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Special Relativity

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Overview

- The principle of special relativity
- Lorentz transformation and consequences
- Space-time
- 4-vectors: position, velocity, momentum, invariants, covariance.
- Derivation of $E=mc^2$
- Examples of the use of 4-vectors
- Inter-relation between β and γ, momentum and energy
- An accelerator problem in relativity
- Motion faster than speed of light

Reading

- W. Rindler: Introduction to Special Relativity (OUP 1991)
- D. Lawden: An Introduction to Tensor Calculus and Relativity
- N.M.J. Woodhouse: Special Relativity (Springer 2002)
- A.P. French: Special Relativity, MIT Introductory Physics Series (Nelson Thomes)
- Misner, Thorne and Wheeler: Relativity
- C. Prior: Special Relativity, CERN Accelerator School (Zeegse)

Historical background

- Groundwork of Special Relativity laid by Lorentz in studies of electrodynamics, with crucial concepts contributed by Einstein to place the theory on a consistent footing.
- Maxwell's equations (1863) attempted to explain electromagnetism and optics through wave theory
  - light propagates with speed $c = 3\times10^8$ m/s in "ether" but with different speeds in other frames
  - the ether exists solely for the transport of e/m waves
  - Maxwell's equations not invariant under Galilean transformations
  - To avoid setting e/m apart from classical mechanics, assume
    - light has speed c only in frames where source is at rest
    - the ether has a small interaction with matter and is carried along with astronomical objects

Contradicted by:

- Aberration of star light (small shift in apparent positions of distant stars)
- Fizeau's 1859 experiments on velocity of light in liquids
- Michelson-Morley 1907 experiment to detect motion of the earth through ether
- Suggestion: perhaps material objects contract in the direction of their motion

$$L(v) = L_0\left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

This was the last gasp of ether advocates and the germ of Special Relativity led by Lorentz, Minkowski and Einstein.

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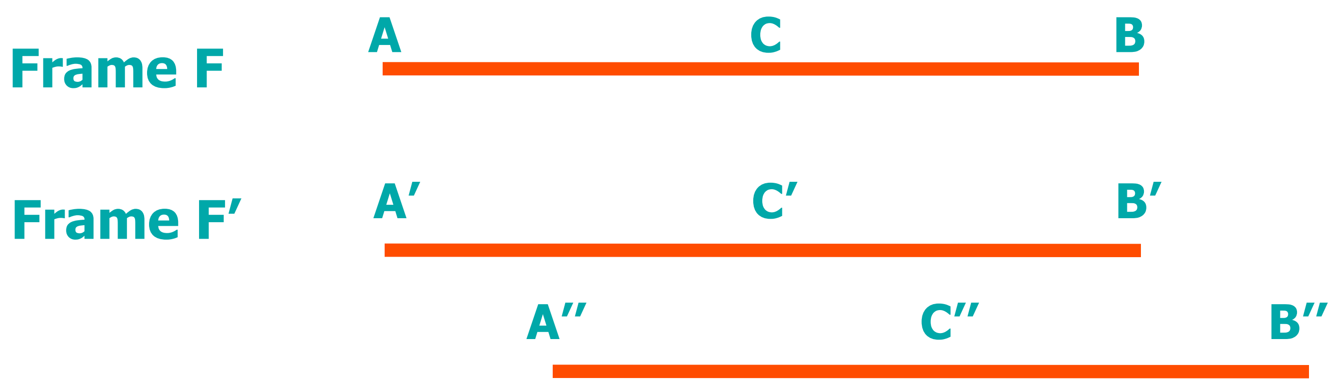
The Principle of Special Relativity

- A frame in which particles under no forces move with constant velocity is *inertial*.
- Consider relations between inertial frames where measuring apparatus (rulers, clocks) can be transferred from one to another: *related frames*.
- Assume:
  - Behaviour of apparatus transferred from F to F' is independent of mode of transfer
  - Apparatus transferred from F to F', then from F' to F'', agrees with apparatus transferred directly from F to F''.
- *The Principle of Special Relativity states that all physical laws take equivalent forms in related inertial frames, so that we cannot distinguish between the frames.*

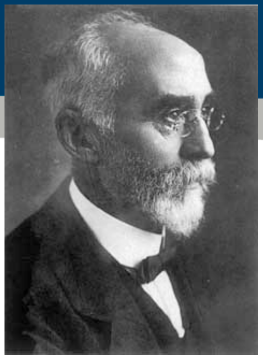
Simultaneity

- Two clocks A and B are synchronised if light rays emitted at the same time from A and B meet at the mid-point of AB

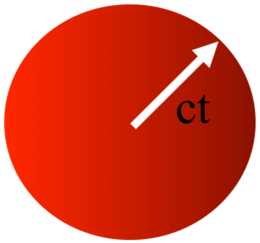


- Frame F' moving with respect to F. Events simultaneous in F cannot be simultaneous in F'.
- Simultaneity is **not** absolute but frame dependent.

