Beam Transfer Lines

- Distinctions between transfer lines and circular machines
- Linking machines together
- Trajectory correction
- Emittance and mismatch measurement
- Blow-up from steering errors, optics mismatch and thin screens
- Phase-plane exchange
Injection, extraction and transfer

- An accelerator has limited dynamic range.
- Chain of stages needed to reach high energy
- Periodic re-filling of storage rings, like LHC
- External experiments, like CNGS

Transfer lines transport the beam between accelerators, and onto targets, dumps, instruments etc.

**Abbreviations:**

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<th>Abbreviation</th>
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<tr>
<td>LHC</td>
<td>Large Hadron Collider</td>
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<tr>
<td>SPS</td>
<td>Super Proton Synchrotron</td>
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<tr>
<td>AD</td>
<td>Antiproton Decelerator</td>
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<td>ISOLDE</td>
<td>Isotope Separator Online Device</td>
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<td>PSB</td>
<td>Proton Synchrotron Booster</td>
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<td>PS</td>
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<td>CNGS</td>
<td>CERN Neutrino to Gran Sasso</td>
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Normalised phase space

- Transform real transverse coordinates $x, x'$ by

$$\begin{bmatrix} \bar{X} \\ \bar{X}' \end{bmatrix} = \mathbf{N} \cdot \begin{bmatrix} x \\ x' \end{bmatrix} = \sqrt{\frac{1}{\beta_s}} \cdot \begin{bmatrix} 1 & 0 \\ \alpha_s & \beta_s \end{bmatrix} \cdot \begin{bmatrix} x \\ x' \end{bmatrix}$$

$$\bar{X} = \sqrt{\frac{1}{\beta_s}} \cdot x$$

$$\bar{X}' = \sqrt{\frac{1}{\beta_s}} \cdot \alpha_s x + \sqrt{\beta_s} x'$$
Normalised phase space

\[
\varepsilon = \gamma \cdot x^2 + 2 \alpha \cdot x \cdot x' + \beta \cdot x'^2
\]

\[
\varepsilon = \bar{X}^2 + \bar{X}'^2
\]
General transport

Beam transport: moving from $s_1$ to $s_2$ through $n$ elements, each with transfer matrix $M_i$

\[
\begin{bmatrix}
x_2 \\
x_2'
\end{bmatrix} = M_{1\to 2} \cdot \begin{bmatrix}
x \\
x'
\end{bmatrix} = \begin{bmatrix}
C & S \\
C' & S'
\end{bmatrix} \cdot \begin{bmatrix}
x \\
x'
\end{bmatrix}
\]

\[M_{1\to 2} = \prod_{i=1}^{n} M_i\]

Twiss parameterisation $M_{1\to 2} = 
\begin{bmatrix}
\sqrt{\frac{\beta_2}{\beta_1}} (\cos \Delta \mu + \alpha_1 \sin \Delta \mu) & \sqrt{\beta_1 \beta_2} \sin \Delta \mu \\
\sqrt{\frac{1}{\beta_1 \beta_2}} [(\alpha_1 - \alpha_2) \cos \Delta \mu - (1 + \alpha_1 \alpha_2) \sin \Delta \mu] & \sqrt{\frac{\beta_1}{\beta_2}} (\cos \Delta \mu - \alpha_2 \sin \Delta \mu)
\end{bmatrix}
The solution is periodic.

Periodicity condition for one turn (closed ring) imposes $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $D_1 = D_2$.

This condition uniquely determines $\alpha(s)$, $\beta(s)$, $\mu(s)$, $D(s)$ around the whole ring.
Circular Machine

- Periodicity of the structure leads to regular motion
  - Map single particle coordinates on each turn at any location
  - Describes an ellipse in phase space, defined by one set of $\alpha$ and $\beta$ values $\Rightarrow$ Matched Ellipse (for this location)

\[
x'_{\text{max}} = \sqrt{a \frac{1 + \alpha^2}{\beta}}
\]

\[
A = \pi a
\]

\[
x_{\text{max}} = \sqrt{a \sqrt{\beta}}
\]

\[
a = \gamma \cdot x^2 + 2\alpha \cdot x \cdot x' + \beta \cdot x'^2
\]

\[
\gamma = \frac{1 + \alpha^2}{\beta}
\]
Circular Machine

- For a location with matched ellipse \((\alpha, \beta)\), an injected beam of emittance \(\varepsilon\), characterised by a different ellipse \((\alpha^*, \beta^*)\) generates (via filamentation) a large ellipse with the original \(\alpha, \beta\), but larger \(\varepsilon\).
Transfer line

