Introduction to „Transverse Beam Dynamics“

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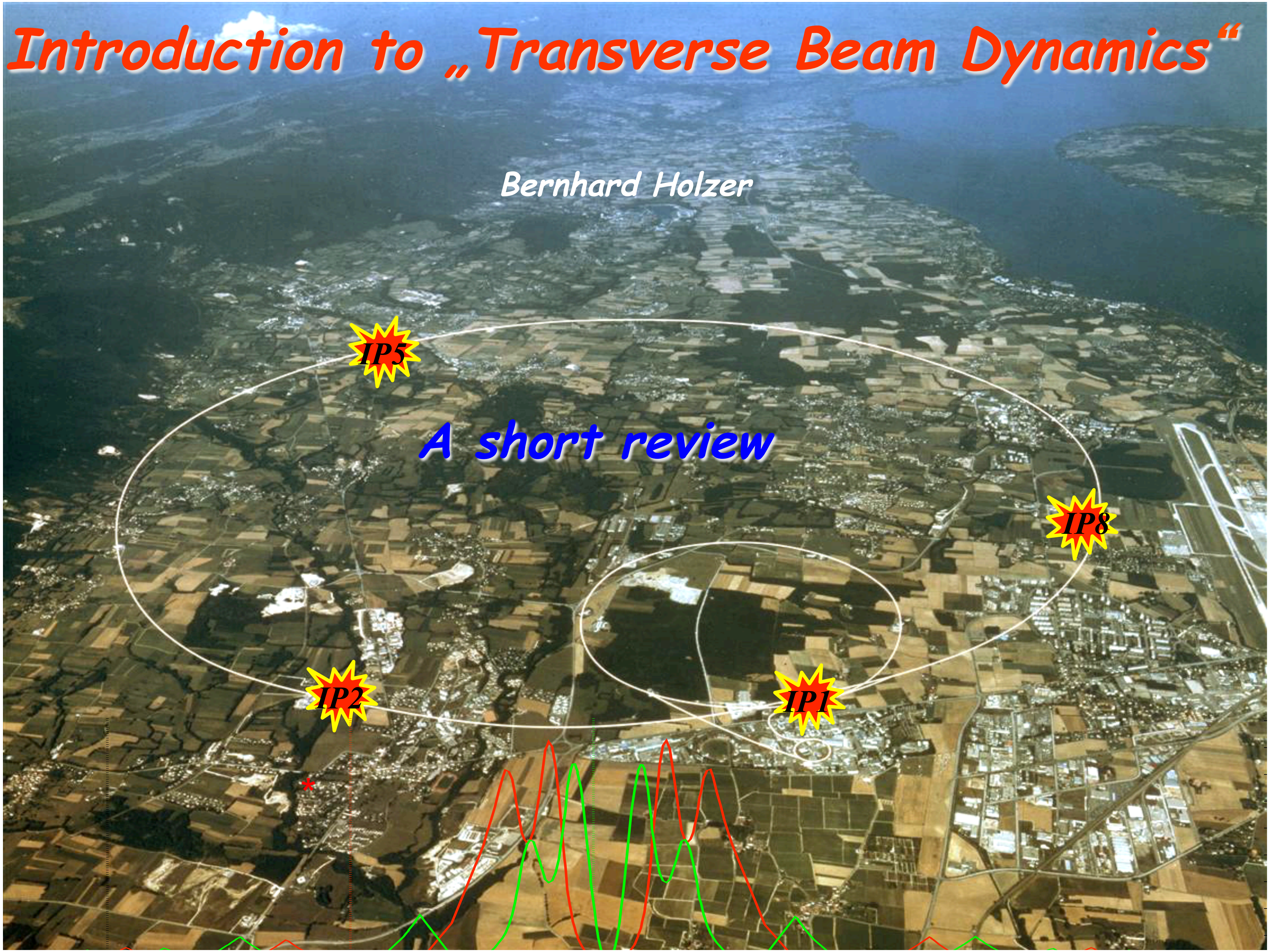
A short review



1.) Introduction and Basic Ideas

„ ... in the end and after all we have to control the geometry of the accelerator or storage ring

→ need transverse deflecting force acting on the parrticle trajectories

Lorentz force $\vec{F} = q * (\vec{E} + \vec{v} \times \vec{B})$

typical velocity in high energy machines: $v \approx c \approx 3*10^8 \, m/s$

Example:

$$B = 1\,T \quad \rightarrow \quad F = q * 3 * 10^8 \frac{m}{s} * 1 \frac{Vs}{m^2}$$

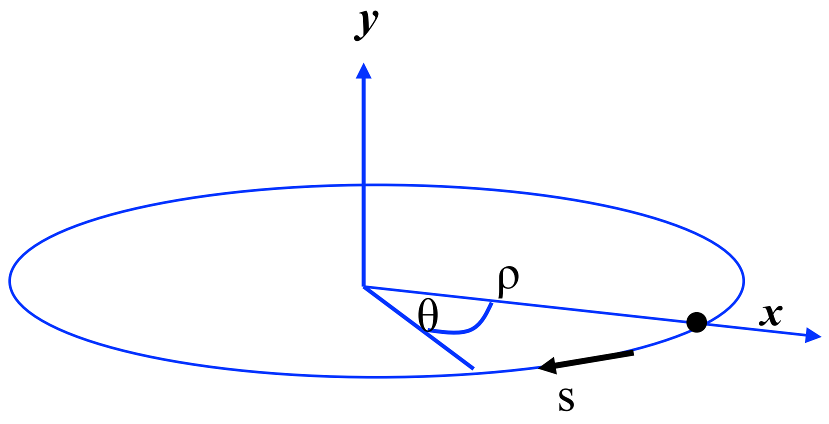
$$F = q * 300 \frac{MV}{m}$$

equivalent electrical field: E

Technical limit for electrical fields:

$$E \leq 1 \frac{MV}{m}$$

Magnetic fields are much stronger than electric ones as soon as the particle velocity is „high enough“.



circular coordinate system

The ideal circular orbit

condition for circular orbit:

Lorentz force $F_L = e v B$

centrifugal force $F_{centr} = \frac{\gamma m_0 v^2}{\rho}$

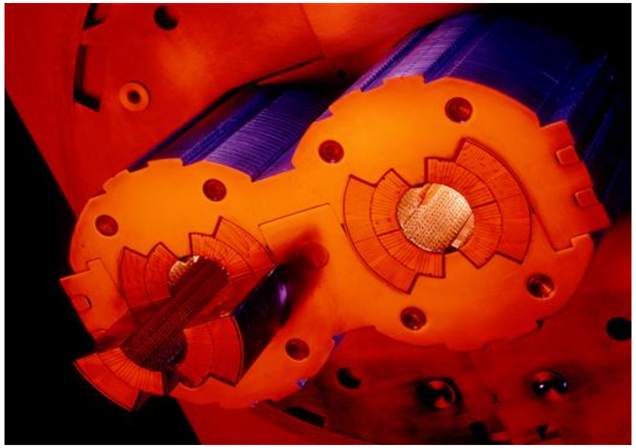
$$\frac{\gamma m_0 \cancel{v}^2}{\rho} = e \cancel{v} B$$

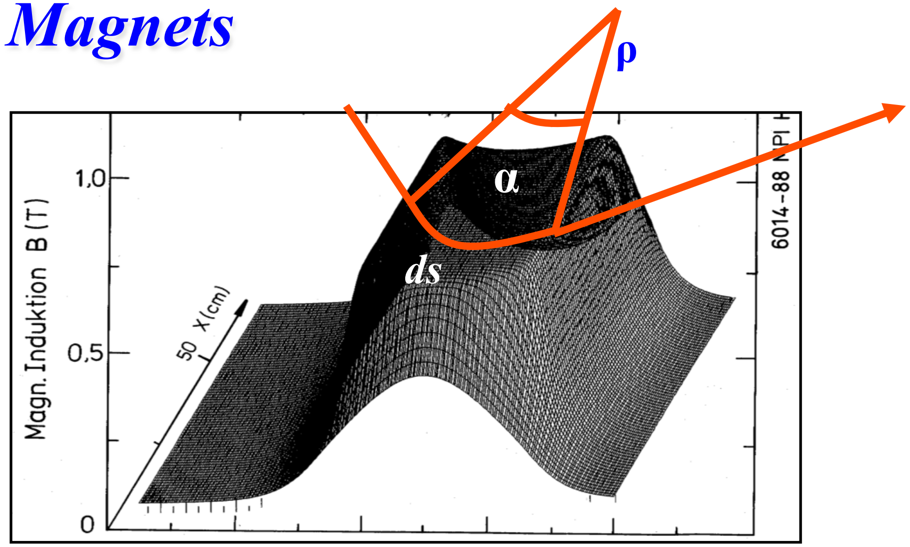
$$\frac{p}{e} = B \rho$$

$B\rho =$ ***"beam rigidity"***

The beam rigidity tells us about the effect of a magnetic field on a particle. Which is valid whenever we deflect the trajectory of a charge particle. ... be it in a storage ring or in a transferline.

The Magnetic Guide Field: Bending Magnets





field map of a storage ring dipole magnet

$$\frac{p}{e} = B\rho \longrightarrow \rho = \frac{p}{B*e} \qquad B \approx 1 \ldots 8\ T$$

The bending radius ... and so the size of the machine is determined by the dipole field and the particle momentum

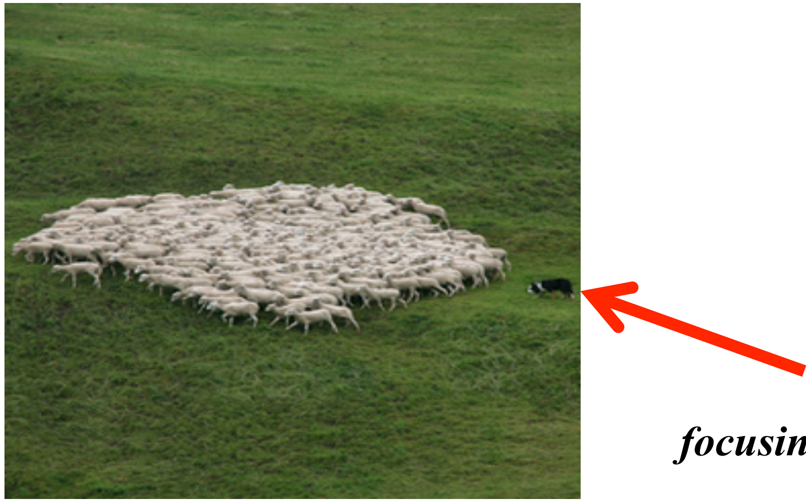
Example LHC, in convenient units:

$B = 8.3\ T\,[Vs/m^2]$ *p=7000 GeV/c* → *ρ=2.83 km*

In case of a storage ring or synchrotron the dipole magnets create a circle (... better polygon) of circumference 2πρ and define the maximum momentum of the particle beam.

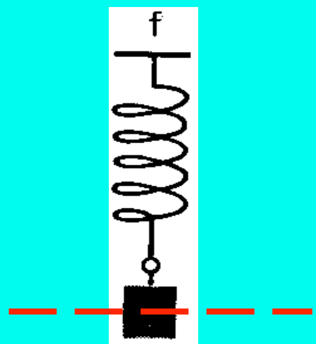
2.) Focusing Properties – Transverse Beam Optics

... keeping the flocs together:
In addition to the pure bending of the beam
we have to keep 10^{11} particles close together



focusing force

classical mechanics:
pendulum



there is a restoring force, proportional to the elongation x:

$$m * \frac{d^2 x}{dt^2} = -c * x$$

general solution: free harmonic oszillation $x(t) = A * \cos(\omega t + \varphi)$

this is how grandma's Kuckuck's clock is working!!!

Quadrupole Magnets:

required: *focusing forces to keep trajectories in vicinity of the ideal orbit*

linear increasing Lorentz force

linear increasing magnetic field $B_y = g\,x \quad , \quad B_x = g\,y$

normalised quadrupole field:

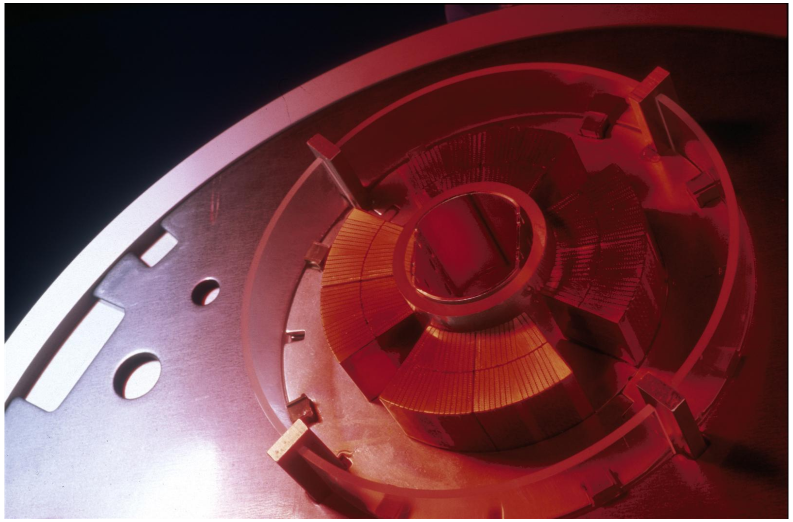
gradient of a quadrupole magnet:

$$g = \frac{2\mu_0 n I}{r^2}$$

$$\longrightarrow \quad k = \frac{g}{p/e}$$

simple rule:

$$k = 0.3\,\frac{g(T/m)}{p(GeV/c)}$$



LHC main quadrupole magnet

$$g \approx 25 \;...\; 220 \;\; T/m$$

what about the vertical plane:
... Maxwell

$$\vec{\nabla} \times \vec{B} = \cancel{\vec{j}} + \cancel{\frac{\partial \vec{E}}{\partial t}} = 0 \qquad \Rightarrow \qquad \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$$

Focusing forces and particle trajectories:

normalise magnet fields to momentum
(remember: $B*\rho = p/q$ *)*

Dipole Magnet

$$\frac{B}{p/q} = \frac{B}{B\rho} = \frac{1}{\rho}$$

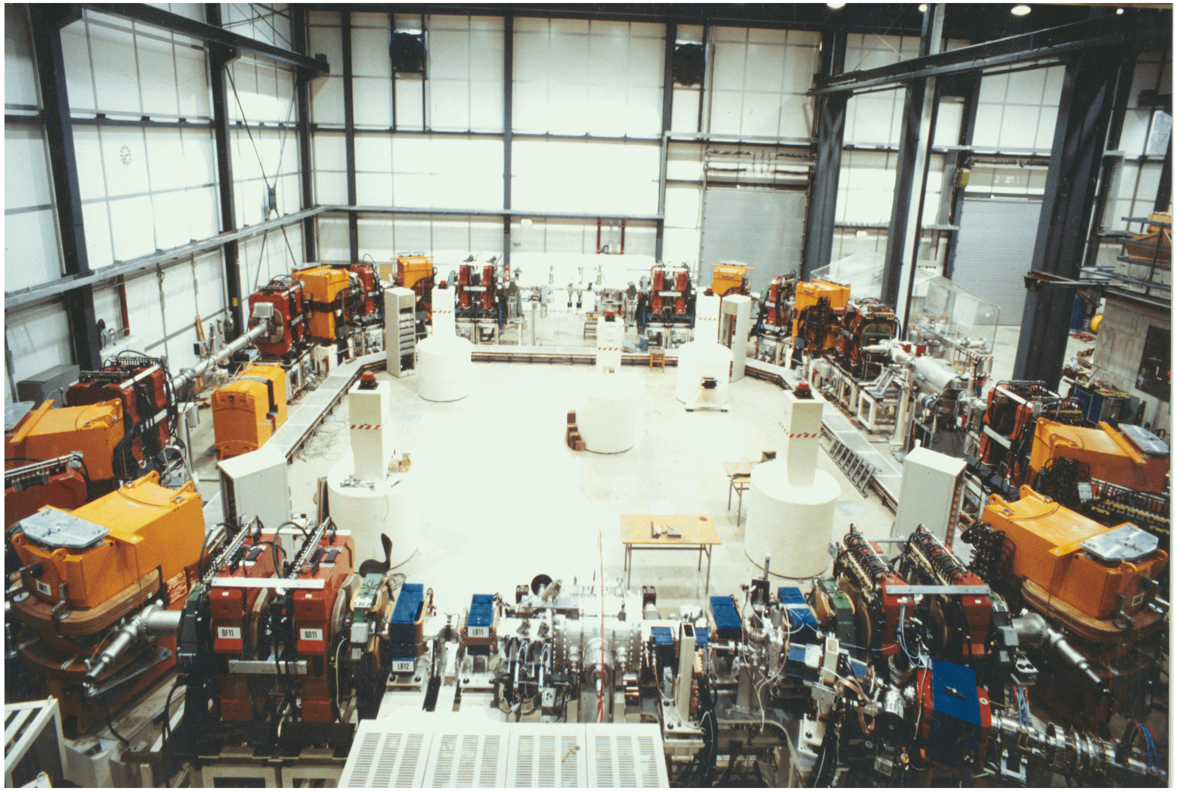
Quadrupole Magnet

$$k := \frac{g}{p/q}$$

The Equation of Motion:

$$\frac{B(x)}{p/e} = \frac{1}{\rho} + k\,x + \cancel{\frac{1}{2!}\,m\,x^2} + \cancel{\frac{1}{3!}\,n\,x^3} + \ldots$$

only terms linear in x, y taken into account ***dipole fields***
quadrupole fields



