1) SPACE-CHARGE

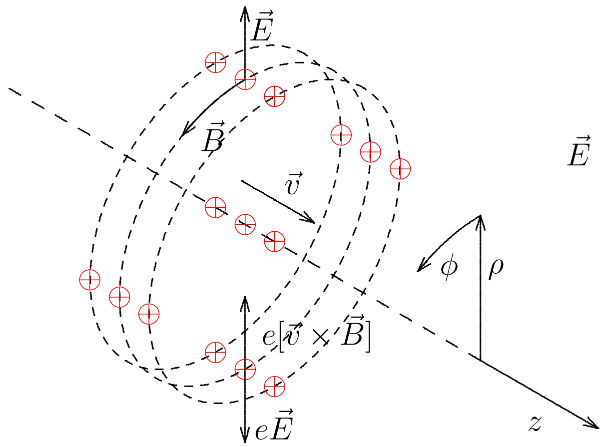
Introduction

The many charged particles in a high intensity beam represent a space-charge and produce electromagnetic self-fields which affect the beam dynamics being otherwise determined by the guide fields of the magnetic lattice and RF-system. Assuming weak self-fields we treat their effects as a perturbation and concentrate on the transverse case where this shifts the betatron frequencies (tunes).

For the **direct space charge effect** the conducting vacuum chamber is neglected, E and B-fields are obtained directly. The E-field is repelling and defocuses while the Lorentz force of the B-field focuses. The balance between them becomes more perfect as the particle velocity v approaches c.

Conducting boundaries modify the field giving an **indirect space-charge effect** which is calculated with image charges. Here the balance between E and B-effects is perturbed and this effect is important also for $v \rightarrow c$

For a rigid, **coherent**, oscillation of the beam as a whole, the direct-space charge represents an internal force which does not influence this motion, however the indirect wall effect does.



Direct space-charge effect

Fields and forces

Continuous (unbunched) beam of circular cross section, radius a, uniform charge/current densities η, $\vec{J} = \eta\beta c$ with total charge per unit length $\lambda = \pi a^2 \eta$ and current $I = \beta c \lambda$, produces cylindrically symmetric fields $\vec{E} = [E_\rho,\, 0,\, 0]$ and $\vec{B} = [0,\, E_\phi,\, 0]$ at radial distance ρ:

$$\boxed{\begin{aligned} \text{div}\,\vec{E} &= \eta/\epsilon_0 \\ \iiint \text{div}\,\vec{E}\,\mathrm{d}V &= \iint \vec{E}\,\mathrm{d}\vec{S}_E \end{aligned}} \qquad \boxed{\begin{aligned} \text{curl}\,\vec{B} &= \mu_0 \vec{J} \\ \oint \vec{B}\cdot\mathrm{d}\vec{s} &= \iint \text{curl}\,\vec{B}\cdot\mathrm{d}\vec{S}_J \end{aligned}}$$

$\mathrm{d}\vec{S}_E = 2\pi\rho[\mathrm{d}\rho,\, 0,\, 0]$, $\mathrm{d}\vec{S}_J = 2\pi\rho\mathrm{d}\rho[0,\, 0,\, 1]$,
$\mathrm{d}\vec{s} = \rho[0,\, \mathrm{d}\phi,\, 0]$, $\mathrm{d}V = 2\pi\rho\mathrm{d}s\mathrm{d}\rho$. Integrate $\int_0^\rho \eta(\rho')\mathrm{d}\rho'$

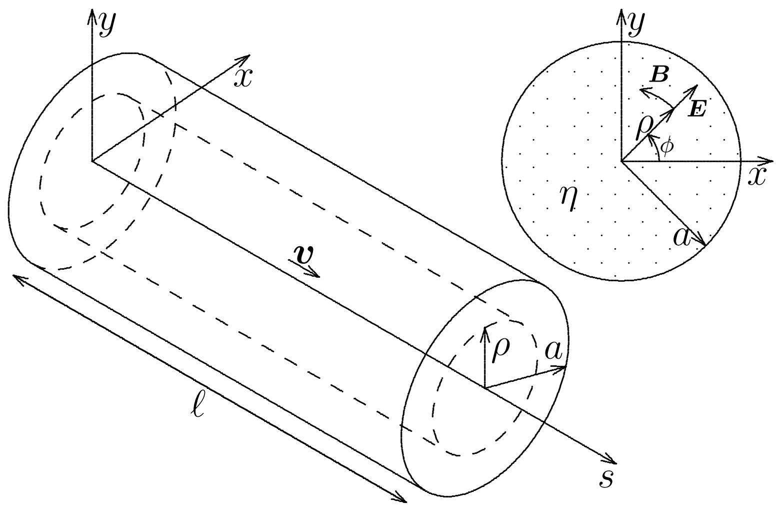
	in $\rho \le a$	
$2\pi\rho\ell E_\rho = \pi a^2 \ell\eta/\epsilon_0$, $\quad E_\rho = \dfrac{\eta\rho}{2\epsilon_0} = \dfrac{\lambda}{2\pi\epsilon_0}\dfrac{\rho}{a^2}$	in $\rho \le a$	$2\pi\rho B_\phi = \pi a^2 \mu_0 J_s$, $\quad B_\phi = \dfrac{\beta\eta\rho}{2\epsilon_0 c} = \dfrac{\mu_0 I}{2\pi}\dfrac{\rho}{a^2}$
$2\pi\rho\ell E_\rho = \pi \rho^2 \ell\eta/\epsilon_0$, $\quad E_\rho = \dfrac{\eta a^2}{2\epsilon_0\rho} = \dfrac{\lambda}{2\pi\epsilon_0}\dfrac{1}{\rho}$	out $\rho \ge a$	$2\pi\rho B_\phi = \pi \rho^2 \mu_0 J_s$, $\quad B_\phi = \dfrac{\beta\eta a^2}{2\epsilon_0 c\rho} = \dfrac{\mu_0 I}{2\pi}\dfrac{1}{\rho}$

.

Fields inside beam, $\rho \le a$, relevant for direct space-charge, only charges $\rho' \le \rho$ contribute. Force on a particle

$$\vec{F} = F_E + F_B = e\left(\vec{E} + [\vec{v}\times\vec{B}]\right) = \frac{e\eta}{2\epsilon_0}(1-\beta^2)\vec{\rho} = \frac{eI}{2\pi\epsilon_0 c\beta\gamma^2}\frac{\vec{\rho}}{a^2}.$$

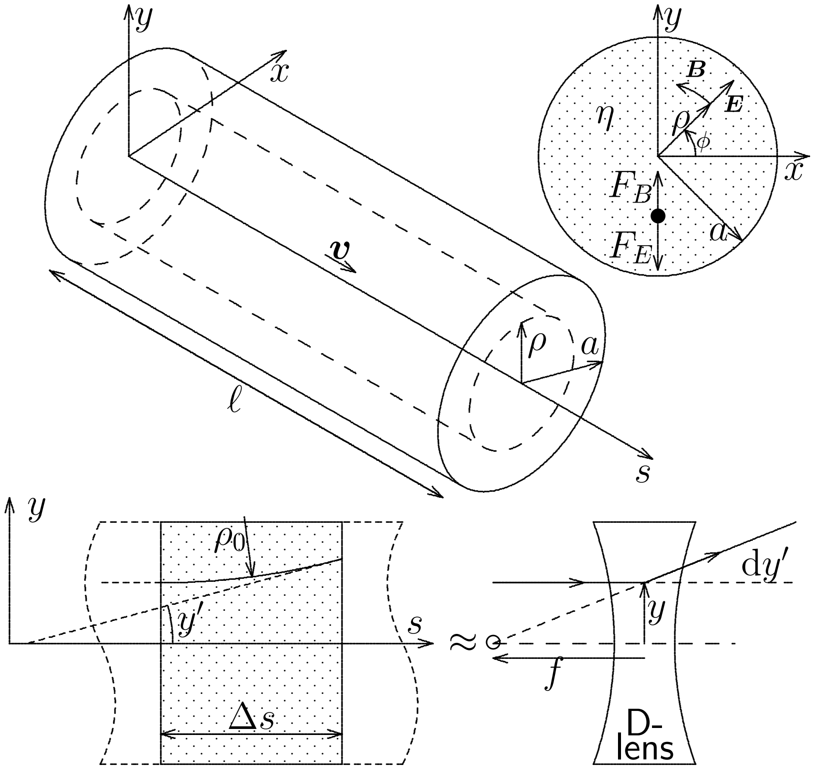
$\vec{F} \propto \vec{\rho}$ gives linear defocusing being $\propto 1/\gamma^2$ and vanishes as $\beta \to 1$.



Space-charge defocusing

Uniform space-charge force on particle is linear, radial, repulsive and defocuses beam in x- and y-plane, changing tunes Q_x, $/Q_y$, (taking y):

$$\vec{F} = F_E + F_B = e\left(\vec{E} + [\vec{v} \times \vec{B}]\right)$$
$$= \frac{e\eta}{2\epsilon_0}(1-\beta^2)\vec{\rho} = \frac{eI}{2\pi\epsilon_0 c\beta\gamma^2}\frac{\vec{\rho}}{a^2}.$$



Force deflects by angle $\Delta y' = \propto y$

$$F_y \approx m_0\gamma \mathrm{d}^2 y/\mathrm{d}t^2 = m_0 c^2\beta^2 c^2 \mathrm{d}y'/\mathrm{d}s$$
$$\frac{\mathrm{d}y'}{y} = \mathrm{d}\left(\frac{1}{f}\right) = \frac{eI\mathrm{d}s}{2\pi\epsilon_0 a^2 m_0 c^3\beta^3\gamma^3} = \frac{2r_0 I ds}{ec\beta^3\gamma^3 a^2}$$
$$r_0 = \frac{e^2}{4\pi\epsilon_0 m_0 c^2} = \begin{matrix} 1.54\cdot 10^{-18}\,\mathrm{m\,protons} \\ 2.82\cdot 10^{-15}\,\mathrm{m\,electrons}\end{matrix}$$

$y' = \mathrm{d}y/\mathrm{d}s \approx \dot{y}/(\beta c) \ll 1$, focusing strength $1/f$, classical particle radius r_0. Tune change by element of length Δs, strength $1/f$.

$$\mathrm{d}Q_y = \frac{\beta_y(s)}{4\pi}\mathrm{d}\left(\frac{1}{f}\right) = \frac{-r_0 I}{2\pi c e\beta^3\gamma^3}\frac{\beta_y(s)\mathrm{d}s}{a^2(s)}$$
$$\Delta Q_y = \frac{-r_0 I}{2\pi c e\beta^3\gamma^3}\oint\frac{\beta_y(s)\mathrm{d}s}{a^2(s)} = \frac{-r_0 I R}{c e\beta^3\gamma^3\mathcal{E}_y}$$

using invariant emittance $\mathcal{E}_y \approx a^2/\beta_y$. Tune shift by local space-charge depends on $\mathcal{E}_y$, not on β_y and a separately. Small β_y gives small a and strong force but reduced effect.
Approx.: $\mathcal{E}_y \approx a^2/\beta_y$; no change of β_y.

Elliptic beam cross section

Uniform η and elliptic cross section with half-axes a, b give fields and forces inside (L. Teng)

$$\vec{E} = [E_x,\, E_y] \;=\; \frac{I}{\pi\epsilon_0(a+b)\beta c}\left[\frac{x}{a}, \frac{y}{b}\right]$$

$$\vec{B} = [B_x,\, B_y] \;=\; \frac{\mu_0 I}{\pi(a+b)}\left[-\frac{y}{b}, \frac{x}{a}\right]$$

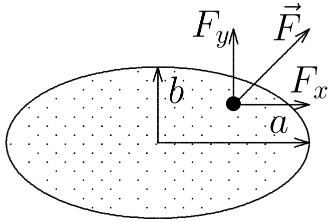
which satisfies $\operatorname{div}\vec{E} = \eta/\epsilon_0$, $\operatorname{curl}\vec{B} = \mu_0\vec{J}$.

$$\vec{F} = e\left[\vec{E} + [\vec{v}\times\vec{B}]\right] = \frac{I\,[(x/a),\,(y/b)]\,.}{\pi\epsilon_0\beta c\gamma^2(a+b)}$$

This force is $F_x \propto x$, $F_y \propto y$ and gives linear defocusing in the two directions.

$$\Delta Q_x = \frac{-r_0 I}{\pi e c\beta^3\gamma^3\mathcal{E}_x}\oint\frac{a}{a+b}\mathrm{d}s$$

$$\Delta Q_y = \frac{-r_0 I}{\pi e c\beta^3\gamma^3\mathcal{E}_y}\oint\frac{b}{a+b}\mathrm{d}s$$



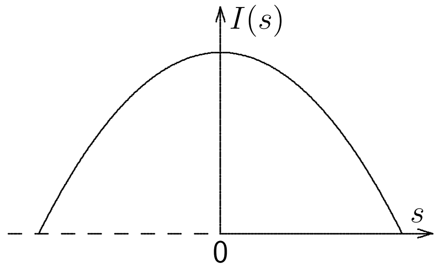
Since a/b depends on s the local tune shift contribution depends also weakly on s.

