\boldsymbol{E} = \hat{y} E_0 \sin \omega t$:

$$\frac{dv}{dt} \simeq \frac{-eE_0}{m_e} \sin \omega t$$

$$\rightarrow v = \frac{eE_0}{m_e \omega} \cos \omega t = v_{\text{os}} \cos \omega t \qquad (12)$$

Dimensionless oscillation amplitude, or 'quiver' velocity:

$$a_0 \equiv \frac{v_{\text{os}}}{c} \equiv \frac{p_{\text{os}}}{m_e c} \equiv \frac{eE_0}{m_e \omega c} \qquad (13)$$

Relativistic intensity

The laser intensity I_L and wavelength λ_L are related to E_0 and ω by:

$$I_L = \frac{1}{2}\varepsilon_0 c E_0^2; \quad \lambda_L = \frac{2\pi c}{\omega}$$

Substituting these into (13) we find :

$$a_0 \simeq 0.85(I_{18}\lambda_\mu^2)^{1/2}, \qquad (14)$$

Exercise

where

$$I_{18} = \frac{I_L}{10^{18}\ \mathrm{Wcm}^{-2}}; \quad \lambda_\mu = \frac{\lambda_L}{\mu m}.$$

Implies that for $I_L \geq 10^{18}$ Wcm^{-2}, $\lambda_L \simeq 1\ \mu$m, we will have relativistic electron velocities, or $a_0 \sim 1$.

Further reading

1. F. F. Chen, *Plasma Physics and Controlled Fusion*, 2nd Ed. (Springer, 2006)
2. R.O. Dendy (ed.), *Plasma Physics, An Introductory Course*, (Cambridge University Press, 1993)
3. J. D. Huba, *NRL Plasma Formulary*, (NRL, Washington DC, 2007) http://www.nrl.navy.mil/ppd/content/nrl-plasma-formulary

Lecture 2: Wave propagation in plasmas

Model hierarchy

1. First principles N-body molecular dynamics
2. Phase-space methods – Vlasov-Boltzmann
3. 2-fluid equations
4. Magnetohydrodynamics (single, magnetised fluid)

The 2-fluid model

Many plasma phenomena can be analysed by assuming that each charged particle component with density n_s and velocity $\boldsymbol{u}_s$ behaves in a fluid-like manner, interacting with other species (s) via the electric and magnetic fields. The rigorous way to derive the governing equations in this approximation is via *kinetic theory*, which is beyond the scope of this lecture.

We therefore begin with the 2-fluid equations for a plasma assumed to be:

- thermal: $T_e > 0$
- collisionless: $\nu_{\mathrm{ie}} \simeq 0$
- and non-relativistic: velocities $u \ll c$.

The 2-fluid model (2)

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \boldsymbol{u}_s) = 0 \qquad (15)$$

$$n_s m_s \frac{d\boldsymbol{u}_s}{dt} = n_s q_s (\boldsymbol{E} + \boldsymbol{u}_s \times \boldsymbol{B}) - \nabla P_s \qquad (16)$$

$$\frac{d}{dt}(P_s n_s^{-\gamma_s}) = 0 \qquad (17)$$

P_s is the thermal pressure of species s; γ_s the specific heat ratio, or $(2+N)/N$), where N is the number of degrees of freedom.

Continuity equation

The continuity equation (Eq. 15) tells us that (in the absence of ionization or recombination) the number of particles *of each species* is conserved.

Noting that the charge and current densities can be written $\rho_s = q_s n_s$ and $\boldsymbol{J}_s = q_s n_s \boldsymbol{u}_s$ respectively, Eq. (15) can also be written:

$$\frac{\partial \rho_s}{\partial t} + \nabla \cdot \boldsymbol{J}_s = 0, \quad (18)$$

which expresses the conservation of *charge*.

Momentum equation

(Eq. 16) governs the motion of a fluid element of species s in the presence of electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$.