The Lorentz Transformation



- Must be linear to agree with standard Galilean transformation in low velocity limit
- Preserves wave fronts of pulses of light,



$$\text{i.e.} \quad P \equiv x^2 + y^2 + z^2 - c^2 t^2 = 0$$

$$\text{whenever} \quad Q \equiv x'^2 + y'^2 + z'^2 - c^2 t'^2 = 0$$

- Solution is the **Lorentz transformation** from frame F (*t,x,y,z)* to frame F'(*t',x',y',z')* moving with velocity *v* along the *x*-axis:

$$\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ x' &= \gamma\left(x - vt\right) \qquad \text{where} \quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \\ y' &= y \\ z' &= z \end{aligned}$$

Outline of Derivation

Set $t' = \alpha t + \beta x$

$$x' = \gamma x + \delta t$$

$$y' = \varepsilon y$$

$$z' = \varsigma z$$

Then $P = kQ$

$$\Leftrightarrow c^2 t'^2 - x'^2 - y'^2 - z'^2 = k\left(c^2 t^2 - x^2 - y^2 - z^2\right)$$

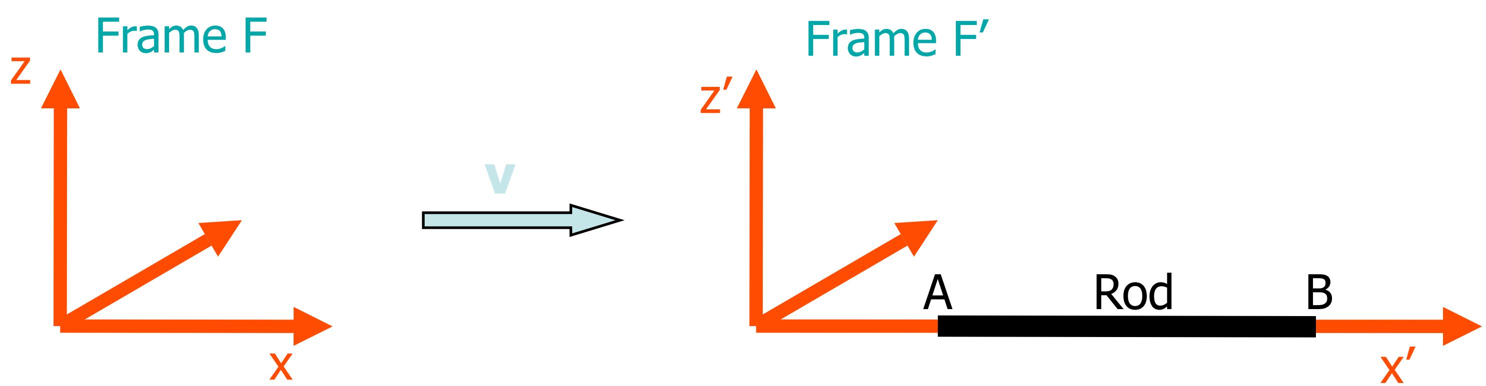
$$\Rightarrow c^2(\alpha t + \beta x)^2 - (\gamma x + \delta t)^2 - \varepsilon^2 y^2 - \varsigma^2 z^2 = k\left(c^2 t^2 - x^2 - y^2 - z^2\right)$$

Equate coefficients of x, y, z, t.

Isotropy of space $\Rightarrow$ $k = k(\vec{v}) = k(|\vec{v}|) = \pm 1$

Apply some common sense (e.g. $\varepsilon, \varsigma, k = +1$ and not -1)

Consequences: length contraction



Rod AB of length L' fixed in F' at x'_A, x'_B. What is its length measured in F?

Must measure positions of ends in F at the same time, so events in F are (t,x_A) and (t,x_B). From Lorentz:

$$x'_A = \gamma\left(x_A - vt\right) \qquad x'_B = \gamma\left(x_B - vt\right)$$

$$L' = x'_B - x'_A = \gamma\left(x_B - x_A\right) = \gamma L > L$$

Moving objects appear contracted in the direction of the motion

Consequences: time dilation

- Clock in frame *F* at point with coordinates (*x*,*y*,*z*) at different times t_A and t_B
- In frame *F'* moving with speed *v*, Lorentz transformation gives

$$t'_A = \gamma\left(t_A - \frac{vx}{c^2}\right) \qquad t'_B = \gamma\left(t_B - \frac{vx}{c^2}\right)$$