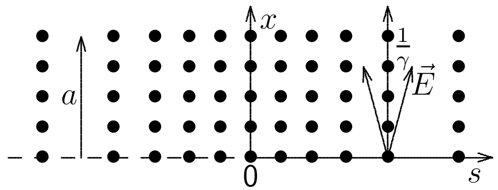
Bunched beams

Current $I(s)$ depends on longitudinal distance s from bunch center. Relativistic field has small opening angle $\approx 1/\gamma$ and depends on local $I(s)$ if this changes little over $\Delta s = a/\gamma$

$$\Delta Q_y \approx -\frac{r_0 R I(s)}{ec\mathcal{E}_y\beta^3\gamma^3}\,, \quad \Delta Q_x \approx -\frac{r_0 R I(s)}{ec\mathcal{E}_x\beta^3\gamma^3},$$

Tune shift depends on particle position s in bunch giving to a tune spread, and, through synchrotron oscillations, to a tune modulation.





Non-uniform distribution

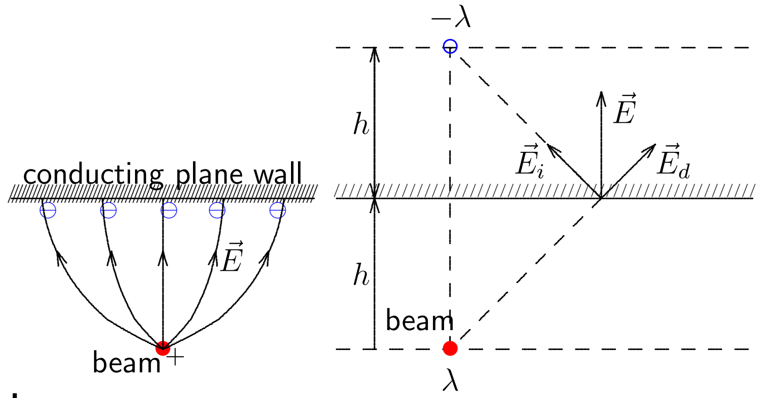
General charge distribution is not uniform, has radial dependence $\eta(\rho)$ giving non-linear force, making tune shift depend on betatron oscillation amplitude and leading to a tune spread.

Indirect space-charge effect — influence of the chamber wall

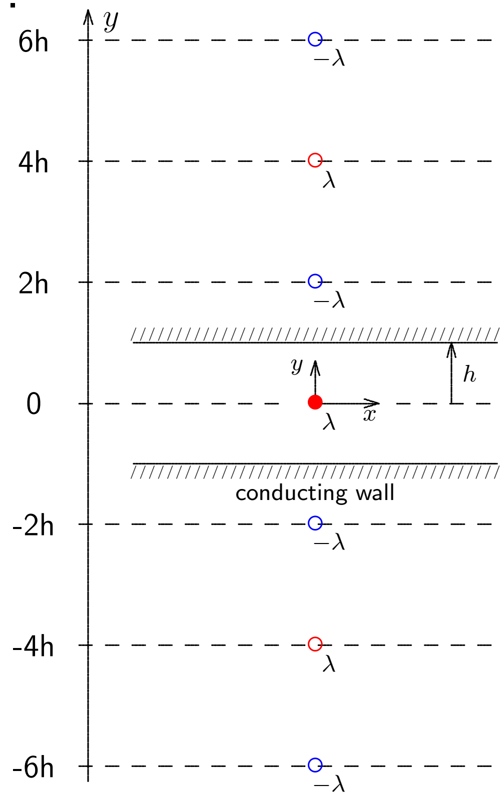
Conducting boundary imposes $E_{\parallel} = 0$ with only $E_{\perp}$. To calculate field we introduce image charge $-\lambda$ at distance h behind wall which cancels $E_{\parallel}$ on surface. Have fields:

direct $\vec{E}_d$, image $\vec{E}_i$, surface $\vec{E}_{d\parallel} = -\vec{E}_{i\parallel}$, $\vec{E}_{\parallel} = 0$

inside: $\vec{E} = \vec{E}_d + \vec{E}_i$, div$\vec{E}_d = \eta/\epsilon_0$, div$\vec{E}_i = 0$



Conducting plates at $\pm h$. To get there $E_{\parallel} = 0$, need image charges of beam and of images. Field close to beam, first order in x, y (quadrupole field) of n-th image pair at $\pm 2nh$ and sum over n



$$E_{iny} = \frac{(-1)^n \lambda}{2\pi\epsilon_0}\left(\frac{1}{2nh+y} - \frac{1}{2nh-y}\right) \approx -\frac{\lambda y}{4\pi\epsilon_0 h^2}\frac{(-1)^n}{n^2}$$

$$E_{iy} = \sum_1^{\infty} E_{iny} = \frac{\lambda y}{4\pi\epsilon_0 h^2}\frac{\pi^2}{12}, \quad \text{div}\vec{E}_i = 0 \to E_{i_x} = -\frac{\lambda x}{4\pi\epsilon_0 h^2}\frac{\pi^2}{12},$$

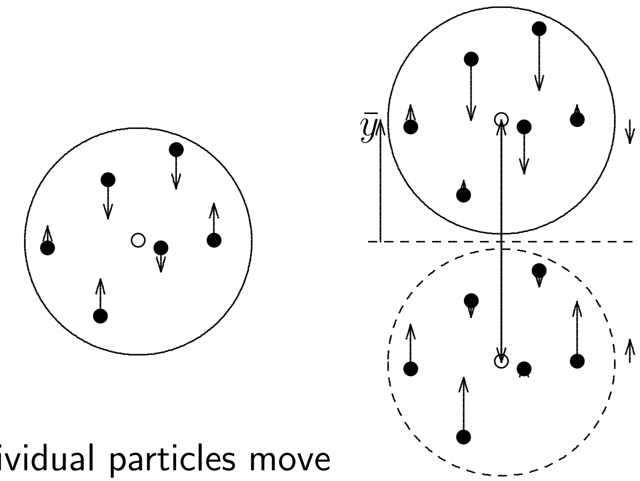
$$F_x = \frac{2e\lambda x}{2\pi\epsilon_0}\left(\frac{1}{2a^2\gamma^2} - \frac{\pi^2}{48h^2}\right), \quad F_y = \frac{2e\lambda y}{2\pi\epsilon_0}\left(\frac{1}{2a^2\gamma^2} + \frac{\pi^2}{48h^2}\right)$$

$$\Delta Q_{x/y} = -\frac{2r_0 I R\langle\beta_{x/y}\rangle}{ec\beta^3\gamma}\left(\frac{1}{2a^2\gamma^2} \mp \frac{\pi^2}{48h^2}\right), \quad \text{with } I = \lambda\beta c.$$

B field not affected, no relativistic E/B-force compensation.

Incoherent and coherent motion

Direct space-charge effect



Individual particles move center-of-mass not: **incoherent** motion

Center of mass moves **coherent** motion

For incoherent motion particles have space-charge tune shift

$$\Delta Q_{\text{inc.}} = -\frac{r_0 I R \beta_y}{c e a^2 \beta^3 \gamma^3}$$

In coherent motion space-charge force is intern, moves with beam, no effect on center-of-mass motion $\Delta Q_{\text{coh.}} = 0$

Indirect space-charge effect

Space-charge field with a conducting wall at distance h was obtained by image line charge at h behind wall. A coherent beam motion by $\bar{y}$ moves first images to $\pm 2h - \bar{y}$ with a field at the beam

$$E_{c1y} = \frac{-\lambda}{2\pi\epsilon_0}\left(\frac{1}{2h+2\bar{y}} - \frac{1}{2h-2\bar{y}}\right).$$

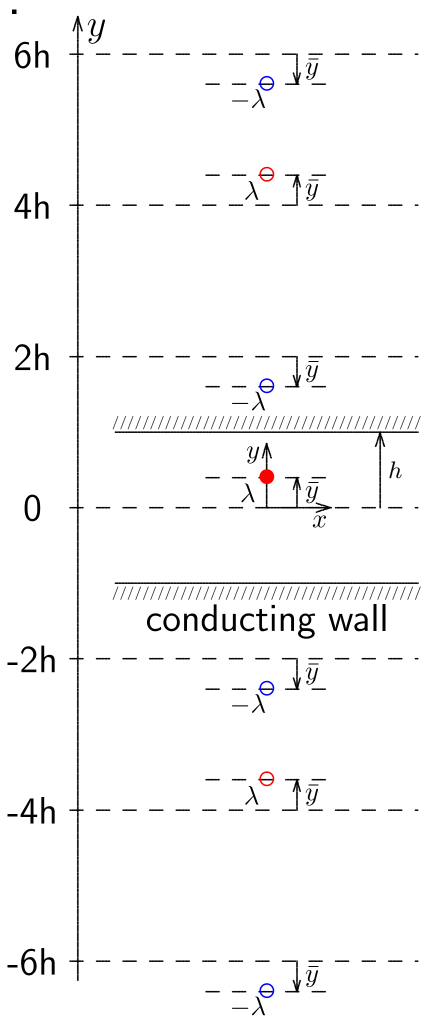
Equidistant 2nd images cancel, general

$$E_{cny} = -\frac{(-1)^n \lambda \bar{y}}{4\pi\epsilon_0 h^2}\left(\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right)$$

$$E_{cy} = \sum_1^\infty E_{cny} = \frac{\lambda \bar{y}}{4\pi\epsilon_0 h^2}\left(\frac{\pi^2}{12} + \frac{\pi^2}{6}\right)$$

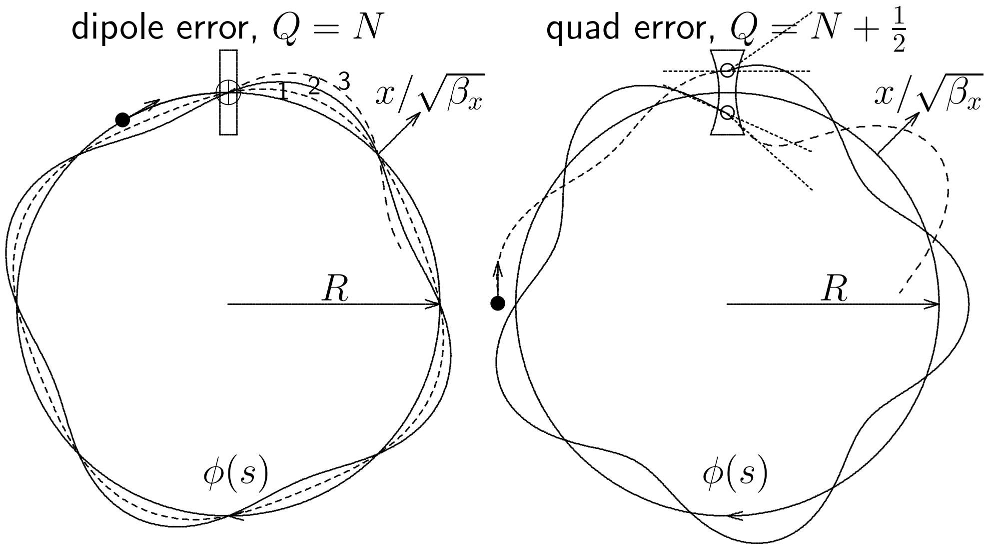
$$Q_{ycoh.} = Q_0 - \frac{\pi^2}{16}\frac{2 r_0 I R \langle\beta_y\rangle}{e c \beta^3 \gamma h^2}$$

$$Q_{ycoh.} - Q_{yinc.} = \frac{2 r_0 I R \langle\beta_y\rangle}{e c \beta^3 \gamma}\left(\frac{1}{2a^2\gamma^2} - \frac{\pi^2}{24 h^2}\right).$$