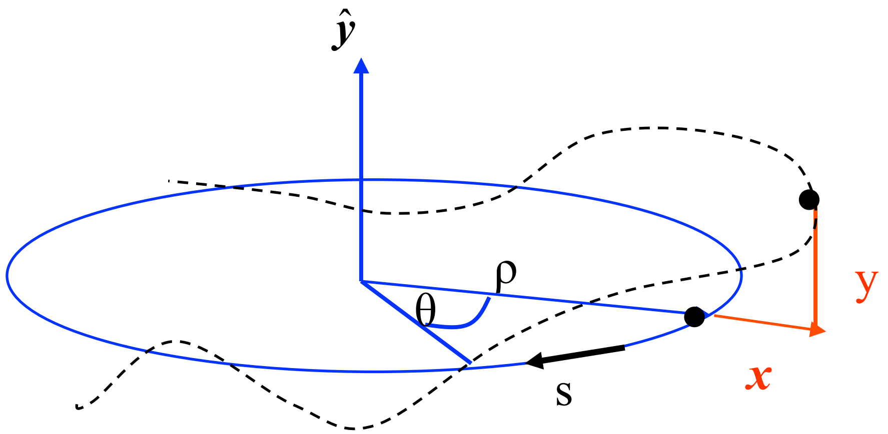
Separate Function Machines:

Split the magnets and optimise them according to their job:

bending, focusing etc

Example:
heavy ion storage ring TSR

The Equation of Motion:



* *Equation for the horizontal motion:*

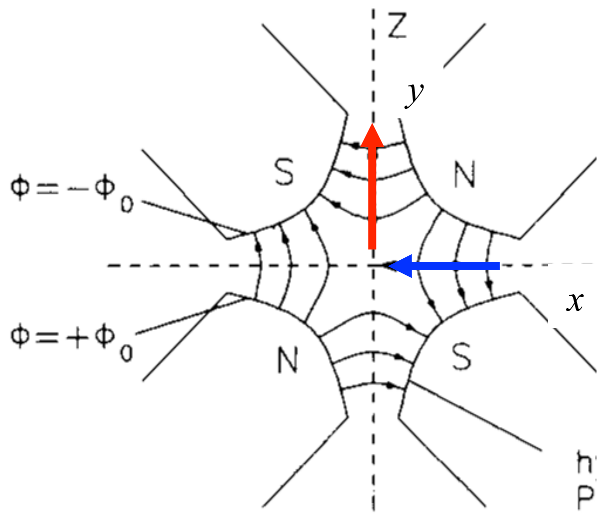
$$x'' + x\left(\frac{1}{\rho^2} + k\right) = 0$$

***x* = particle amplitude**
***x'* = angle of particle trajectory (wrt ideal path line)**

* *Equation for the vertical motion:*

$\frac{1}{\rho^2} = 0$ ***no dipoles … in general …***

$k \leftrightarrow -k$ ***quadrupole field changes sign***



$$y'' - k\,y = 0$$

Remark: *… there seems to be a focusing even without a quadrupole gradient*

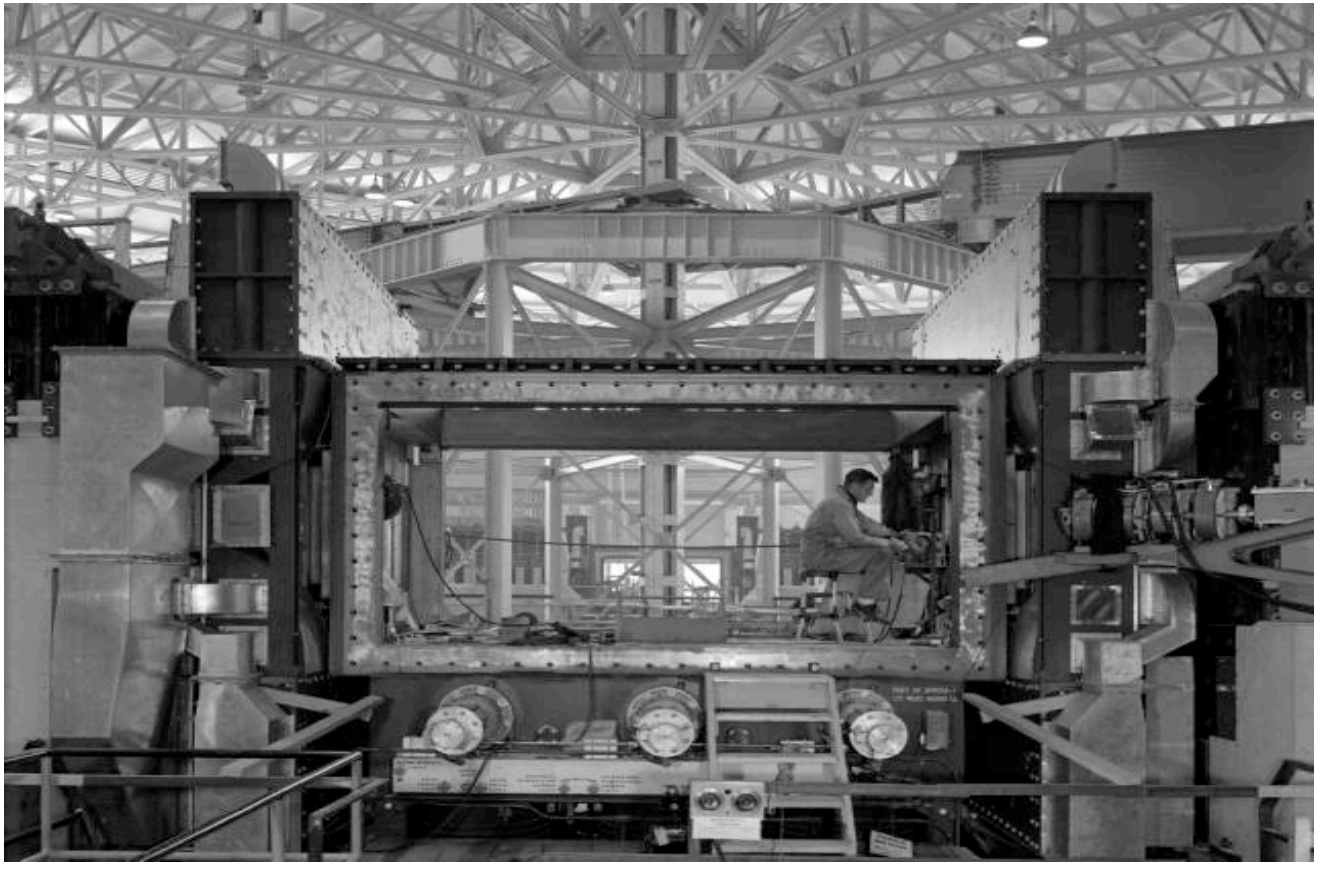
„weak focusing of dipole magnets"

$$x'' + (\frac{1}{\rho^2} - k) \cdot x = 0 \qquad \boldsymbol{k} = 0 \quad \Rightarrow \quad \boldsymbol{x}'' = -\frac{1}{\boldsymbol{\rho}^2}\,\boldsymbol{x}$$

even without quadrupoles there is a retriving force (i.e. focusing) in the bending plane of the dipole magnets
… however ... in large machines it is weak. (!)

The last weak focusing high energy machine … BEVATRON

→ *large apertures needed*
→ *very expensive magnets*



4.) Solution of Trajectory Equations

Define … hor. plane: K= 1/ρ² + k

… vert. Plane: K = - k

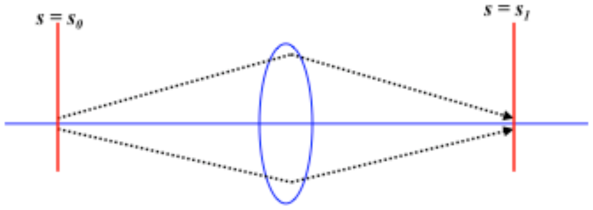
$$x'' + K\,x = 0$$

Differential Equation of harmonic oscillator … with spring constant K

Ansatz: *Hor. Focusing Quadrupole K > 0:*

$$x(s) = x_0 \cdot \cos(\sqrt{|K|}s) + x'_0 \cdot \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s)$$

$$x'(s) = -x_0 \cdot \sqrt{|K|} \cdot \sin(\sqrt{|K|}s) + x'_0 \cdot \cos(\sqrt{|K|}s)$$



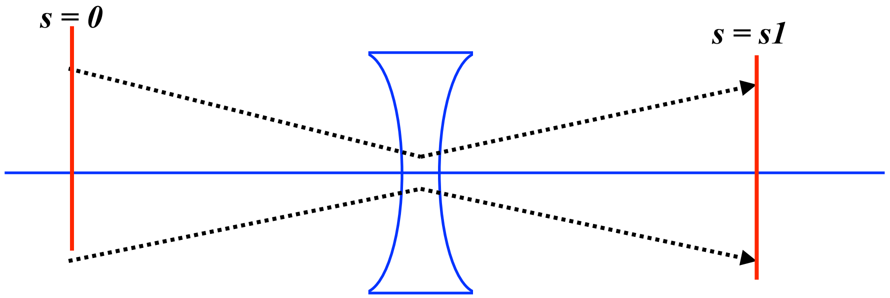
For convenience expressed in matrix formalism:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M_{foc} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$

$$M_{foc} = \begin{pmatrix} \cos\left(\sqrt{|K|}l\right) & \frac{1}{\sqrt{|K|}} \sin\left(\sqrt{|K|}l\right) \\ -\sqrt{|K|}\sin\left(\sqrt{|K|}l\right) & \cos\left(\sqrt{|K|}l\right) \end{pmatrix}$$

hor. defocusing quadrupole:

$$x'' - K\,x = 0$$



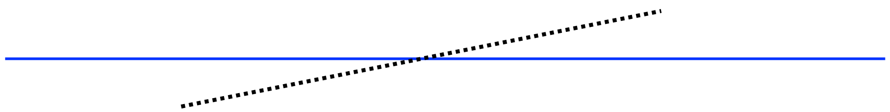
Ansatz: ***Remember from school***

$$x(s) = a_1 \cdot \cosh(\omega s) + a_2 \cdot \sinh(\omega s)$$

$$M_{defoc} = \begin{pmatrix} \cosh\sqrt{|K|}\,l & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}\,l \\ \sqrt{|K|}\sinh\sqrt{|K|}\,l & \cosh\sqrt{|K|}\,l \end{pmatrix}$$

drift space:

$$K = 0$$



$$x(s) = x_0' * s$$

$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

! ***with the assumptions made, the motion in the horizontal and vertical planes are independent „ ... the particle motion in x & y is uncoupled“***

Combining the two planes:

Clear enough (hopefully ... ?) : a quadrupole magnet that is focussing o-in one plane acts as defocusing lens in the other plane ... et vice versa.

hor foc. quadrupole lens

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}s \\ -\sqrt{|K|}\sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}$$

matrix of the same magnet in the vert. plane:

$$M_{defoc} = \begin{pmatrix} \cosh\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}l \\ \sqrt{|K|}\sinh\sqrt{|K|}l & \cosh\sqrt{|K|}l \end{pmatrix}$$

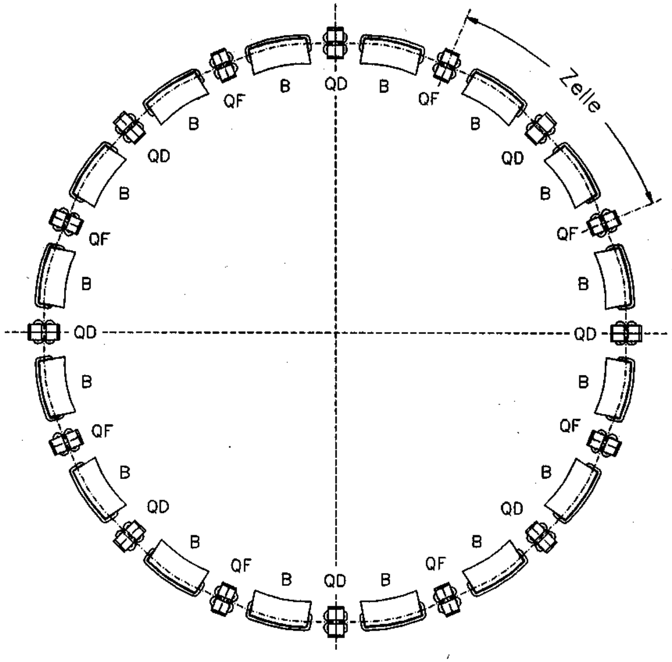
$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_f = \begin{pmatrix} \cos(\sqrt{|k|}s) & \frac{1}{\sqrt{|k|}}\sin(\sqrt{|k|}s) & 0 & 0 \\ -\sqrt{|k|}\sin(\sqrt{|k|}s) & \cos(\sqrt{|k|}s) & 0 & 0 \\ 0 & 0 & \cosh(\sqrt{|k|}s) & \frac{1}{\sqrt{|k|}}\sinh(\sqrt{|k|}s) \\ 0 & 0 & \sqrt{|k|}\sinh(\sqrt{|k|}s) & \cosh(\sqrt{|k|}s) \end{pmatrix} * \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_i$$

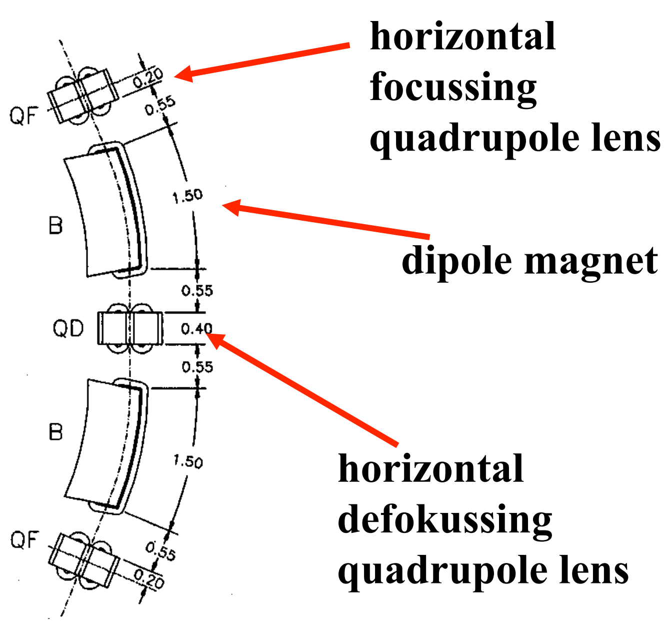
„veni vidi vici ...“ .... or in english .... „we got it !“

* we can calculate the trajectory of a single particle, inside a storage ring magnet (lattice element)
* for arbitrary initial conditions x_0, x'_0
* we can combine these trajectory parts (also mathematically) and so get the complete transverse trajectory around the storage ring

$$M_{total} = M_{QF} * M_D * M_{QD} * M_{Bend} * M_{D*} \dots$$

Beispiel: Speichering für Fußgänger (Wille)



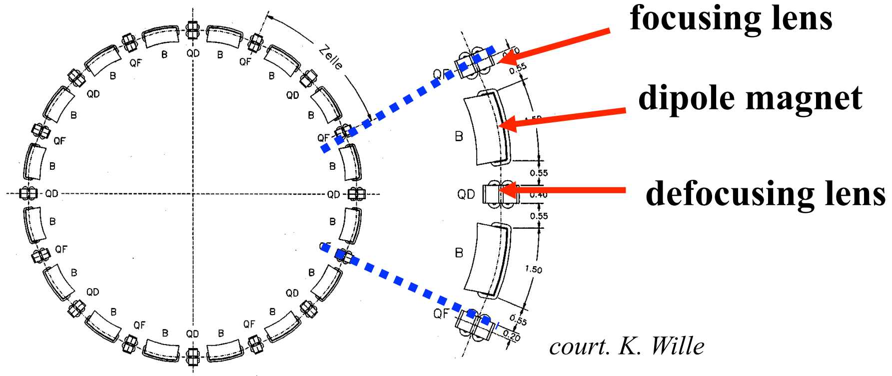


Transformation through a system of lattice elements

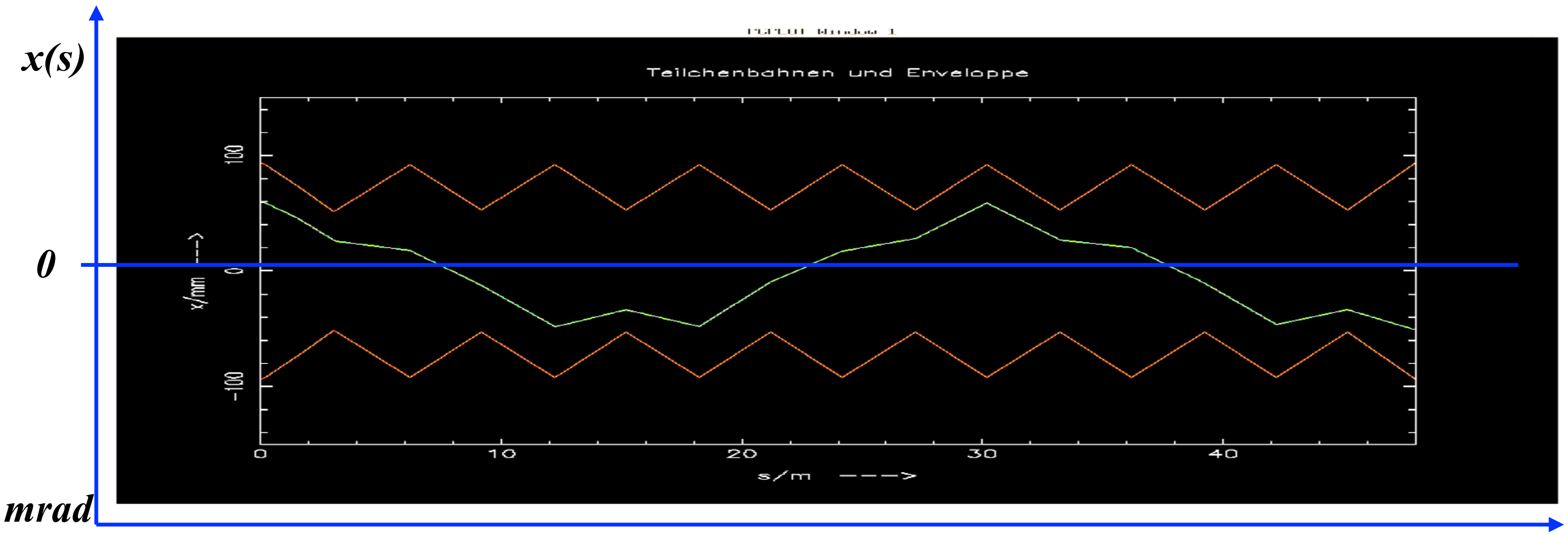
combine the single element solutions by multiplication of the matrices

$$M_{total} = M_{QF} * M_D * M_{QD} * M_{Bend} * M_{D*}.....$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s_2, s_1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$



in each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator ! ! !