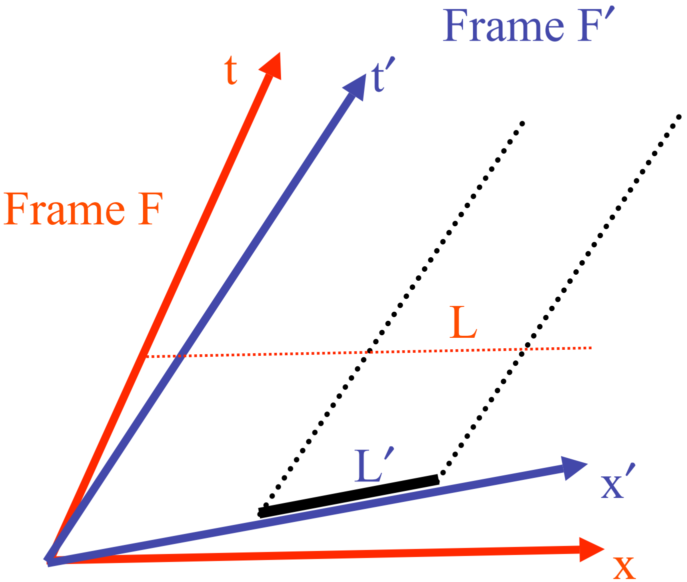
- So

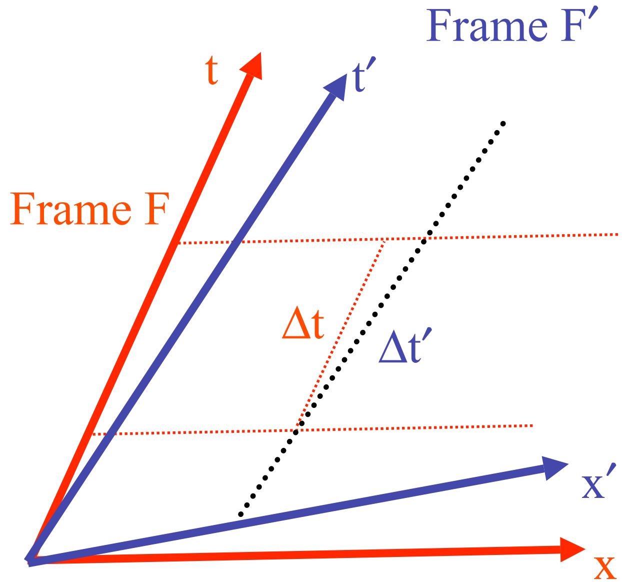
$$\Delta t' = t'_B - t'_A = \gamma\Big(t_B - t_A\Big) = \gamma\Delta t > \Delta t$$

Moving clocks appear to run slow

$$x' = \gamma(x - vt) \quad t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

Representation of the Lorentz Transformation





Length contraction L<L′

Rod at rest in F′. Measurement in F at fixed time t, along a line parallel to x-axis

Time dilatation: Δt<Δt′

Clock at rest in F. Time difference in F′ from line parallel to x′-axis

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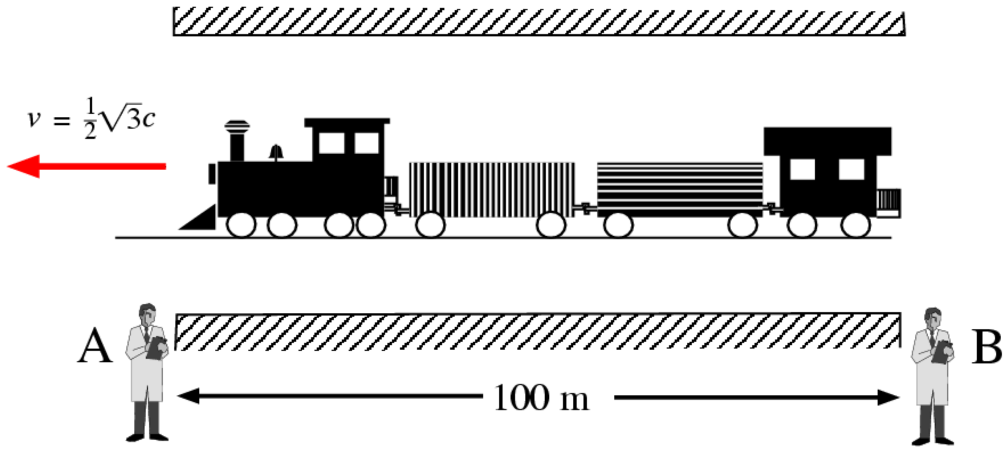
Example: High Speed Train



All clocks synchronised.

A's clock and driver's clock read 0 as front of train emerges from tunnel.

- Observers A and B at exit and entrance of tunnel say the train is moving, has contracted and has length

$$\frac{100}{\gamma} = 100 \times \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} = 100 \times \left(1 - \frac{3}{4}\right)^{\frac{1}{2}} = 50\mathrm{m}$$

- But the tunnel is moving relative to the driver and guard on the train and they say the train is 100 m in length but the tunnel has contracted to 50 m

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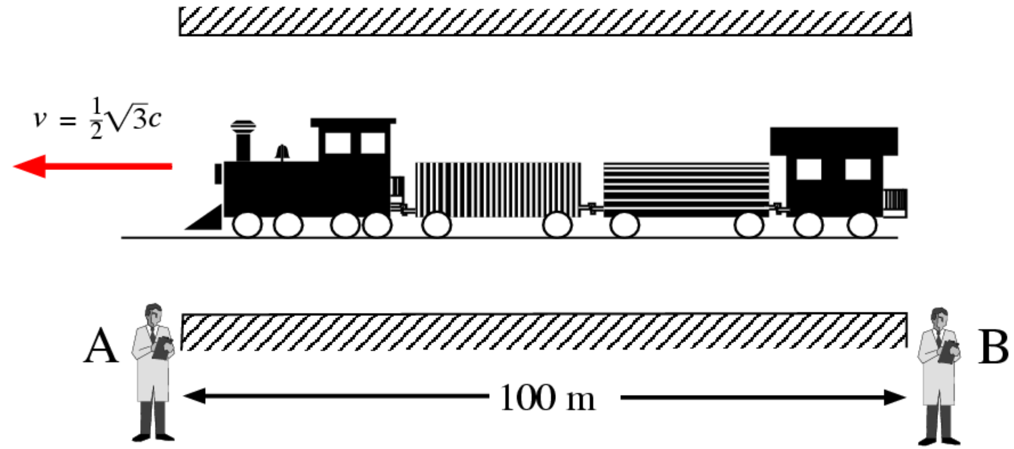
Question 1

A’s clock (and the driver's clock) reads zero as the driver exits tunnel. What does B’s clock read when the guard goes in?