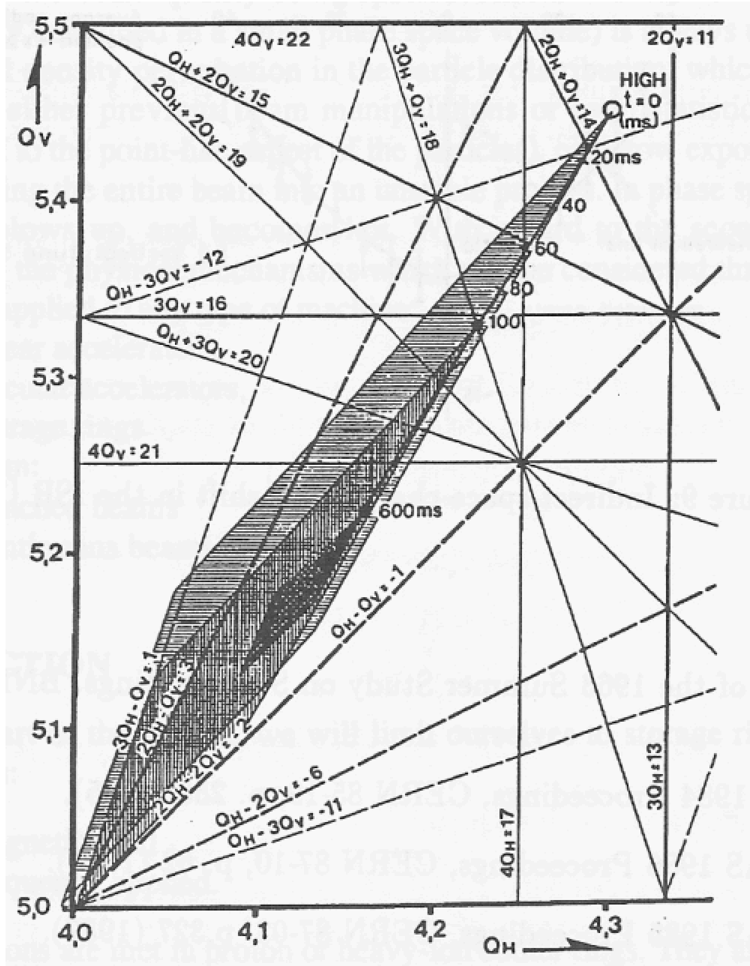
Problems caused by space-charge in rings

In rings space-charge can shift tunes into resonances where $Q = N/M$ is a simple rational fraction. Dipole imperfection deflects particle each turn in phase if Q =integer and for a quadrupole error this happens if Q =half integer. Since space-charge shifts coherent and incoherent tunes differently and produces spread it may be difficult to avoid all resonances.



Related effects

Beam-beam effect: electric and magnetic forces ad.
Ions and electron clouds: don't move, no B-force.

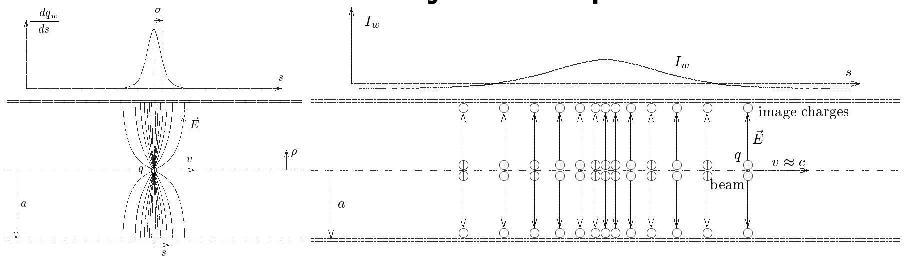


Direct space-charge tune shift and spread at different γ during acceleration (shaded areas), CERN booster, E. Brouzet, K.H. Schindl.

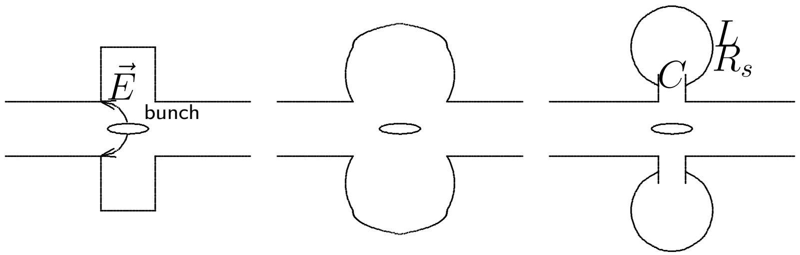
2) IMPEDANCES AND WAKE FUNCTIONS

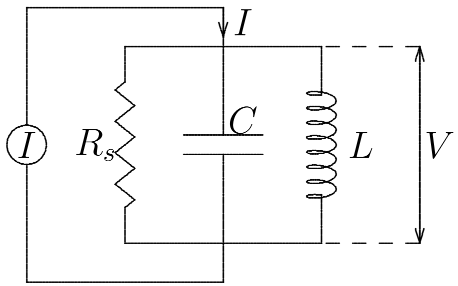
Resonator

For space charge a perfectly conducting wall of uniform cross section and electrostatic methods were used. General cross sections have resonances described by an impedance.



Beam induces wall current $I_w = -(I_b - \langle I_b \rangle)$



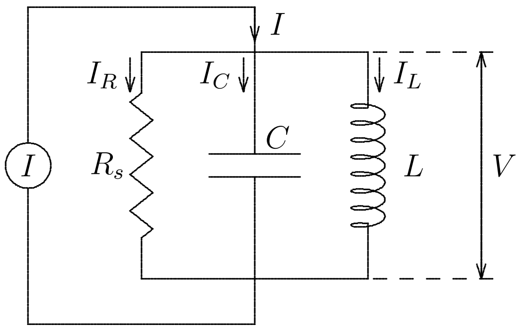


Cavities have narrow band oscillation modes which can drive coupled bunch instabilities. Each resembles an **RCL - circuit** and can, in good approximation, be treated as such. This circuit has a shunt impedance R_s, an inductance L and a capacity C. In a real cavity these parameters cannot easily be separated and we use others which can be measured directly: The **resonance frequency** ω_r, the **quality factor** Q and the **damping rate** α:

$$\omega_r = \frac{1}{\sqrt{LC}}, \quad Q = R_s\sqrt{\frac{C}{L}} = \frac{R_s}{L\omega_r} = R_s C \omega_r$$

$$\alpha = \frac{\omega_r}{2Q}, \quad L = \frac{R_s}{Q\omega_r}, \quad C = \frac{Q}{\omega_r R_s}.$$

Driving this circuit with a current I gives the voltages and currents across the elements



$$\begin{aligned} V_R &= I_R R_s \\ V_C &= \frac{1}{C}\int I_C dt \\ V_L &= L\frac{dI_L}{dt} \end{aligned}$$

$$V_R = V_C = V_L = V, \;\; I_R + I_C + I_L = I$$

$$\dot{I} = \dot{I}_R + \dot{I}_C + \dot{I}_L = \dot{V}/R_s + C\ddot{V} + V/L.$$

Using $L = R_s/(\omega_r Q)$, $C = Q/(\omega_r R_s)$ gives

differential eqn. $\ddot{V} + \dfrac{\omega_r}{Q}\dot{V} + \omega_r^2 V = \dfrac{\omega_r R_s}{Q}\dot{I}$

Homogeneous solution is damped oscillation

$$V(t) = \mathrm{e}^{-\alpha \mathrm{t}}\left(\mathrm{A}\cos\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\mathrm{t}\right)\right.$$
$$\left. + B\sin\left(\omega_r\sqrt{1-\frac{1}{4Q^2}}t\right)\right), \;\; \alpha = \frac{\omega_r}{2Q}$$

Wake/Green – function, pulse response

$I(t) = q\delta t$, charge q gives capacity voltage

$$V(0^+) = \frac{q}{C} = \frac{\omega_r R_s}{Q}q \text{ using } C = \frac{Q}{\omega_r R_s}$$

Energy stored in C = energy lost by q

$$U = \frac{q^2}{2C} = \frac{\omega_r R_s}{2Q}q^2 = \frac{V(0^+)}{2}q = k_{pm}q^2$$

parasitic mode loss factor $k_{pm} = \omega_r R_s/2Q$

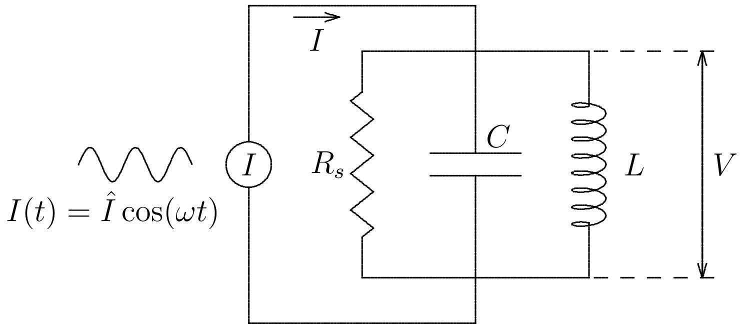
Capacitor discharges first through resistor

$$-\dot{V}(0^+) = \frac{\dot{q}}{C} = \frac{I_R}{C} = \frac{V(0^+)}{CR_s} = -\frac{2\omega_r k_{pm}}{Q}q.$$

$$V(0^+),\;\; \dot{V}(0^+) \rightarrow A = 2qk_{pm},\;\; B = \frac{-A}{\sqrt{4Q^2-1}}$$

$$V(t) = 2qk_{pm}\mathrm{e}^{-\alpha \mathrm{t}}\left(\cos\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\mathrm{t}\right)\right.$$
$$\left. - \frac{\sin\left(\omega_r\sqrt{1-\frac{1}{4Q^2}}t\right)}{\sqrt{4Q^2-1}}\right) \approx 2qk_{pm}\mathrm{e}^{-\alpha \mathrm{t}}\cos(\omega_\mathrm{r}\mathrm{t}).$$

Impedance



A **harmonic** excitation of circuit with current $I = \hat{I}\cos(\omega t)$ gives differential equation

$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q}\dot{I} = -\frac{\omega_r R_s}{Q}\hat{I}\omega\sin(\omega t).$$

Homogeneous solution damps leaving particular one $V(t) = A\cos(\omega t) + B\sin(\omega t)$. Put into differential equation, separating cosine and sine

$$\begin{aligned} -(\omega^2 - \omega_r^2)A + \frac{\omega_r\omega}{Q}B &= 0 \\ (\omega^2 - \omega_r^2)B + \frac{\omega_r\omega}{Q}A &= \frac{\omega_r\omega R_s}{Q}\hat{I}. \end{aligned}$$

Voltage induced by current $\hat{I}\cos(\omega t)$ is

$$V(t) = \hat{I}R_s \frac{\cos(\omega t) + Q\frac{\omega^2-\omega_r^2}{\omega_r\omega}\sin(\omega t)}{1 + Q^2\left(\frac{\omega^2-\omega_r^2}{\omega_r\omega}\right)^2}$$

Cosine term is **in phase** with exciting current, absorbs energy, **resistive**. Sine term is **out of phase**, does not absorb energy, **reactive**. Voltage/current ratio is **impedance** as **function of frequency** ω

$$\begin{aligned} Z_r(\omega) &= R_s \frac{1}{1 + Q^2\left(\frac{\omega_r^2-\omega^2}{\omega_r\omega}\right)^2} \\ Z_i(\omega) &= -R_s \frac{Q\frac{\omega^2-\omega_r^2}{\omega_r\omega}}{1 + Q^2\left(\frac{\omega^2-\omega_r^2}{\omega_r\omega}\right)^2}. \end{aligned}$$

Resistive part $Z_r(\omega) \geq 0$, reactive part $Z_i(\omega)$ positive below, negative above ω_r.

