1) SPACE-CHARGE

Introduction

The many charged particles in a high intensity beam represent a space-charge and produce electromagnetic self-fields which affect the beam dynamics being otherwise determined by the guide fields of the magnetic lattice and RF-system. Assuming weak self-fields we treat their effects as a perturbation and concentrate on the transverse case where this shifts the betatron frequencies (tunes).

For the direct space charge effect the conducting vacuum chamber is neglected, \( E \) and \( B \)-fields are obtained directly. The \( E \)-field is repelling and defocusses while the Lorentz force of the \( B \)-field focuses. The balance between them becomes more perfect as the particle velocity \( v \) approaches \( c \).

Conducting boundaries modify the field giving an indirect space-charge effect which is calculated with image charges. Here the balance between \( E \) and \( B \)-effects is perturbed and this effect is important also for \( v \to c \).

For a rigid, coherent, oscillation of the beam as a whole, the direct-space charge represents an internal force which does not influence this motion, however the indirect wall effect does.
Direct space-charge effect

Fields and forces

Continuous (unbunched) beam of circular cross section, radius \( a \), uniform charge/current densities \( \eta \), \( \vec{J} = \eta \beta c \) with total charge per unit length \( \lambda = \pi a^2 \eta \) and current \( I = \beta c \lambda \), produces cylindrically symmetric fields \( \vec{E} = [E_\rho, 0, 0] \) and \( \vec{B} = [0, E_\phi, 0] \) at radial distance \( \rho \):

\[
\begin{align*}
\text{div } \vec{E} &= \eta / \epsilon_0 \\
\frac{1}{\rho} \text{ div } \vec{E} \, d\rho = \oint \vec{E} \, d\vec{S}_E \\
\text{curl } \vec{B} &= \mu_0 \vec{J} \\
\oint \vec{B} \cdot d\vec{s} &= \oint \text{curl } \vec{B} \cdot d\vec{S}_J \\
d\vec{S}_E &= 2\pi \rho \, [d\rho, 0, 0], \quad d\vec{S}_J = 2\pi \rho \, d\rho \, [0, 0, 1], \quad d\vec{s} = \rho \, [0, d\phi, 0], \quad dV = 2\pi \rho \, ds \, d\rho.
\end{align*}
\]

Integrate \( \int_0^\rho \eta(\rho')d\rho' \)

\[
\begin{align*}
2\pi \rho \ell E_\rho &= \pi a^2 \ell \eta / \epsilon_0 \\
E_\rho &= \frac{\eta \rho}{2\epsilon_0} = \frac{\lambda \rho}{2\pi \epsilon_0 a^2} \\
2\pi \rho \ell E_\rho &= \pi \rho^2 \ell \eta / \epsilon_0 \\
E_\rho &= \frac{\eta a^2}{2\epsilon_0 \rho} = \frac{\lambda}{2\pi \epsilon_0 \rho}
\end{align*}
\]

For \( \rho \leq a \), relevant for direct space-charge, only charges \( \rho' \leq \rho \) contribute. Force on a particle \( \vec{F} = F_E + F_B = e \left( \vec{E} + [\vec{v} \times \vec{B}] \right) \)

\[
= \frac{e\eta}{2\epsilon_0} (1 - \beta^2) \vec{\rho} = \frac{eI}{2\pi \epsilon_0 c \beta \gamma^2 a^2} \vec{\rho}.
\]

\( \vec{F} \propto \vec{\rho} \) gives linear defocusing being \( \propto 1/\gamma^2 \) and vanishes as \( \beta \to 1 \).
**Space-charge defocusing**

Uniform space-charge force on particle is linear, radial, repulsive and defocuses beam in \(x\)- and \(y\)-plane, changing tunes \(Q_x, \; /Q_y\), (taking \(y\)):

\[
\vec{F} = F_E + F_B = e \left( \vec{E} + [\vec{v} \times \vec{B}] \right) = \frac{e\eta}{2\epsilon_0} (1 - \beta^2) \vec{\rho} = \frac{eI}{2\pi\epsilon_0 c} \gamma^2 \vec{\rho}.
\]

Force deflects by angle \(\Delta y' = \propto y\)

\[
F_y \approx m_0 \gamma d^2 y/dt^2 = m_0 c^2 \beta^2 \gamma^2 dy'/ds
\]

\[
\frac{dy'}{y} = d \left( \frac{1}{f} \right) = \frac{eI ds}{2\pi\epsilon_0 a^2 m_0 c^2 \beta^3 \gamma^3} = \frac{2r_0 I ds}{ec \beta^3 \gamma^3 a^2}
\]

\[
r_0 = \frac{e^2}{4\pi\epsilon_0 m_0 c^2} = 1.54 \times 10^{-18} \text{ m protons}
\]

\[
r_0 = 2.82 \times 10^{-15} \text{ m electrons}
\]

\[
y' = dy'/ds \approx \dot{y}/(\beta c) \ll 1, \text{ focusing strength } 1/f, \text{ classical particle radius } r_0. \text{ Tune change by element of length } \Delta s, \text{ strength } 1/f.
\]

\[
\frac{\text{d}Q_y}{4\pi} = \frac{\beta_y(s) d \left( \frac{1}{f} \right)}{2\pi\epsilon_0 c \beta^3 \gamma^3 a^2(s)} = \frac{-r_0 I \beta_y(s) ds}{2\pi\epsilon_0 c \beta^3 \gamma^3 a^2(s)}
\]

\[
\Delta Q_y = \frac{-r_0 I}{2\pi\epsilon_0 c \beta^3 \gamma^3} \int \beta_y(s) ds \frac{a^2(s)}{a^2(s)} = \frac{-r_0 IR}{ce \beta^3 \gamma^3 \mathcal{E}_y}
\]

using invariant emittance \(\mathcal{E}_y \approx a^2/\beta_y\). Tune shift by local space-charge depends on \(\mathcal{E}_y\), not on \(\beta_y\) and \(a\) separately. Small \(\beta_y\) gives small \(a\) and strong force but reduced effect.