One pass:

\[
\begin{bmatrix}
X_2' \\
X_2
\end{bmatrix} = M_{1 \rightarrow 2} \cdot \begin{bmatrix}
X' \\
X
\end{bmatrix}
\]

\[
M_{1 \rightarrow 2} = \begin{bmatrix}
\frac{\sqrt{\beta_2}}{\beta_1} (\cos \Delta \mu + \alpha_1 \sin \Delta \mu) & \sqrt{\beta_1 \beta_2} \sin \Delta \mu \\
\sqrt{\frac{1}{\beta_1 \beta_2}} \left[ (\alpha_1 - \alpha_2) \cos \Delta \mu - (1 + \alpha_1 \alpha_2) \sin \Delta \mu \right] & \frac{\sqrt{\beta_2}}{\beta_1} (\cos \Delta \mu - \alpha_2 \sin \Delta \mu)
\end{bmatrix}
\]

- No periodic condition exists
- The Twiss parameters are simply propagated from beginning to end of line
- At any point in line, \( \alpha(s) \) \( \beta(s) \) are functions of \( \alpha_1 \beta_1 \)
Transfer line

• On a single pass there is no regular motion
  – Map single particle coordinates at entrance and exit.
  – Infinite number of equally valid possible starting ellipses for single particle
    ……transported to infinite number of final ellipses…

$$M_{0 \rightarrow L}$$
Transfer Line

- Initial $\alpha$, $\beta$ defined for transfer line by beam shape at entrance

  \[ \alpha^*, \beta^* \]

- Propagation of this beam ellipse depends on line elements

- A transfer line optics is different for different input beams
• The optics functions in the line depend on the initial values

- Design $\beta_x$ functions in a transfer line
  - $\beta_x$ functions with different initial conditions

• Same considerations are true for Dispersion function:
  - Dispersion in ring defined by periodic solution $\rightarrow$ ring elements
  - Dispersion in line defined by initial D and D’ and line elements
Another difference… unlike a circular ring, a change of an element in a line affects only the downstream Twiss values (including dispersion).
• Beams have to be transported from extraction of one machine to injection of next machine
  – Trajectories must be matched, ideally in all 6 geometric degrees of freedom
    \((x,y,z,\theta,\phi,\psi)\)
• Other important constraints can include
  – Minimum bend radius, maximum quadrupole gradient, magnet aperture, cost, geology
The Twiss parameters can be propagated when the transfer matrix $\mathbf{M}$ is known:

$$
\begin{bmatrix}
  x_2 \\
  x_2'
\end{bmatrix} = \mathbf{M}_{1 \rightarrow 2} \cdot \begin{bmatrix}
  x \\
  x'
\end{bmatrix} = \begin{bmatrix}
  C & S \\
  C' & S'
\end{bmatrix} \cdot \begin{bmatrix}
  x \\
  x'
\end{bmatrix}
$$

$\alpha_2, \beta_2, \alpha_1, \beta_1$

$$
\begin{bmatrix}
  \beta_2 \\
  \alpha_2 \\
  \gamma_2
\end{bmatrix} = \begin{bmatrix}
  C^2 & -2CS & S^2 \\
  -CC' & CS' + SC' & -SS' \\
  C'n^2 & -2C'S' & S'^2
\end{bmatrix} \cdot \begin{bmatrix}
  \beta_1 \\
  \alpha_1 \\
  \gamma_1
\end{bmatrix}
$$
Linking Machines

• Linking the optics is a complicated process
  – Parameters at start of line have to be propagated to matched parameters at the end of the line
  – Need to “match” 8 variables \((\alpha_x \beta_x D_x D'_x \text{ and } \alpha_y \beta_y D_y D'_y)\)
  – Maximum \(\beta\) and \(D\) values are imposed by magnet apertures
  – Other constraints can exist
    • phase conditions for collimators,
    • insertions for special equipment like stripping foils
  – Need to use a number of independently powered (“matching”) quadrupoles
  – Matching with computer codes and relying on mixture of theory, experience, intuition, trial and error, …
Linking Machines

- For long transfer lines we can simplify the problem by designing the line in separate sections
  - Regular central section – e.g. FODO or doublet, with quads at regular spacing, (+ bending dipoles), with magnets powered in series
  - Initial and final matching sections – independently powered quadrupoles, with sometimes irregular spacing.