Moving train length 50m, so driver has still 50m to travel before he exits and his clock reads 0. A's clock and B's clock are synchronised. Hence the reading on B's clock is

$$-\frac{50}{v} = -\frac{100}{\sqrt{3}c} \approx -200\,\mathrm{ns}$$

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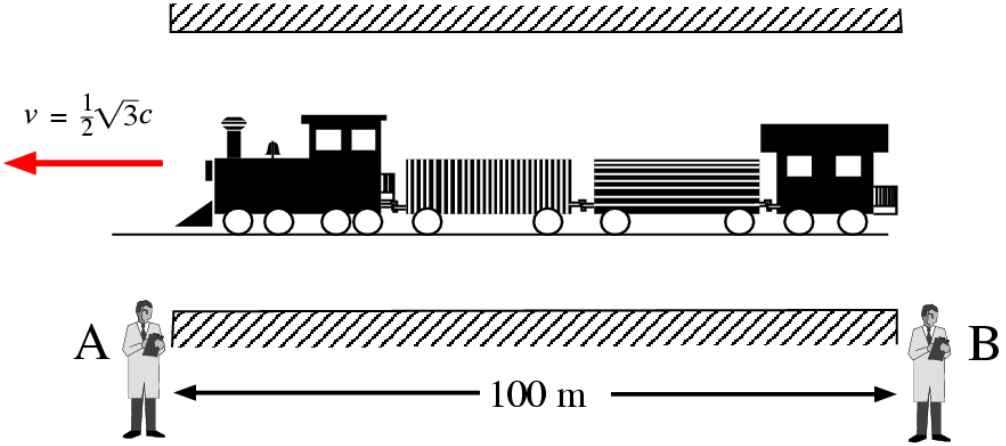
Question 2

What does the guard's clock read as he goes in?



To the guard, tunnel is only 50m long, so driver is 50m past the exit as guard goes in. Hence clock reading is

$$+\frac{50}{v} = +\frac{100}{\sqrt{3}c} \approx +200\,\mathrm{ns}$$

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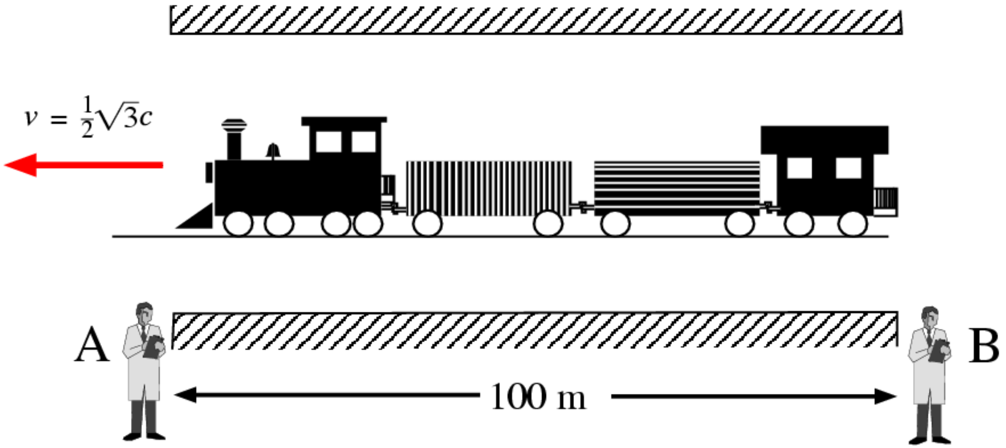
Question 3

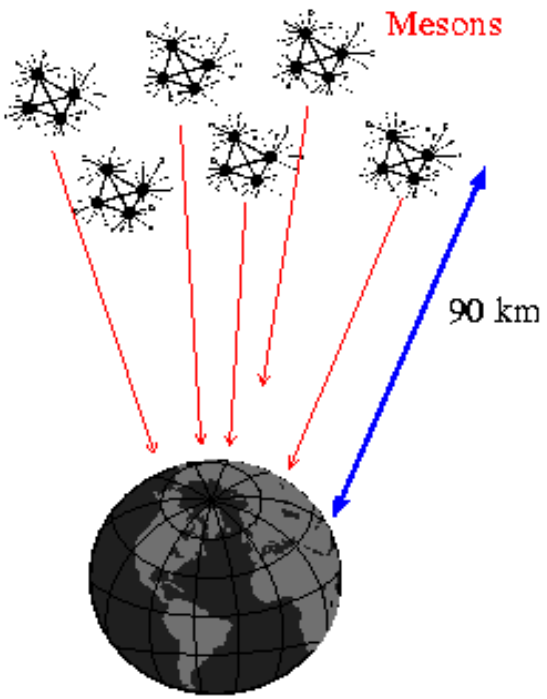
Where is the guard when his clock reads 0?



Guard's clock reads 0 when driver's clock reads 0, which is as driver exits the tunnel. To guard and driver, tunnel is 50m, so guard is 50m from the entrance in the train's frame, or 100m in tunnel frame.

So the guard is 100m from the entrance to the tunnel when his clock reads 0.