Approx.: \(\mathcal{E}_y \approx a^2/\beta_y\); no change of \(\beta_y\).
 Elliptic beam cross section
Uniform $\eta$ and elliptic cross section with half-axes $a$, $b$ give fields and forces inside (L. Teng)

$$\vec{E} = [E_x, E_y] = \frac{I}{\pi \varepsilon_0 (a + b) \beta c} \left[ \frac{x}{a}, \frac{y}{b} \right]$$

$$\vec{B} = [B_x, B_y] = \frac{\mu_0 I}{\pi (a + b)} \left[ \frac{y}{b}, \frac{x}{a} \right]$$

which satisfies $\text{div} \vec{E} = \eta / \varepsilon_0$, $\text{curl} \vec{B} = \mu_0 \vec{J}$.

$$\vec{F} = e \left[ \vec{E} + [\vec{v} \times \vec{B}] \right] = \frac{I \left[ (x/a), (y/b) \right]}{\pi \varepsilon_0 \beta c \gamma^2 (a + b)}.$$

This force is $F_x \propto x$, $F_y \propto y$ and gives linear defocusing in the two directions.

$$\Delta Q_x = -\frac{r_0 I}{\pi c \beta^3 \gamma^3 E_x} \int \frac{a}{a + b} \text{d}s$$

$$\Delta Q_y = -\frac{r_0 I}{\pi c \beta^3 \gamma^3 E_y} \int \frac{b}{a + b} \text{d}s$$

Since $a/b$ depends on $s$ the local tune shift contribution depends also weakly on $s$.

Bunched beams
Current $I(s)$ depends on longitudinal distance $s$ from bunch center. Relativistic field has small opening angle $\approx 1/\gamma$ and depends on local $I(s)$ if this changes little over $\Delta s = a/\gamma$.

$$\Delta Q_x \approx -\frac{r_0 RI(s)}{ec \gamma^3 \beta^3}, \quad \Delta Q_y \approx -\frac{r_0 RI(s)}{ec \gamma^3 \beta^3 E_y}.$$ 

Tune shift depends on particle position $s$ in bunch giving to a tune spread, and, through synchrotron oscillations, to a tune modulation.

Non-uniform distribution
General charge distribution is not uniform, has radial dependence $\eta(\rho)$ giving non-linear force, making tune shift depend on betatron oscillation amplitude and leading to a tune spread.
Indirect space-charge effect — influence of the chamber wall

Conducting boundary imposes $E_\parallel = 0$ with only $E_\perp$. To calculate field we introduce image charge $-\lambda$ at distance $h$ behind wall which cancels $E_\parallel$ on surface. Have fields:

- direct $\vec{E}_d$, image $\vec{E}_i$, surface $\vec{E}_d\| = -\vec{E}_i\|$, $\vec{E}_\parallel = 0$
- inside: $\vec{E} = \vec{E}_d + \vec{E}_i$, $\text{div}\vec{E}_d = \eta/\epsilon_0$, $\text{div}\vec{E}_i = 0$

Conducting plates at $\pm h$. To get there $E_\parallel = 0$, need image charges of beam and of images. Field close to beam, first order in $x$, $y$ (quadrupole field) of $n$-th image pair at $\pm 2nh$ and sum over $n$

$$E_{iny} = \frac{(-1)^n \lambda}{2\pi \epsilon_0} \left( \frac{1}{2nh + y} - \frac{1}{2nh - y} \right) \approx -\frac{\lambda y}{4\pi \epsilon_0 h^2} \frac{(-1)^n}{n^2}$$

$$E_{iy} = \sum_{1}^{\infty} E_{iny} = \frac{\lambda y \frac{\pi^2}{4\pi \epsilon_0 h^2}}{12}, \quad \text{div}\vec{E}_i = 0 \rightarrow E_{i_x} = -\frac{\lambda x}{4\pi \epsilon_0 h^2} \frac{\pi^2}{12},$$

$$F_x = \frac{2e \lambda x}{2\pi \epsilon_0} \left( \frac{1}{2a^2 \gamma^2} - \frac{\pi^2}{48h^2} \right), \quad F_y = \frac{2e \lambda y}{2\pi \epsilon_0} \left( \frac{1}{2a^2 \gamma^2} + \frac{\pi^2}{48h^2} \right)$$

$$\Delta Q_{x/y} = -\frac{2r_0 I R \langle \beta_{x/y} \rangle}{e c \beta^3 \gamma} \left( \frac{1}{2a^2 \gamma^2} \pm \frac{\pi^2}{48h^2} \right), \quad \text{with } I = \lambda \beta c.$$