![Diagram showing the transfer line between SPS and LHC](image_url)
• Magnet misalignments, field and powering errors cause the trajectory to deviate from the design

• Use small independently powered dipole magnets (correctors) to steer the beam

• Measure the response using monitors (pick-ups) downstream of the corrector ($\pi/2, 3\pi/2, \ldots$)

• Horizontal and vertical elements are separated

• $H$-correctors and pick-ups located at $F$-quadrupoles (large $\beta_x$)

• $V$-correctors and pick-ups located at $D$-quadrupoles (large $\beta_y$)
Trajectory correction

- Global correction can be used which attempts to minimise the RMS offsets at the BPMs, using all or some of the available corrector magnets.
- Steering in matching sections, extraction and injection region requires particular care
  - D and $\beta$ functions can be large $\rightarrow$ bigger beam size
  - Often very limited in aperture
  - Injection offsets can be detrimental for performance
Trajectory correction

**Uncorrected trajectory.**

$y$ growing as a result of random errors in the line.

The RMS at the BPMs is 3.4 mm, and $y_{\text{max}}$ is 12.0 mm

**Corrected trajectory.**

The RMS at the BPMs is 0.3 mm and $y_{\text{max}}$ is 1 mm
Trajectory correction

- Sensitivity to BPM errors is an important issue
  - If the BPM phase sampling is poor, the loss of a few key BPMs can allow a very bad trajectory, while all the monitor readings are \( \sim \) zero

Correction with some monitors disabled

With poor BPM phase sampling the correction algorithm produces a trajectory with 185mm \( y_{\text{max}} \)

Note the change of vertical scale
Steering (dipole) errors

- Precise delivery of the beam is important.
  - To avoid injection oscillations and emittance growth in rings
  - For stability on secondary particle production targets
- Convenient to express injection error in $\sigma$ (includes $x$ and $x'$ errors)

$$\Delta a [\sigma] = \sqrt{(x^2 + x'^2)/\varepsilon} = \sqrt{(\gamma^2 + 2\alpha xx' + \beta x'^2)/\varepsilon)$$
Steering (dipole) errors

- Static effects (e.g. from errors in alignment, field, calibration, …) are dealt with by trajectory correction (steering).
- But there are also dynamic effects, from:
  - Power supply ripples
  - Temperature variations
  - Non-trapezoidal kicker waveforms
- These dynamic effects produce a variable injection offset which can vary from batch to batch, or even within a batch.
- An injection damper system is used to minimise effect on emittance
Blow-up from steering error

- Consider a collection of particles with amplitudes $A$
- The beam can be injected with an error in angle and position.
- For an injection error $\Delta a_y$ (in units of sigma = $\sqrt{\beta \varepsilon}$) the mis-injected beam is offset in normalised phase space by $d = \Delta a_x \sqrt{\varepsilon}$
Blow-up from steering error

- The new particle coordinates in normalised phase space are

\[
\mathbf{X}_{\text{new}} = \mathbf{X}_0 + d \cos \theta \\
\mathbf{X}'_{\text{new}} = \mathbf{X}_0' + d \sin \theta
\]

- For a general particle distribution, where \( A \) denotes amplitude in normalised phase space

\[
A^2 = \mathbf{X}^2 + \mathbf{X}'^2
\]

\[
\epsilon = \langle A^2 \rangle / 2
\]
Blow-up from steering error

- So if we plug in the new coordinates....

\[
A_{new}^2 = (\overline{X}_0 + d\cos\theta)^2 + (\overline{X}'_0 + d\sin\theta)^2
\]

\[
= \overline{X}_0^2 + \overline{X}'_0^2 + 2d(\overline{X}_0\cos\theta + \overline{X}'_0\sin\theta) + d^2
\]

\[
\langle A_{new}^2 \rangle = \langle \overline{X}_0^2 \rangle + \langle \overline{X}'_0^2 \rangle + 2d \langle \overline{X}_c\cos\theta + \overline{X}'_c\sin\theta \rangle + \langle d^2 \rangle
\]

\[
= 2\varepsilon_0 + 2d \langle \overline{X}_0\cos\theta + \overline{X}'_0\sin\theta \rangle + d^2
\]

\[
= 2\varepsilon_0 + d^2
\]

- Giving for the emittance increase

\[
\varepsilon_{new} = \frac{\langle A_{new}^2 \rangle}{2} = \varepsilon_0 + d^2 / 2
\]

\[
= \varepsilon_0 \left(1 + \Delta a^2 / 2\right)
\]
Blow-up from steering error

A numerical example....