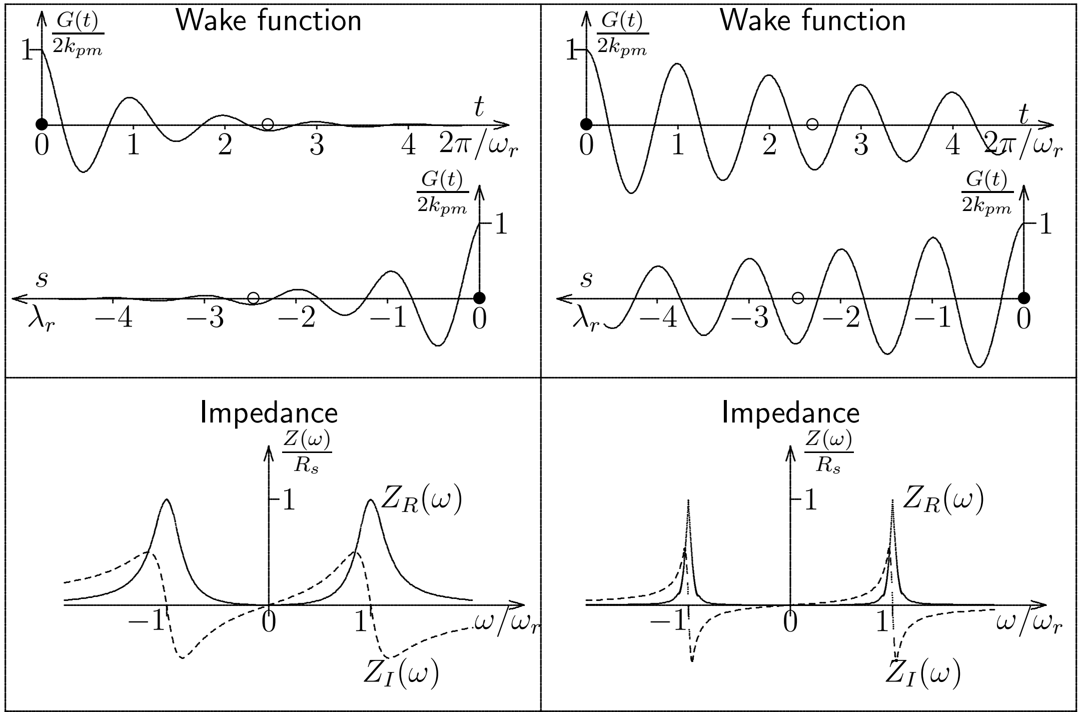
$$\hat{I}\cos(\omega t) \rightarrow V = \hat{I}[Z_r\cos(\omega t) - Z_i\sin(\omega t)]$$
$$\hat{I}\sin(\omega t) \rightarrow V = \hat{I}[Z_r\sin(\omega t) + Z_i\cos(\omega t)]$$

Complex notation

Excite: $I(t) = \hat{I}\cos(\omega t) = \hat{I}\dfrac{e^{j\omega t} + e^{-j\omega t}}{2}$

with $0 \leq \omega \leq \infty$

$I(t) = \hat{I}e^{j\omega t}/2$ with $-\infty \leq \omega \leq \infty$



$Q = 3.0$ $Q = 15.0$

$$Z(\omega) = R_s \frac{1 - jQ\frac{\omega^2-\omega_r^2}{\omega\omega_r}}{1 + Q^2\left(\frac{\omega^2-\omega_r^2}{\omega\omega_r}\right)^2} = Z_r + jZ_i$$

$$\approx R_s \frac{1 - j2Q\Delta\omega/\omega_r}{1 + 4Q^2\left(\Delta\omega/\omega_r\right)^2} \quad \text{for } Q \gg 1$$

$\omega \approx \omega_r, |\omega - \omega_r|/\omega_r = |\Delta\omega|/\omega_r \ll 1.$

Resonator impedance properties:

$\text{at}\,\omega = \omega_r \;\rightarrow\; Z_r(\omega_r)\ \text{max.},\; Z_i(\omega_r) = 0$

$0 < \omega < \omega_r \;\rightarrow\; Z_i(\omega) > 0\ \text{(inductive)}$

$\omega > \omega_r \;\rightarrow\; Z_i(\omega) < 0\ \text{(capacitive)}$

General impedance or wake properties

$Z_r(\omega) = Z_r(-\omega)\ ,\ \ Z_i(\omega) = -Z_i(-\omega)$

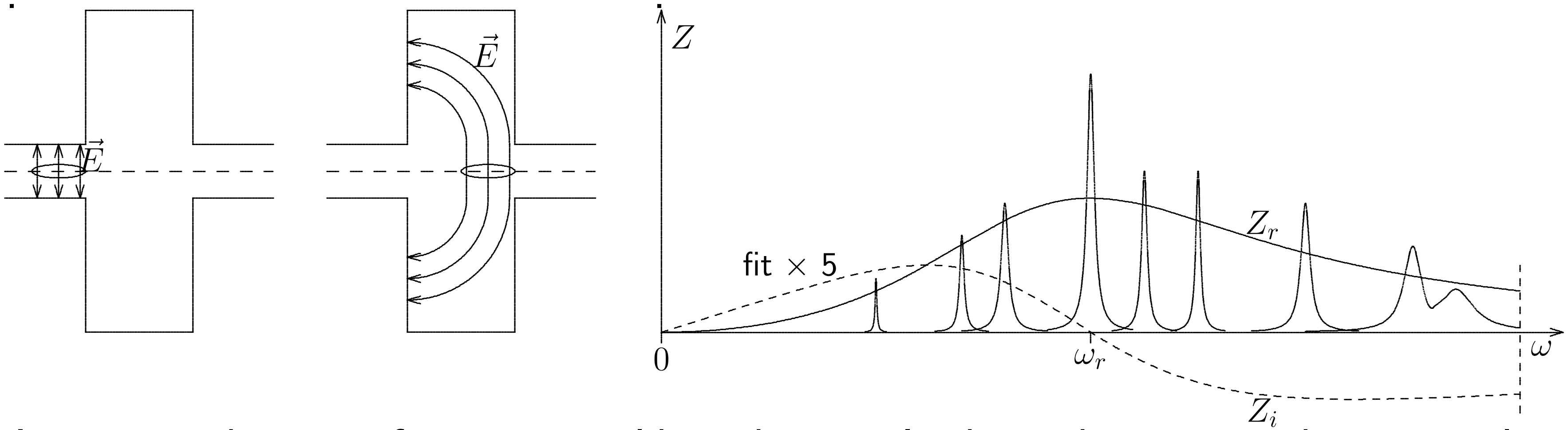
$Z(\omega) = \int_{-\infty}^{\infty} G(t)e^{-j\omega t}dt$

$Z(\omega) \propto$ Fourier transform of $G(t)$

$\text{for}\, t < 0 \;\rightarrow\; G(t) = 0,$

no fields before particle arrives, $\beta \approx 1$.

Typical impedance of a ring



Aperture changes form cavity-like objects with $\omega_r,\ R_s$ and Q and impedance $Z(\omega)$ developed for $\omega < \omega_r$, where it is inductive

$$Z(\omega) = R_s \frac{1 - jQ\frac{\omega^2-\omega_r^2}{\omega\omega_r}}{1+\left(Q\frac{\omega^2-\omega_r^2}{\omega\omega_r}\right)^2} \approx j\frac{R_s\omega}{Q\omega_r} + ...$$

Sum impedance at $\omega \ll \omega_{rk}$ divided by mode number $n = \omega/\omega_0$ is with inductance L

$$\left|\frac{Z}{n}\right|_0 = \sum_k \frac{R_{sk}\omega_0}{Q_k\omega_{rk}} = L\omega_0 = L\frac{\beta c}{R}.$$

It depends on impedance per length, $\approx$ 15 Ω in older, 1 Ω in newer rings. The shunt impedances R_{sk} increase with ω up to cut-off frequency where wave propagation starts and become wider and smaller. A broad band resonator fit helps to characterize impedance giving $Z_r,\ Z_i,\ G(t)$ useful for single traversal effects. However, for multi-traversal instabilities narrow resonances at ω_{rk} must be used.

3) LONGITUDINAL INSTABILITIES

Longitudinal dynamics

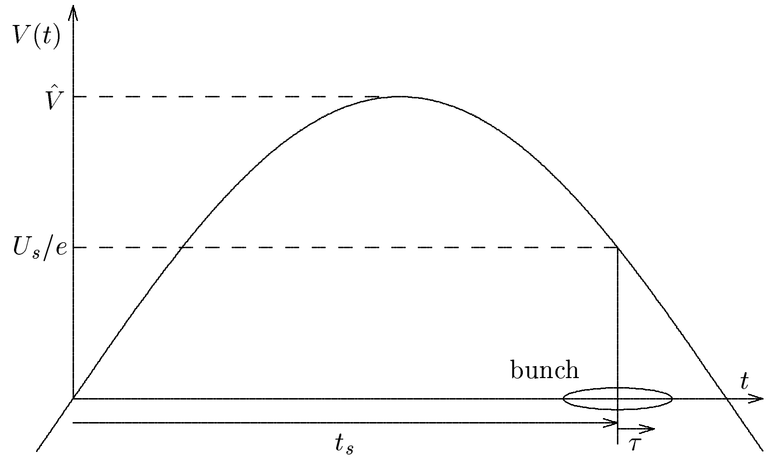
A particle with momentum deviation Δp has different orbit length L, revolution time T_0 and frequency ω_0

$$\frac{\Delta L}{L} = \alpha_c \frac{\Delta p}{p} = \frac{\alpha_c}{\beta^2}\frac{\Delta E}{E}$$
$$\frac{\Delta T}{T} = -\frac{\Delta \omega_0}{\omega_0} = \left(\alpha_c - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p} = \eta_c \frac{\Delta p}{p}$$

with momentum compaction $\alpha_c = 1/\gamma_T^2$, slip factor η_c. At transition energy $m_0 c^2 \gamma_T$ the ω_0-dependence on Δp changes sign

$$E > E_T \to \frac{1}{\gamma^2} < \alpha_c \to \eta_c > 1,\ \frac{\Delta\omega_0}{\Delta E} < 0$$
$$E < E_T \to \frac{1}{\gamma^2} > \alpha_c \to \eta_c < 1,\ \frac{\Delta\omega_0}{\Delta E} > 0\,.$$

For $\gamma \gg 1 \to \Delta p/p \approx \Delta E/E = \epsilon$, $\eta_c \approx \alpha_c$.



RF-cavity of voltage $\hat{V}$, frequency $\omega_{\text{RF}} = h\omega_0$, SR energy loss U the energy gain or loss of a particle in one turn $\delta\epsilon = \delta E/E$ is

$$\delta E = e\hat{V}\sin(h\omega_0(t_s + \tau)) - U$$

t_s= synchronous arrival time at the cavity, τ= deviation from it, synchronous phase $\phi_s = h\omega_0 t_s$. For $h\omega_0\tau \ll 1$ we develop

$$\delta E = e\hat{V}\sin(\phi_s) + h\omega_0 e\hat{V}\cos\phi_s \tau - U.$$

For $\delta E/E \ll 1$ use smooth approximation $\dot{E} \approx \delta E/T_0$, $\dot{\tau} = \Delta T/T_0 = \eta_c \Delta E/E$

$$\dot{E} = \frac{\omega_0 e\hat{V}\sin\phi_s}{2\pi} + \frac{\omega_0^2 he\hat{V}\cos\phi_s}{2\pi}\tau - \frac{\omega_0}{2\pi}U.$$

Use $T_0 = 2\pi/\omega_0$, relative energy $\epsilon = \Delta E/E$

$$\dot{\epsilon} = \frac{\omega_0 e\hat{V}\sin\phi_s}{2\pi E} + \frac{\omega_0^2 he\hat{V}\cos\phi_s}{2\pi E}\tau - \frac{\omega_0}{2\pi}\frac{U}{E}.$$

Energy loss U may depend on E

$$U(\epsilon,\tau) \approx U_0 + \frac{\partial U}{\partial E}\Delta E$$

giving for the derivative of the energy loss

$$\begin{aligned}\dot{\epsilon} &= \frac{\omega_0^2 he\hat{V}\cos\phi_s}{2\pi E}\tau - \frac{\omega_0}{2\pi}\frac{\partial U}{\partial E}\epsilon \\ \dot{\tau} &= \eta_c\epsilon\end{aligned}$$

where we used that for synchronous particle $\epsilon = 0\,,\ \tau = 0$ we have $U_0 = e\hat{V}\sin\phi_s$

Combining these into a second order equation

$$\ddot{\tau} + \frac{\omega_0}{2\pi}\frac{\partial U}{\partial E}\dot{\tau} + \omega_{s0}^2\tau = 0,$$

$$\omega_{s0}^2 = \frac{-\omega_0^2 h\eta_c e\hat{V}\cos\phi_s}{2\pi E}\ ,\ \alpha_s = \frac{1}{2}\frac{\omega_0}{2\pi}\frac{\partial U}{\partial E}$$

$$\omega_{s1}^2 = \omega_{s0}^2 - \alpha_s^2 \approx \omega_{s0}^2$$

$$\ddot{\tau} + 2\alpha_s\dot{\tau} + \omega_{s0}^2\tau = 0$$

$$\tau = \hat{\tau}\mathrm{e}^{-\alpha_s t}\cos(\omega_{s1}t)\ ,\ \epsilon = \hat{\epsilon}\mathrm{e}^{-\alpha_s t}\sin(\omega_{s1}t)$$

From $\dot{\tau} = \eta_c\epsilon$ we get $\hat{\epsilon} = \omega_{s0}\hat{\tau}/\eta_c$.
To get real ω_{s0} we need $\cos\phi_s \le 0$ above transition where $\eta_c > 0$ and vice versa.
To get a stable (decaying) solution we need an energy loss which increases with E

$$\alpha_s = \frac{\omega_0}{4\pi}\frac{\partial U}{\partial E} = \frac{\omega_0}{4\pi E}\frac{\partial U}{\partial \epsilon} > 0.$$