$B$ field not affected, no relativistic $E/B$-force compensation.
Incoherent and coherent motion

Direct space-charge effect

For incoherent motion particles have space-charge tune shift

\[ \Delta Q_{\text{inc.}} = -\frac{r_0 IR \beta_y}{ce\alpha^2 \beta^3 \gamma^3} \]

In coherent motion space-charge force is intern, moves with beam, no effect on center-of-mass motion \( \Delta Q_{\text{coh.}} = 0 \)

Indirect space-charge effect

Space-charge field with a conducting wall at distance \( h \) was obtained by image line charge at \( h \) behind wall. A coherent beam motion by \( \bar{y} \) moves first images to \( \pm 2h - \bar{y} \) with a field at the beam

\[ E_{c1y} = \frac{-\lambda}{2\pi \epsilon_0} \left( \frac{1}{2h + 2\bar{y}} - \frac{1}{2h - 2\bar{y}} \right) \]

Equidistant 2nd images cancel, general

\[ E_{cy} = \sum_{\lambda=1}^{\infty} E_{cny} = \frac{\lambda \bar{y}}{4\pi \epsilon_0 h^2} \left( \frac{\pi^2}{12} + \frac{\pi^2}{6} \right) \]

\[ Q_{ycoh.} = Q_0 - \frac{\pi^2}{16} \frac{2r_0 IR \langle \beta_y \rangle}{ec \beta^3 \gamma h^2} \]

\[ Q_{ycoh.} - Q_{yinc.} = \frac{2r_0 IR \langle \beta_y \rangle}{ec \beta^3 \gamma} \left( \frac{1}{2a^2 \gamma^2} - \frac{\pi^2}{24h^2} \right) \]
Problems caused by space-charge in rings
In rings space-charge can shift tunes into resonances where \( Q = \frac{N}{M} \) is a simple rational fraction. Dipole imperfection deflects particle each turn in phase if \( Q \) =integer and for a quadrupole error this happens if \( Q \) =half integer. Since space-charge shifts coherent and incoherent tunes differently and produces spread it may be difficult to avoid all resonances.

**Related effects**
Beam-beam effect: electric and magnetic forces ad. Ions and electron clouds: don’t move, no \( B \)-force.
2) IMPEDANCES AND WAKE FUNCTIONS

Resonator

For space charge a perfectly conducting wall of uniform cross section and electrostatic methods were used. General cross sections have resonances described by an impedance.

Beam induces wall current \( I_w = - (I_b - \langle I_b \rangle) \)

Cavities have narrow band oscillation modes which can drive coupled bunch instabilities. Each resembles an RCL - circuit and can, in good approximation, be treated as such. This circuit has a shunt impedance \( R_s \), an inductance \( L \) and a capacity \( C \). In a real cavity these parameters cannot easily be separated and we use others which can be measured directly: The resonance frequency \( \omega_r \), the quality factor \( Q \) and the damping rate \( \alpha \):

\[
\omega_r = \frac{1}{\sqrt{LC}}, \quad Q = R_s \sqrt{\frac{C}{L}} = \frac{R_s}{L \omega_r} = R_s C \omega_r
\]

\[
\alpha = \frac{\omega_r}{2Q}, \quad L = \frac{R_s}{Q \omega_r}, \quad C = \frac{Q}{\omega_r R_s}
\]
Driving this circuit with a current $I$ gives the voltages and currents across the elements

\[
\begin{align*}
V_R &= I_R R_s, \\
V_C &= \frac{1}{C} \int I_C dt, \\
V_L &= L \frac{dI_L}{dt}
\end{align*}
\]

\[V_R = V_C = V_L = V, \quad I_R + I_C + I_L = I\]
\[I = I_R + I_C + I_L = \dot{V} / R_s + C \ddot{V} + V / L.\]

Using $L = R_s / (\omega_r Q)$, $C = Q / (\omega_r R_s)$ gives
differential eqn. \[
\ddot{V} + \frac{\omega_r}{Q} \dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q} \dot{I}
\]

Homogeneous solution is damped oscillation
\[V(t) = e^{-\alpha t} \left( A \cos \left( \omega_r \sqrt{1 - \frac{1}{4Q^2}} t \right) + B \sin \left( \omega_r \sqrt{1 - \frac{1}{4Q^2}} t \right) \right), \quad \alpha = \frac{\omega_r}{2Q}\]

Wake/Green – function, pulse response

$I(t) = q \delta t$, charge $q$ gives capacity voltage
\[V(0^+) = \frac{q}{C} = \frac{\omega_r R_s}{Q} q \quad \text{using} \quad C = \frac{Q}{\omega_r R_s}\]

Energy stored in $C = \text{energy lost by } q$
\[U = \frac{q^2}{2C} = \frac{\omega_r R_s}{2Q} q^2 = \frac{V(0^+)}{2} q = k_{pm} q^2\]

parasitic mode loss factor $k_{pm} = \omega_r R_s / 2Q$

Capacitor discharges first through resistor
\[-\dot{V}(0^+) = \frac{q}{C} = \frac{I_R}{I} = \frac{V(0^+)}{C R_s} = -\frac{2 \omega_r k_{pm}}{Q} q.\]