Consider an offset $\Delta a$ of 0.5 sigma for injected beam

$$\varepsilon_{new} = \varepsilon_0 \left(1 + \frac{\Delta a^2}{2}\right)$$

$$= 1.125 \varepsilon_0$$
Blow-up from betatron mismatch

- Optical errors occur in transfer line and ring, such that the beam can be injected with a mismatch.
- Filamentation will produce an emittance increase.
- In normalised phase space, consider the matched beam as a circle, and the mismatched beam as an ellipse.
Blow-up from betatron mismatch

General betatron motion – coordinates of particles on mismatched ellipse

\[ x_2 = \sqrt{\varepsilon_2 \beta_2} \sin(\phi + \phi_0), \quad x'_2 = \sqrt{\varepsilon_2 / \beta_2} \left[ \cos(\phi + \phi_0) - \alpha_2 \sin(\phi + \phi_0) \right] \]

applying the normalising transformation for the matched beam (subscript 1)

\[
\begin{bmatrix}
\bar{x}_2 \\
\bar{x}'_2
\end{bmatrix} = \sqrt{\frac{1}{\beta_1}} \cdot \begin{bmatrix}
1 & 0 \\
\alpha_1 & \beta_1
\end{bmatrix} \cdot \begin{bmatrix}
x_2 \\
x'_2
\end{bmatrix}
\]

an ellipse is obtained in normalised phase space

\[
A^2 = \bar{x}_2^2 \left[ \frac{\beta_1 + \beta_2}{\beta_1} \left( \alpha_1 - \alpha_2 \frac{\beta_1}{\beta_2} \right)^2 \right] + \bar{x}'_2^2 \frac{\beta_2}{\beta_1} - 2 \bar{x}_2 \bar{x}'_2 \left[ \frac{\beta_2}{\beta_1} \left( \alpha_1 - \alpha_2 \frac{\beta_1}{\beta_2} \right) \right]
\]

characterised by \( \gamma_{\text{new}}, \beta_{\text{new}} \) and \( \alpha_{\text{new}} \), where

\[
\alpha_{\text{new}} = -\frac{\beta_2}{\beta_1} \left( \alpha_1 - \alpha_2 \frac{\beta_1}{\beta_2} \right), \quad \beta_{\text{new}} = \frac{\beta_2}{\beta_1}, \quad \gamma_{\text{new}} = \frac{\beta_1}{\beta_2} + \frac{\beta_2}{\beta_1} \left( \alpha_1 - \alpha_2 \frac{\beta_1}{\beta_2} \right)^2
\]
Blow-up from betatron mismatch

From the general ellipse properties

\[ a = \frac{A}{\sqrt{2}} \left( \sqrt{H+1} + \sqrt{H-1} \right), \quad b = \frac{A}{\sqrt{2}} \left( \sqrt{H+1} - \sqrt{H-1} \right) \]

where

\[ H = \frac{1}{2} (\gamma_{\text{new}} + \beta_{\text{new}}) \]

\[ = \frac{1}{2} \left( \frac{\beta_1}{\beta_2} + \frac{\beta_2}{\beta_1} \left( \alpha_1 - \alpha_2 \right)^2 + \frac{\beta_2}{\beta_1} \right) \]

giving

\[ \lambda = \frac{1}{\sqrt{2}} \left( \sqrt{H+1} + \sqrt{H-1} \right), \quad \frac{1}{\lambda} = \frac{1}{\sqrt{2}} \left( \sqrt{H+1} - \sqrt{H-1} \right) \]

\[ \overline{X}_{\text{new}} = \lambda \cdot A \sin(\phi + \phi_f), \quad \overline{X'}_{\text{new}} = \frac{1}{\lambda} A \cos(\phi + \phi_f) \]

\[ a = \bar{A} / \lambda \]
\[ b = A \cdot \lambda \]
Blow-up from betatron mismatch

We can evaluate the square of the distance of a particle from the origin as

\[ A_{\text{new}}^2 = \overline{X}_{\text{new}}^2 + \overline{X}'_{\text{new}}^2 = \lambda^2 \cdot A_0^2 \sin^2(\phi + \phi_t) + \frac{1}{\lambda^2} A_0^2 \cos^2(\phi + \phi_t) \]

The new emittance is the average over all phases

\[ \varepsilon_{\text{new}} = \frac{1}{2} \left( \langle A_{\text{new}}^2 \rangle \right) = \frac{1}{2} \left( \lambda^2 \langle A_0^2 \sin^2(\phi + \phi_t) \rangle + \frac{1}{\lambda^2} \langle A_0^2 \cos^2(\phi + \phi_t) \rangle \right) \]

\[ = \frac{1}{2} \left( \langle A_0^2 \rangle \left( \lambda^2 \langle \sin^2(\phi + \phi_t) \rangle + \frac{1}{\lambda^2} \langle \cos^2(\phi + \phi_t) \rangle \right) \right) \]

\[ = \frac{1}{2} \varepsilon_0 \left( \lambda^2 + \frac{1}{\lambda^2} \right) \]