Remark: In the absence of fields, and assuming strict quasineutrality ($n_e = Zn_i = n$; $\boldsymbol{u}_e = \boldsymbol{u}_i = \boldsymbol{u}$), we recover the *Navier-Stokes* equations. Exercise

In the plasma accelerator context we will usually deal with *un*magnetised plasmas, and stationary ions $\boldsymbol{u}_i = 0$, in which case the momentum equation reads:

$$n_e m_e \frac{d\boldsymbol{u}_e}{dt} = -en_e\boldsymbol{E} - \nabla P_e \quad (19)$$

Note that $\boldsymbol{E}$ can include both external and internal field components (via charge-separation).

Longitudinal plasma waves

A characteristic property of plasmas is their ability to transfer momentum and energy via collective motion. One of the most important examples of this is the oscillation of the electrons against a stationary ion background, or *Langmuir wave*.

Returning to the 2-fluid model, we can simplify Eqs.(15-17) by setting $\boldsymbol{u}_i = 0$, restricting the electron motion to one dimension (x) and taking $\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$:

$$\begin{aligned} \frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e u_e) &= 0 \\ n_e\left(\frac{\partial u_e}{\partial t} + u_e \frac{\partial u_e}{\partial x}\right) &= -\frac{e}{m} n_e E - \frac{1}{m}\frac{\partial P_e}{\partial x} \\ \frac{d}{dt}\left(\frac{P_e}{n_e^{\gamma_e}}\right) &= 0 \end{aligned} \qquad (20)$$

Longitudinal plasma waves (2)

Poisson's equation

The above system (20) has 3 equations and 4 unknowns.

To close it we need an expression for the electric field, which, since $\boldsymbol{B} = 0$, can be found from Gauss' law (Poisson's equation) with $Zn_i = n_0 = \text{const}$:

$$\frac{\partial E}{\partial x} = \frac{e}{\varepsilon_0}(n_0 - n_e) \qquad (21)$$

Longitudinal plasma waves (3)

1D electron fluid equations

$$\begin{aligned} \frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e u_e) &= 0 \\ n_e \left(\frac{\partial u_e}{\partial t} + u_e \frac{\partial u_e}{\partial x} \right) &= -\frac{e}{m} n_e E - \frac{1}{m} \frac{\partial P_e}{\partial x} \\ \frac{d}{dt} \left(\frac{P_e}{n_e^{\gamma_e}} \right) &= 0 \\ \frac{\partial E}{\partial x} &= \frac{e}{\varepsilon_0} (n_0 - n_e) \end{aligned} \qquad (22)$$

Longitudinal plasma waves (4)

Linearization

This system is nonlinear, and apart from a few special cases, cannot be solved exactly. A common technique for analyzing waves in plasmas therefore is to *linearize* the equations, assuming the perturbed amplitudes are small compared to the equilibrium values:

$$\begin{aligned} n_e &= n_0 + n_1, \\ u_e &= u_1, \\ P_e &= P_0 + P_1, \\ E &= E_1, \end{aligned}$$

where $n_1 \ll n_0$, $P_1 \ll P_0$. These expressions are substituted into (22) and all products $n_1 \partial_t u_1$, $u_1 \partial_x u_1$ etc. are neglected to get a set of linear equations for the perturbed quantities...

Linearized equations

Exercise

$$\begin{aligned}
\frac{\partial n_1}{\partial t} + n_0 \frac{\partial u_1}{\partial x} &= 0, \\
n_0 \frac{\partial u_1}{\partial t} &= -\frac{e}{m} n_0 E_1 - \frac{1}{m}\frac{\partial P_1}{\partial x}, \\
\frac{\partial E_1}{\partial x} &= -\frac{e}{\varepsilon_0} n_1, \\
P_1 &= 3 k_B T_e n_1 .
\end{aligned} \tag{23}$$

N.B. Expression for P_1 results from specific heat ratio $\gamma_e = 3$ and assuming isothermal background electrons, $P_0 = k_B T_e n_0$ (ideal gas) – see Kruer (1988).