$V(0^+), \dot{V}(0^+) \rightarrow A = 2q k_{pm}, \quad B = \frac{-A}{\sqrt{4Q^2 - 1}}$

\[V(t) = 2q k_{pm} e^{-\alpha t} \left( \cos \left( \omega_r \sqrt{1 - \frac{1}{4Q^2}} t \right) - \frac{\sin \left( \omega_r \sqrt{1 - \frac{1}{4Q^2}} t \right)}{\sqrt{4Q^2 - 1}} \right) \approx 2q k_{pm} e^{-\alpha t} \cos(\omega_r t).\]
**Impedance**

A **harmonic** excitation of circuit with current $I = \hat{I} \cos(\omega t)$ gives differential equation

$$\ddot{V} + \frac{\omega_r}{Q} \dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q} \hat{I} = -\frac{\omega_r R_s}{Q} \hat{I} \omega \sin(\omega t).$$

Homogeneous solution damps leaving particular one $V(t) = A \cos(\omega t) + B \sin(\omega t)$. Put into differential equation, separating cosine and sine

$$-(\omega^2 - \omega_r^2) A + \frac{\omega_r \omega}{Q} B = 0$$

$$(\omega^2 - \omega_r^2) B + \frac{\omega_r \omega}{Q} A = \frac{\omega_r \omega R_s}{Q} \hat{I}.$$

Voltage induced by current $\hat{I} \cos(\omega t)$ is

$$V(t) = \hat{I} R_s \frac{\cos(\omega t) + Q \frac{\omega^2 - \omega_r^2}{\omega_r \omega} \sin(\omega t)}{1 + Q^2 \left(\frac{\omega^2 - \omega_r^2}{\omega_r \omega}\right)^2}.$$

Cosine term is **in phase** with exciting current, absorbs energy, **resistive**. Sine term is **out of phase**, does not absorb energy, **reactive**. Voltage/current ratio is **impedance** as function of frequency $\omega$

$$Z_r(\omega) = R_s \frac{1}{1 + Q^2 \left(\frac{\omega^2 - \omega_r^2}{\omega_r \omega}\right)^2};$$

$$Z_i(\omega) = -R_s \frac{Q \frac{\omega^2 - \omega_r^2}{\omega_r \omega}}{1 + Q^2 \left(\frac{\omega^2 - \omega_r^2}{\omega_r \omega}\right)^2}.$$

Resistive part $Z_r(\omega) \geq 0$, reactive part $Z_i(\omega)$ positive below, negative above $\omega_r$.

$\hat{I} \cos(\omega t) \rightarrow V = \hat{I} [Z_r \cos(\omega t) - Z_i \sin(\omega t)]$

$\hat{I} \sin(\omega t) \rightarrow V = \hat{I} [Z_r \sin(\omega t) + Z_i \cos(\omega t)]$
**Complex notation**

Excite: $I(t) = \hat{I} \cos(\omega t) = \hat{I} e^{j\omega t} + e^{-j\omega t}$

with $0 \leq \omega \leq \infty$

$I(t) = \hat{I} e^{j\omega t} / 2$ with $-\infty \leq \omega \leq \infty$

\[ Z(\omega) = R_s \frac{1 - jQ\frac{\omega^2 - \omega_r^2}{\omega}}{1 + Q^2 \left(\frac{\omega^2 - \omega_r^2}{\omega_r}\right)^2} = Z_r + jZ_i \]

\[ \approx R_s \frac{1 - j2Q\Delta\omega/\omega_r}{1 + 4Q^2 \left(\Delta\omega/\omega_r\right)^2} \quad \text{for} \ Q \gg 1 \]

\[ \omega \approx \omega_r, |\omega - \omega_r|/\omega_r = |\Delta\omega|/\omega_r \ll 1. \]

Resonator impedance properties:

- at $\omega = \omega_r \rightarrow Z_r(\omega_r)$ max., $Z_i(\omega_r) = 0$
  - $0 < \omega < \omega_r \rightarrow Z_i(\omega) > 0$ (inductive)
  - $\omega > \omega_r \rightarrow Z_i(\omega) < 0$ (capacitive)

General impedance or wake properties

$Z_r(\omega) = Z_r(-\omega)$, $Z_i(\omega) = -Z_i(-\omega)$

$Z(\omega) = \int_{-\infty}^{\infty} G(t)e^{-j\omega t} dt$

$Z(\omega) \propto$ Fourier transform of $G(t)$

for $t < 0 \rightarrow G(t) = 0$, no fields before particle arrives, $\beta \approx 1$. 
**Typical impedance of a ring**

Aperture changes form cavity-like objects with $\omega_r$, $R_s$ and $Q$ and impedance $Z(\omega)$ developed for $\omega < \omega_r$, where it is inductive

$$Z(\omega) = R_s \frac{1 - jQ\frac{\omega^2 - \omega_r^2}{\omega\omega_r}}{1 + (Q\frac{\omega^2 - \omega_r^2}{\omega\omega_r})^2} \approx j\frac{R_s\omega}{Q\omega_r} + \ldots$$

Sum impedance at $\omega \ll \omega_{rk}$ divided by mode number $n = \omega/\omega_0$ is with inductance $L$

$$\left|\frac{Z}{n}\right|_0 = \sum_k \frac{R_{sk}\omega_0}{Q_k\omega_{rk}} = L\omega_0 = L\frac{\beta c}{R}.$$