If we’re feeling diligent, we can substitute back for \( \lambda \) to give

\[ \varepsilon_{\text{new}} = \frac{1}{2} \varepsilon_0 \left( \lambda^2 + \frac{1}{\lambda^2} \right) = H \varepsilon_0 = \frac{1}{2} \varepsilon_0 \left( \frac{\beta_1}{\beta_2} + \frac{\beta_2}{\beta_1} \right) \left( \alpha_1 - \alpha_2 \frac{\beta_1}{\beta_2} \right)^2 + \frac{\beta_2}{\beta_1} \]

where subscript 1 refers to matched ellipse, 2 to mismatched ellipse.
Blow-up from betatron mismatch

A numerical example….consider $b = 3a$ for the mismatched ellipse

$$\lambda = \sqrt{\frac{b}{a}} = \sqrt{3}$$

Then

$$\varepsilon_{new} = \frac{1}{2} \varepsilon_0 \left( \lambda^2 + \frac{1}{\lambda^2} \right)$$

$$= 1.67 \varepsilon_0$$
Emittance and mismatch measurement

- A profile monitor is need to measure the beam size
  - E.g. beam screen (luminescent) provides 2D density profile of the beam
- Profile fit gives transverse beam sizes $\sigma$.
- In a ring, $\beta$ is ‘known’ so $\varepsilon$ can be calculated from a single screen
Emittance and mismatch measurement

- Emittance and optics measurement in a line needs 3 profile measurements in a dispersion-free region.
- Measurements of $\sigma_0, \sigma_1, \sigma_2$, plus the two transfer matrices $M_{01}$ and $M_{12}$ allows determination of $\varepsilon$, $\alpha$ and $\beta$.

\[
\varepsilon = \frac{\sigma_0^2}{\beta_0} = \frac{\sigma_1^2}{\beta_1} = \frac{\sigma_2^2}{\beta_2}
\]
We have
\[
\begin{bmatrix}
\beta_1 \\
\alpha_1 \\
\gamma_1
\end{bmatrix} = \begin{bmatrix}
C_1^2 & -2C_1S_1 & S_1^2 \\
-C_1C_1' & C_1S_1'+S_1C_1' & -S_1S_1' \\
C_1'^2 & -2C_1'S_1' & S_1'^2
\end{bmatrix} \begin{bmatrix}
\beta_0 \\
\alpha_0 \\
\gamma_0
\end{bmatrix}
\]

where
\[
\begin{bmatrix}
C_1 & S_1 \\
C_1' & S_1'
\end{bmatrix} = \begin{bmatrix}
\sqrt{\frac{\beta_2}{\beta_1}} (\cos \Delta \mu + \alpha_1 \sin \Delta \mu) & \sqrt{\beta_1 \beta_2} \sin \Delta \mu \\
\sqrt{\frac{1}{\beta_1 \beta_2}} ([\alpha_1 - \alpha_2] \cos \Delta \mu - (1 + \alpha_1 \alpha_2) \sin \Delta \mu) & \sqrt{\frac{1}{\beta_2}} (\cos \Delta \mu - \alpha_2 \sin \Delta \mu)
\end{bmatrix}
\]

so that
\[
\beta_1 = C_1^2 \beta_0 - 2C_1S_1 \alpha_0 + \frac{S_1^2}{\beta_0} (1 + \alpha_0^2), \quad \beta_2 = C_2^2 \beta_0 - 2C_2S_2 \alpha_0 + \frac{S_2^2}{\beta_0} (1 + \alpha_0^2)
\]

Using
\[
\beta_0 = \frac{\sigma_0^2}{\epsilon}, \quad \beta_1 = \left(\frac{\sigma_1}{\sigma_0}\right)^2 \beta_0, \quad \beta_2 = \left(\frac{\sigma_2}{\sigma_0}\right)^2 \beta_0
\]

we find
\[
\alpha_0 = \frac{1}{2} \beta_0 W
\]

where
\[
W = \frac{(\sigma_2 / \sigma_0)^2 / S_2^2 - (\sigma_1 / \sigma_0)^2 / S_1^2 - (C_2 / S_2)^2 + (C_1 / S_1)^2}{(C_1 / S_1) - (C_2 / S_2)}
\]
Emittance and mismatch measurement

Some (more) algebra with above equations gives

\[ \beta_0 = 1 / \sqrt{\left( \frac{\sigma_2}{\sigma_0} \right)^2 / S_2^2 - \left( \frac{C_2}{S_2} \right)^2 + W \left( \frac{C_2}{S_2} \right)^2 - W^2 / 4} \]