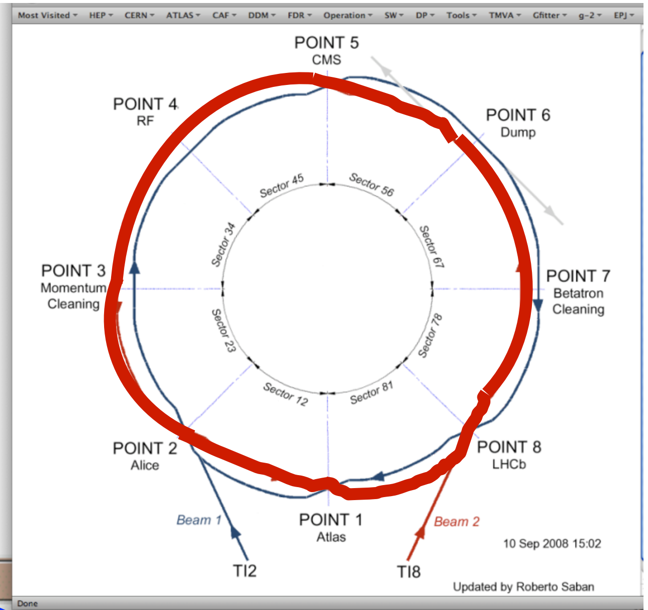


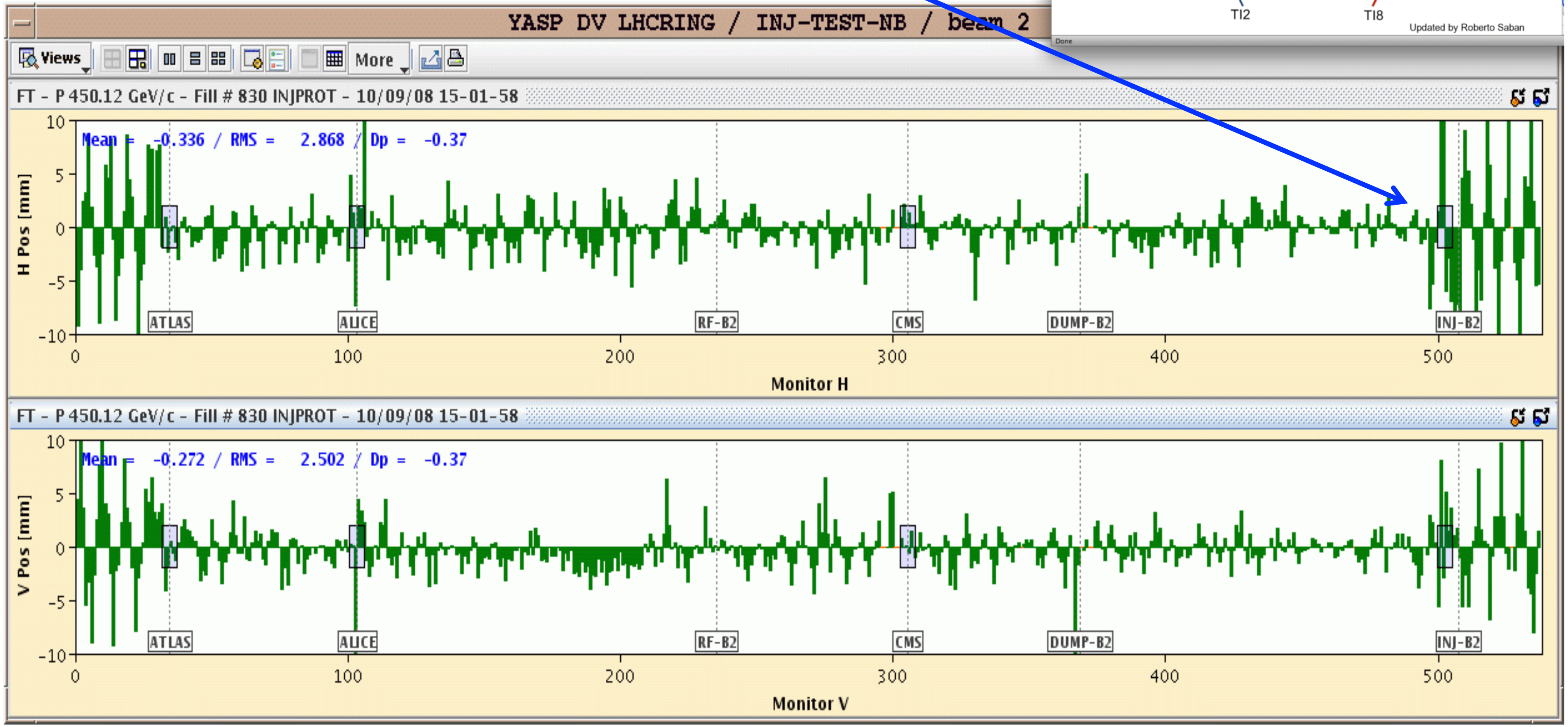
typical values
in a strong
foc. machine:
x ≈ mm, x´ ≤ mrad

LHC Operation: Beam Commissioning

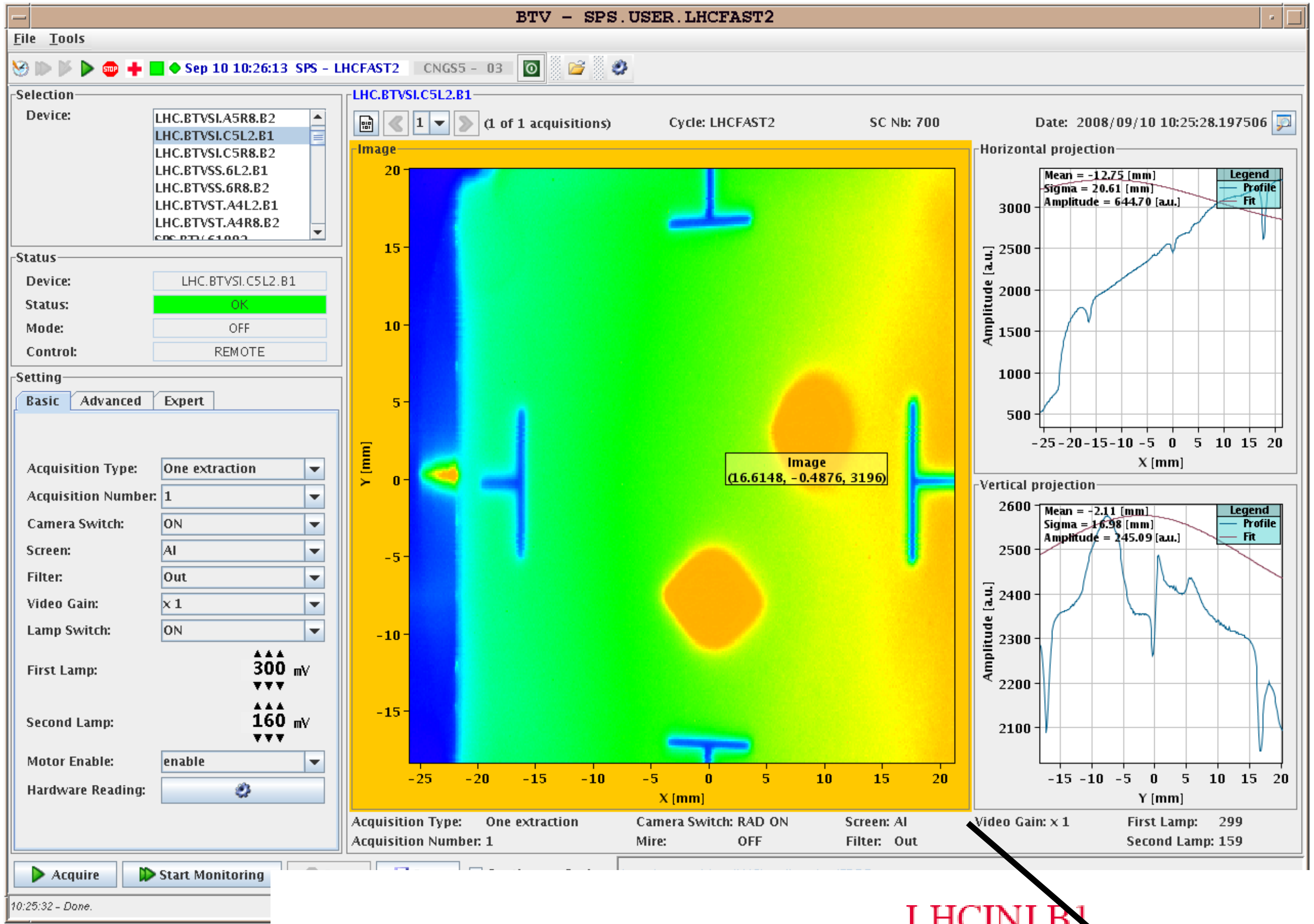
First turn steering "sector by sector:"

Treat the machine as transferline !!



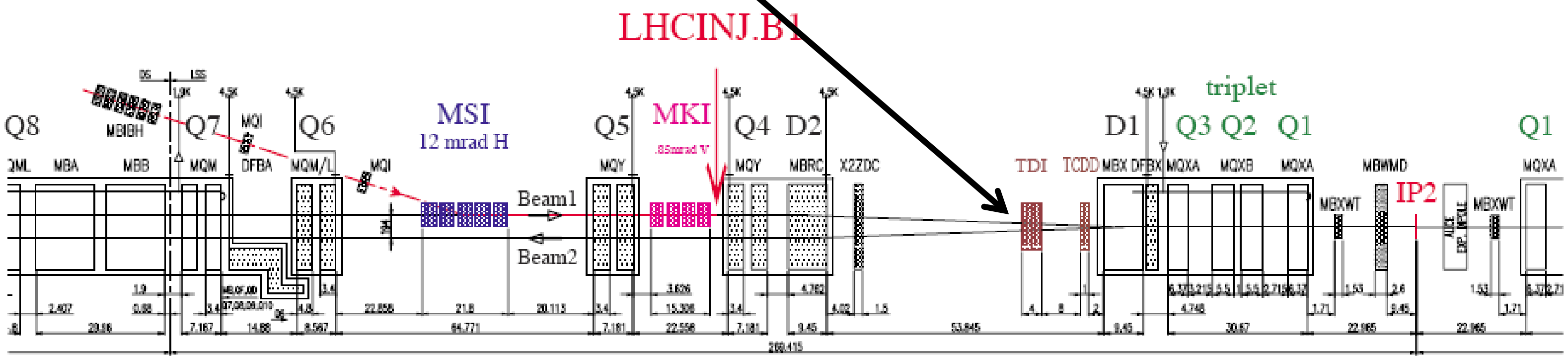


LHC Operation: the First Beam



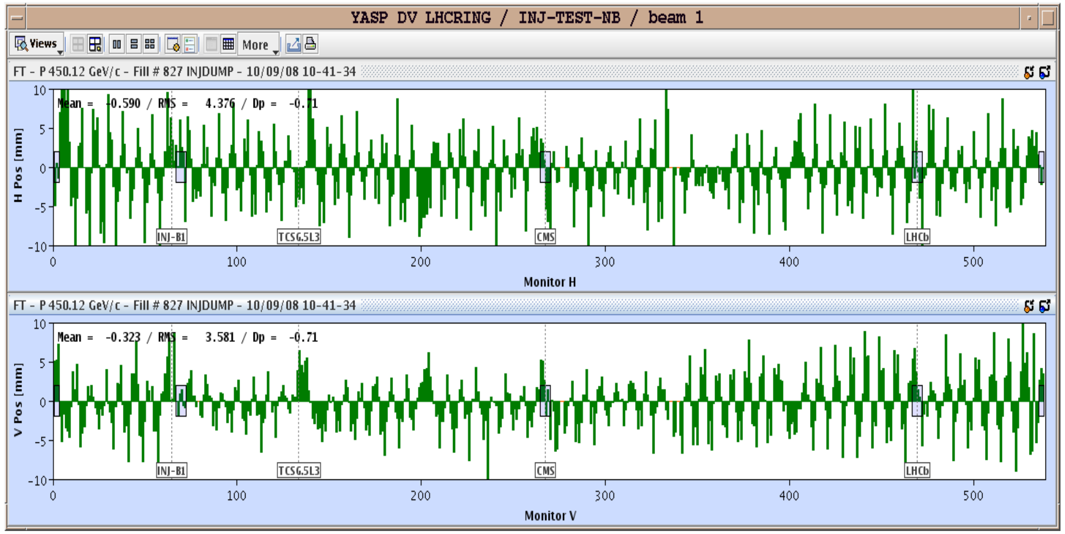
Beam 1 on OTR screen
1st and 2nd turn

!!! this is NOT (yet)
a storage ring !!!