It depends on impedance per length, $\approx 15\ \Omega$ in older, $1\ \Omega$ in newer rings. The shunt impedances $R_{sk}$ increase with $\omega$ up to cutoff frequency where wave propagation starts and become wider and smaller. A broad band resonator fit helps to characterize impedance giving $Z_r$, $Z_i$, $G(t)$ useful for single traversal effects. However, for multi-traversal instabilities narrow resonances at $\omega_{rk}$ must be used.
3) LONGITUDINAL INSTABILITIES

Longitudinal dynamics

A particle with momentum deviation \( \Delta p \) has different orbit length \( L \), revolution time \( T_0 \) and frequency \( \omega_0 \)

\[
\frac{\Delta L}{L} = \frac{\Delta p}{p} = \frac{\alpha_c \Delta E}{p} \quad \frac{\Delta T}{T} = -\frac{\Delta \omega_0}{\omega_0} = \left( \alpha_c - \frac{1}{\gamma^2} \right) \frac{\Delta p}{p} = \eta_c \frac{\Delta p}{p}
\]

with momentum compaction \( \alpha_c = 1/\gamma_T^2 \), slip factor \( \eta_c \). At transition energy \( m_0 c^2 \gamma T \) the \( \omega_0 \)-dependence on \( \Delta p \) changes sign

\( E > E_T \to \frac{1}{\gamma^2} < \alpha_c \to \eta_c > 1, \quad \frac{\Delta \omega_0}{\Delta E} < 0 \)

\( E < E_T \to \frac{1}{\gamma^2} > \alpha_c \to \eta_c < 1, \quad \frac{\Delta \omega_0}{\Delta E} > 0 \).

For \( \gamma \gg 1 \to \Delta p/p \approx \Delta E/E = \epsilon, \eta_c \approx \alpha_c \).

RF-cavity of voltage \( \hat{V} \), frequency \( \omega_{RF} = h\omega_0 \), SR energy loss \( U \) the energy gain or loss of a particle in one turn \( \delta \epsilon = \delta E/E \) is

\[
\delta E = e\hat{V} \sin(h\omega_0(t_s + \tau)) - U
\]

\( t_s = \) synchronous arrival time at the cavity, \( \tau = \) deviation from it, synchronous phase \( \phi_s = h\omega_0 t_s \). For \( h\omega_0 \tau \ll 1 \) we develop

\[
\delta E = e\hat{V} \sin(\phi_s) + h\omega_0 e\hat{V} \cos \phi_s \tau - U.
\]
For $\delta E/E \ll 1$ use smooth approximation

$$\dot{E} \approx \delta E/T_0, \quad \dot{\tau} = \Delta T/T_0 = \eta_c \Delta E/E$$

$$\dot{E} = \frac{\omega_0 \hat{V} \sin \phi_s}{2\pi} + \frac{\omega_0^2 h \hat{V} \cos \phi_s}{2\pi} \tau - \frac{\omega_0}{2\pi} U.$$  

Use $T_0 = 2\pi/\omega_0$, relative energy $\epsilon = \Delta E/E$

$$\dot{\epsilon} = \frac{\omega_0 \hat{V} \sin \phi_s}{2\pi E} + \frac{\omega_0^2 h \hat{V} \cos \phi_s}{2\pi E} \tau - \frac{\omega_0 U}{2\pi E}.$$  

Energy loss $U$ may depend on $E$

$$U(\epsilon, \tau) \approx U_0 + \frac{\partial U}{\partial E} \Delta E$$

giving for the derivative of the energy loss

$$\dot{\epsilon} = \frac{\omega_0^2 h \hat{V} \cos \phi_s}{2\pi E} \tau - \frac{\omega_0}{2\pi} \frac{\partial U}{\partial E}$$

$$\dot{\tau} = \eta_c \epsilon$$

where we used that for synchronous particle $\epsilon = 0, \quad \tau = 0$ we have $U_0 = e\hat{V} \sin \phi_s$

Combining these into a second order equation

$$\ddot{\epsilon} + \frac{\omega_0}{2\pi} \frac{\partial U}{\partial E} \dot{\epsilon} + \omega_{s0}^2 \epsilon = 0,$$

$$\omega_{s0}^2 = \frac{-\omega_0^2 h \eta_c \hat{V} \cos \phi_s}{2\pi E}, \quad \alpha_s = \frac{1}{2\pi} \frac{\omega_0}{\partial U}$$

$$\omega_{s1} = \omega_{s0}^2 - \alpha_s^2 \approx \omega_{s0}^2$$

$$\ddot{\tau} + 2\alpha_s \dot{\tau} + \omega_{s0}^2 \tau = 0$$

$$\tau = \hat{\tau} e^{-\alpha_s t} \cos(\omega_{s1} t), \quad \epsilon = \hat{\epsilon} e^{-\alpha_s t} \sin(\omega_{s1} t)$$

From $\dot{\tau} = \eta_c \epsilon$ we get $\hat{\epsilon} = \omega_{s0} \hat{\tau}/\eta_c$.

To get real $\omega_{s0}$ we need $\cos \phi_s \leq 0$ above transition where $\eta_c > 0$ and vice versa.