And finally we are in a position to evaluate \( \varepsilon \) and \( \alpha_0 \)

\[ \varepsilon = \sigma_0^2 \beta_0 \quad \alpha_0 = \frac{1}{2} \beta_0 W \]

Comparing measured \( \alpha_0, \beta_0 \) with expected values gives numerical measurement of mismatch
Blow-up from thin scatterer

- Scattering elements are sometimes required in the beam
  - Thin beam screens ($\text{Al}_2\text{O}_3$, Ti) used to generate profiles.
  - Metal windows also used to separate vacuum of transfer lines from vacuum in circular machines.
  - Foils are used to strip electrons to change charge state

- The emittance of the beam increases when it passes through, due to multiple Coulomb scattering.

\[
\text{rms angle increase: } \sqrt{\langle \theta_s^2 \rangle [\text{mrad}]} = \frac{14.1}{ \beta_c p[\text{MeV} / \text{c}]} Z_{\text{inc}} \sqrt{\frac{L}{L_{\text{rad}}}} \left( 1 + 0.11 \cdot \log_{10} \frac{L}{L_{\text{rad}}} \right)
\]

$\beta_c = v/c$, $p = \text{momentum}$, $Z_{\text{inc}} = \text{particle charge} / e$, $L = \text{target length}$, $L_{\text{rad}} = \text{radiation length}$
Blow-up from thin scatterer

Each particle gets a random angle change $\theta_s$ but there is no effect on the positions at the scatterer

$$\begin{align*}
\bar{X}_{\text{new}} &= \bar{X}_0 \\
\bar{X}'_{\text{new}} &= \bar{X}'_c + \sqrt{\beta \theta_s}
\end{align*}$$

After filamentation the particles have different amplitudes and the beam has a larger emittance

$$\epsilon = \langle A_{\text{new}}^2 \rangle / 2$$
Blow-up from thin scatterer

\[
A_{\text{new}}^2 = \bar{X}_{\text{new}}^2 + \bar{X}'_{\text{new}}^2
\]

\[
= \bar{X}_0^2 + (\bar{X}'_0 + \sqrt{\beta \theta_s})^2
\]

\[
= \bar{X}_0^2 + \bar{X}'_0^2 + 2\sqrt{\beta} \langle \bar{X}'_0 \theta_s \rangle + \beta \theta_s^2
\]

\[
\langle A_{\text{new}}^2 \rangle = \langle \bar{X}_0^2 \rangle + \langle \bar{X}'_0^2 \rangle + 2\sqrt{\beta} \langle \bar{X}'_0 \theta_s \rangle + \beta \langle \theta_s^2 \rangle
\]

\[
= 2\varepsilon_0 + 2\sqrt{\beta} \langle \bar{X}'_0 \theta_s \rangle + \beta \langle \theta_s^2 \rangle
\]

\[
= 2\varepsilon_0 + \beta \langle \theta_s^2 \rangle
\]

\[
\varepsilon_{\text{new}} = \varepsilon_0 + \frac{\beta}{2} \langle \theta_s^2 \rangle
\]

Need to keep \( \beta \) small to minimise blow-up (small \( \beta \) means large spread in angles in beam distribution, so additional angle has small effect on distn.)
Blow-up from charge stripping foil

- For LHC heavy ions, $\text{Pb}^{53+}$ is stripped to $\text{Pb}^{82+}$ at 4.25GeV/u using a 0.8mm thick Al foil, in the PS to SPS line.
- $\Delta \varepsilon$ is minimised with low-$\beta$ insertion ($\beta_{xy} \sim 5 \text{ m}$) in the transfer line.
- Emittance increase expected is about 8%.
Acceptances of circular accelerators tend to be larger in horizontal plane (bending dipole gap height small as possible)

Several multturn extraction process produce beams which have emittances which are larger in the *vertical* plane $\rightarrow$ larger losses

We can overcome this by exchanging the H and V phase planes (emittance exchange)

In the following, remember that the matrix is our friend…
Emittance exchange

Phase-plane exchange requires a transformation of the form:

\[
\begin{pmatrix}
    x_1 \\
    x'_1 \\
    y_1 \\
    y'_1
\end{pmatrix}
= \begin{pmatrix}
    0 & 0 & m_{13} & m_{14} \\
    0 & 0 & m_{23} & m_{24} \\
    m_{31} & m_{32} & 0 & 0 \\
    m_{41} & m_{42} & 0 & 0
\end{pmatrix}
\begin{pmatrix}
    x_0 \\
    x'_0 \\
    y_0 \\
    y'_0
\end{pmatrix}
\]

A skew quadrupole is a normal quadrupole rotated by an angle \( \theta \).