Example: Cosmic Rays



- μ-mesons are created in the upper atmosphere, 90km from earth. Their half life is τ=2 μs, so they can travel at most 2 $\times 10^{-6}$c=600m before decaying. So how do more than 50% reach the earth's surface?
- Mesons see distance contracted by γ, so

$$v\tau \approx \frac{90}{v} km$$

- Earthlings say mesons' clocks run slow so their half-life is $\gamma\tau$ and

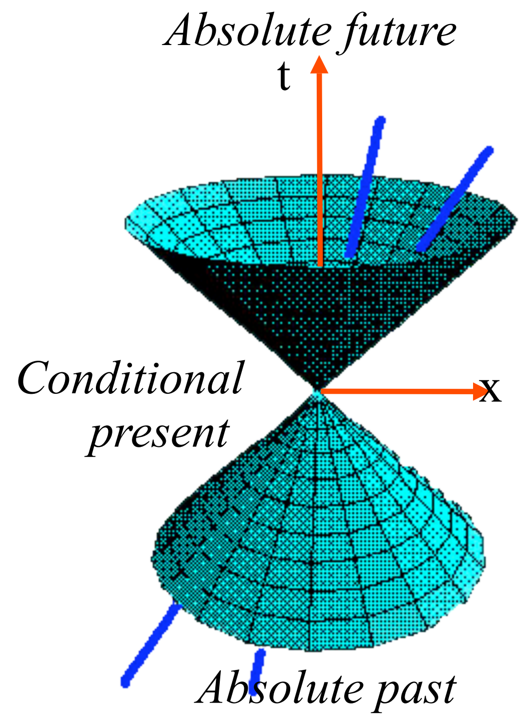
$$v(\gamma\tau) \approx 90\, km$$

- Both give

$$\frac{\gamma v}{c} = \frac{90\, km}{c\tau} = 150, \quad v \approx c, \quad \gamma \approx 150$$

Space-time

- An invariant is a quantity that has the same value in all inertial frames.
- Lorentz transformation is based on invariance of
$$c^2t^2 - (x^2 + y^2 + z^2) = (ct)^2 - \vec{x}^2$$
- 4D space with coordinates (t,x,y,z) is called space-time and the point $(t, x, y, z) = (t, \vec{x})$ is called an event.
- Fundamental invariant (preservation of speed of light):



$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = c^2\Delta t^2 \left(1 - \frac{\Delta x^2 + \Delta y^2 + \Delta z^2}{c^2\Delta t^2}\right) = c^2\Delta t^2 \left(1 - \frac{v^2}{c^2}\right) = c^2 \left(\frac{\Delta t}{\gamma}\right)^2$$

$\tau = \int \frac{dt}{\gamma}$ is called proper time, time in instantaneous rest frame, an invariant. $\Delta s = c\Delta\tau$ is called the separation between two events

4-Vectors

The Lorentz transformation can be written in matrix form as

$$\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned} \quad \Longrightarrow \quad \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{\gamma v}{c} & 0 & 0 \\ -\frac{\gamma v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Lorentz matrix *L*

Position 4-vector *X*

An object made up of 4 elements which transforms like X is called a 4-vector

(analogous to the 3-vector of classical mechanics)

$$X' = LX$$

Invariants

Basic invariant

$$c^2t^2 - x^2 - y^2 - z^2 = (ct, x, y, z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = X^T g X = X \cdot X$$

Inner product of two 4-vectors $A = (a_0, \vec{a}), \quad B = (b_0, \vec{b})$

$$A \cdot A = A^T g A = a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3 = a_0 b_0 - \vec{a} \cdot \bar{b}$$

Invariance:

$$A' \cdot A' = (LA)^T g (LA) = A^T (L^T g L) A = A^T g A = A \cdot A$$

Similarly $\quad A' \cdot B' = A \cdot B$

4-Vectors in S.R. Mechanics

- **Velocity:** $$V = \frac{\mathrm{d}X}{\mathrm{d}\tau} = \gamma \frac{\mathrm{d}X}{\mathrm{d}t} = \gamma \frac{\mathrm{d}}{\mathrm{d}t}(ct, \vec{x}) = \gamma(c, \vec{v})$$
- **Note invariant** $$V \cdot V = \gamma^2(c^2 - \vec{v}^2) = \frac{c^2 - \vec{v}^2}{1 - \vec{v}^2/c^2} = c^2$$
- **Momentum:** $$P = m_0 V = m_0 \gamma(c, \vec{v}) = (mc, \vec{p})$$

$m = m_0\gamma$ is the relativistic mass

$p = m_0\gamma\vec{v} = m\vec{v}$ is the relativistic 3-momentum

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4-Force

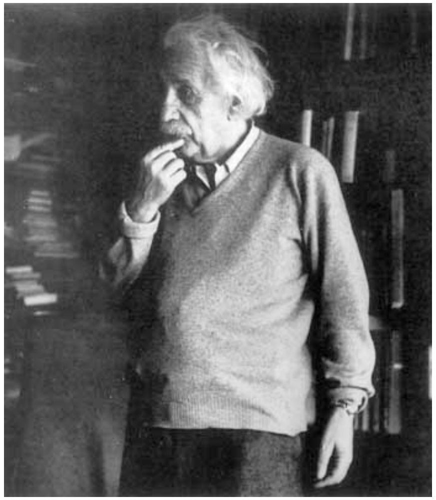
From Newton's 2nd Law expect 4-Force given by

$$
\begin{aligned}
F &= \frac{\mathrm{d}P}{\mathrm{d}\tau} = \gamma\frac{\mathrm{d}P}{\mathrm{d}t} \\
&= \gamma\frac{\mathrm{d}}{\mathrm{d}t}(mc,\vec{p}) = \gamma\left(c\frac{\mathrm{d}m}{\mathrm{d}t},\frac{\mathrm{d}\vec{p}}{\mathrm{d}t}\right) \\
&= \gamma\left(c\frac{\mathrm{d}m}{\mathrm{d}t},\vec{f}\right)
\end{aligned}
$$