Wave equation

We can now eliminate E_1, P_1 and u_1 from (23) to get:

$$\left(\frac{\partial^2}{\partial t^2} - 3 v_{te}^2 \frac{\partial^2}{\partial x^2} + \omega_p^2 \right) n_1 = 0, \tag{24}$$

with $v_{te}^2 = k_B T_e / m_e$ and ω_p given by (8) as before.

Finally, we look for plane wave solutions of the form $A = A_0 e^{i(\omega t - kx)}$, so that our derivative operators become: $\frac{\partial}{\partial t} \to i\omega$; $\frac{\partial}{\partial x} \to -ik$.

Substitution into (24) yields finally:

Bohm-Gross dispersion relation for electron plasma waves

$$\omega^2 = \omega_p^2 + 3 k^2 v_{te}^2 \tag{25}$$

Electromagnetic waves

To describe *transverse* electromagnetic (EM) waves, we need two more of Maxwells equations: Faraday's law (35) and Ampère's law (36), which we come to in their usual form later.

To simplify things, taking our cue from the previous analysis of small-amplitude, longitudinal waves, we linearize and again apply the harmonic approximation $\frac{\partial}{\partial t} \rightarrow i\omega$:

$$\nabla \times \boldsymbol{E}_1 = -i\omega \boldsymbol{B}_1, \quad (26)$$

$$\nabla \times \boldsymbol{B}_1 = \mu_0 \boldsymbol{J}_1 + i\varepsilon_0\mu_0\omega \boldsymbol{E}_1, \quad (27)$$

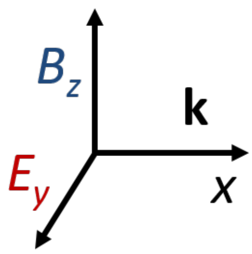
where the transverse current density is given by:

$$\boldsymbol{J}_1 = -n_0 e \boldsymbol{u}_1. \quad (28)$$

Electromagnetic waves (2)

Ohm's law

We now look for pure EM plane-wave solutions with $\boldsymbol{E}_1 \perp \boldsymbol{k}$. Also note that the group and phase velocities $v_p, v_g \gg v_{te}$, so that we can assume a *cold* plasma with $P_e = n_0 k_B T_e = 0$.



The linearized electron fluid velocity and corresponding current are then:

$$\boldsymbol{u}_1 = -\frac{e}{i\omega m_e}\boldsymbol{E}_1,$$

$$\boldsymbol{J}_1 = \frac{n_0 e^2}{i\omega m_e}\boldsymbol{E}_1 \equiv \sigma \boldsymbol{E}_1, \quad (29)$$

where σ is the *AC electrical conductivity*.

Electromagnetic waves (3)
Dielectric function

By analogy with dielectric media (see eg: Jackson), in which Ampere's law is usually written $\nabla\times\boldsymbol{B}_1 = \mu_0\partial_t\boldsymbol{D}_1$, by substituting (29) into (36), can show that

$$\boldsymbol{D}_1 = \varepsilon_0\varepsilon\boldsymbol{E}_1$$

with

$$\varepsilon = 1 + \frac{\sigma}{i\omega\varepsilon_0} = 1 - \frac{\omega_p^2}{\omega^2}. \tag{30}$$

Electromagnetic waves (4)
Refractive index

From (30) it follows immediately that:

Refractive index

$$\eta \equiv \sqrt{\varepsilon} = \frac{ck}{\omega} = \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} \tag{31}$$

with

Dispersion relation

$$\omega^2 = \omega_p^2 + c^2k^2 \tag{32}$$

Exercise

The above expression can also be found directly by elimination of $\boldsymbol{J}_1$ and $\boldsymbol{B}_1$ from Eqs. (26)-(29).

Propagation characteristics
Underdense plasmas

From the dispersion relation (32) a number of important features of EM wave propagation in plasmas can be deduced.