The transfer matrix \( S \) obtained by a rotation of the normal transfer matrix \( M_q \):

\[
S = R^{-1}M_qR
\]

where \( R \) is the rotation matrix

\[
\begin{pmatrix}
    \cos \theta & 0 & \sin \theta & 0 \\
    0 & \cos \theta & 0 & \sin \theta \\
    -\sin \theta & 0 & \cos \theta & 0 \\
    0 & -\sin \theta & 0 & \cos \theta
\end{pmatrix}
\]

(you can convince yourself of what \( R \) does by checking that \( x_0 \) is transformed to \( x_1 = x_0 \cos \theta + y_0 \sin \theta \), \( y_0 \) into \( -x_0 \sin \theta + y_0 \cos \theta \), etc.)
Emittance exchange

For a thin-lens approximation

\[
M_q = \begin{pmatrix}
1 & 0 & 0 & 0 \\
\delta & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -\delta & 1
\end{pmatrix}
\]

(where \( \delta = kl = 1/f \) is the quadrupole strength)

So that

\[
S = R^{-1} M_q R = \begin{pmatrix}
\cos \theta & 0 & -\sin \theta & 0 \\
0 & \cos \theta & 0 & -\sin \theta \\
\sin \theta & 0 & \cos \theta & 0 \\
0 & \sin \theta & 0 & \cos \theta
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
\delta & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -\delta & 1
\end{pmatrix}
\begin{pmatrix}
\cos \theta & 0 & \sin \theta & 0 \\
0 & \cos \theta & 0 & \sin \theta \\
-\sin \theta & 0 & \cos \theta & 0 \\
0 & -\sin \theta & 0 & \cos \theta
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
\delta \cos 2\theta & 1 & \delta \sin 2\theta & 0 \\
0 & 0 & 1 & 0 \\
\delta \sin 2\theta & 0 & -\delta \cos 2\theta & 1
\end{pmatrix}
\]

For the case of \( \theta = 45^\circ \),

this reduces to

\[
S = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & \delta & 0 \\
0 & 0 & 1 & 0 \\
\delta & 0 & 0 & 1
\end{pmatrix}
\]
Emittance exchange

The transformation required can be achieved with 3 such skew quads in a lattice, of strengths $\delta_1, \delta_2, \delta_3$, with transfer matrices $S_1, S_2, S_3$.

The transfer matrix without the skew quads is $C = BA$.

$$C = \begin{pmatrix} C_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C_y \end{pmatrix}$$

$$C_x = \begin{pmatrix} \sqrt{\frac{\beta_{x2}}{\beta_{x1}}} \cos \Delta \phi_x + \alpha_{x1} \sin \Delta \phi_x & \sqrt{\beta_{x1} \beta_{x2}} \sin \Delta \phi_x \\ \left(\alpha_{x2} - \alpha_{x1}\right) \cos \Delta \phi_x - \left(1 + \alpha_{x1} \alpha_{x2}\right) \sin \Delta \phi_x & \sqrt{\beta_{x1} \beta_{x2}} \cos \Delta \phi_x - \alpha_{x2} \sin \Delta \phi_x \end{pmatrix}$$

and similar for $C_y$. 

[Diagram of skew quad arrangement]
With the skew quads the overall matrix is \( M = S_3 B S_2 A S_1 \)

\[
\begin{bmatrix}
  c_{11} + b_{12} a_{34} \delta_1 \delta_2 & c_{12} & c_{12} \delta_1 + b_{12} a_{33} \delta_2 & b_{12} a_{34} \delta_2 \\
  c_{21} + b_{22} a_{34} \delta_1 \delta_2 + \delta_3 (c_{34} \delta_1 + b_{34} a_{11} \delta_2) & c_{22} + b_{34} a_{12} \delta_2 \delta_3 & c_{22} \delta_1 + b_{22} a_{33} \delta_2 + \delta_3 (c_{33} + b_{34} a_{12} \delta_1 \delta_2) & b_{22} a_{34} \delta_2 + c_{34} \delta_3 \\
  c_{34} \delta_1 + b_{34} a_{11} \delta_2 & a_{12} b_{34} \delta_2 & c_{33} + b_{34} a_{12} \delta_1 \delta_2 & c_{34} \\
  \delta_3 (c_{11} + b_{12} a_{34} \delta_1 \delta_2) + c_{44} \delta_1 + b_{22} a_{34} \delta_1 \delta_2 & c_{12} \delta_3 + b_{44} a_{12} \delta_2 & \delta_3 (c_{12} \delta_1 + b_{12} a_{33} \delta_2) + c_{43} + b_{44} a_{12} \delta_1 \delta_2 & c_{44} + b_{12} a_{34} \delta_2 \delta_3 \\
\end{bmatrix}
\]