Note: 3-force equation: $\vec{f} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = m_0\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\vec{v})$

Einstein's Relation

- Momentum invariant $P \cdot P = m_0^2 V \cdot V = m_0^2 c^2$
- Differentiate $P \cdot \frac{\mathrm{d}P}{\mathrm{d}\tau} \implies V \cdot \frac{\mathrm{d}P}{\mathrm{d}\tau} = 0 \implies V \cdot F = 0$



$$\implies \gamma(c, \vec{v}) \cdot \gamma \left(c \frac{\mathrm{d}m}{\mathrm{d}t}, \vec{f} \right) = 0$$

$$\implies \frac{\mathrm{d}}{\mathrm{d}t}(mc^2) - \vec{v} \cdot \vec{f} = 0$$

$$\begin{aligned} \vec{v} \cdot \vec{f} &= \text{rate at which force does work} \\ &= \text{rate of change of kinetic energy} \end{aligned}$$

Therefore kinetic energy is

$$T = mc^2 + \text{constant} = m_0 c^2 (\gamma - 1)$$

$E = mc^2$ is total energy

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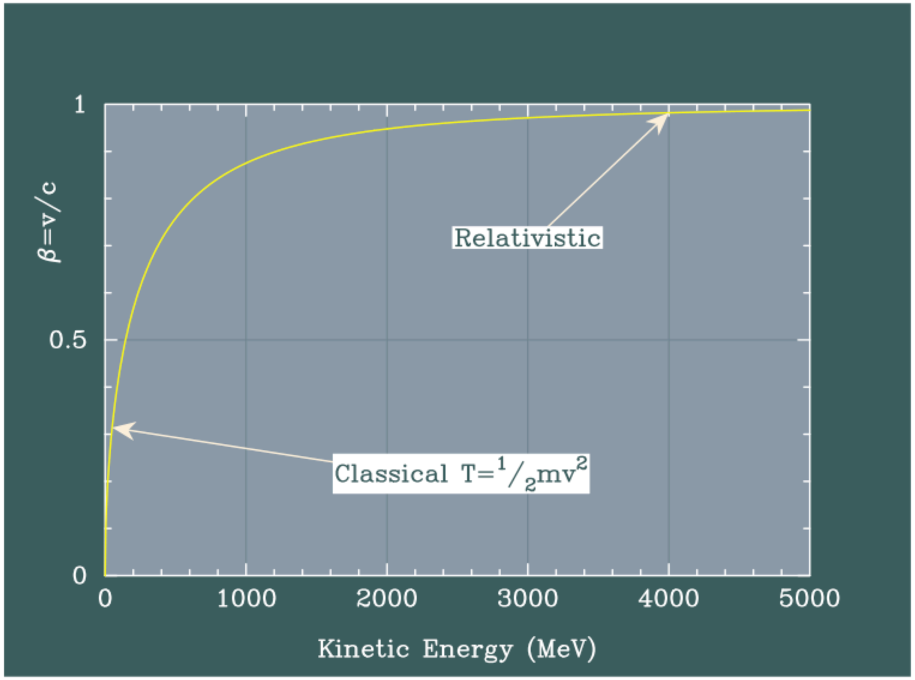
Basic Quantities used in Accelerator Calculations

$$\begin{array}{lrcl} \text{Relative velocity} & \beta & = & \dfrac{v}{c} \\ \text{Velocity} & v & = & \beta c \\ \text{Momentum} & p & = & mv = m_0\gamma\beta c \\ \text{Kinetic energy} & T & = & (m - m_0)c^2 = m_0c^2(\gamma - 1) \end{array}$$

$$\gamma \quad = \quad \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = \left(1 - \beta^2\right)^{-\frac{1}{2}}$$

$$\Longrightarrow \quad (\beta\gamma)^2 \quad = \quad \frac{\gamma^2 v^2}{c^2} = \gamma^2 - 1 \quad \Longrightarrow \quad \beta^2 = \frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2}$$

Velocity *v.* Energy



$$T = m_0(\gamma - 1)c^2$$

$$\gamma = 1 + \frac{T}{m_0c^2}$$

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}}$$

$$p = m_0\beta\gamma c$$

For $v \ll c$, $$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \ldots$$

so $$T = m_0c^2(\gamma - 1) \approx \frac{1}{2}m_0v^2$$

Energy-Momentum Invariant

$$P \cdot P = m_0^2 V \cdot V = m_0^2 c^2$$

$$\downarrow$$

$$\begin{aligned}\frac{E^2}{c^2} - p^2 \implies p^2c^2 &= E^2 - (m_0c^2)^2 = E^2 - E_0^2 \\ &= (E - E_0)(E + E_0) \\ &= T(T + 2E_0)\end{aligned}$$

Example: ISIS 800 MeV protons (E_0=938 MeV)