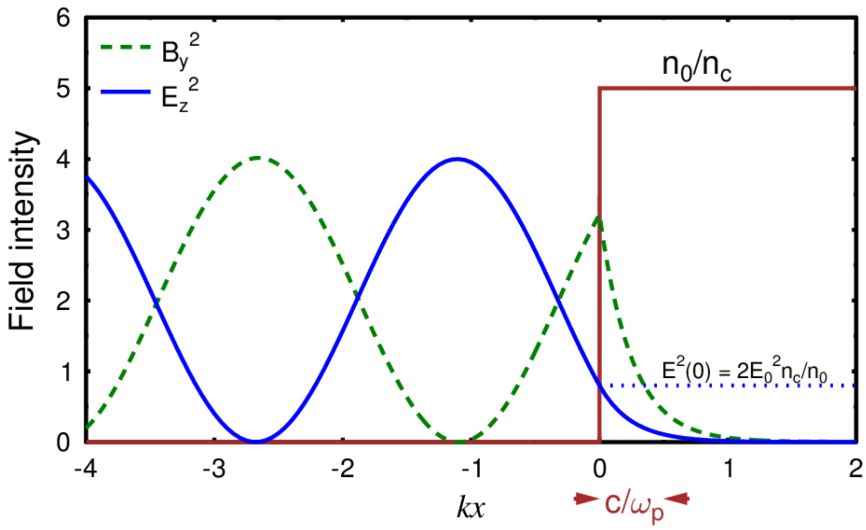
For *underdense* plasmas ($n_e \ll n_c$):

$$\text{Phase velocity} \quad v_p = \frac{\omega}{k} \simeq c\left(1 + \frac{\omega_p^2}{2\omega^2}\right) > c$$

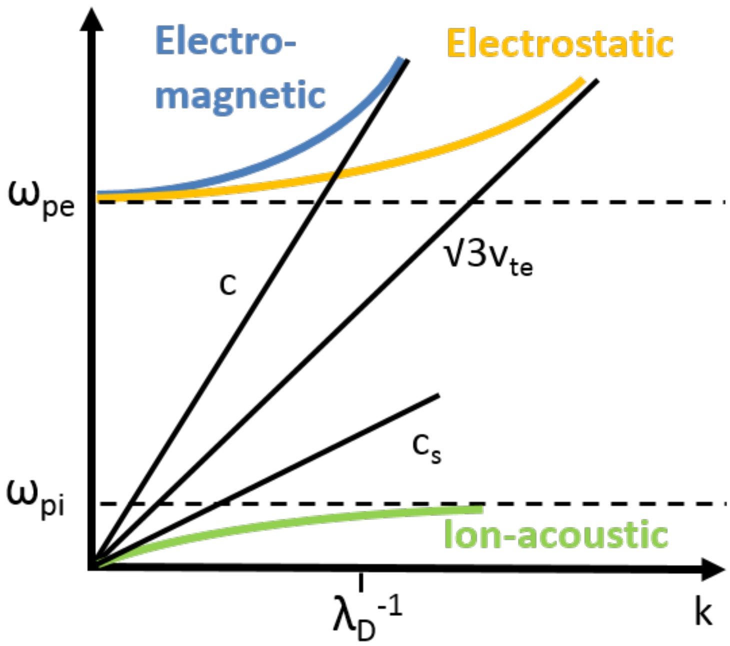
$$\text{Group velocity} \quad v_g = \frac{\partial\omega}{\partial k} \simeq c\left(1 - \frac{\omega_p^2}{2\omega^2}\right) < c$$

Propagation characteristics (2)
Overdense plasmas

In the opposite case, $n_e > n_c$, the refractive index η becomes imaginary, and the wave can no longer propagate, becoming evanescent instead, with a decay length determined by the *collisionless skin depth* c/ω_p.



Summary: dispersion curves



Nonlinear wave propagation

The starting point for most analyses of nonlinear wave propagation phenomena is the Lorentz equation of motion for the electrons in a *cold* ($T_e = 0$), unmagnetized plasma, together with Maxwell's equations.