To get a stable (decaying) solution we need an energy loss which increases with $E$

$$\alpha_s = \frac{\omega_0}{4\pi} \frac{\partial U}{\partial E} = \frac{\omega_0}{4\pi E} \frac{\partial U}{\partial \epsilon} > 0.$$
**Induced voltage and energy loss by a stationary bunch**

Circulating symmetric bunch ($N_b$ particles) has current

$$I(t) = \sum_{-\infty}^{\infty} I(t - kT_0)$$

$$I(t) = I_0 + 2 \sum_{1}^{\infty} I_p \cos(p\omega_0 t), \quad I_p = \int_{0}^{T_0} I(t) \cos(p\omega_0 t) \, dt$$

In impedance $Z(\omega)$ it induces voltage

$$V(t) = 2 \sum I_p [Z_r(p\omega_0) \cos(p\omega_0 t) - Z_i(p\omega_0) \sin(p\omega_0 t)]$$

Energy lost per particles and turn $U = \int_{0}^{T_0} I(t)V(t) \, dt/N_b$

$$U = \frac{2T_0}{N_b} \sum_{1}^{\infty} I_p^2 Z_r(p\omega_0) = \frac{2e}{I_0} \sum_{1}^{\infty} I_p^2 Z_r(p\omega_0)$$

using

$$\int_{0}^{T_0} \cos(p'\omega_0 t) \sin(p\omega_0 t) \, dt = 0, \quad I_0 = eN_b/T_0$$

$$\int_{0}^{T_0} \cos(p'\omega_0 t) \cos(p\omega_0 t) \, dt = \begin{cases} T_0/2 & \text{for } p' = p \\ 0 & \text{for } p' \neq p \end{cases}$$
Robinson instability

Qualitative treatment

Important longitudinal instability of a bunch interacting with an narrow impedance, called Robinson instability. In a qualitative approach we take single bunch and a narrow-band cavity of resonance frequency \(\omega_r\) and impedance \(Z(\omega)\) taking only its resistive part \(Z_r\). The revolution frequency \(\omega_0\) depends on energy deviation \(\Delta E\)

\[
\frac{\Delta \omega_0}{\omega_0} = -\eta_c \frac{\Delta p}{p}.
\]

While the bunch is executing a coherent dipole mode oscillation \(\epsilon(t) = \hat{\epsilon} \cos(\omega_s t)\) its energy and revolution frequency are modulated. Above transition \(\omega_0\) is small when the energy is high and \(\omega_0\) is large when the energy is small. If the cavity is tuned to a resonant frequency slightly smaller than the RF-frequency \(\omega_r < p\omega_0\) the bunch sees a higher impedance and loses more energy when it has an energy excess and it loses less energy when it has a lack of energy. This leads to a damping of the oscillation. If \(\omega_r > p\omega_0\) this is reversed and leads to an instability. Below transition energy the dependence of the revolution frequency is reversed which changes the stability criterion.
Qualitative understanding

\[
\begin{align*}
\epsilon &= \hat{e} e^{-\alpha_s t} \sin(\omega_s t), \text{ damping if } \alpha_s > 0 \\
\alpha_s &= \frac{\omega_s p I_p^2 (Z_r(\omega_{p+}) - Z_r(\omega_{p-}))}{2 I_0 h \tilde{V} \cos \phi_s} \\
\gamma > \gamma_T, \cos \phi_s < 0, \text{ stable } Z_r(\omega_{p-}) > Z_r(\omega_{p+}) \\
\text{Damping rate } \propto Z_r \text{ difference at side-bands.}
\end{align*}
\]

RF-cavity:
\[\alpha_s \approx \frac{I_0 (Z_r(\omega_{p+}) - Z_r(\omega_{p-}))}{\omega_s 0} \frac{2V}{2V \cos \phi_s} \approx \frac{p I_p^2 (Z_r(\omega_{p+}) - Z_r(\omega_{p-}))}{\omega_s 0} \frac{2I_0 h \tilde{V} \cos \phi_s}{2I_0 h \tilde{V} \cos \phi_s} \]

general:
\[\alpha_s = \sum_p \frac{p I_p^2 (Z_r(\omega_{p+}) - Z_r(\omega_{p-}))}{\omega_s 0} \frac{2I_0 h \tilde{V} \cos \phi_s}{2I_0 h \tilde{V} \cos \phi_s} \]

Cavity field induced by the two sidebands

Phase motion of the bunch center

Narrow band \(\rightarrow\) long memory, vice-versa
Potential well bunch lengthening

At low frequency wall is inductive with $L \omega_0 = |Z/n|_0$:

We take a parabolic bunch form

$$I_b(\tau) = \hat{I} \left(1 - \frac{\tau^2}{\hat{\tau}^2}\right) = \frac{3\pi I_0}{2\omega_0 \hat{\tau}} \left(1 - \frac{\tau^2}{\hat{\tau}^2}\right)$$

$$\frac{dI_b}{d\tau} = -\frac{3\pi I_0 \tau}{\omega_0 \hat{\tau}^3}, \quad I_0 = \langle I_b \rangle,$$

$$V = \hat{V} (\sin \phi_s + h\omega_0 \cos \phi_s \tau) + \frac{3\pi I_0 L \tau}{\omega_0 \hat{\tau}^3}$$

$$V = \hat{V} \left[\sin \phi_s + \cos \phi_s h\omega_0 \left(1 + \frac{3\pi |Z/n|_0 I_0}{h \hat{V} \cos \phi_s (\omega_0 \hat{\tau})^3}\right)\right]$$