5.) Orbit & Tune:

as soon as we close the orbit, we enter the world of
"closed orbits",
synchrotrons,
storage rings.
in other words: periodic conditions

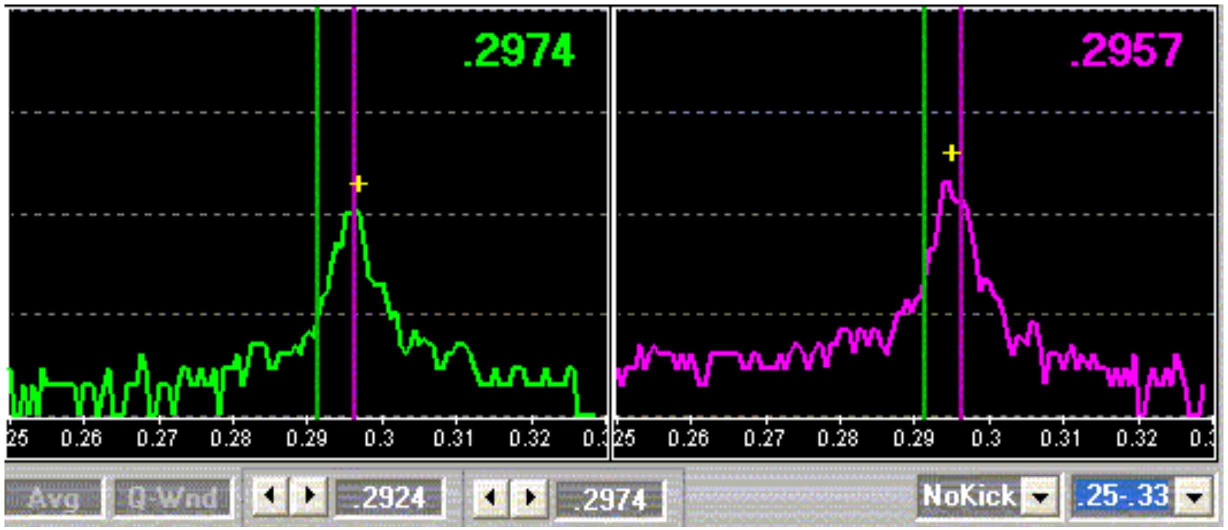


Tune: number of oscillations per turn

$Q_x = 64.31, \quad Q_y = 59.32$

LHC revolution frequency: 11.3 kHz

$0.31 * 11.3 = 3.5 kHz$



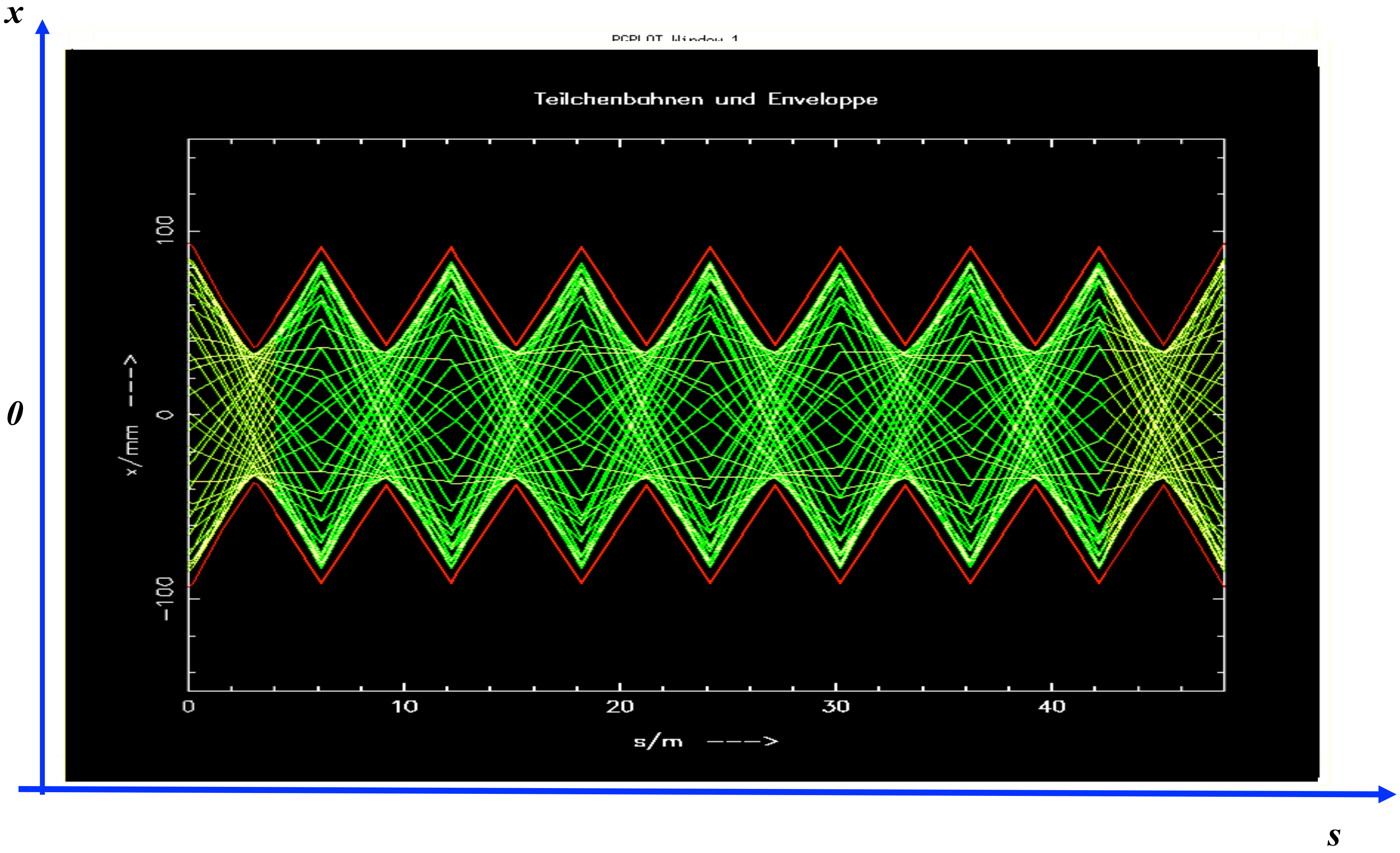
... and the tunes in x and y are different.

i.e. we can apply different focusing forces in the two planes

i.e. we can create different beam sizes in the two planes

Question: ***what will happen, if the particle performs a second turn ?***

... or a third one or ... 10^{10} turns



Astronomer Hill:

differential equation for motions with periodic focusing properties
„Hill 's equation"

Example: particle motion with periodic coefficient

equation of motion: $$x''(s) - k(s)x(s) = 0$$

restoring force ≠ const,
$k(s)$ = depending on the position s
$k(s+L) = k(s)$, periodic function

we expect a kind of quasi harmonic oscillation: amplitude & phase will depend on the position s in the ring.

6.) The Beta Function

„it is convenient to see"... after some beer

... we make two statements:

1.) There exists a mathematical function, that defines the envelope of all particle trajectories and so can act as measure for the beam size. We call it the β – function.

2.) Whow !!

A particle oscillation can then be written in the form

$$x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s) + \phi)$$

ε, Φ = integration constants determined by initial conditions

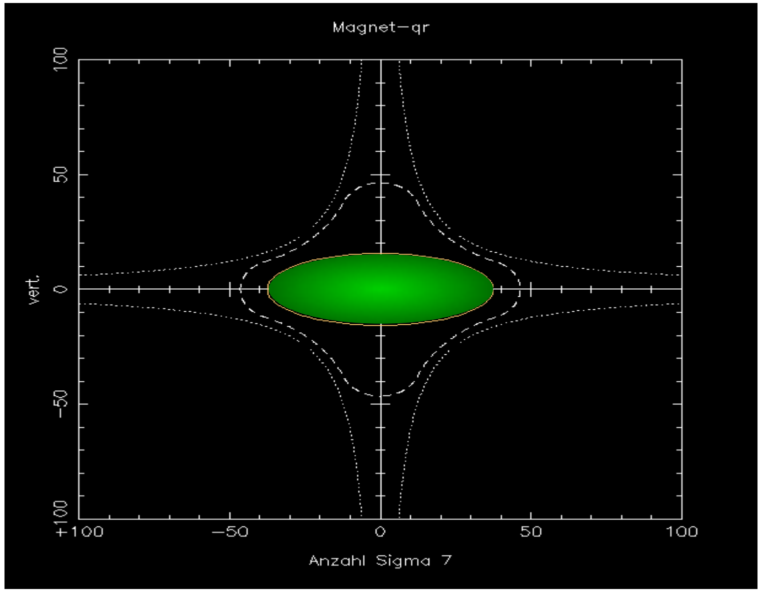
β(s) periodic function given by focusing properties of the lattice ↔ quadrupoles

$$\beta(s+L) = \beta(s)$$

ε beam emittance = woozilycity of the particle ensemble, intrinsic beam parameter,
cannot be changed by the foc. properties.
scientifiquely spoken: area covered in transverse x, x´ phase space

… and it is constant !!!

The Beta Function

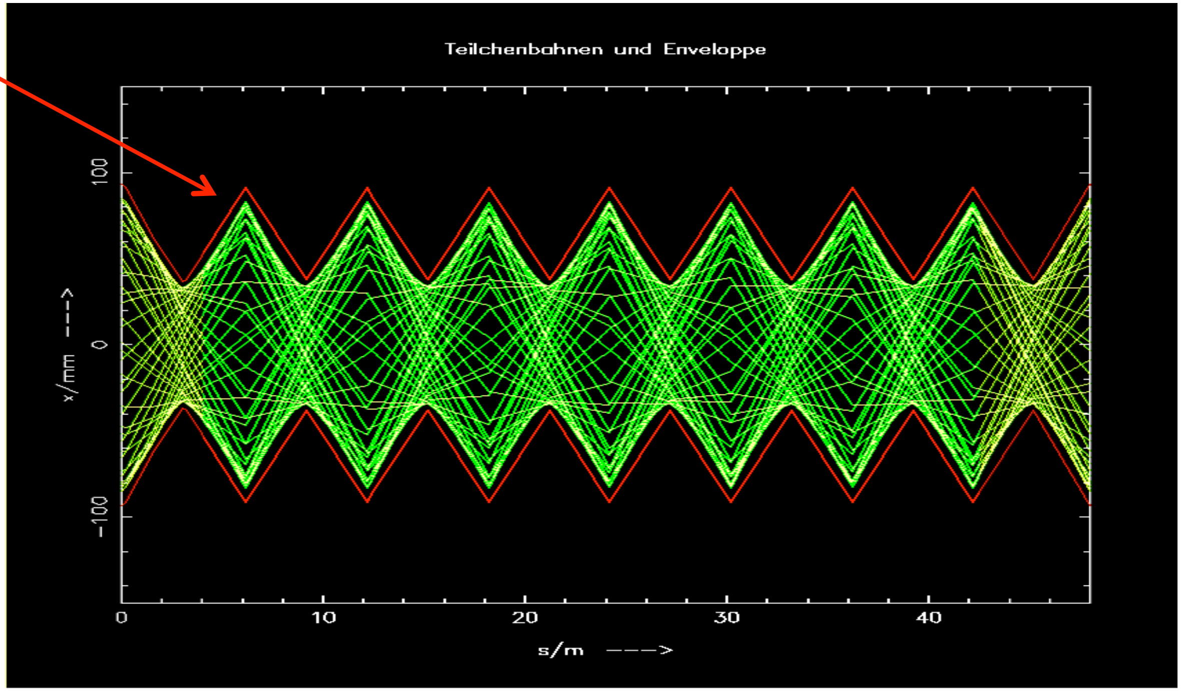


If we obtain the x, x' coordinates of a particle trajectory via

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s_2, s_1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$

The maximum size of any particle amplitude at a position "s" is given by

$$\hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$



β determines the beam size
(... the envelope of all particle trajectories at a given position "s" in the storage ring.

It reflects the periodicity of the magnet structure.

7.) Beam Emittance and Phase Space Ellipse

general solution of Hill equation

$$(1) \qquad x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\,\cos(\psi(s)+\phi)$$

$$(2) \qquad x'(s) = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\,\{\alpha(s)\cos(\psi(s)+\phi)+\sin(\psi(s)+\phi)\}$$

from (1) we get

$$\cos(\psi(s)+\phi) = \frac{x(s)}{\sqrt{\varepsilon}\,\sqrt{\beta(s)}}$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

Insert into (2) and solve for ε

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$

***** *ε* **is a** *constant* **of the motion** **…** *it is independent of „s“*
***** **parametric representation of an** *ellipse in the x x ʻ space*
***** **shape and orientation of ellipse are given by α, β, γ**

Phase Space Ellipse

particel trajectory: $x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\}$

max. Amplitude: $\hat{x}(s) = \sqrt{\varepsilon\beta}$ → *x´ at that position …?*

… put $\hat{x}(s)$ *into* $\boldsymbol{\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s)\, x'^2(s)}$ *and solve for x´*

$$\varepsilon = \gamma \cdot \varepsilon\beta + 2\alpha\sqrt{\varepsilon\beta}\cdot x' + \beta x'^2$$

$$\rightarrow \quad x' = -\alpha\cdot\sqrt{\varepsilon/\beta}$$

* ***A high β-function means a large beam size and a small beam divergence.*** **!**
  … et vice versa !!!

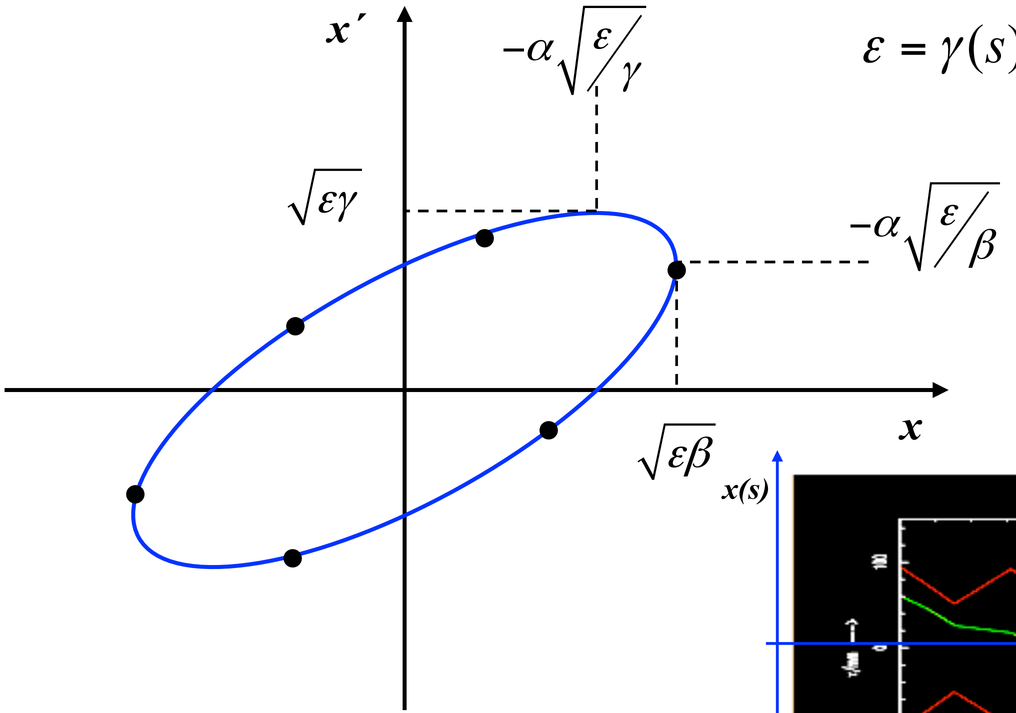
* ***In the middle of a quadrupole β = maximum,***
  α = zero } $x' = 0$

… and the ellipse is flat

Beam Emittance and Phase Space Ellipse

In phase space x, x' a particle oscillation, observed at a given position "s" in the ring is running on an ellipse ... making Q revolutions per turn.

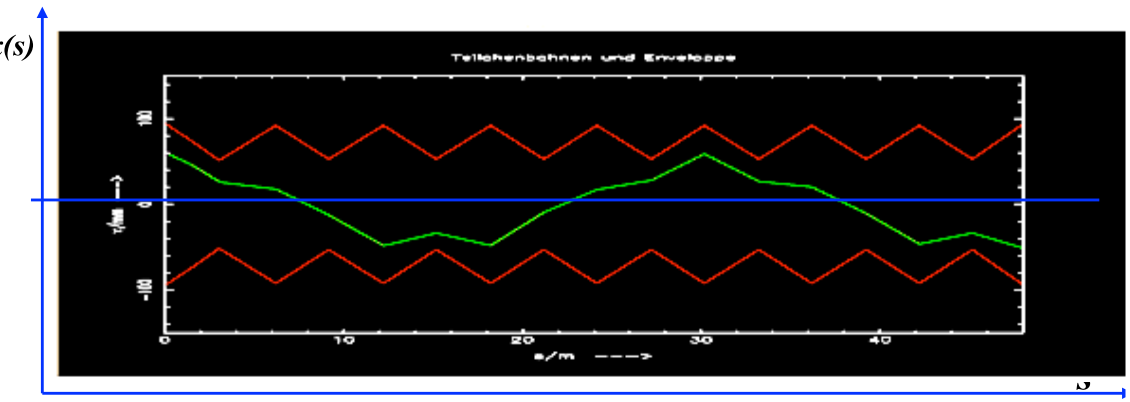
$$x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s) + \varphi)$$



$$\varepsilon = \gamma(s) * x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$

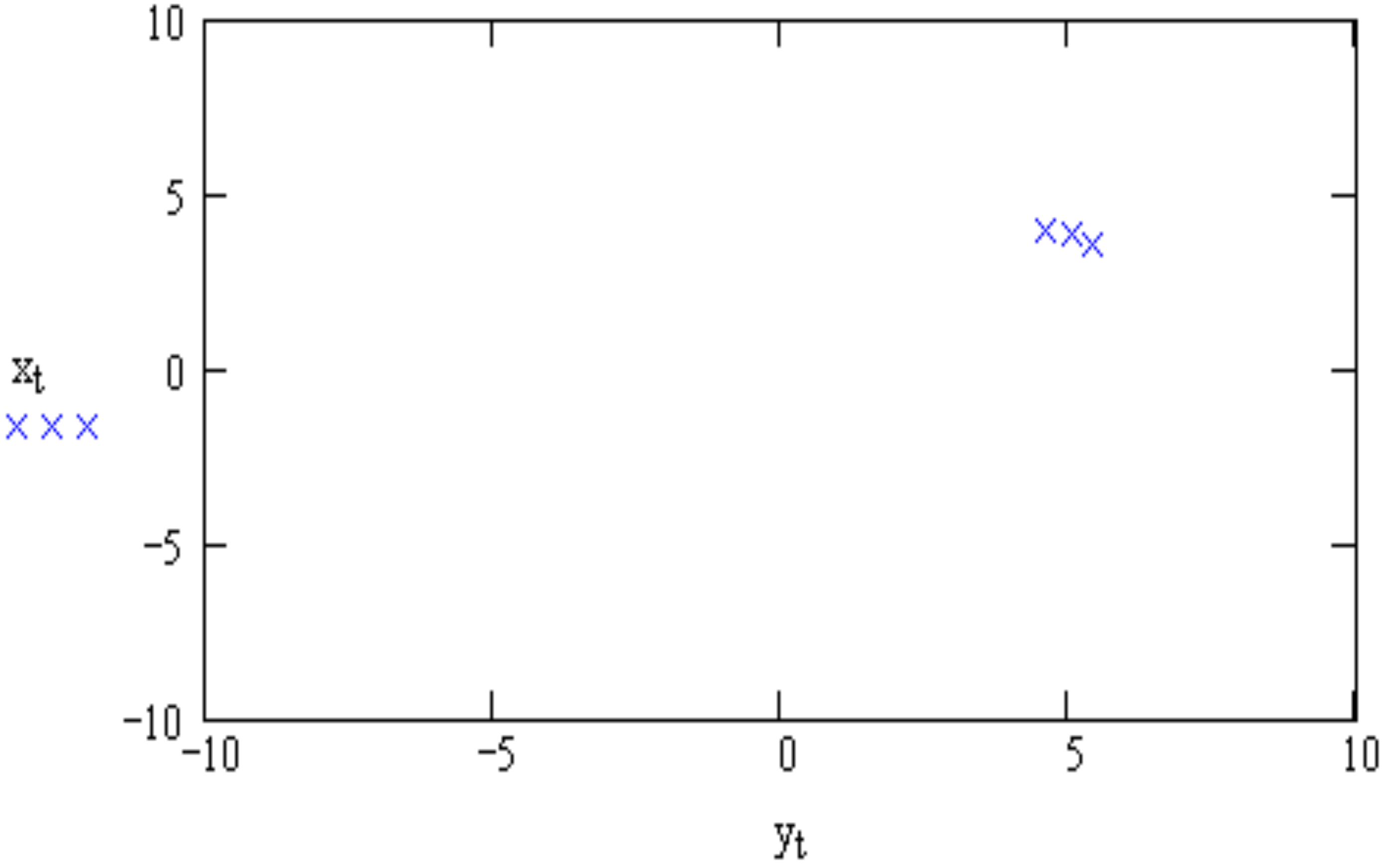
Liouville:* in reasonable storage rings *area in phase space is constant.