Induced voltage and energy loss by a stationary bunch

Circulating symmetric bunch (N_b particles) has current

$$I(t) = \sum_{-\infty}^{\infty} I(t - kT_0)$$

$$I(t) = I_0 + 2\sum_{1}^{\infty} I_p \cos(p\omega_0 t),\ \ I_p = \int_0^{T_0} I(t)\cos(p\omega_0 t)dt$$

In impedance $Z(\omega)$ it induces voltage

$$V(t) = 2\sum I_p\left[Z_r(p\omega_0)\cos(p\omega_0 t - Z_i(p\omega_0)\sin(p\omega_0 t)\right]$$

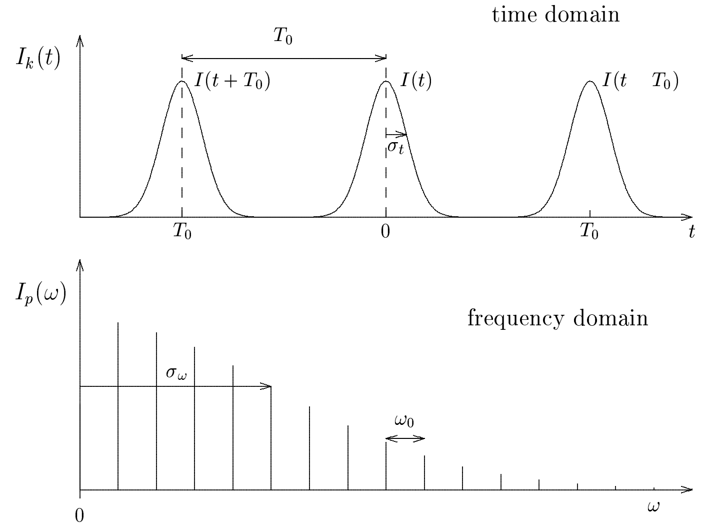
Energy lost per particles and turn $U = \int_0^{T_0} I(t)V(t)dt/N_b$

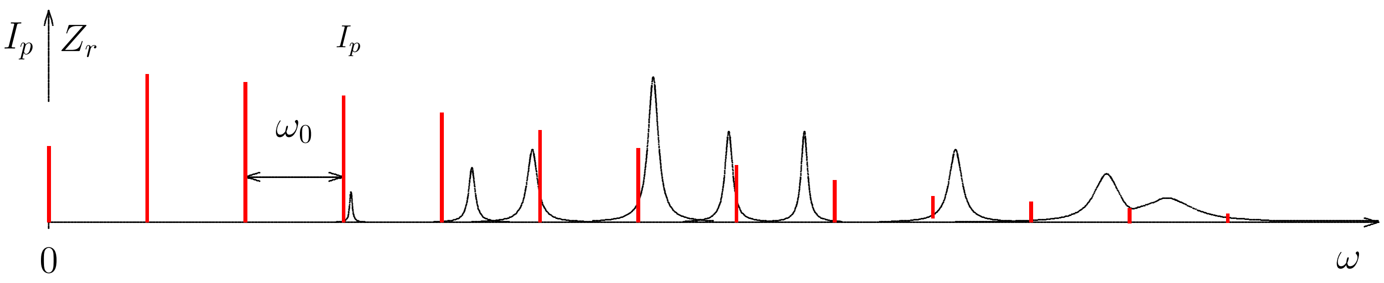
$$U = \frac{2T_0}{N_b}\sum_{1}^{\infty} I_p^2 Z_r(p\omega_0) = \frac{2e}{I_0}\sum_{1}^{\infty} I_p^2 Z_r(p\omega_0)$$

using $\int_0^{T_0} \cos(p'\omega_0 t)\sin(p\omega_0 t)dt = 0,\ \ I_0 = eN_b/T_0$

$$\int_0^{T_0} \cos(p'\omega_0 t)\cos(p\omega_0 t)dt = \begin{matrix} T_0/2 & \text{for } p' = p \\ 0 & \text{for } p' \neq p \end{matrix}$$

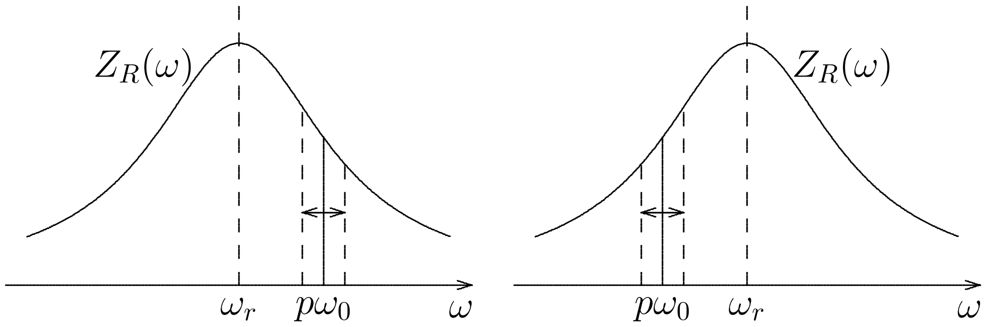
circulating single bunch





Robinson instability

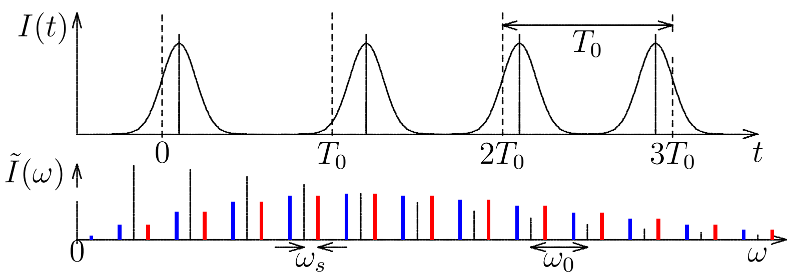
Qualitative treatment

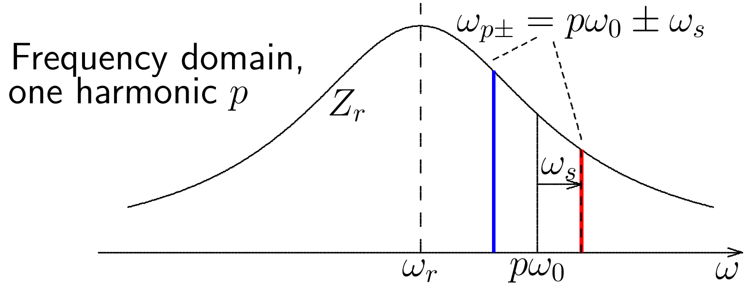


Important longitudinal instability of a bunch interacting with an narrow impedance, called **Robinson** instability. In a qualitative approach we take single bunch and a narrow-band cavity of resonance frequency ω_r and impedance $Z(\omega)$ taking only its resistive part Z_r. The revolution frequency ω_0 depends on energy deviation ΔE

$$\frac{\Delta\omega_0}{\omega_0} = -\eta_c \frac{\Delta p}{p}.$$

While the bunch is executing a coherent dipole mode oscillation $\epsilon(t) = \hat{\epsilon}\cos(\omega_s t)$ its energy and revolution frequency are modulated. **Above transition** ω_0 is **small** when the **energy is high** and ω_0 is **large** when the **energy is small**. If the cavity is tuned to a resonant frequency slightly smaller than the RF-frequency $\omega_r < p\omega_0$ the bunch sees a higher impedance and **loses more energy** when it has an **energy excess** and it **loses less energy** when it has a **lack of energy**. This leads to a **damping** of the oscillation. If $\omega_r > p\omega_0$ this is reversed and leads to an **instability**. Below transition energy the dependence of the revolution frequency is reversed which changes the stability criterion.





$$\epsilon = \hat{\epsilon} e^{-\alpha_s t} \sin(\omega_s t), \text{ damping if } \alpha_s > 0$$

$$\alpha_s = \frac{\omega_{s0} p I_p^2 (Z_r(\omega_{p+}) - Z_r(\omega_{p-}))}{2 I_0 h \hat{V} \cos\phi_s}$$

$\gamma > \gamma_T$, $\cos\phi_s < 0$, stable $Z_r(\omega_{p-}) > Z_r(\omega_{p+})$

Damping rate $\propto Z_r$ difference at side-bands.

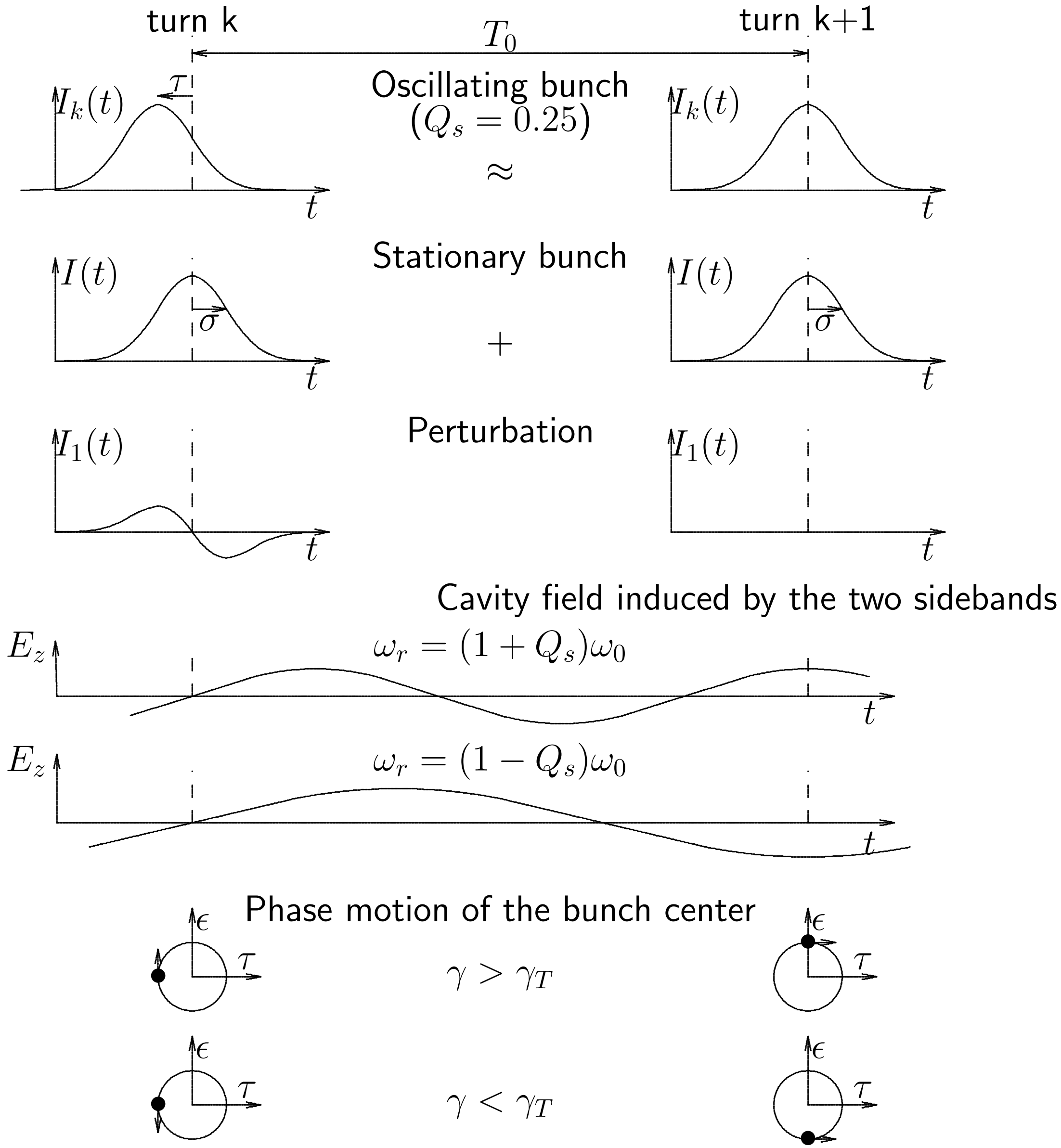
RF-cavity: $p = h$, $I_p \approx I_0$.

$$\frac{\alpha_s}{\omega_{s0}} \approx \frac{I_0 (Z_r(\omega_{p+}) - Z_r(\omega_{p-}))}{2 \hat{V} \cos\phi_s}$$

general:

$$\frac{\alpha_s}{\omega_{s0}} = \sum_p \frac{p I_p^2 (Z_r(\omega_{p+}) - Z_r(\omega_{p-}))}{2 I_0 h \hat{V} \cos\phi_s}$$