We also make two assumptions:

1. The ions are initially assumed to be singly charged ($Z = 1$) and are treated as a immobile ($v_i = 0$), homogeneous background with $n_0 = Zn_i$.
2. Thermal motion is neglected – justified for underdense plasmas because the temperature remains small compared to the typical oscillation energy in the laser field ($v_{os} \gg v_{te}$).

Lorentz-Maxwell equations

Starting equations (SI units) are as follows

$$\frac{\partial \boldsymbol{p}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{p} = -e(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}), \qquad (33)$$

$$\nabla\cdot\boldsymbol{E} = \frac{e}{\varepsilon_0}(n_0 - n_e), \qquad (34)$$

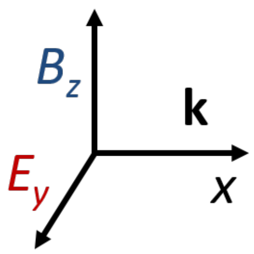
$$\nabla\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \qquad (35)$$

$$c^2\nabla\times\boldsymbol{B} = -\frac{e}{\varepsilon_0}n_e\boldsymbol{v} + \frac{\partial \boldsymbol{E}}{\partial t}, \qquad (36)$$

$$\nabla\cdot\boldsymbol{B} = 0, \qquad (37)$$

where $\boldsymbol{p} = \gamma m_e \boldsymbol{v}$ and $\gamma = (1 + p^2/m_e^2c^2)^{1/2}$.

Electromagnetic waves



To simplify matters we first assume a plane-wave geometry like that above, with the transverse electromagnetic fields given by $\boldsymbol{E}_L = (0, E_y, 0)$; $\boldsymbol{B}_L = (0, 0, B_z)$.
From Eq. (33) the transverse electron momentum is then simply given by:

Exercise

$$p_y = eA_y, \qquad (38)$$

where $E_y = \partial A_y/\partial t$.
This relation expresses conservation of canonical momentum.

The EM wave equation I

Substitute $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$; $\mathbf{B} = \nabla \times \mathbf{A}$ into Ampère Eq.(36):

$$c^2\nabla \times (\nabla \times \mathbf{A}) + \frac{\partial^2 \mathbf{A}}{\partial t^2} = \frac{\mathbf{J}}{\varepsilon_0} - \nabla\frac{\partial\phi}{\partial t},$$

where the current $\mathbf{J} = -en_e\mathbf{v}$.
Now we use a bit of vectorial magic, splitting the current into rotational (solenoidal) and irrotational (longitudinal) parts:

$$\mathbf{J} = \mathbf{J}_\perp + \mathbf{J}_{||} = \nabla \times \mathbf{\Pi} + \nabla\Psi$$

from which we can deduce (see Jackson!):

$$\mathbf{J}_{||} - \frac{1}{c^2}\nabla\frac{\partial\phi}{\partial t} = 0.$$

The EM wave equation II

Now apply Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ and $v_y = eA_y/\gamma$ from (38), to finally get:

EM wave

$$\frac{\partial^2 A_y}{\partial t^2} - c^2\nabla^2 A_y = \mu_0 J_y = -\frac{e^2 n_e}{\varepsilon_0 m_e \gamma}A_y. \quad (39)$$

The nonlinear source term on the RHS contains two important bits of physics:

$n_e = n_0 + \delta n \quad \rightarrow \quad$ Coupling to plasma waves

$\gamma = \sqrt{1 + \mathbf{p}^2/m_e^2c^2} \quad \rightarrow \quad$ Relativistic effects

Electrostatic (Langmuir) waves I

Taking the *longitudinal* (x)-component of the momentum equation (33) gives:

$$\frac{d}{dt}(\gamma m_e v_x) = -eE_x - \frac{e^2}{2m_e\gamma}\frac{\partial A_y^2}{\partial x}$$