$$\omega_{s0}^2 = -\frac{\omega_0^2 h \eta_c e \hat{V} \cos \phi_s}{2\pi E}$$

$$\omega_s^2 = \omega_{s0}^2 \left[1 + \frac{3\pi |Z/n|_0 I_0}{h \hat{V}_{RF} \cos \phi_s (\omega_0 \hat{\tau})^3}\right]$$

$$\Delta \omega_s = \omega_s - \omega_{s0} \approx \frac{3\pi |Z/n|_0 I_0}{2h \hat{V}_{RF} \cos \phi_s (\omega_0 \hat{\tau}_0)^3}$$
Decreasing $\omega_s$ reduces longitudinal focusing, increases bunch length $\hat{\tau}$. Relative energy spread $\hat{\epsilon} = \hat{\tau}_s \omega_s / \eta_c$ is given for electrons by synchrotron radiation, for protons the product (emittance) $\hat{\tau} \hat{\epsilon} = \text{const.}$

\[
\frac{\omega_s^2}{\omega_{s0}^2} = 1 + \frac{3\pi |Z/n|_0 I_0}{h \hat{V}_{RF} \cos \phi_s (\omega_0 \hat{\tau})^3}
\]

\[
\frac{\omega_s - \omega_{s0}}{\omega_{s0}} = \frac{\Delta \omega_s}{\omega_s} \approx \frac{3\pi |Z/n|_0 I_0}{2h \hat{V} \cos \phi_s (\omega_0 \hat{\tau}_0)^3}
\]

Only incoherent frequency $\omega_s$ of single particles is changed (reduced $\gamma > \gamma_T$, increased $\gamma < \gamma_T$), not coherent dipole (rigid bunch) frequency $\omega_{s1}$. The two get separated.

Electron $\frac{\Delta \hat{\tau}}{\hat{\tau}_0} = -\frac{\Delta \omega_s}{\omega_{s0}}$, proton $\frac{\Delta \hat{\tau}}{\hat{\tau}_0} = -\frac{\Delta \omega_s}{2\omega_{s0}}$

From observed bunch lengthening impedance is estimated.

Frequency measurement would be better, but $\omega_s$ is invisible and $\omega_{s1}$ does not move, however, quadrupole mode can be used

\[
\frac{\omega_{s2} - 2\omega_{s0}}{2\omega_{s0}} = \frac{\Delta \omega_{s2}}{\omega_{s2}} \approx \frac{1}{4} \frac{\Delta \omega_s}{\omega_{s0}}.
\]
Separation of coherent and incoherent frequencies

The wall inductance, and most reactive impedances, separate coherent and incoherent frequencies. A swing with a non-rigid frame can illustrate this mechanism. A coherent, center-of-mass, motion moves the frame and changes the frequency, this is not the case if oscillate at a different phases, leaving th incoherent frequency unchanged. For space-charge this causes mainly problems with resonances, here a loss of a stabilization mechanism, called Landau damping, is more important. A spread in individual particle frequencies produces phase mixing which reduces the center-of-mass, coherent, amplitude and gives some stabilization. A separation between coherent and incoherent frequencies makes this ineffective.
4) TRANSVERSE INSTABILITIES

Transverse impedance

Field excited by \( Ix = D = \dot{D} \cos(\omega t) \)
\[
\frac{\partial E_z}{\partial x} = -kIx, \quad E_z(x) = -kIx^2
\]
\[Z_L(x) = -\int E_z dz/I = -E_z\ell/I = k\ell x^2\]
\[
\int B d\vec{a} = -\int \vec{E}d\vec{s}, \quad \dot{B}_y x\ell = E_z\ell = -k\ell Dx
\]
\[\dot{B}_y = -k\dot{D} \cos(\omega t), \quad B = -k\dot{D} \sin(\omega t)/\omega\]
field \( B \) out of phase with \( D = lx \)
\[\dot{B}_y = -k\dot{D}/\omega, \quad \text{Lorentz force } \vec{F} \approx -ec\dot{B}_y\]
\[
Z_T = -\frac{F_x \ell}{eD} = \frac{ck\ell}{\omega} = \frac{cZ_L}{x^2\omega} = \frac{c}{2\omega} \frac{d^2Z_L}{dx^2}, \quad \left[ \Omega \right]/\left[ \text{m} \right]
\]

Used special case to define transverse impedance and its relation to second derivative of the longitudinal impedance of same mode. In general we have the impedances long.: integrated field/current; trans.: integrated defl. field/ dipole moment. On resonance, \( E_z \) is in, \( B_y \) out of phase of \( I \). General deflecting mode, using \( x = \hat{x}e^{j\omega t} \)
\[
Z_T(\omega) = j \frac{\int \left( \vec{E}(\omega) + [\vec{v} \times \vec{B}(\omega)] \right) \cdot ds}{Ix(\omega)}
\]

Relation \( Z_L \) to \( Z_T \) of different modes:
In ring of global and vacuum chamber radii \( R \) and \( b \) the impedances, averaged for different modes, have semi-empirical ratio
\[
Z_T(\omega) \approx \frac{2RZ_L(\omega)}{b^2 \omega/\omega_0}
\]

From area available for the wall current we expect \( Z_L \propto 1/b \), therefore \( Z_T \propto 1/b^3 \).
Transverse instability of a single, rigid bunch

A bunch $p$ traverses a cavity with off-set $x$, excites a field $-E_z$ which converts after $T_r/4$ into field $-B_y$, then into $E_z$ and after into $B_y$.