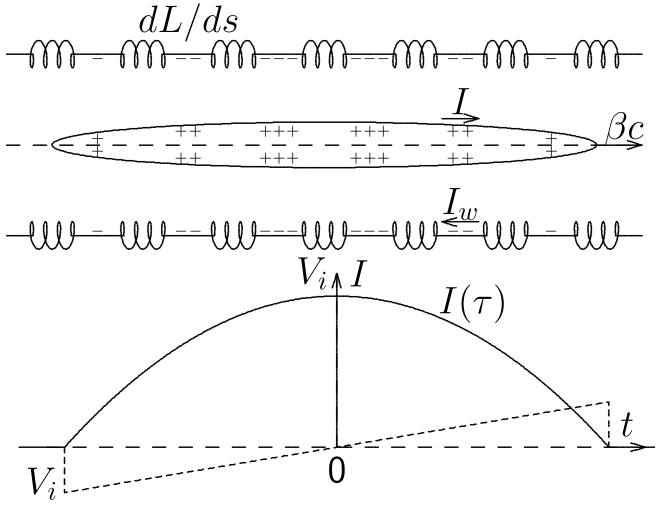
Qualitative understanding



Narrow band $\longrightarrow$ long memory, vice-versa

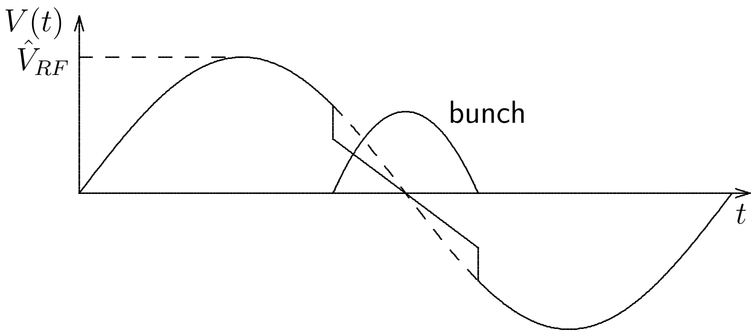
Potential well bunch lengthening

At low frequency wall is inductive with $L\omega_0 = |Z/n|_0$:



$$E_z = -\frac{\mathrm{d}L}{\mathrm{d}z}\frac{\mathrm{d}I_w}{\mathrm{d}t} = \frac{\mathrm{d}L}{\mathrm{d}z}\frac{\mathrm{d}I_b}{\mathrm{d}t}$$

$$V = -\int E_z dz = -L\frac{\mathrm{d}I_b}{\mathrm{d}\tau}$$



We take a parabolic bunch form

$$I_b(\tau) = \hat{I}\left(1-\frac{\tau^2}{\hat{\tau}^2}\right) = \frac{3\pi I_0}{2\omega_0\hat{\tau}}\left(1-\frac{\tau^2}{\hat{\tau}^2}\right)$$

$$\frac{\mathrm{d}I_b}{\mathrm{d}\tau} = -\frac{3\pi I_0\tau}{\omega_0\hat{\tau}^3},\ I_0 = \langle I_b\rangle,$$

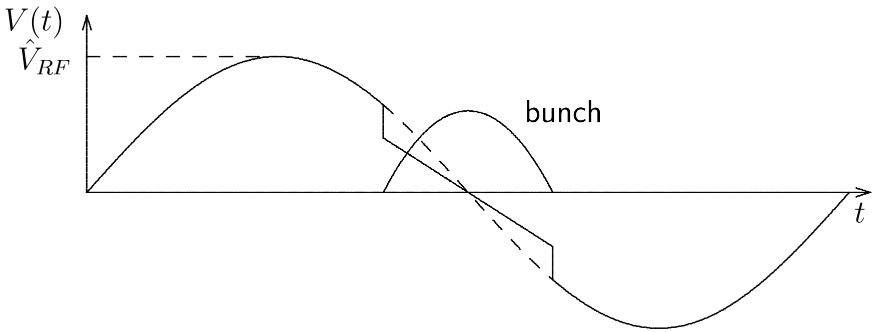
$$V = \hat{V}(\sin\phi_s + h\omega_0\cos\phi_s\tau) + \frac{3\pi I_0 L\tau}{\omega_0\hat{\tau}^3}$$

$$V = \hat{V}\left[\sin\phi_s + \cos\phi_s h\omega_0\left(1+\frac{3\pi|Z/n|_0 I_0}{h\hat{V}\cos\phi_s(\omega_0\hat{\tau})^3}\right)\tau\right]$$

$$\omega_{s0}^2 = -\frac{\omega_0^2 h\eta_c e\hat{V}\cos\phi_s}{2\pi E}$$

$$\omega_s^2 = \omega_{s0}^2\left[1+\frac{3\pi|Z/n|_0 I_0}{h\hat{V}_{RF}\cos\phi_s(\omega_0\hat{\tau})^3}\right]$$

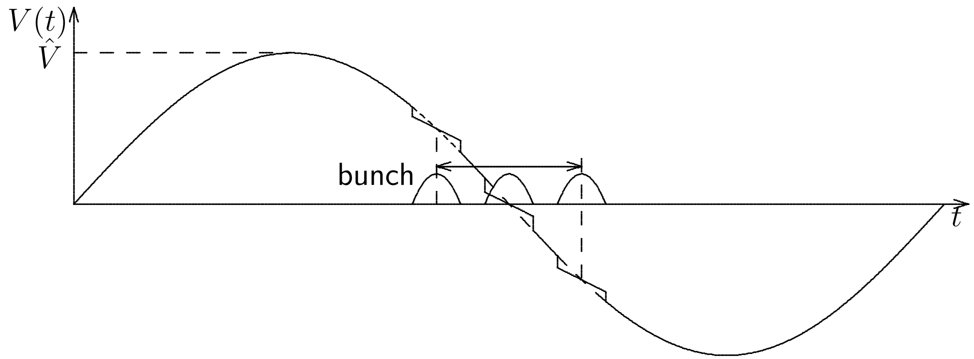
$$\frac{\Delta\omega_s}{\omega_0} = \frac{\omega_s-\omega_{s0}}{\omega_{s0}} \approx \frac{3\pi|Z/n|_0 I_0}{2h\hat{V}_{RF}\cos\phi_s(\omega_0\hat{\tau}_0)^3}$$



$$\frac{\omega_s^2}{\omega_{s0}^2} = 1 + \frac{3\pi |Z/n|_0 I_0}{h\hat{V}_{RF}\cos\phi_s(\omega_0\hat{\tau})^3}$$

$$\frac{\omega_s - \omega_{s0}}{\omega_{s0}} = \frac{\Delta\omega_s}{\omega_s} \approx \frac{3\pi |Z/n|_0 I_0}{2h\hat{V}\cos\phi_s(\omega_0\hat{\tau}_0)^3}$$

Only incoherent frequency ω_s of single particles is changed (reduced $\gamma > \gamma_T$, increased $\gamma < \gamma_T$), not coherent dipole (rigid bunch) frequency ω_{s1}. The two get separated.



Decreasing ω_s reduces longitudinal focusing, increases bunch length $\hat{\tau}$. Relative energy spread $\hat{\epsilon} = \hat{\tau}\omega_s/\eta_c$ is given for electrons by synchrotron radiation, for protons the product (emittance) $\hat{\tau}\hat{\epsilon}$=const.

electron $\frac{\Delta\hat{\tau}}{\hat{\tau}_0} = -\frac{\Delta\omega_s}{\omega_{s0}}$, proton $\frac{\Delta\hat{\tau}}{\hat{\tau}_0} = -\frac{\Delta\omega_s}{2\omega_{s0}}$

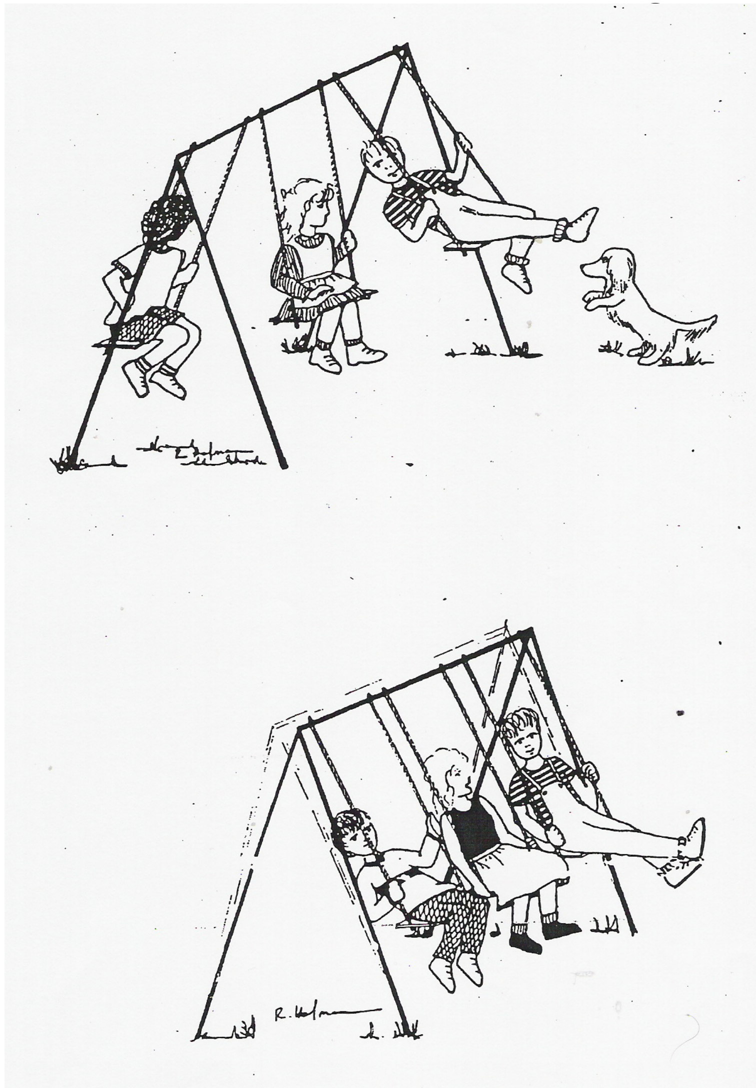
From observed bunch lengthening impedance is estimated.

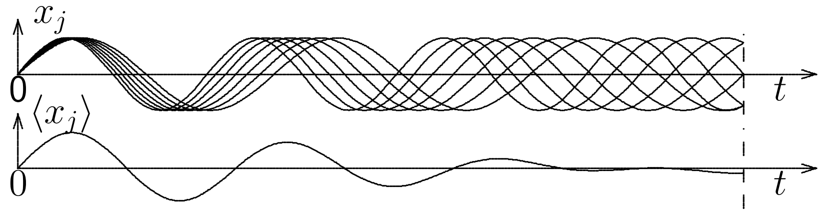
Frequency measurement would be better, but ω_s is invisible and ω_{s1} doe not move, however, quadrupole mode can be used

$$\frac{\omega_{s2} - 2\omega_{s0}}{2\omega_{s0}} = \frac{\Delta\omega_{s2}}{\omega_{s2}} \approx \frac{1}{4}\frac{\Delta\omega_s}{\omega_{s0}}.$$

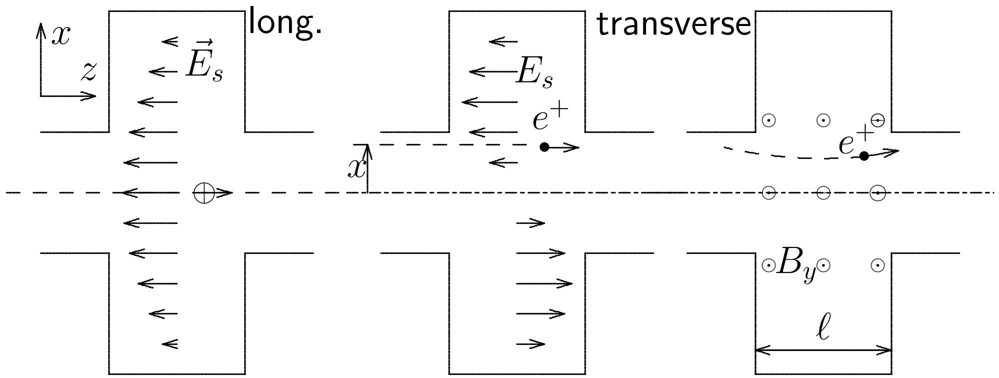
Separation of coherent and incoherent frequencies

The wall inductance, and most reactive impedances, separate coherent and incoherent frequencies. A swing with a non-rigid frame can illustrate this mechanism. A coherent, center-of-mass, motion moves the frame and changes the frequency, this is not the case if oscillate at a different phases, leaving th incoherent frequency unchanged. For space-charge this causes mainly problems with resonances, here a loss of a stabilization mechanism, called Landau damping, is more important. A spread in individual particle frequencies produces phase mixing which reduces the center-of-mass, coherent, amplitude and gives some stabilization. A separation between coherent and incoherent frequencies makes this ineffective.