We can eliminate v_x using Ampère's law $(36)_x$:

$$0 = -\frac{e}{\varepsilon_0} n_e v_x + \frac{\partial E_x}{\partial t},$$

while the electron density can be determined via Poisson's equation (34):

$$n_e = n_0 - \frac{\varepsilon_0}{e}\frac{\partial E_x}{\partial x}.$$

Electrostatic (Langmuir) waves II

The above (closed) set of equations can in principle be solved numerically for arbitrary pump strengths. For the moment, we simplify things by linearizing the *plasma* fluid quantities:

$$\begin{aligned} n_e &\simeq n_0 + n_1 + ... \\ v_x &\simeq v_1 + v_2 + ... \end{aligned}$$

and neglect products like $n_1 v_1$ etc. This finally leads to:

Driven plasma wave

$$\left(\frac{\partial^2}{\partial t^2} + \frac{\omega_p^2}{\gamma_0}\right) E_x = -\frac{\omega_p^2 e}{2m_e\gamma_0^2}\frac{\partial}{\partial x} A_y^2 \qquad (40)$$

The driving term on the RHS is the *relativistic ponderomotive force*, with $\gamma_0 = (1 + a_0^2/2)^{1/2}$.

Cold plasma fluid equations: summary

Electromagnetic wave

$$\frac{\partial^2 A_y}{\partial t^2} - c^2\nabla^2 A_y = \mu_0 J_y = -\frac{e^2 n_e}{\varepsilon_0 m_e \gamma} A_y$$

Electrostatic (Langmuir) wave

$$\left(\frac{\partial^2}{\partial t^2} + \frac{\omega_p^2}{\gamma_0}\right) E_x = -\frac{\omega_p^2 e}{2 m_e \gamma_0^2}\frac{\partial}{\partial x} A_y^2$$

Cold plasma fluid equations: outlook

These coupled fluid equations and their fully non-linear variations describe a vast range of nonlinear laser-plasma interaction phenomena:

- plasma wake generation: Bingham, Assmann
- blow-out regime: Silva
- laser self-focussing and channelling: Najmudin, Cros
- parametric instabilities
- harmonic generation, ...

Plasma-accelerated particle *beams*, on the other hand, cannot be treated with fluid theory and require a more sophisticated kinetic approach. – see Pukhov

Further reading

1. J. Boyd and J. J. Sanderson, *The Physics of Plasmas*
2. W. Kruer, *The Physics of Laser Plasma Interactions*, Addison-Wesley, 1988
3. P. Gibbon, *Short Pulse Laser Interactions with Matter: An Introduction*, Imperial College Press, 2005
4. J. D. Jackson, *Classical Electrodynamics*, Wiley 1975/1998
5. J. P. Dougherty in Chapter 3 of R. Dendy *Plasma Physics*, 1993

Constants

Name	Symbol	Value (SI)	Value (cgs)
Boltzmann constant	k_B	1.38×10^{-23} JK^{-1}	1.38×10^{-16} erg K^{-1}
Electron charge	e	1.6×10^{-19} C	4.8×10^{-10} statcoul
Electron mass	m_e	9.1×10^{-31} kg	9.1×10^{-28} g
Proton mass	m_p	1.67×10^{-27} kg	1.67×10^{-24} g
Planck constant	h	6.63×10^{-34} Js	6.63×10^{-27} erg-s
Speed of light	c	3×10^{8} ms^{-1}	3×10^{10} cms^{-1}
Dielectric constant	ε_0	8.85×10^{-12} Fm^{-1}	—
Permeability constant	μ_0	$4\pi \times 10^{-7}$	—
Proton/electron mass ratio	m_p/m_e	1836	1836
Temperature = 1eV	e/k_B	11604 K	11604 K
Avogadro number	N_A	6.02×10^{23} mol^{-1}	6.02×10^{23} mol^{-1}
Atmospheric pressure	1 atm	1.013×10^{5} Pa	1.013×10^{6} dyne cm^{-2}