The bunch oscillates with tune $Q$ having a fractional part $q = 1/4$ seen as sidebands at $\omega_0(\text{integer} \pm q)$ by a stationary observer.

A) Cavity is tuned to upper sideband. Next turn bunch traverses in situation 'A', $t = T_r/4$ with velocity in $-x$-direction and gets by $B_y$ force in $+x$-direction which damps oscillation.

B) Cavity is tuned to lower sideband, bunch traverses next in situation 'B', $t = T_r3/4 = T_r(1 - 1/4)$ with negative velocity and force in same direction, increases velocity, instability.

damping rate $a = \frac{e\omega_0\beta_x}{4\pi m_0 c^2 \gamma I_0} \sum_{\omega > 0} \left( I_{p+}^2 Z_{Tr}(\omega_p^+) - I_{p-}^2 Z_{Tr}(\omega_p^-) \right), \ \omega_{p\pm} = \omega_0(p \pm q)$.
Transverse instability of many rigid bunches

$M$ bunches can oscillate in $M$ independent modes $n = M\Delta\phi/2\pi$, phase $\Delta\phi$ between them seen in global view. Locally, bunches pass with increasing time delay shown as bullets fitted by upper (solid) and lower (dashed) side-band frequency. Higher frequencies can be fitted and spectrum repeats every $4\omega_0$. $\omega_{p\pm} = \omega_0(pM \pm (n + q))$

Spectrum $n = 3$, $q = 1/4$

$\Delta\phi = \pi$, $n = 2$

$\Delta\phi = 3\pi/2$, $n = 3$
Non-rigid bunch - head-tail modes, $Q' = 0$

Particle distribution in a bunch

Phase-space distr. $\psi$ rotates with $\omega_s$, not visible, but projection $\lambda(\tau) = \int \psi(\tau, \Delta E) dE$ or current $I = q\beta c \lambda$. Study motion by selecting particles with fixed synch. osc. amplitude $\hat{\tau}$ rotating in phase-space, moving from head to tail and vice versa while executing at same time vertical betatron oscillation $y = \dot{y} \cos(Q_y \omega_0 t)$. With $Q' \approx dQ/(dE/E) = 0$ tune is constant during synchrotron motion.

Mode $m=0$, all in phase, rigid bunch

Mode $m=1$, head and tail in opposite phase, not rigid

A very high impedance can couple these modes and give a Transverse Mode Coupling Instability, TMCI.
Head-tail mode \( m = 0 \) for \( Q' \neq 0 \)

Synchrotron oscillation in \( \Delta E \) affect transverse motion via chromaticity \( Q' = dQ/(dp/p) \). For \( \gamma > \gamma_T \) has excess energy moving from head to tail and lack going from tail to head. For \( Q' > 0 \), phase advances in first, lags in second step; vice versa for \( Q' < 0 \) or \( \gamma < \gamma_T \). Figure shows motion for \( T_\beta = T_s/8 \), for \( Q' = 0 \) and \( Q' < 0 \) in 4 steps of \( T_\beta/8 \).

\[
\begin{align*}
Q' &= 0 \quad Q' > 0 \\
\text{CERN booster; Gareyte, Sacherer.}
\end{align*}
\]
**Model of head-tail instability**

Above transition energy:

\( Q' = 0 \): Going from head to tail or vice versa has same phase change. Phase lag and advance interchange, giving neither damping nor growth.

\( Q' < 0 \): Going from head to tail there is a loss in phase, going from tail to head a gain (picture), giving a systematic phase advance between head and tail and in average growth.

\( Q' > 0 \): Going from head to tail there is a phase gain, going back a loss, giving a systematic phase lag between head and tail and in average damping.

Below transition this situation is reversed.

Head tail spectrum:

\[
y I_p \quad \omega_0 \quad Q' < 0
\]
A merry-go-round, having vertically moving horses, can illustrate transverse modes:
Coupled bunch modes, real space $y = f(\theta, t)$
Head-Tail modes, phase-space $y = f(\Delta E, \tau, t)$

**Summary**

Present instability treatment, invented by K. Robinson and generalized to nearly all cases by Frank Sacherer.

This demands resistive impedance at upper, $Z^+$, and lower, $Z^-$, side-band to fulfill **stability conditions**:

<table>
<thead>
<tr>
<th></th>
<th>above transition</th>
<th>below transition</th>
</tr>
</thead>
<tbody>
<tr>
<td>longitudinal, stability</td>
<td>$Z^+_r &lt; Z^-_r$</td>
<td>$Z^+_r &gt; Z^-_r$</td>
</tr>
<tr>
<td>transverse $Q' = 0$, stability</td>
<td>$Z^+<em>{Tr} &gt; Z^-</em>{Tr}$</td>
<td>$Z^+<em>{Tr} &gt; Z^-</em>{Tr}$</td>
</tr>
<tr>
<td>transverse head-tail, stability</td>
<td>$Q' &gt; 0$</td>
<td>$Q' &lt; 0$</td>
</tr>
</tbody>
</table>

Ken Robinson

Frank Sacherer