4) TRANSVERSE INSTABILITIES
Transverse impedance



Field excited by $Ix = D = \hat{D}\cos(\omega t)$

$$\frac{\partial E_z}{\partial x} = -kIx,\ E_z(x) = -kIx^2$$

$$Z_L(x) = -\int E_z dz/I = -E_z\ell/I = k\ell x^2$$

$$\int \dot{\vec{B}}\mathrm{d}\vec{a} = -\oint \vec{E}\mathrm{d}\vec{s},\ \dot{B}_y x\ell = E_z\ell = -k\ell D x$$

$$\dot{B}_y = -k\hat{D}\cos(\omega t),\ B = -k\hat{D}\sin(\omega t)/\omega$$

field B out of phase with $D = Ix$

$$\hat{B}_y = -k\hat{D}/\omega,\ \text{Lorentz force } \hat{F} \approx -ec\hat{B}_y$$

$$Z_T = -\frac{F_x\ell}{e\hat{D}} = \frac{ck\ell}{\omega} = \frac{cZ_L}{x^2\omega} = \frac{c}{2\omega}\frac{\mathrm{d}^2 Z_L}{\mathrm{d}x^2},\ \left[\frac{\Omega}{\mathrm{m}}\right].$$

Used special case to define transverse impedance and its relation to second derivative of the longitudinal impedance of same mode. In General we have the impedances long.: integrated field/current; trans.: integrated defl. field/ dipole moment On resonance, E_z is in, B_y out of phase of I. General deflecting mode, using $x = \hat{x}\mathrm{e}^{j\omega t}$

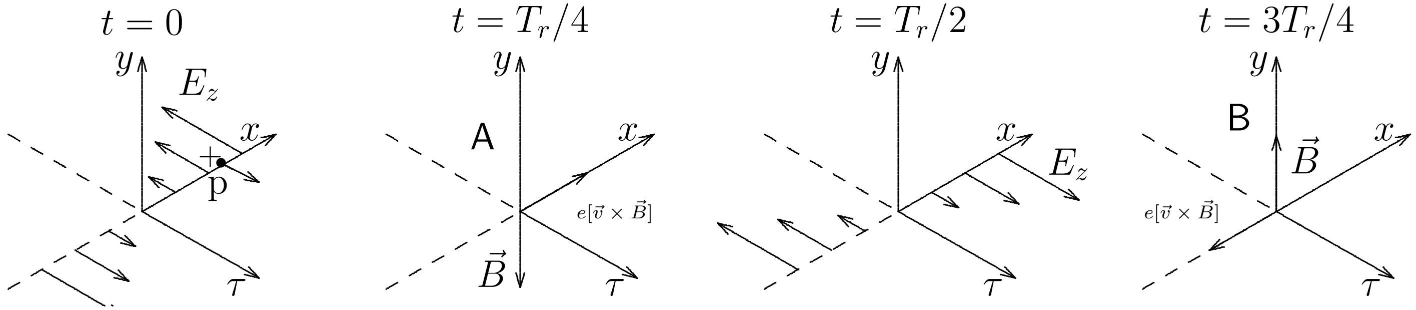
$$Z_T(\omega) = j\frac{\int\left(\vec{E}(\omega) + [\vec{v}\times\vec{B}(\omega)]\right)_T \mathrm{d}s}{Ix(\omega)}$$

Relation Z_L to Z_T of different modes:
In ring of global and vacuum chamber radii R and b the impedances, averaged for **different** modes, have semi-empirical ratio

$$Z_T(\omega) \approx \frac{2R}{b^2}\frac{Z_L(\omega)}{\omega/\omega_0}$$

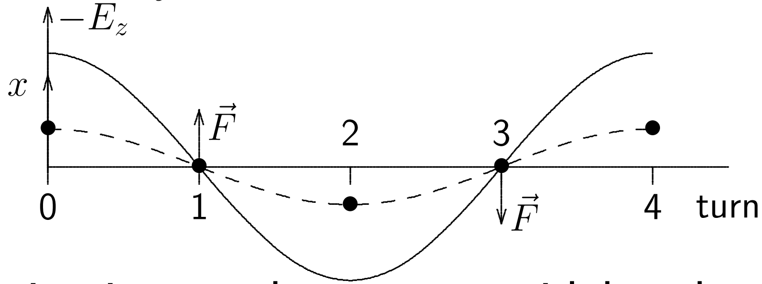
From area available for the wall current we expect $Z_L \propto 1/b$, therefore $Z_T \propto 1/b^3$.

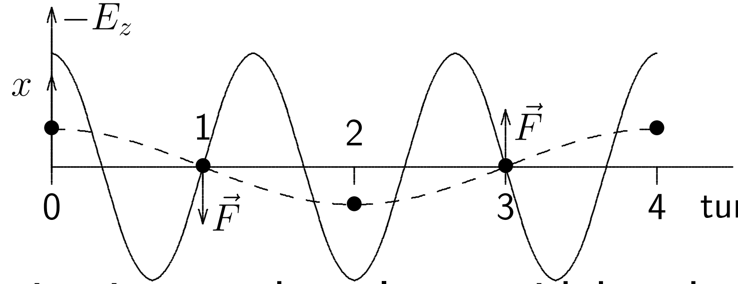
Transverse instability of a single, rigid bunch



A bunch p traverses a cavity with off-set x, excites a field $-E_z$ which converts after $T_r/4$ into field $-B_y$, then into E_z and after into B_y.

The bunch oscillates with tune Q having a fractional part $q = 1/4$ seen as sidebands at $\omega_0(\text{integer} \pm q)$ by a stationary observer.





A) Cavity is tuned to upper sideband. Next turn bunch traverses in situation 'A', $t = T_r/4)$ with velocity in $-x$-direction and gets by B_y force in $+x$-direction which damps oscillation.

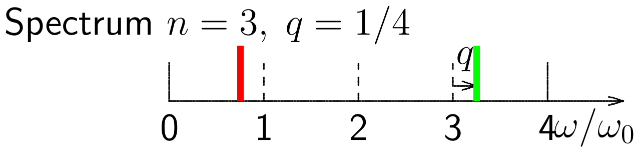
B) Cavity is tuned to lower sideband, bunch traverses next in situation 'B', $t = T_r 3/4 = T_r(1-1/4)$ with negative velocity and force in same direction, increases velocity, instability.

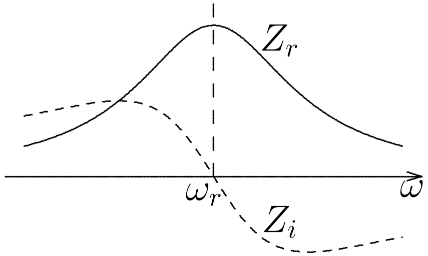
$$\text{damping rate } a = \frac{e\omega_0\beta_x}{4\pi m_0 c^2 \gamma I_0} \sum_{\omega>0} \left(I_{p+}^2 Z_{Tr}(\omega_p^+) - I_{p-}^2 Z_{Tr}(\omega_p^-) \right), \ \omega_{p\pm} = \omega_0(p \pm q).$$

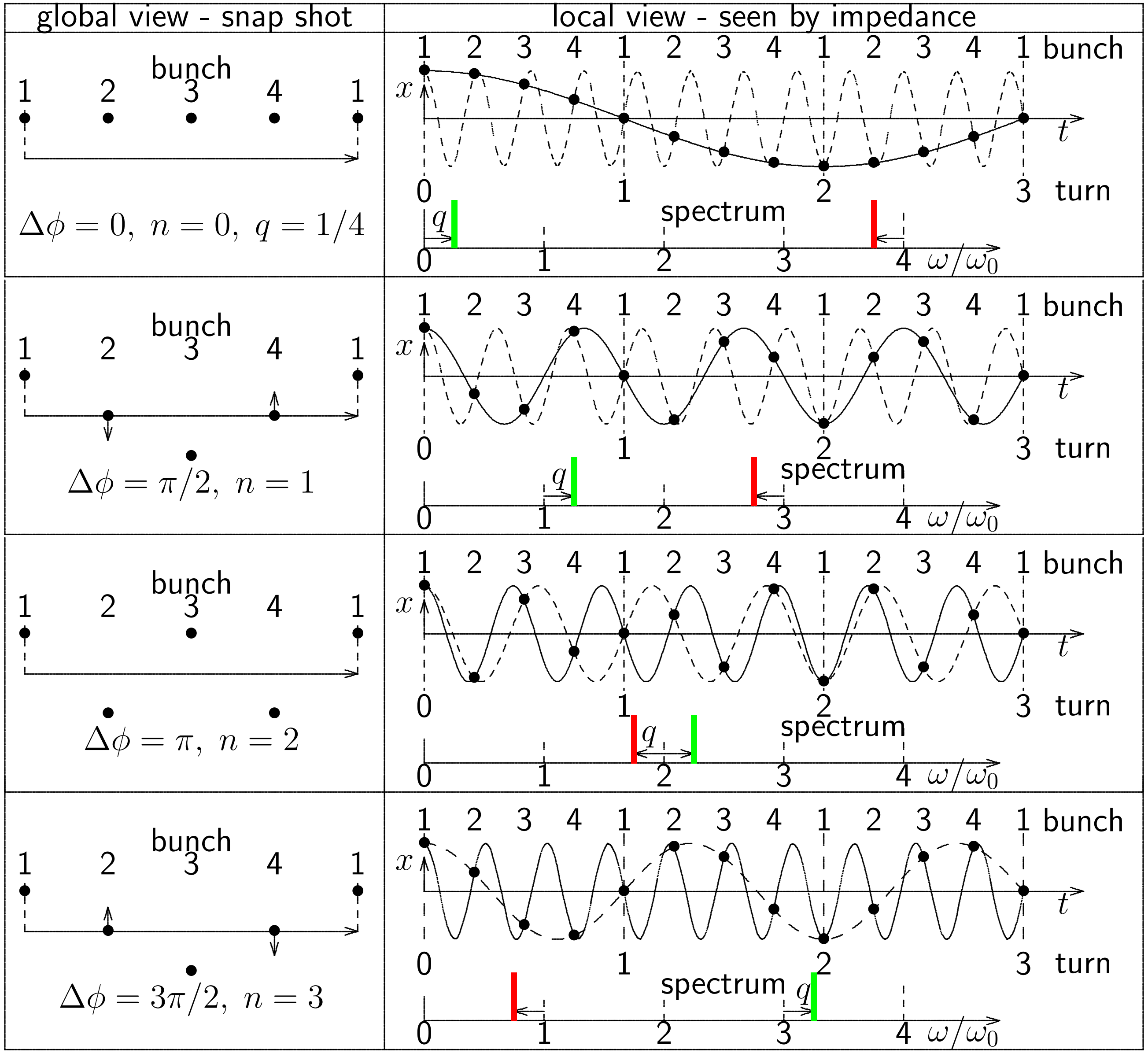
Transverse instability of many rigid bunches

M bunches can oscillate in M independent modes $n = M\Delta\phi/2\pi$, phase $\Delta\phi$ between them seen in global view. Locally, bunches pass with increasing time delay shown as bullets fitted by upper (solid) and lower (dashed) side-band frequency. Higher frequencies can be fitted and spectrum repeats every $4\omega_0$.