Formulae

Name	Symbol	Formula (SI)	Formula (cgs)
Debye length	λ_D	$\left(\frac{\varepsilon_0 k_B T_e}{e^2 n_e}\right)^{\frac{1}{2}}$ m	$\left(\frac{k_B T_e}{4\pi e^2 n_e}\right)^{\frac{1}{2}}$ cm
Particles in Debye sphere	N_D	$\frac{4\pi}{3}\lambda_D^3$	$\frac{4\pi}{3}\lambda_D^3$
Plasma frequency (electrons)	ω_{pe}	$\left(\frac{e^2 n_e}{\varepsilon_0 m_e}\right)^{\frac{1}{2}}$ s^{-1}	$\left(\frac{4\pi e^2 n_e}{m_e}\right)^{\frac{1}{2}}$ s^{-1}
Plasma frequency (ions)	ω_{pi}	$\left(\frac{Z^2 e^2 n_i}{\varepsilon_0 m_i}\right)^{\frac{1}{2}}$ s^{-1}	$\left(\frac{4\pi Z^2 e^2 n_i}{m_i}\right)^{\frac{1}{2}}$ s^{-1}
Thermal velocity	$v_{te} = \omega_{pe}\lambda_D$	$\left(\frac{k_B T_e}{m_e}\right)^{\frac{1}{2}}$ ms^{-1}	$\left(\frac{k_B T_e}{m_e}\right)^{\frac{1}{2}}$ cms^{-1}
Electron gyrofrequency	ω_c	eB/m_e s^{-1}	eB/m_e s^{-1}
Electron-ion collision frequency	ν_{ei}	$\frac{\pi^{\frac{3}{2}} n_e Z e^4 \ln\Lambda}{2^{\frac{1}{2}}(4\pi\varepsilon_0)^2 m_e^2 v_{te}^3}$ s^{-1}	$\frac{4(2\pi)^{\frac{1}{2}} n_e Z e^4 \ln\Lambda}{3 m_e^2 v_{te}^3}$ s^{-1}
Coulomb-logarithm	$\ln\Lambda$	$\ln\frac{9N_D}{Z}$	$\ln\frac{9N_D}{Z}$

Useful formulae

Plasmafrequency: $\omega_{pe} = 5.64 \times 10^4 n_e^{\frac{1}{2}}$ s^{-1}

Critical density: $n_c = 10^{21}\lambda_L^{-2}$ cm^{-3}

Debye length: $\lambda_D = 743\, T_e^{\frac{1}{2}} n_e^{-\frac{1}{2}}$ cm

Skin depth: $\delta = c/\omega_p = 5.31 \times 10^5 n_e^{-\frac{1}{2}}$ cm

Elektron-ion collision frequency: $\nu_{ei} = 2.9 \times 10^{-6} n_e T_e^{-\frac{3}{2}} \ln\Lambda$ s^{-1}

Ion-ion collision frequency: $\nu_{ii} = 4.8 \times 10^{-8} Z^4 \left(\frac{m_p}{m_i}\right)^{\frac{1}{2}} n_i T_i^{-\frac{3}{2}} \ln\Lambda$ s^{-1}

Quiver amplitude: $a_0 \equiv \frac{p_{osc}}{m_e c} = \left(\frac{I\lambda_L^2}{1.37 \times 10^{18}\mathrm{Wcm}^{-2}\mu\mathrm{m}^2}\right)^{\frac{1}{2}}$

Relativistic focussing threshold: $P_c = 17\left(\frac{n_c}{n_e}\right)$ GW

T_e in eV; n_e, n_i in cm^{-3}, wavelength λ_L in μm

Maxwell's Equations

Name	(SI)	(cgs)
Gauss' law	$\boldsymbol{\nabla}.\boldsymbol{E} = \rho/\varepsilon_0$	$\boldsymbol{\nabla}.\boldsymbol{E} = 4\pi\rho$
Gauss' magnetism law	$\boldsymbol{\nabla}.\boldsymbol{B} = 0$	$\boldsymbol{\nabla}.\boldsymbol{B} = 0$
Ampère	$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}$	$\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}$
Faraday	$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$	$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}$
Lorentz force per unit charge	$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}$	$\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}$