Equating the terms with our target matrix form

\[
\begin{bmatrix}
  0 & 0 & m_{13} & m_{14} \\
  0 & 0 & m_{23} & m_{24} \\
  m_{31} & m_{32} & 0 & 0 \\
  m_{41} & m_{42} & 0 & 0 \\
\end{bmatrix}
\]

a list of conditions result which must be met for phase-plane exchange.
Emittance exchange

\[ 0 = c_{12} \]
\[ 0 = c_{34} \]
\[ 0 = c_{11} + b_{12} a_{34} \delta_1 \delta_2 \]
\[ 0 = c_{22} + b_{34} a_{12} \delta_2 \delta_3 \]
\[ 0 = c_{33} + b_{34} a_{12} \delta_1 \delta_2 \]
\[ 0 = c_{44} + b_{12} a_{34} \delta_2 \delta_3 \]
\[ 0 = c_{21} + b_{22} a_{34} \delta_1 \delta_2 + \delta_3 (c_{34} \delta_1 + b_{34} a_{11} \delta_2) \]
\[ 0 = c_{43} + b_{44} a_{12} \delta_1 \delta_2 + \delta_3 (c_{12} \delta_1 + b_{12} a_{33} \delta_2) \]

The simplest conditions are \( c_{12} = c_{34} = 0 \).

Looking back at the matrix \( \mathbf{C} \), this means that \( \Delta \phi_x \) and \( \Delta \phi_y \) need to be integer multiples of \( \pi \) (i.e. the phase advance from first to last skew quad should be \( 180^\circ, 360^\circ, \ldots \))

\[
\delta_1 \delta_2 = -\frac{c_{11}}{b_{12} a_{34}} = -\frac{c_{33}}{b_{34} a_{12}}
\]

\[
\delta_2 \delta_3 = -\frac{c_{22}}{b_{34} a_{12}} = -\frac{c_{44}}{b_{12} a_{34}}
\]

We also have for the strength of the skew quads
Several solutions exist which give $M$ the target form.

One of the simplest is obtained by setting all the skew quadrupole strengths the same, and putting the skew quads at symmetric locations in a 90° FODO lattice.

From symmetry $A = B$, and the values of $\alpha$ and $\beta$ at all skew quads are identical.

Therefore $A_x = B_x = \begin{pmatrix} \cos \Delta \phi_x + \alpha_x \sin \Delta \phi_x & \beta_x \sin \Delta \phi_x \\ -\frac{(1-\alpha_x^2) \sin \Delta \phi_x}{\beta_x} & \cos \Delta \phi_x - \alpha_x \sin \Delta \phi_x \end{pmatrix}$ with the same form for $y$

The matrix $C$ is similar, but with phase advances of $2\Delta \phi$
Since we have chose a 90° FODO phase advance, $\Delta \phi_x = \Delta \phi_y = \pi/2$, and $2\Delta \phi_x = 2\Delta \phi_y = \pi$ which means we can now write down $A, B$ and $C$:

$$
A = B = \begin{pmatrix}
\alpha_x & \beta_x & 0 & 0 \\
-\frac{(1-\alpha_x^2)}{\beta_x} & -\alpha_x & 0 & 0 \\
0 & 0 & \alpha_y & \beta_y \\
0 & 0 & -\frac{(1-\alpha_y^2)}{\beta_y} & -\alpha_y
\end{pmatrix}
$$

$$
C = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
$$

i.e. 180° across the insertion in both planes.

we can then write down the skew lens strength as $\delta_1 = \delta_2 = \delta_3 = \delta_s = \frac{1}{\sqrt{\beta_x \beta_y}}$

For the 90° FODO with half-cell length $L$, $\delta_F = -\delta_D = \frac{\sqrt{2}}{L}$, $\delta_s = \frac{1}{L\sqrt{2}}$
Summary

- Transfer lines present interesting challenges and differences from circular machines
  - No periodic condition mean optics is defined by transfer line element strengths and by initial beam ellipse
  - Matching at the extremes is subject to many constraints
  - Trajectory correction is rather simple compared to circular machine
  - Emittance blow-up is an important consideration, and arises from several sources
  - Phase-plane rotation is sometimes required - skew quads
Keywords for related topics

- Transfer lines
  - Achromat bends
  - Algorithms for optics matching
  - The effect of alignment and gradient errors on the trajectory and optics
  - Trajectory correction algorithms
  - SVD trajectory analysis
  - Kick-response optics measurement techniques in transfer lines
  - Optics measurements including dispersion and $\delta p/p$ with $>3$ screens
  - Different phase-plane exchange insertion solutions