=> pc=1.463 GeV

$$\beta\gamma = \frac{m_0\beta\gamma c^2}{m_0c^2} = \frac{pc}{E_0} = 1.56$$

$$\gamma^2 = (\beta\gamma)^2 + 1 \implies \gamma = 1.85$$

$$\beta = \frac{\beta\gamma}{\gamma} = 0.84$$

Relationships between small variations in parameters ΔE, ΔT, Δp, Δβ, Δγ

$$\begin{aligned} (\beta\gamma)^2 &= \gamma^2 - 1 \\ \Longrightarrow \quad \beta\gamma\Delta(\beta\gamma) &= \gamma\Delta\gamma \\ \Longrightarrow \quad \beta\Delta(\beta\gamma) &= \Delta\gamma \qquad (1) \end{aligned}$$

$$\begin{aligned} \frac{1}{\gamma^2} &= 1 - \beta^2 \\ \Longrightarrow \quad \frac{1}{\gamma^3}\Delta\gamma &= \beta\Delta\beta \qquad (2) \end{aligned}$$

$$\begin{aligned} \frac{\Delta p}{p} &= \frac{\Delta(m_0\beta\gamma c)}{m_0\beta\gamma c} = \frac{\Delta(\beta\gamma)}{\beta\gamma} \\ &= \frac{1}{\beta^2}\frac{\Delta\gamma}{\gamma} = \frac{1}{\beta^2}\frac{\Delta E}{E} \\ &= \gamma^2\frac{\Delta\beta}{\beta} \\ &= \frac{\gamma}{\gamma+1}\frac{\Delta T}{T} \qquad \text{(exercise)} \end{aligned}$$

Note: valid to first order only

	$\dfrac{\Delta\beta}{\beta}$	$\dfrac{\Delta p}{p}$	$\dfrac{\Delta T}{T}$	$\dfrac{\Delta E}{E} = \dfrac{\Delta\gamma}{\gamma}$
$\dfrac{\Delta\beta}{\beta} =$	$\dfrac{\Delta\beta}{\beta}$	$\dfrac{1}{\gamma^2}\dfrac{\Delta p}{p}$ $\dfrac{\Delta p}{p} - \dfrac{\Delta\gamma}{\gamma}$	$\dfrac{1}{\gamma(\gamma+1)}\dfrac{\Delta T}{T}$	$\dfrac{1}{\beta^2\gamma^2}\dfrac{\Delta\gamma}{\gamma}$ $\dfrac{1}{\gamma^2-1}\dfrac{\Delta\gamma}{\gamma}$
$\dfrac{\Delta p}{p} =$	$\gamma^2\dfrac{\Delta\beta}{\beta}$	$\dfrac{\Delta p}{p}$	$\dfrac{\gamma}{\gamma+1}\dfrac{\Delta T}{T}$	$\dfrac{1}{\beta^2}\dfrac{\Delta\gamma}{\gamma}$
$\dfrac{\Delta T}{T} =$	$\gamma(\gamma+1)\dfrac{\Delta\beta}{\beta}$	$\left(1+\dfrac{1}{\gamma}\right)\dfrac{\Delta p}{p}$	$\dfrac{\Delta T}{T}$	$\dfrac{\gamma}{\gamma-1}\dfrac{\Delta\gamma}{\gamma}$
$\dfrac{\Delta E}{E} =$	$(\beta\gamma)^2\dfrac{\Delta\beta}{\beta}$	$\beta^2\dfrac{\Delta p}{p}$	$\left(1-\dfrac{1}{\gamma}\right)\dfrac{\Delta T}{T}$	$\dfrac{\Delta\gamma}{\gamma}$
$\dfrac{\Delta\gamma}{\gamma} =$	$(\gamma^2-1)\dfrac{\Delta\beta}{\beta}$	$\dfrac{\Delta p}{p} - \dfrac{\Delta\beta}{\beta}$		

4-Momentum Conservation

- Equivalent expression for 4-momentum $P = m_0\gamma(c,\vec{v}) = (mc,\vec{p}) = \left(E/_c,\vec{p}\right)$
- Invariant $m_0^2c^2 = P \cdot P = \frac{E^2}{c^2} - \vec{p}^2$ $\boxed{E^2/_{c^2} = m_0^2c^2 + \vec{p}^2}$
- Classical momentum conservation laws → conservation of 4-momentum. Total 3-momentum and total energy are conserved.

$$\sum_{\text{particles, i}} P_i = \text{constant}$$

$$\Rightarrow \sum_{\text{particles, i}} E_i \text{ and } \sum_{\text{particles,i}} \vec{p}_i \text{ constant}$$

Example of use of invariants

- Two particles have equal rest mass m_0.
  - Frame 1: one particle at rest, total energy is E_1.
  - Frame 2: centre of mass frame where velocities are equal and opposite, total energy is E_2.

Problem: Relate E_1 to E_2

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Total energy E_1

(Fixed target experiment)

$$P_1 = \left(\frac{E_1 - m_0 c^2}{c}, \vec{p} \right) \qquad P_2 = \left(m_0 c, \vec{0} \right)$$

Total energy E_2

(Colliding beams expt)

$$P_1 = \left(\frac{E_2}{2c}, \vec{p'} \right) \qquad P_2 = \left(\frac{E_2}{2c}, -\vec{p'} \right)$$

$$\text{Invariant:} \qquad P_2 \cdot (P_1 + P_2)$$

$$m_0 c \times \frac{E_1}{c} - 0 \times p = \frac{E_2}{2c} \times \frac{E_2}{c} + p' \times 0$$