A = π*ε=const

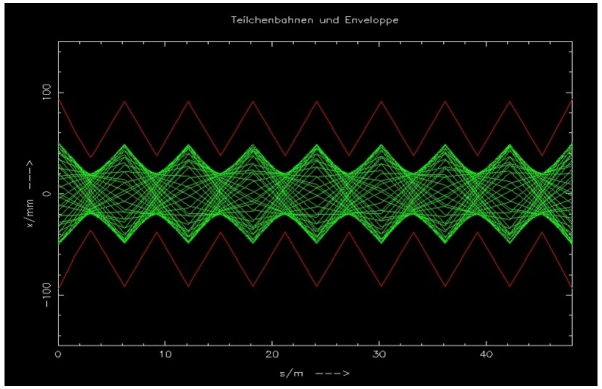


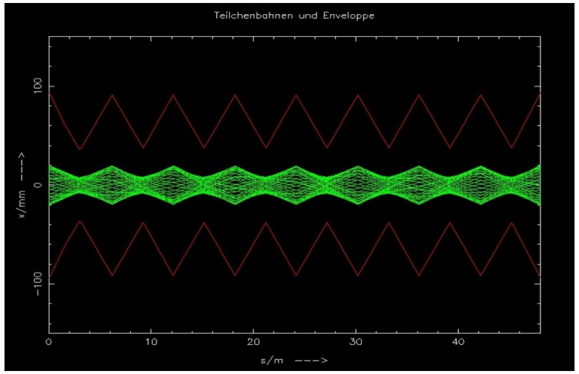
… and now the ellipse:

note for each turn x, x´at a given position „s_1“ and plot in the phase space diagram



Emittance of the Particle Ensemble:

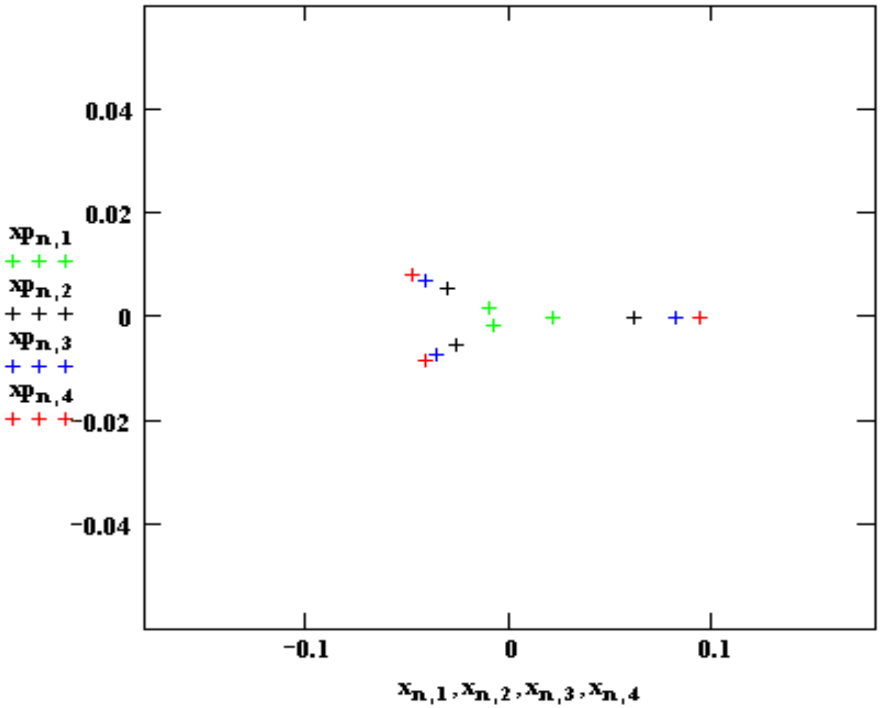




... to be very clear:

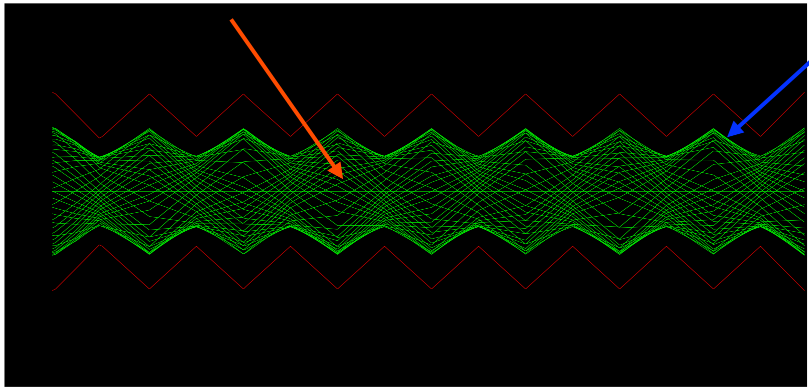
as long as our particle is running on an ellipse in x, x' space everything is alright, the beam is stable and we can sleep well at nights.

If however we have scattering at the rest gas, or non-linear fields, or beam collisions (!) the particle will perform a jump in x' and ε will increase



Emittance of the Particle Ensemble:

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\Psi(s) + \phi) \qquad \hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$



single particle trajectories, N ≈ 10 [11] per bunch

Gauß Particle Distribution:

$$\rho(x) = \frac{N \cdot e}{\sqrt{2\pi}\sigma_x} \cdot e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}}$$

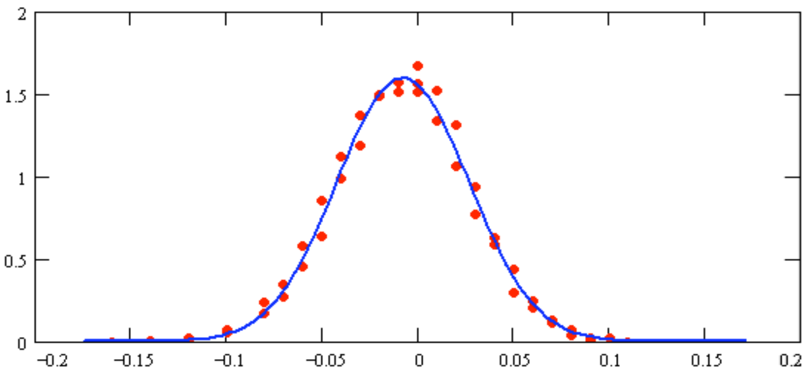
particle at distance 1 σ from centre
↔ 68.3 % of all beam particles

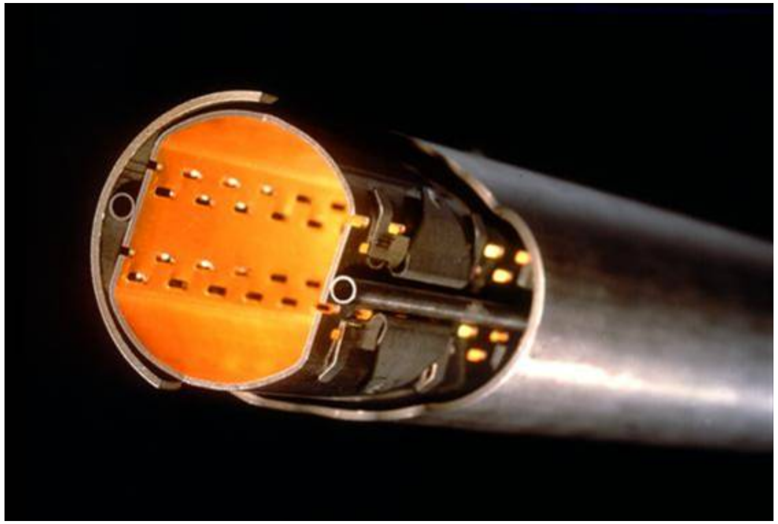
LHC:

$$\beta = 180\,m$$

$$\varepsilon = 5*10^{-10}\,m\,rad$$

$$\sigma = \sqrt{\varepsilon * \beta} = \sqrt{5*10^{-10}m*180m} = 0.3\,mm$$





aperture requirements: $r_0 = 12 * \sigma$

Statistical Interpretation of the beam emittance

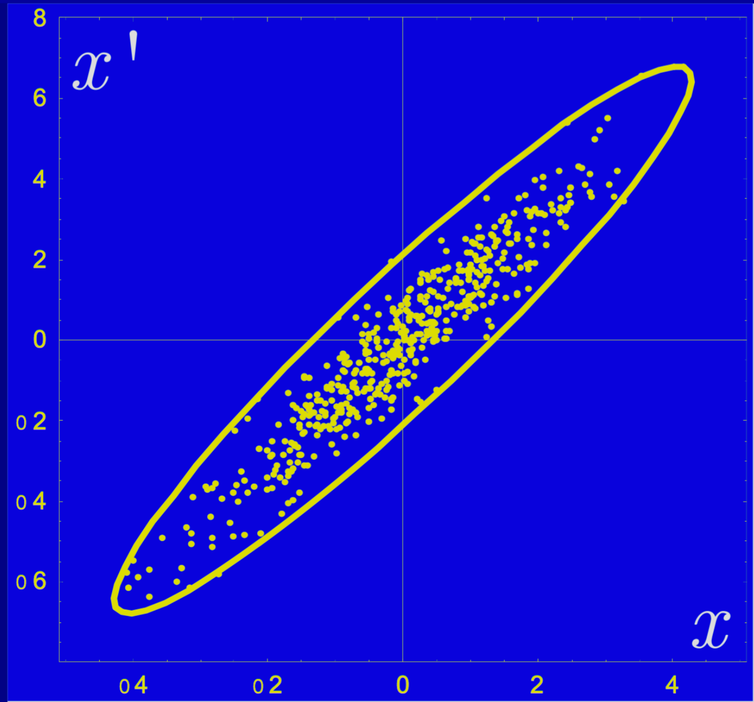
Sometimes we express the rms emittance as determinant of a "beam" or sigma matrix

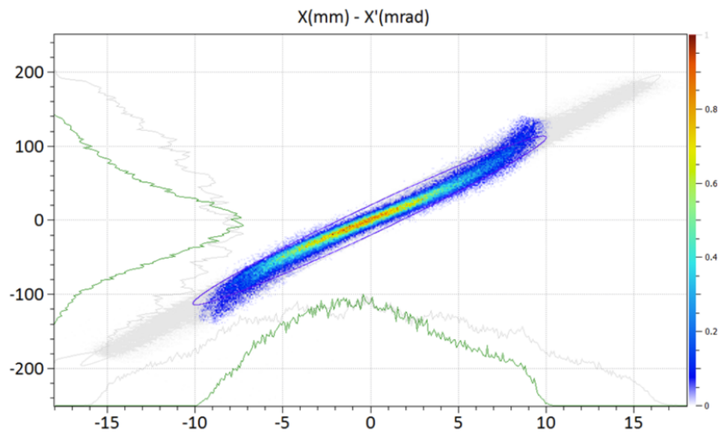
2nd-order moments:

$$\sigma = \begin{pmatrix} \langle x^2 \rangle & \langle x x' \rangle \\ \langle x x' \rangle & \langle x'^2 \rangle \end{pmatrix}$$

$$\varepsilon = \sqrt{|\sigma|}$$

$$= \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2}$$





With the obvious connection to the Twiss parametrisation

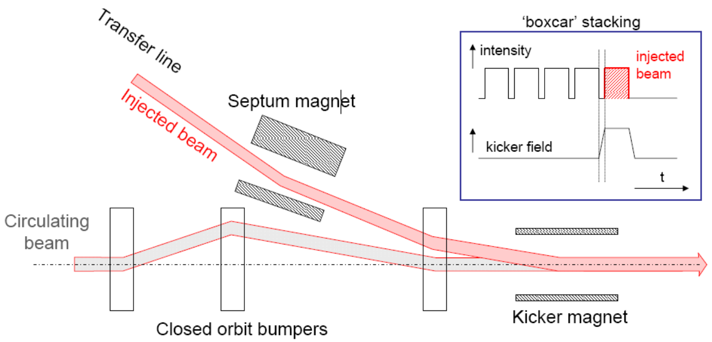
$$\sigma = \begin{pmatrix} \varepsilon\beta & -\varepsilon\alpha \\ -\varepsilon\alpha & \varepsilon(1+\alpha^2)/\beta \end{pmatrix}$$

The phase space area can differ considerably from the ideal ellipse in case of non-linear fields or special initial distributions

for details see e.g. N.Walker in CAS 2005

8.) Transferlines & Injection: Errors & Tolerances

* *quadrupole strengths --> "beta beat" $\Delta\beta / \beta$*
* *alignment of magnets --> orbit distortion in transferline & storage ring*
* *septum & kicker pulses --> orbit distortion & emittance dilution in storage ring*



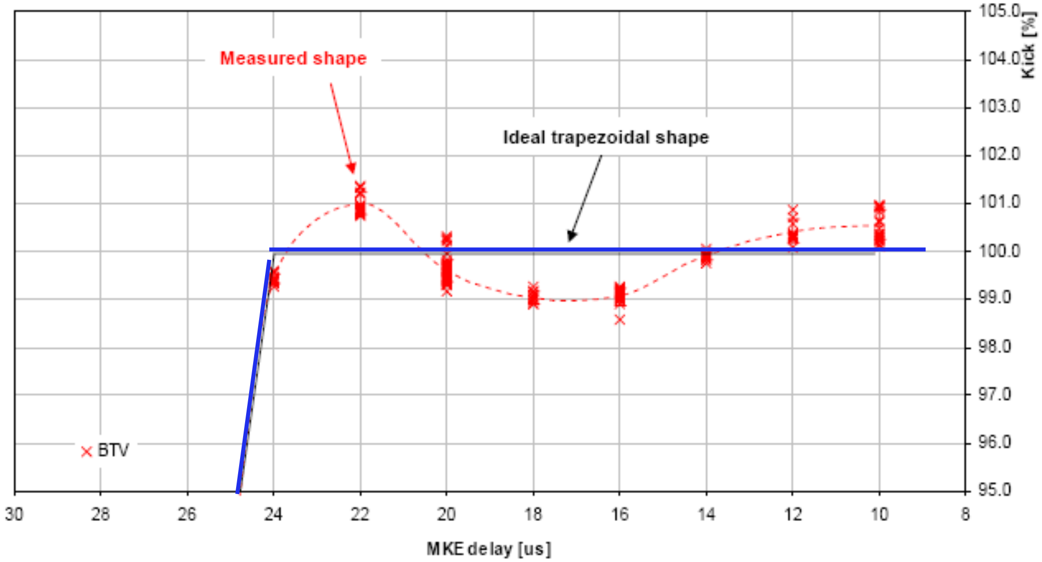
Example: Error in position Δa:

$$\varepsilon_{new} = \varepsilon_0 * (1 + \frac{\Delta a^2}{2})$$

Δa =0.5 σ

$\rightarrow$ $$\varepsilon_{new} = 1.125 * \varepsilon_0$$

And remember: 1 σ < mm

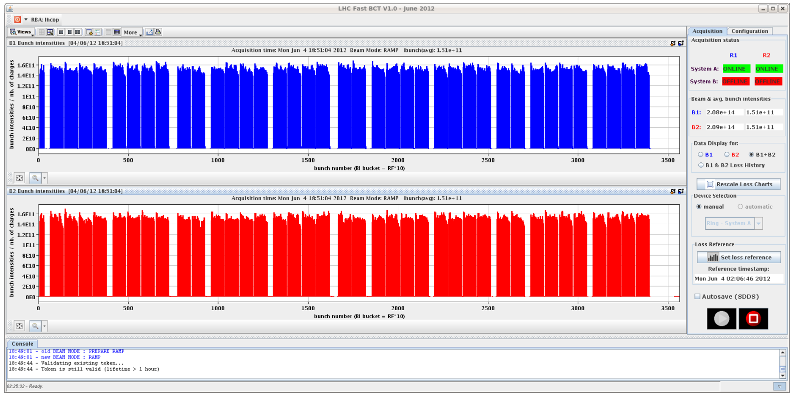


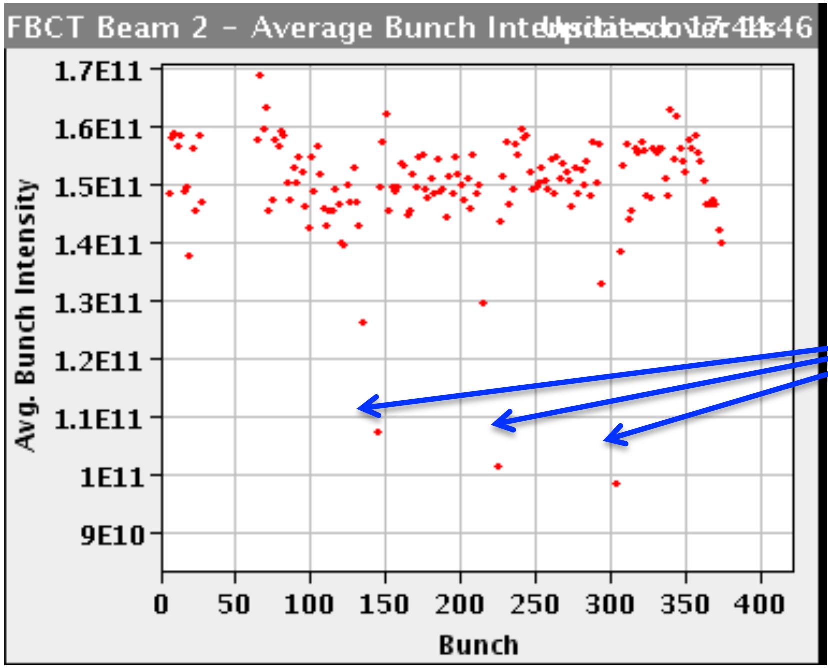
Kicker "plateau" at the end of the PS - SPS transferline measured via injection - oscillations

Problems with Emittance dilution:
it is only too real: *LHC logbook: Sat 9-June "Late-Shift"*

18:18h injection for physics

clean injection !