$\omega_{p\pm} = \omega_0(pM \pm (n + q))$

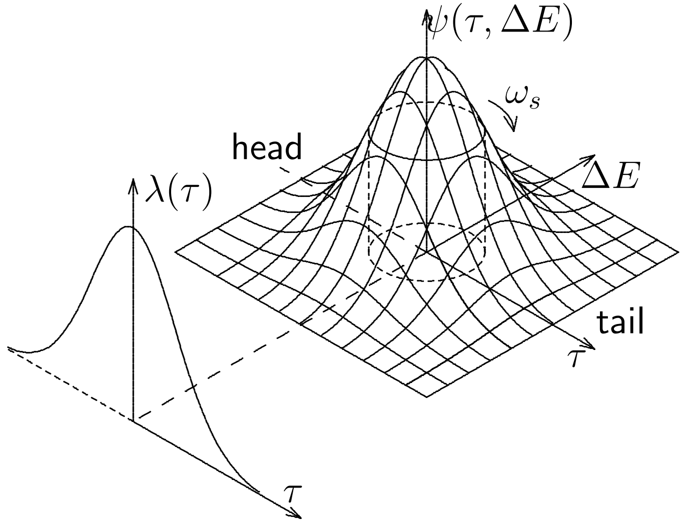






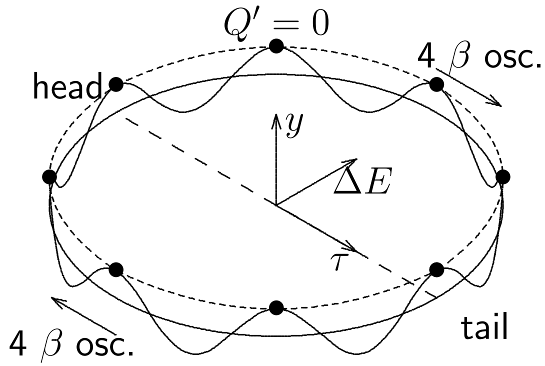
Non-rigid bunch - head-tail modes, $Q' = 0$

Particle distribution in a bunch

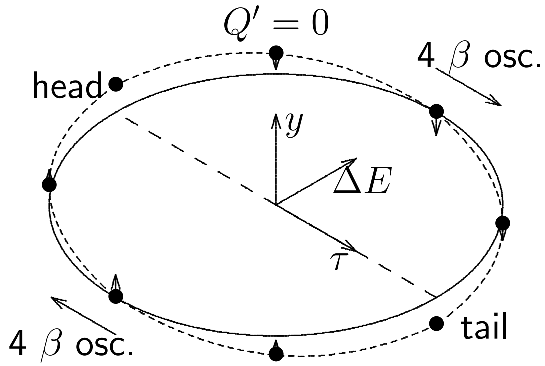


Phase-space distr. ψ rotates with ω_s, not visible, but projection $\lambda(\tau) = \int \psi(\tau, \Delta E) \mathrm{d}E$ or current $I = q\beta c\lambda$. Study motion by selecting particles with fixed synchr. osc. amplitude $\hat{\tau}$ rotating in phase-space, moving from head to tail and vice versa while executing at same time vertical betatron oscillation $y = \hat{y}\cos(Q_y\omega_0 t)$. With $Q' \approx \mathrm{d}Q/(\mathrm{d}E/E) = 0$ tune is constant during synchrotron motion.

Mode m=0, all in phase, rigid bunch



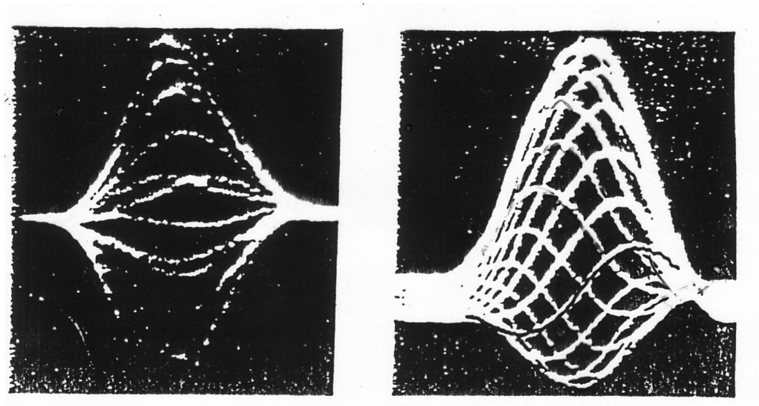
Mode m=1, head and tail in opposite phase, not rigid



A very high impedance can couple these modes and give a Transverse Mode Coupling Instability, TMCI.

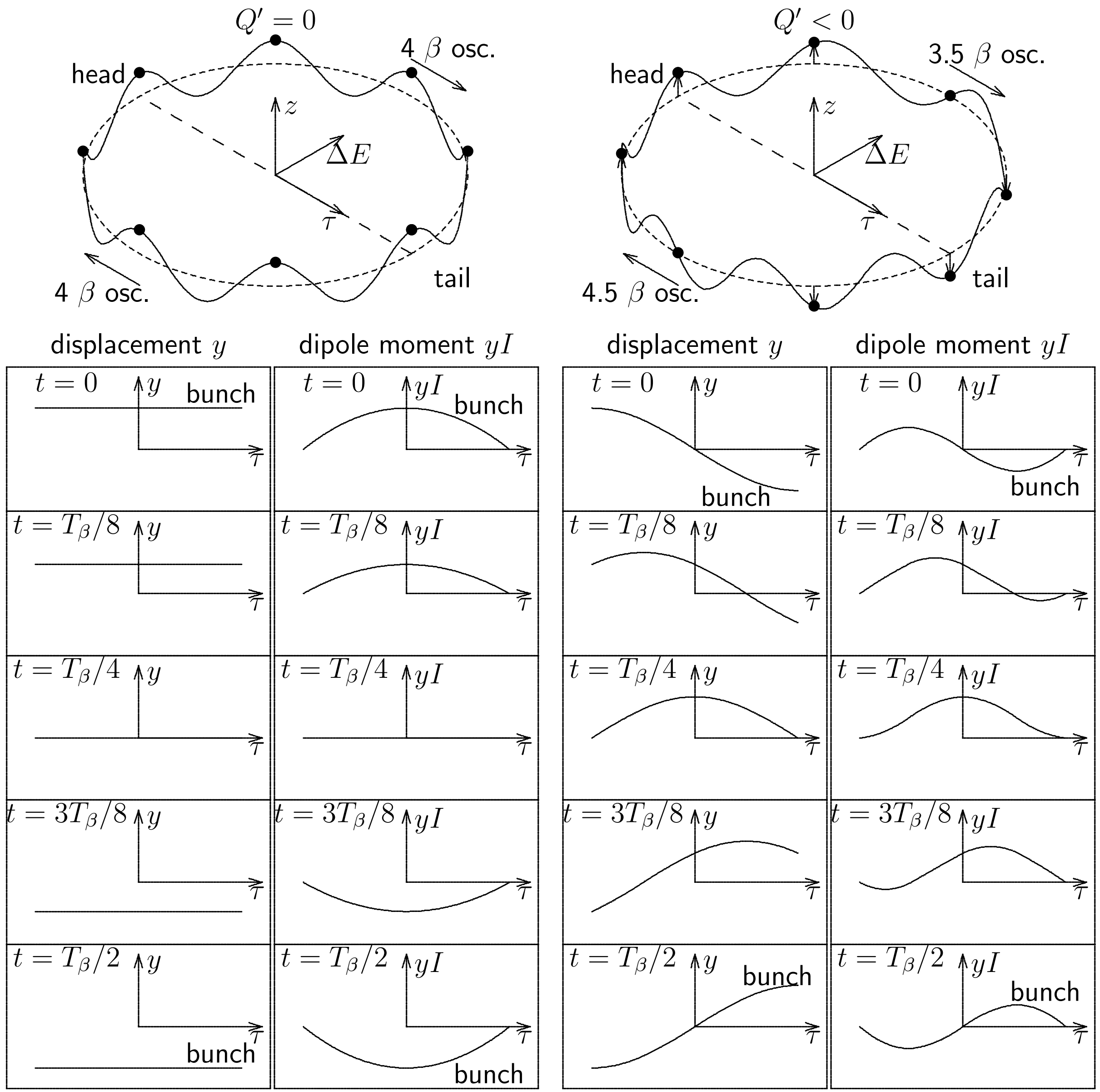
Head-tail mode m=0 for $Q' \neq 0$

Synchrotron oscillation in ΔE affect transverse motion via chromaticity $Q' = dQ/(dp/p)$. For $\gamma > \gamma_T$ has excess energy moving from head to tail and lack going from tail to head. For $Q' > 0$, phase advances in first, lags in second step; vice versa for $Q' < 0$ or $\gamma < \gamma_T$. Figure shows motion for $T_\beta = T_s/8$, for $Q' = 0$ and $Q' < 0$ in 4 steps of $T_\beta/8$.



$Q' = 0$ $\qquad$ $Q' > 0$

CERN booster; Gareyte, Sacherer.



Model of head-tail instability

Above transition energy:

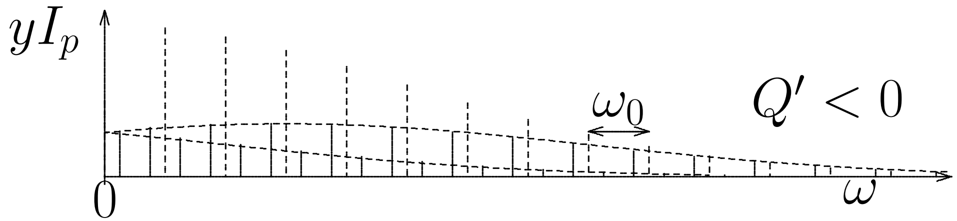
$Q' = 0$: Going from head to tail or vice versa has same phase change. Phase lag and advance interchange, giving neither damping nor growth.

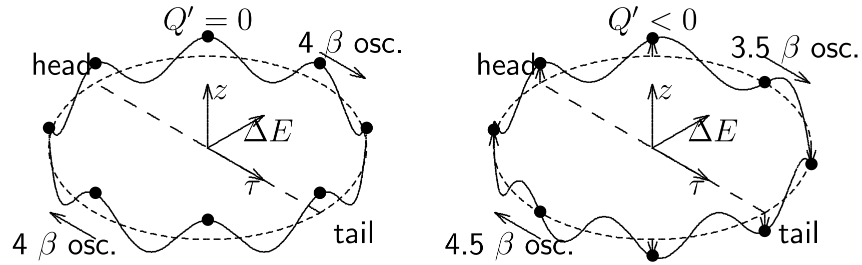
$Q' < 0$: Going from head to tail there is a loss in phase, going from tail to head a gain (picture), giving a systematic phase advance between head and tail and in average growth.

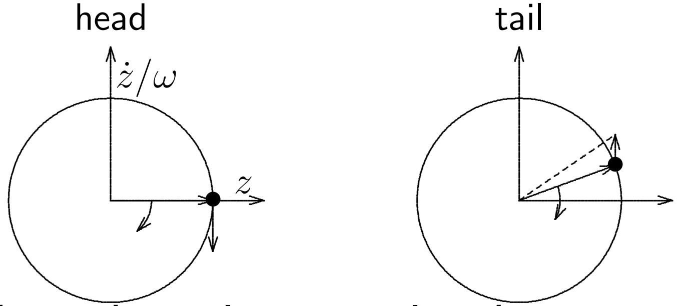
$Q' > 0$: Going from head to tail there is a phase gain, going back a loss, giving a systematic phase lag between head and tail and in average damping.

Below transition this situation is reversed.

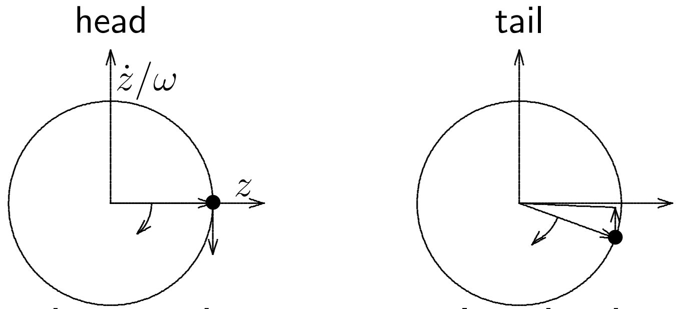
Head tail spectrum:



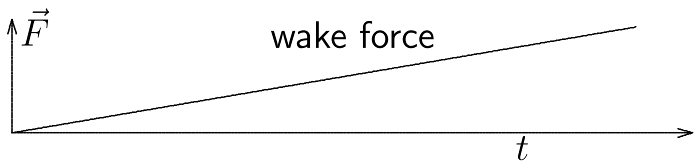




Tail has phase lag, amplitude increasea



Tail has phase advance, amplitude decreasea



A merry-go-round, having vertically moving horses, can illustrate transverse modes:
Coupled bunch modes, real space $y = f(\theta, t)$
Head-Tail modes, phase-space $y = f(\Delta E, \tau, t)$

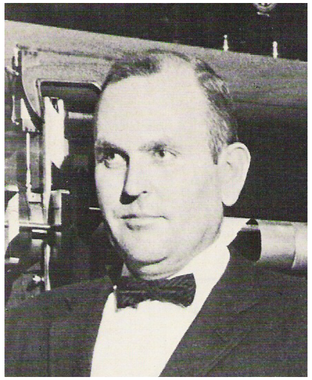
Summary

Present instability treatment, invented by K. Robinson and generalized to nearly all cases by Frank Sacherer.

This demands resistive impedance at upper, Z^+, and lower, Z^-, side-band to fulfill **stability conditions**:

	above transition	below transition
longitudinal, stability	$Z_r^+ < Z_r^-$	$Z_r^+ > Z_r^-$
transverse $Q' = 0$, stability	$Z_{Tr}^+ > Z_{Tr}^-$	$Z_{Tr}^+ > Z_{Tr}^-$
transverse head-tail, stability	$Q' > 0$	$Q' < 0$

Ken Robinson



Frank Sacherer