$$\Rightarrow \qquad 2 m_0 c^2 E_1 = E_2^2$$

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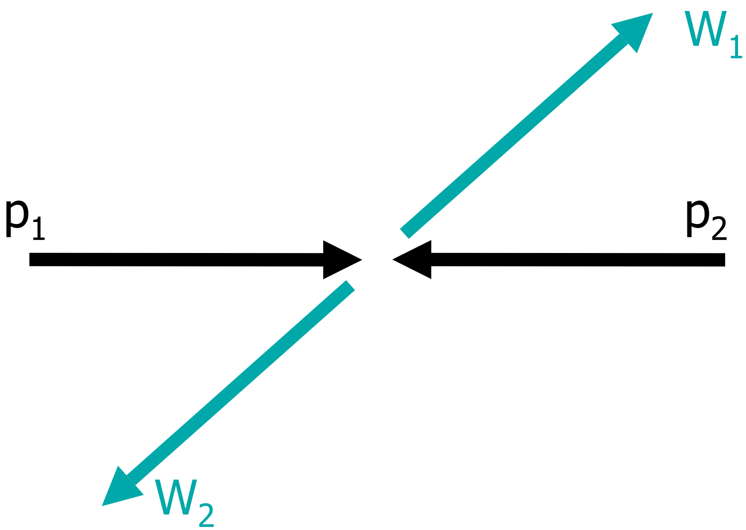
Collider Problem

- In an accelerator, a proton p_1 with rest mass m_0 collides with an anti-proton p_2 (with the same rest mass), producing two particles W_1 and W_2 with equal rest mass $M_0=100m_0$
  - Expt 1: p_1 and p_2 have equal and opposite velocities in the lab frame. Find the minimum energy of p_2 in order for W_1 and W_2 to be produced.
  - Expt 2: in the rest frame of p_1, find the minimum energy E' of p_2 in order for W_1 and W_2 to be produced.

Experiment 1

Note: $E^2/c^2 = \vec{p}^{\,2} + m_0^2 c^2 \Rightarrow$ same m_0, same p mean same E.



Total 3-momentum is zero before collision and so is zero afterwards.

4-momenta before collision:

$$P_1 = \left(E/c, \vec{p}\right) \quad P_2 = \left(E/c, -\vec{p}\right)$$

4-momenta after collision:

$$P_1 = \left(E'/c, \vec{q}\right) \quad P_2 = \left(E'/c, -\vec{q}\right)$$

Energy conservation $\Rightarrow E = E' >$ rest energy $= M_0 c^2 = 100\ m_0 c^2$

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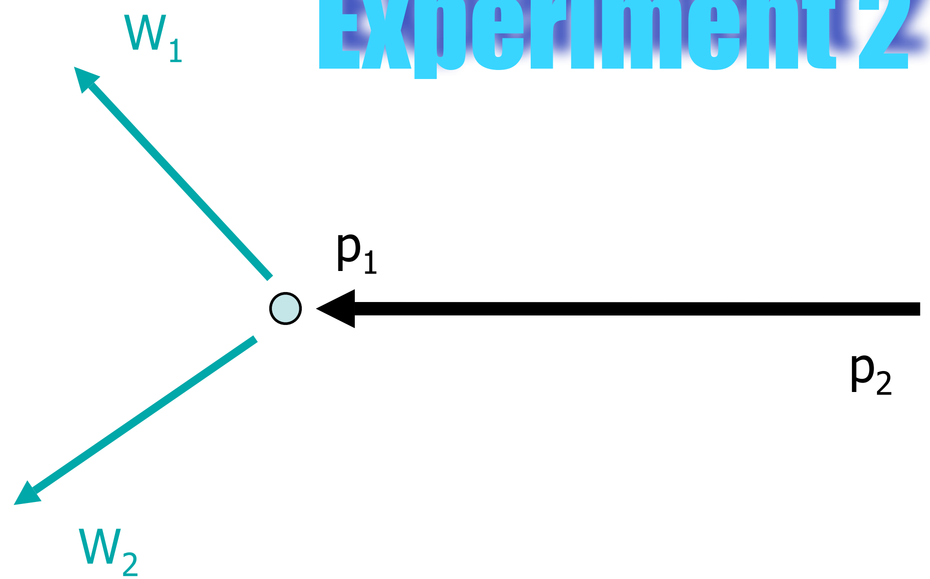
Experiment 2

Before collision:

$$P_1 = \left(m_0 c, \vec{0}\right) \quad P_2 = \left(E'/c, \vec{p}\right)$$



Total energy is

$$E_1 = E' + m_0 c^2$$

Use previous result $2m_0c^2E_1 = E_2^2$ to relate E_1 to total energy E_2 in C.O.M frame

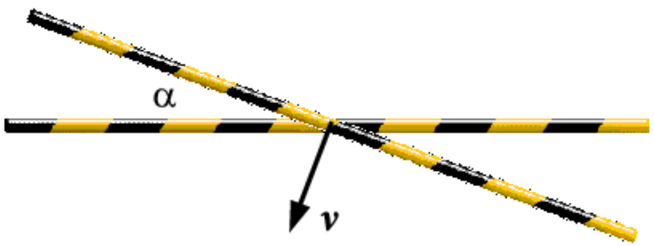
$$2m_0c^2E_1 = E_2^2$$

$$\Rightarrow \quad 2m_0c^2\left(E' + m_0c^2\right) = (2E)^2 > \left(200\, m_0c^2\right)^2$$

$$\Rightarrow \quad E' > \left(2\times 10^4 - 1\right) m_0c^2 \approx 20{,}000\, m_0c^2$$

Motion faster than light

1. Two rods sliding over each other. Speed of intersection point is $v/\sin\alpha$, which can be made greater than c.
2. Explosion of planetary nebula. Observer sees bright spot spreading out. Light from P arrives $t=d\alpha^2/2c$ later.



$$t = \frac{d\alpha^2}{2c} \approx \frac{x}{c}\frac{\alpha}{2} << \frac{x}{c}$$