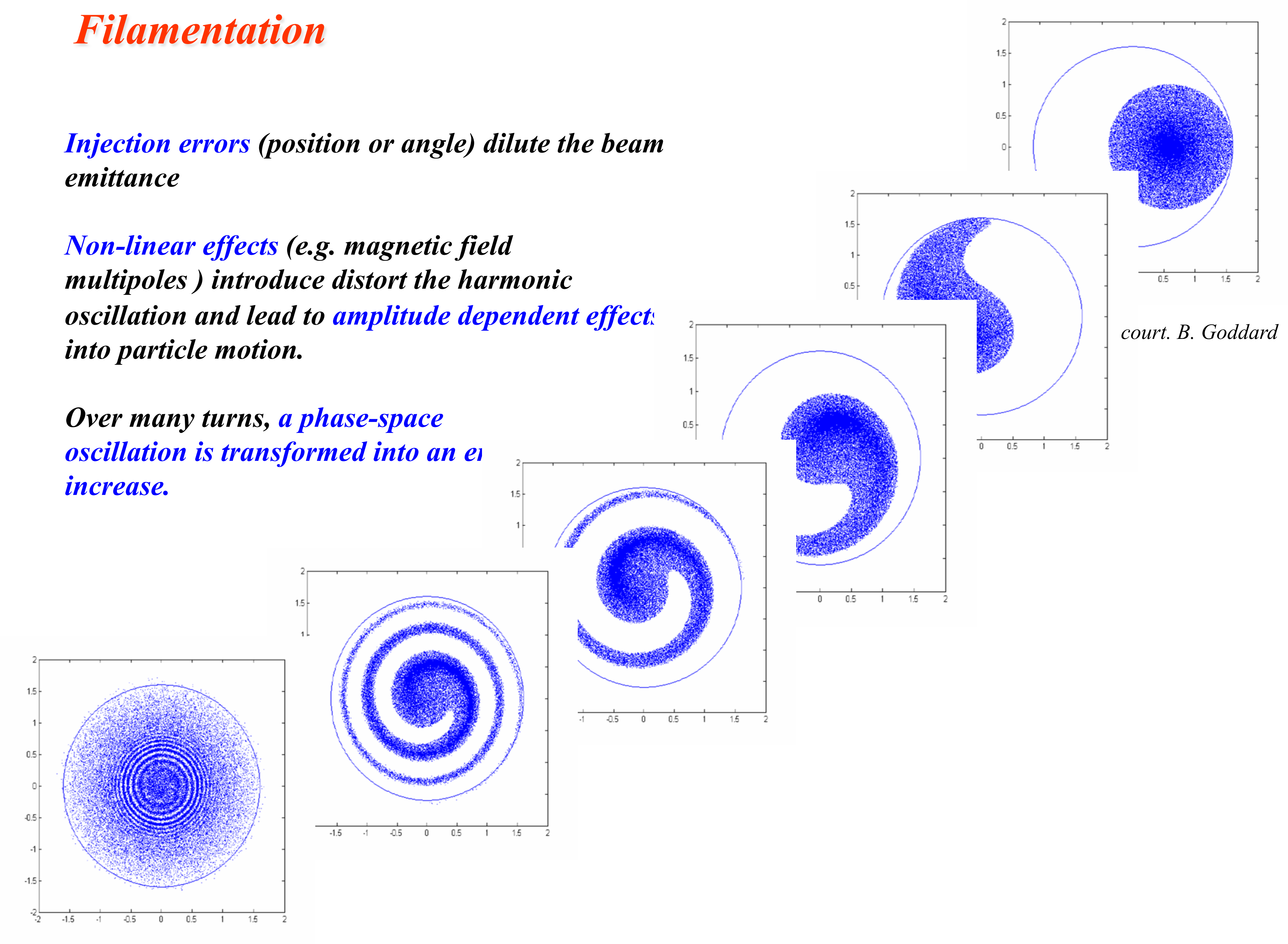
but particle losses in signle bunches
when beams are brought into collision

Filamentation

Injection errors ***(position or angle) dilute the beam emittance***

Non-linear effects ***(e.g. magnetic field multipoles) introduce distort the harmonic oscillation and lead to*** ***amplitude dependent effect*** ***into particle motion.***

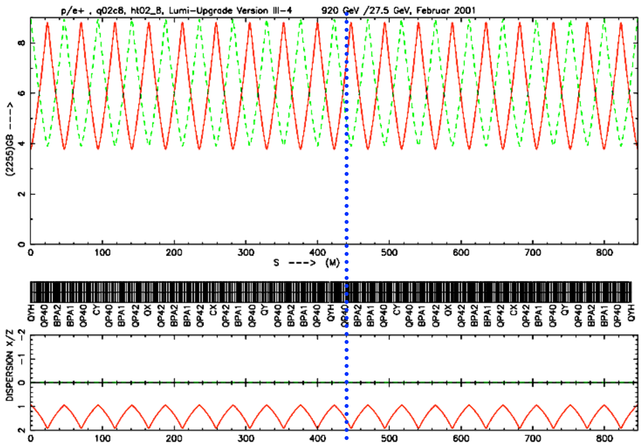
Over many turns, ***a phase-space oscillation is transformed into an e*** ***increase.***



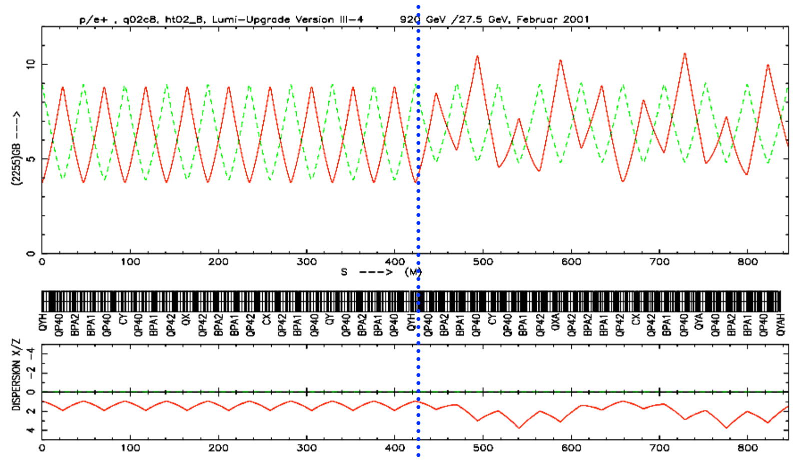
court. B. Goddard

Matched & unmatched Transferline

Example: HERA Arc, FoDo structure



Transferline: ***matched beam optics.***
twiss parameters at start correspond to periodic Twiss

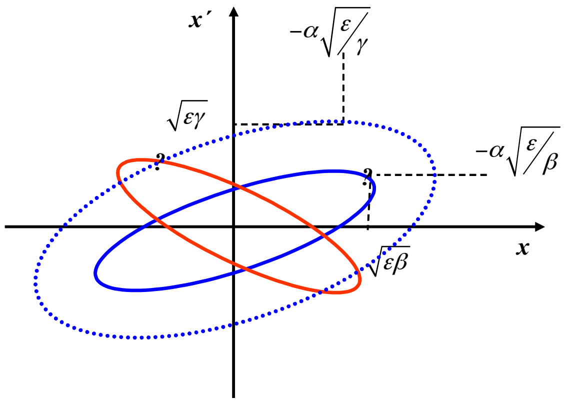


Transferline: un-matched beam optics at half the way:

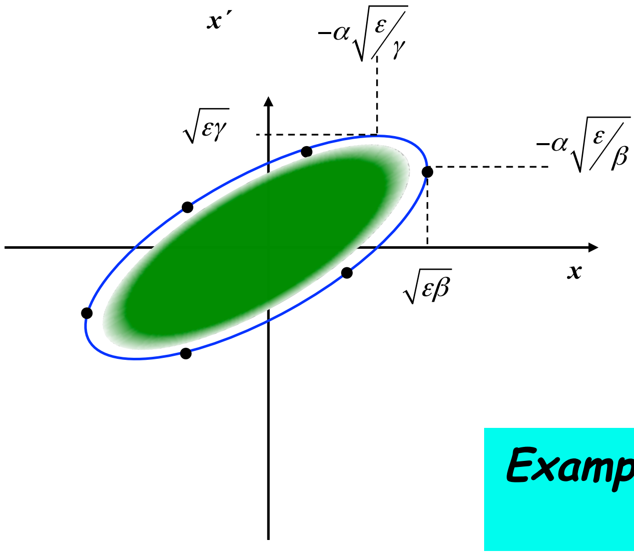
twiss parameters at start correspond to periodic Twiss
quadstrengths reduced by 20 % for second part

→ beta-functions & dispersion are distorted

... and how it looks in phase space



Main task: keep the transferline optically transparent.

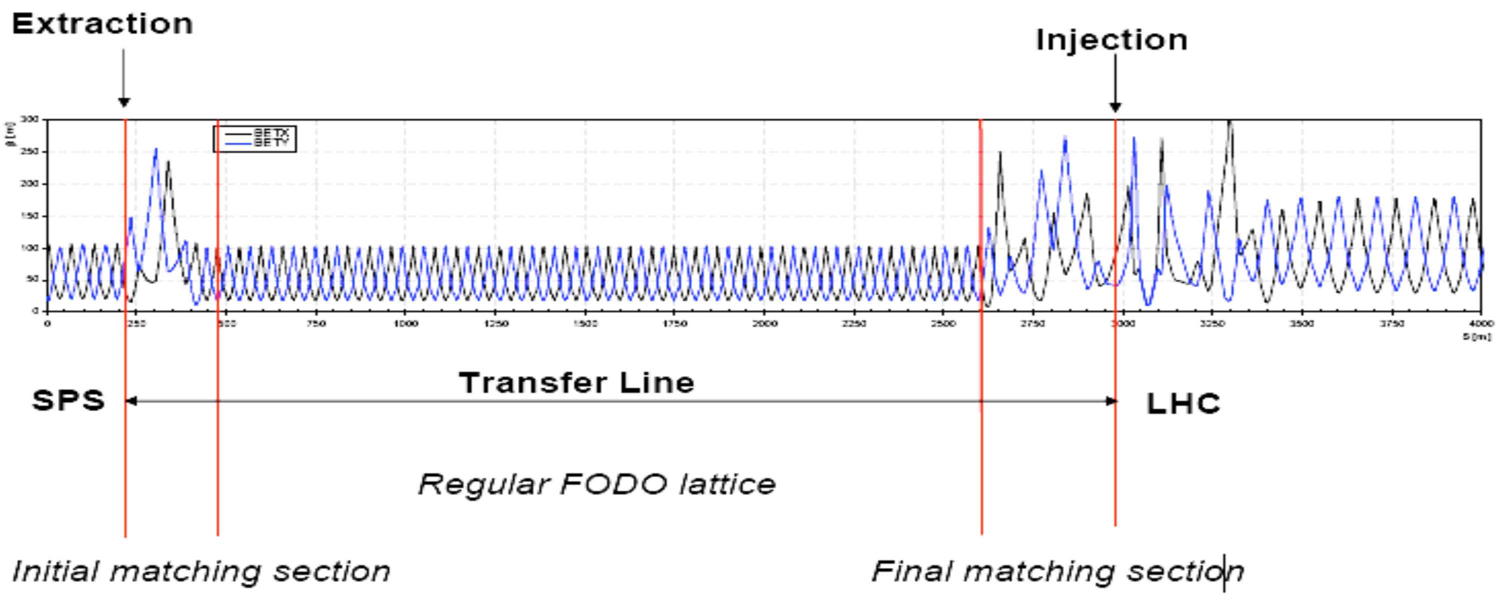


$$\varepsilon = \gamma(s) * x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$

Injected Beam has to be matched to the optics of the storage ring

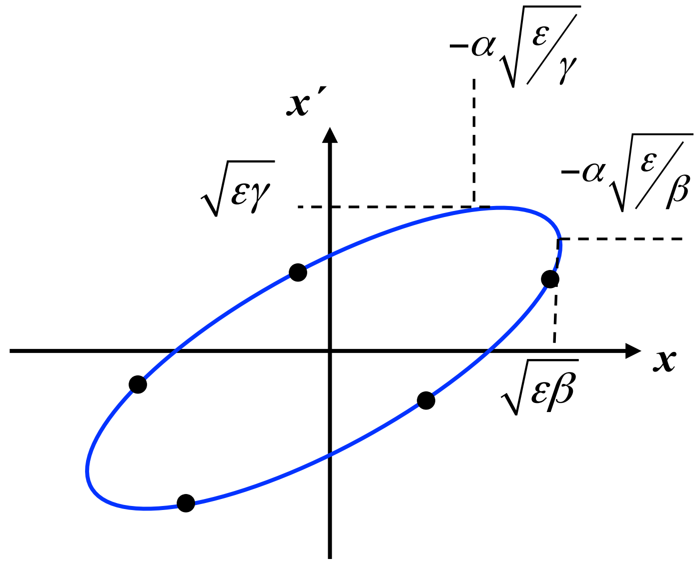
Example:

SPS-match-Transferline-match-LHC



10.) Liouville during Acceleration

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\, x'^2(s)$$



***Beam Emittance** corresponds to the area covered in the*
x, x´ Phase Space Ellipse

***Liouville:** Area in phase space is constant.*

But so sorry ... $\varepsilon \neq$ const !

Classical Mechanics:

***phase space** = diagram of the two canonical variables*
position** & **momentum
x p_x

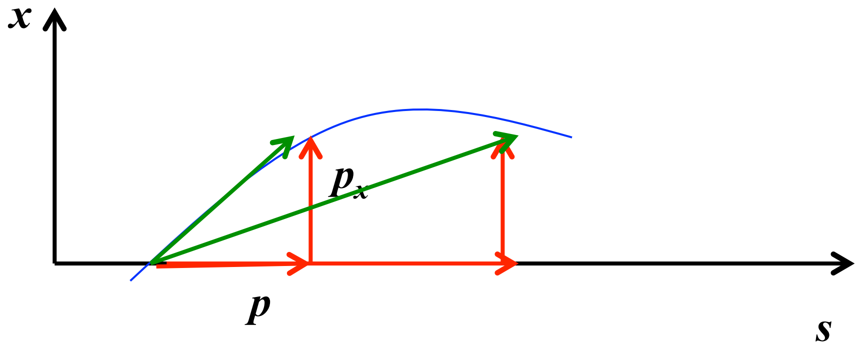
According to Hamiltonian mechanics:
phase space diagram relates the variables q and p

Liouvilles Theorem: $\int p\,dq = const$

$$\int p_x\,dx = const$$

for convenience (i.e. because we are lazy bones) we use in accelerator theory:

$$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{\beta_x}{\beta} = \frac{p_x}{p}$$



$$\underbrace{\int x'\,dx}_{\varepsilon} = \frac{\int p_x\,dx}{p} \propto \frac{const}{m_0 c \cdot \gamma\beta}$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow \quad \varepsilon = \int x'dx \propto \frac{1}{\beta\gamma}$$

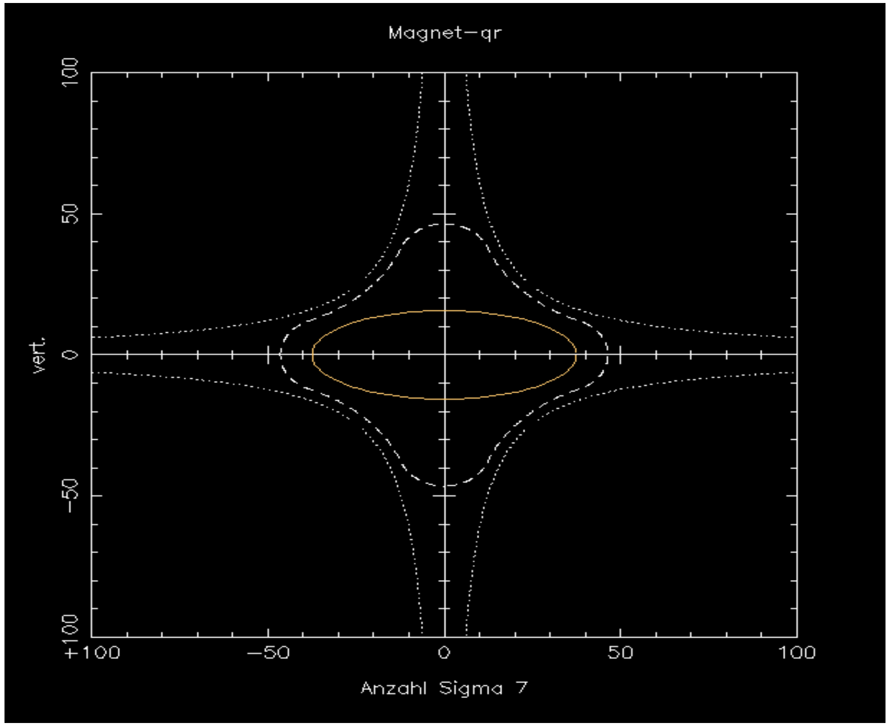
the beam emittance shrinks during acceleration $\varepsilon \sim 1/\gamma$

$$\beta_x = \frac{v_x}{c}$$

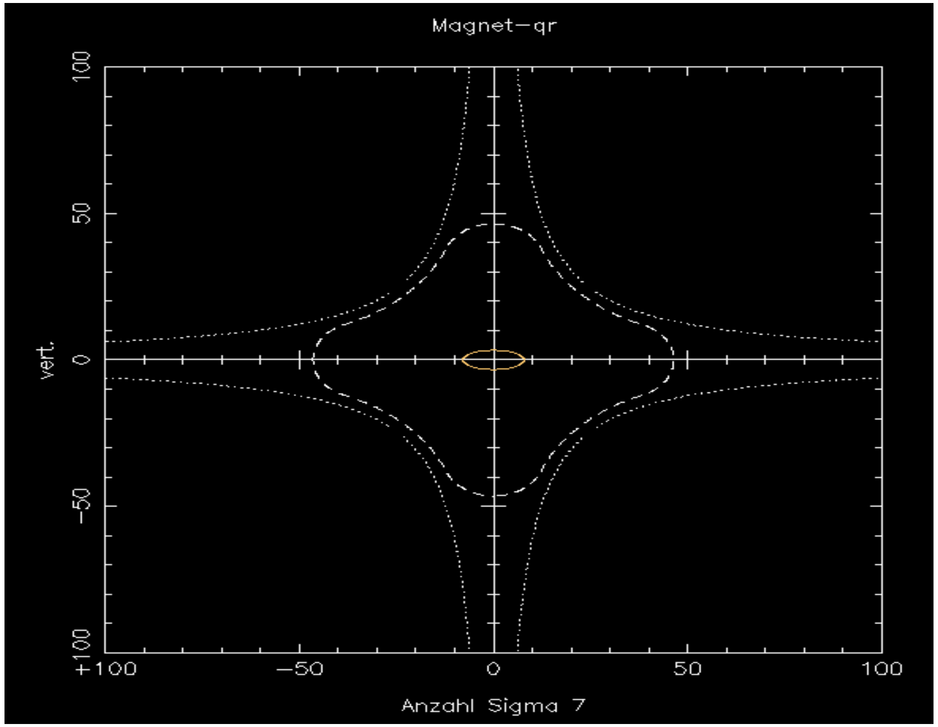
Example: HERA proton ring

injection energy: 40 GeV $\gamma = 43$
flat top energy: 920 GeV $\gamma = 980$

emittance ε *(40GeV)* $= 1.2 * 10^{-7}$
ε *(920GeV)* $= 5.1 * 10^{-9}$



7 σ beam envelope at E = 40 GeV



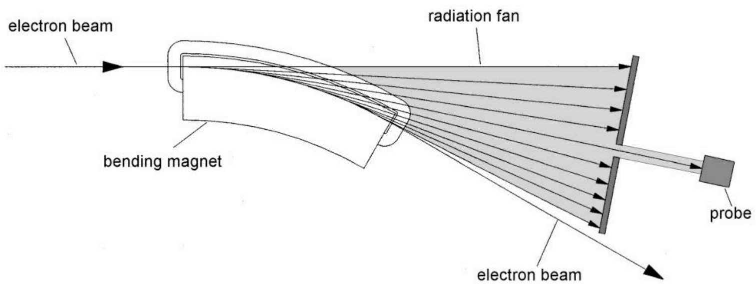
… and at E = 920 GeV

9.) Emittance in an electron ring: $\varepsilon \propto \gamma^2$

One word of caution:

As soon as ε is determined by the radiation process …

i.e. by the fact that the particle looses energy and is thus travelling on on a dispersive orbit we observe a complitely different behavior:



$$P_s = \frac{e^2 c}{6\pi\varepsilon_0} * \frac{1}{\left(m_0 c^2\right)^4} \frac{E^4}{R^4}$$ Synchrotron radiation power

$$\Delta E = \frac{e^2}{3\varepsilon_0 \left(m_0 c^2\right)^4} \frac{E^4}{R}$$ Energy loss per turn

$$\omega_c = \frac{3c\gamma^3}{2R}$$ Critical energy

$$\varepsilon_{x0} \equiv \frac{\sigma_{x\beta}^2}{\beta} = \frac{C_q E^2}{J_x} \cdot \frac{\langle \mathcal{H} \rangle_{mag}}{\rho}$$

$$\mathcal{H} = \gamma D^2 + 2\alpha D D' + \beta D'^2$$

ε is quadratically dependent on the beam energy

... but be aware of the fact that in a linac it still shrinks just as protons do

The „ not so ideal world “

11.) The „ $\Delta p / p \neq 0$“ Problem

ideal accelerator: all particles will see the same accelerating voltage.

→ $\Delta p / p = 0$

„nearly ideal“ accelerator: Cockroft Walton or van de Graaf

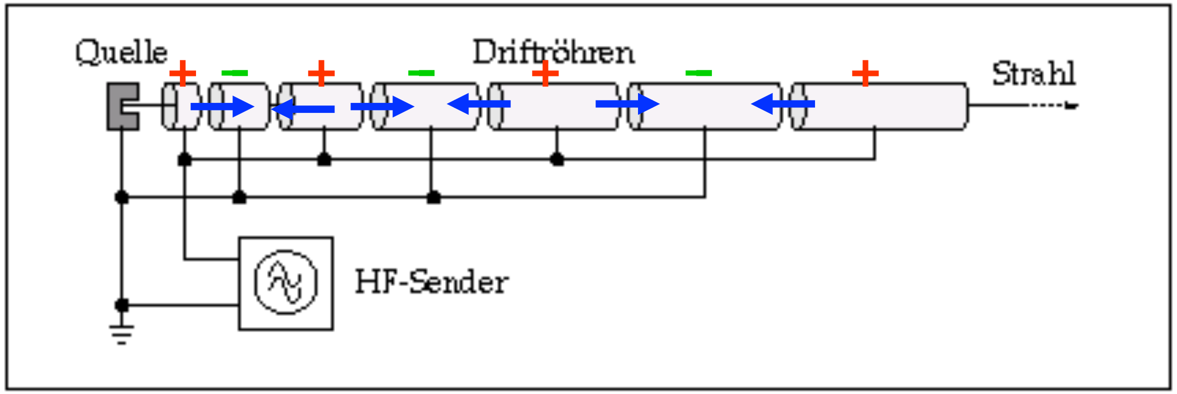
$\Delta p / p \approx 10^{-5}$

Vivitron, Straßbourg, inner structure of the acc. section

MP Tandem van de Graaf Accelerator at MPI for Nucl. Phys. Heidelberg

RF Acceleration

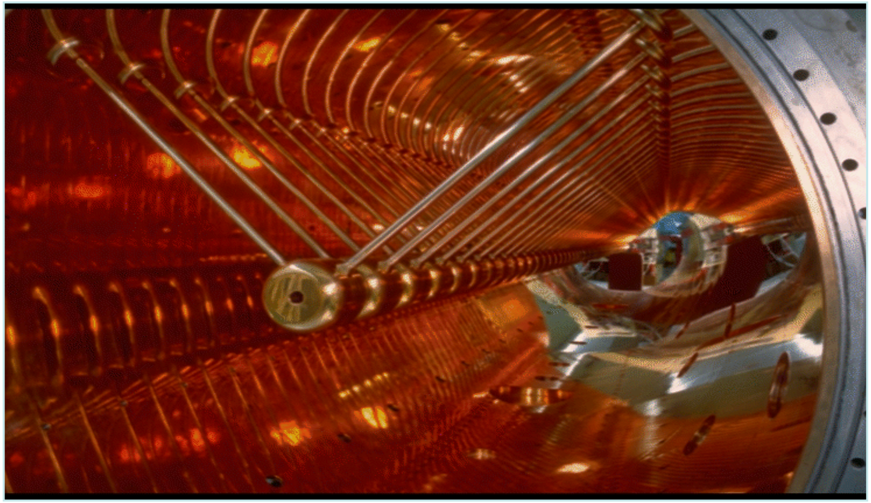
1928, Wideroe



Energy Gain per „Gap":

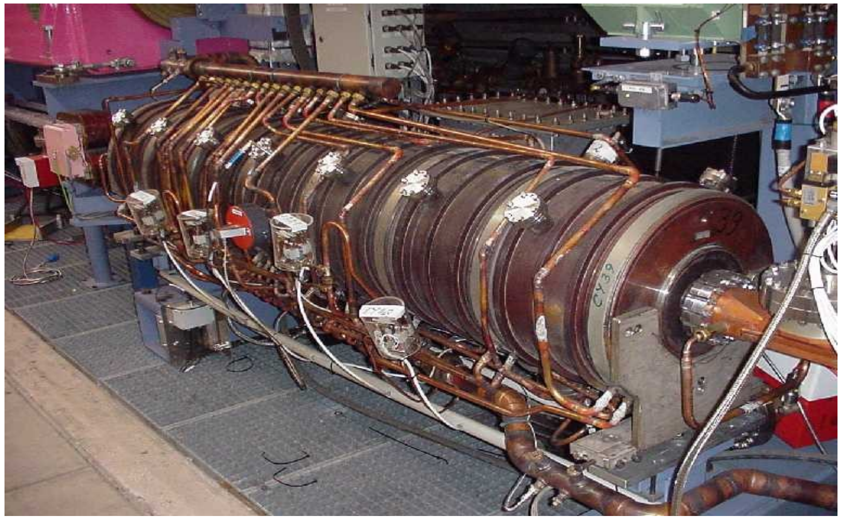
$$W = n * q\, U_0 \sin \omega_{RF} t$$

drift tube structure at a proton linac (GSI Unilac)



n *number of gaps between the drift tubes*
q *charge of the particle*
U_0 *Peak voltage of the RF System*
Ψ_S *synchronous phase of the particle*

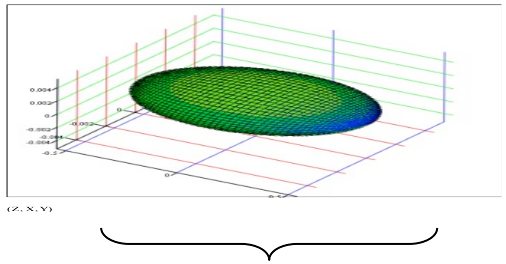
500 MHz cavities in an electron storage ring



* ***RF Acceleration:*** ***multiple application of the same acceleration voltage; brillant idea to gain higher energies***

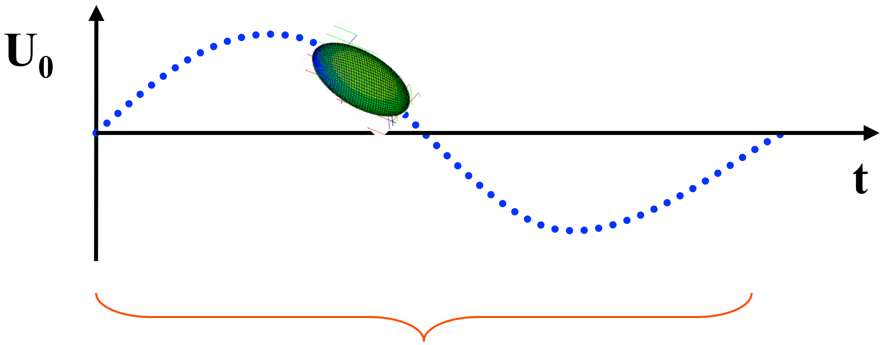
RF Acceleration-Problem: panta rhei !!!

(Heraklit: 540-480 v. Chr.)



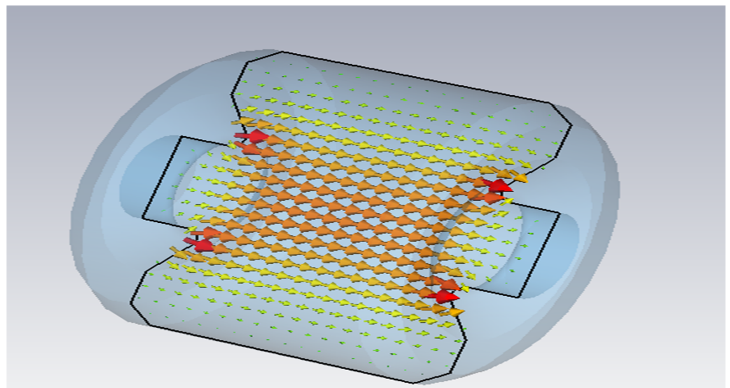
Bunch length of Electrons ≈ 1cm

just a stupid (and nearly wrong) example)



$\lambda = 75\ cm$

$$\nu = 400\,MHz, \quad c = \lambda\,\nu \quad \Rightarrow \quad \lambda = 75\ cm$$



$\sin(90^o) = 1$

$\sin(84^o) = 0.994$

$$\frac{\Delta U}{U} = 6.0\ 10^{-3}$$

typical momentum spread of an electron bunch:

$$\frac{\Delta p}{p} \approx 1.0\ 10^{-3}$$

Dispersive and Chromatic Effects: $\Delta p/p \neq 0$

Are there any Problems ???

Sure there are !!!

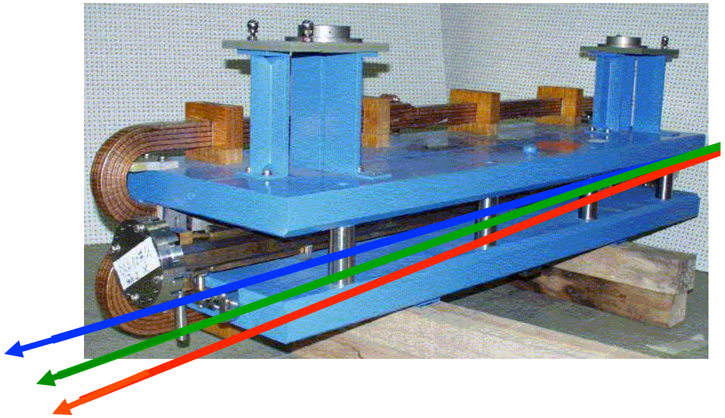
font colors due to pedagogical reasons

12.) Dispersion and Chromaticity:
Magnet Errors for $\Delta p/p \neq 0$

Influence of external fields on the beam: *prop. to magn. field & prop. zu 1/p*

dipole magnet

$$\alpha = \frac{\int B\,dl}{p/e}$$



$$x_D(s) = D(s)\,\frac{\Delta p}{p}$$

focusing lens

$$k = \frac{g}{p/e}$$

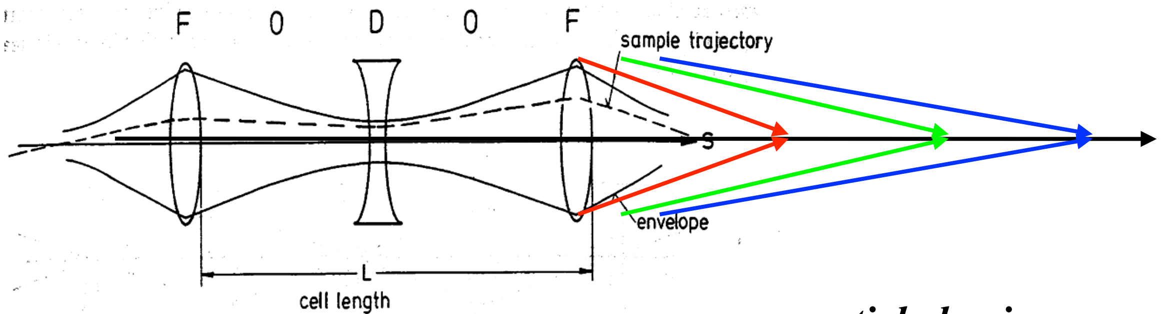


Figure 29: FODO cell

particle having ...

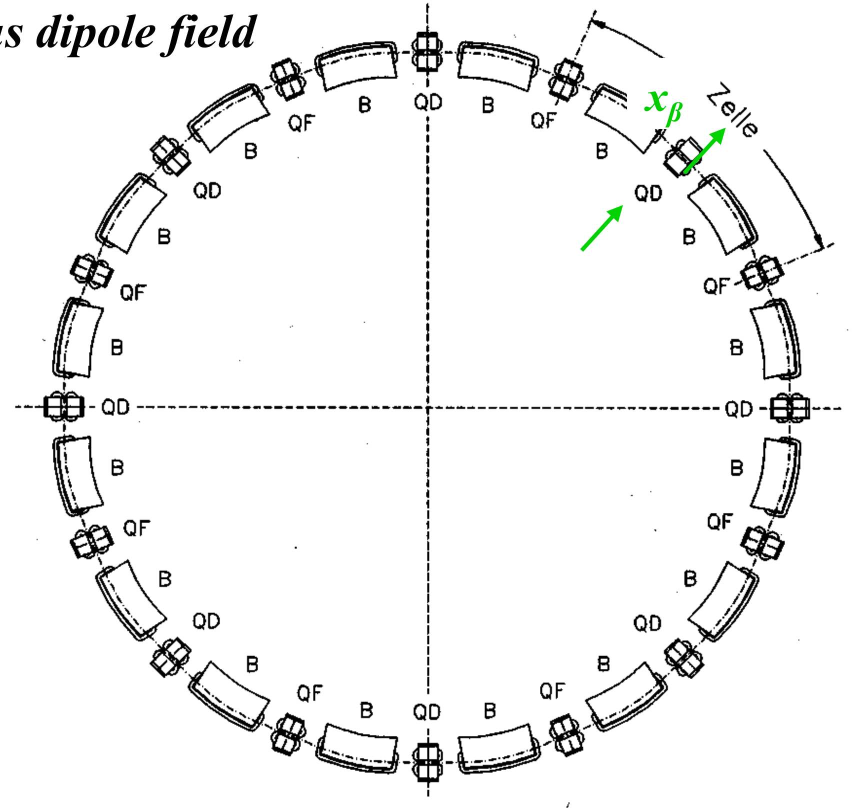
to high energy

to low energy

ideal energy

Dispersion

Example: homogeneous dipole field



it for $\Delta p/p > 0$

$$: D(s)\cdot\frac{\Delta p}{p}$$

Matrix formalism:

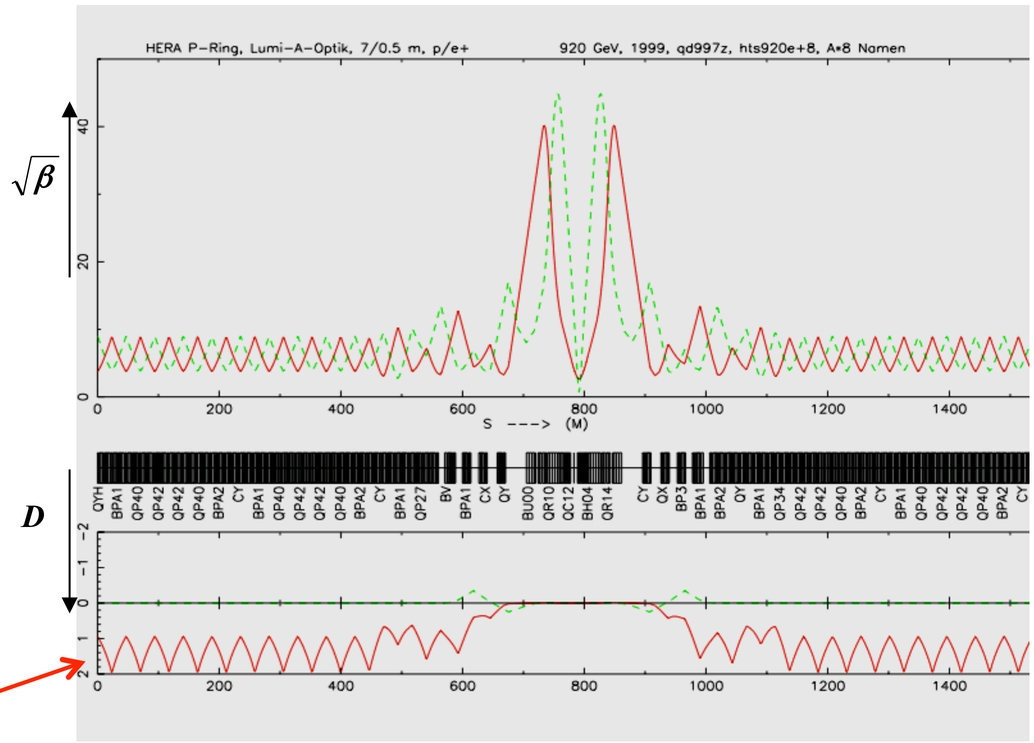
$$x(s) = x_\beta(s) + D(s)\cdot\frac{\Delta p}{p}$$

$$x(s) = C(s)\cdot x_0 + S(s)\cdot x_0' + D(s)\cdot\frac{\Delta p}{p}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p}\begin{pmatrix} D \\ D' \end{pmatrix}_0$$

or expressed as 3x3 matrix

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0$$



Example

$$x_\beta = 1 \ldots 2\, mm$$
$$D(s) \approx 1 \ldots 2\, m$$
$$\Delta p/p \approx 1 \cdot 10^{-3}$$

Amplitude of Orbit oscillation contribution due to Dispersion ≈ beam size

→ Dispersion must vanish at the collision point

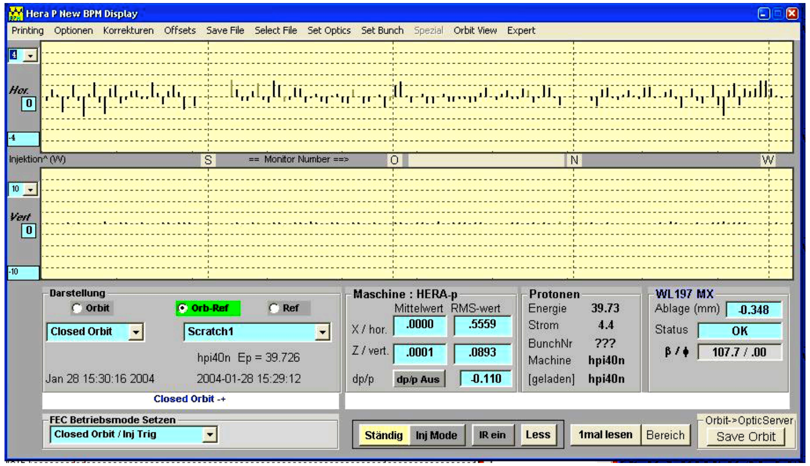
!

Calculate D, D´: ... takes a couple of sunny Sunday evenings !

$$D(s) = S(s) \int_{s_0}^{s_1} \frac{1}{\rho} C(\tilde{s})\, d\tilde{s} - C(s) \int_{s_0}^{s_1} \frac{1}{\rho} S(\tilde{s})\, d\tilde{s}$$

(proof see CAS proc.)

Dispersion is visible



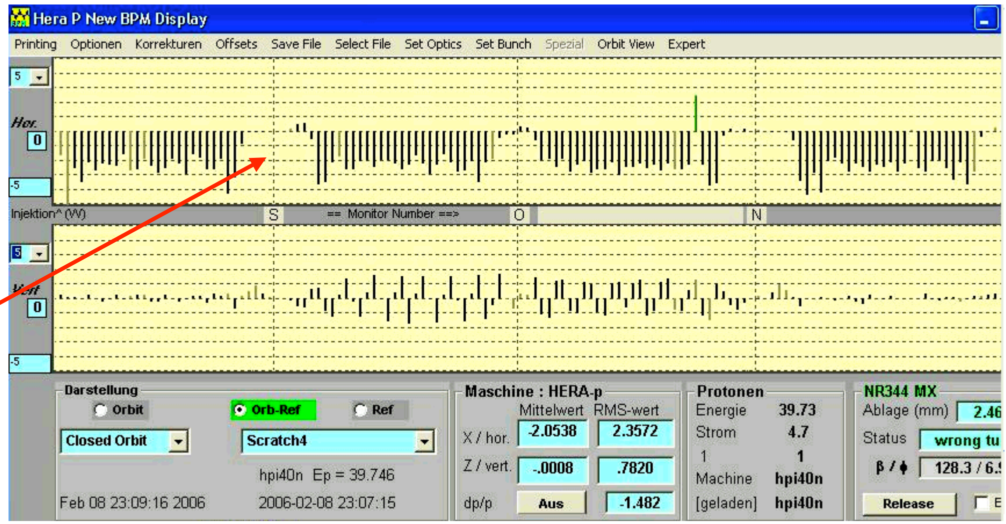
HERA Standard Orbit

HERA Dispersion Orbit

dedicated energy change of the stored beam

→ closed orbit is moved to a dispersions trajectory

$$x_D = D(s) * \frac{\Delta p}{p}$$



Attention: at the Interaction Points we require D=D′= 0

13.) Transfer Matrix M … *yes we had the topic already*

general solution of Hill's equation

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\}$$

$$x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left[\alpha(s)\cos\{\psi(s)+\phi\}+\sin\{\psi(s)+\phi\}\right]$$

remember the trigonometrical gymnastics: sin(a + b) = … etc

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta_s}\left(\cos\psi_s\cos\phi - \sin\psi_s\sin\phi\right)$$

$$x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta_s}}\left[\alpha_s\cos\psi_s\cos\phi - \alpha_s\sin\psi_s\sin\phi + \sin\psi_s\cos\phi + \cos\psi_s\sin\phi\right]$$

starting at point s(0) = s_0, where we put $\Psi(0) = 0$

$$\cos\phi = \frac{x_0}{\sqrt{\varepsilon\beta_0}} \quad ,$$

$$\sin\phi = -\frac{1}{\sqrt{\varepsilon}}\left(x_0'\sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}}\right)$$

inserting above …

$$x(s)=\sqrt{\frac{\beta_s}{\beta_0}}\left\{\cos\psi_s+\alpha_0\sin\psi_s\right\}x_0+\left\{\sqrt{\beta_s\beta_0}\sin\psi_s\right\}x_0'$$

$$x'(s)=\frac{1}{\sqrt{\beta_s\beta_0}}\left\{(\alpha_0-\alpha_s)\cos\psi_s-(1+\alpha_0\alpha_s)\sin\psi_s\right\}x_0+\sqrt{\frac{\beta_0}{\beta_s}}\left\{\cos\psi_s-\alpha_s\sin\psi_s\right\}x_0'$$

which can be expressed ... for convenience ... in matrix form

$$\begin{pmatrix}x\\x'\end{pmatrix}_s=M\begin{pmatrix}x\\x'\end{pmatrix}_0$$

$$M=\begin{pmatrix}\sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s+\alpha_0\sin\psi_s\right) & \sqrt{\beta_s\beta_0}\sin\psi_s\\ \frac{(\alpha_0-\alpha_s)\cos\psi_s-(1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s-\alpha_s\sin\psi_s\right)\end{pmatrix}$$

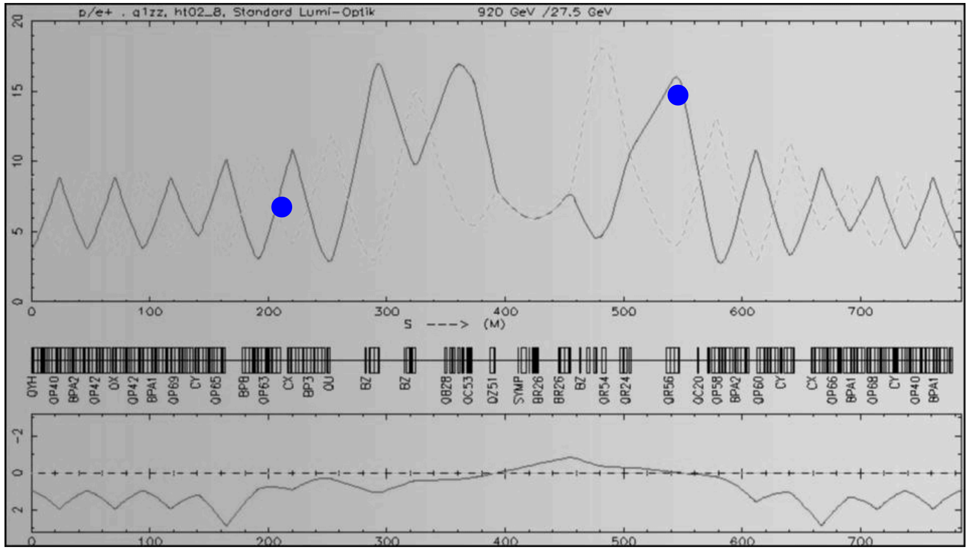
* ***we can calculate the single particle trajectories between two locations in the ring, if we know the α β γ at these positions.***
* ***and nothing but the α β γ at these positions.***
* ***... !***

14.) Transformation of α, β, γ

consider two positions in the storage ring: s_0 , s

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$



Betafunction in a Storage Ring

since ε = const (Liouville):

$$\varepsilon = \beta_s x'^2 + 2\alpha_s x x' + \gamma_s x^2$$

$$\varepsilon = \beta_0 x_0'^2 + 2\alpha_0 x_0 x_0' + \gamma_0 x_0^2$$

... remember W = CS′-SC′ = 1

$$\begin{pmatrix} x \\ x' \end{pmatrix}_0 = M^{-1} * \begin{pmatrix} x \\ x' \end{pmatrix}_s$$

$$M^{-1} = \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix}$$

$$\rightarrow \quad x_0 = m_{22} x - m_{12} x'$$
$$x_0' = -m_{21} x + m_{11} x'$$

... inserting into ε

$$\varepsilon = \beta_0 (m_{11} x' - m_{21} x)^2 + 2\alpha_0 (m_{22} x - m_{12} x')(m_{11} x' - m_{21} x) + \gamma_0 (m_{22} x - m_{12} x')^2$$

sort via x, x′ and compare the coefficients to get ....

The Twiss parameters α, β, γ can be transformed through the lattice via the matrix elements defined above.

$$\beta(s) = m_{11}^2\,\beta_0 - 2m_{11}m_{12}\,\alpha_0 + m_{12}^2\,\gamma_0$$

$$\alpha(s) = -m_{11}m_{21}\,\beta_0 + (m_{12}m_{21} + m_{11}m_{22})\,\alpha_0 - m_{12}m_{22}\,\gamma_0$$

$$\gamma(s) = m_{21}^2\,\beta_0 - 2m_{21}m_{22}\,\alpha_0 + m_{22}^2\,\gamma_0$$

in matrix notation:

$$\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_{s2} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{12}m_{21} + m_{22}m_{11} & -m_{12}m_{22} \\ m_{12}^2 & -2m_{22}m_{21} & m_{22}^2 \end{pmatrix} * \begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_{s1}$$

!

1.) this expression is important

2.) given the twiss parameters α, β, γ at any point in the lattice we can transform them and calculate their values at any other point in the ring.

3.) the transfer matrix is given by the focusing properties of the lattice elements, the elements of M are just those that we used to calculate single particle trajectories.

... and here starts the lattice design !!!

Most simple example: ***drift space***

$$M_{drift} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

particle coordinates

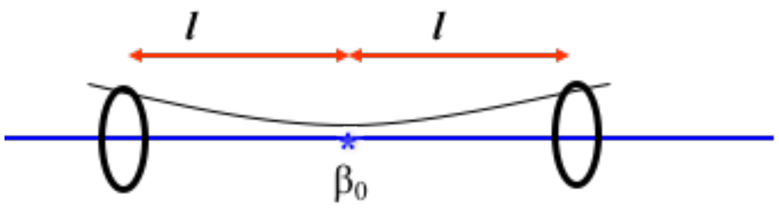
$$\begin{pmatrix} x \\ x' \end{pmatrix}_l = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ x' \end{pmatrix}_0 \quad \rightarrow \quad \begin{aligned} x(l) &= x_0 + l * x_0' \\ x'(l) &= x_0' \end{aligned}$$

transformation of twiss parameters:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_l = \begin{pmatrix} 1 & -2l & l^2 \\ 0 & 1 & -l \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

$$\beta(s) = \beta_0 - 2l * \alpha_0 + l^2 * \gamma_0$$

Special case: symmetric drift

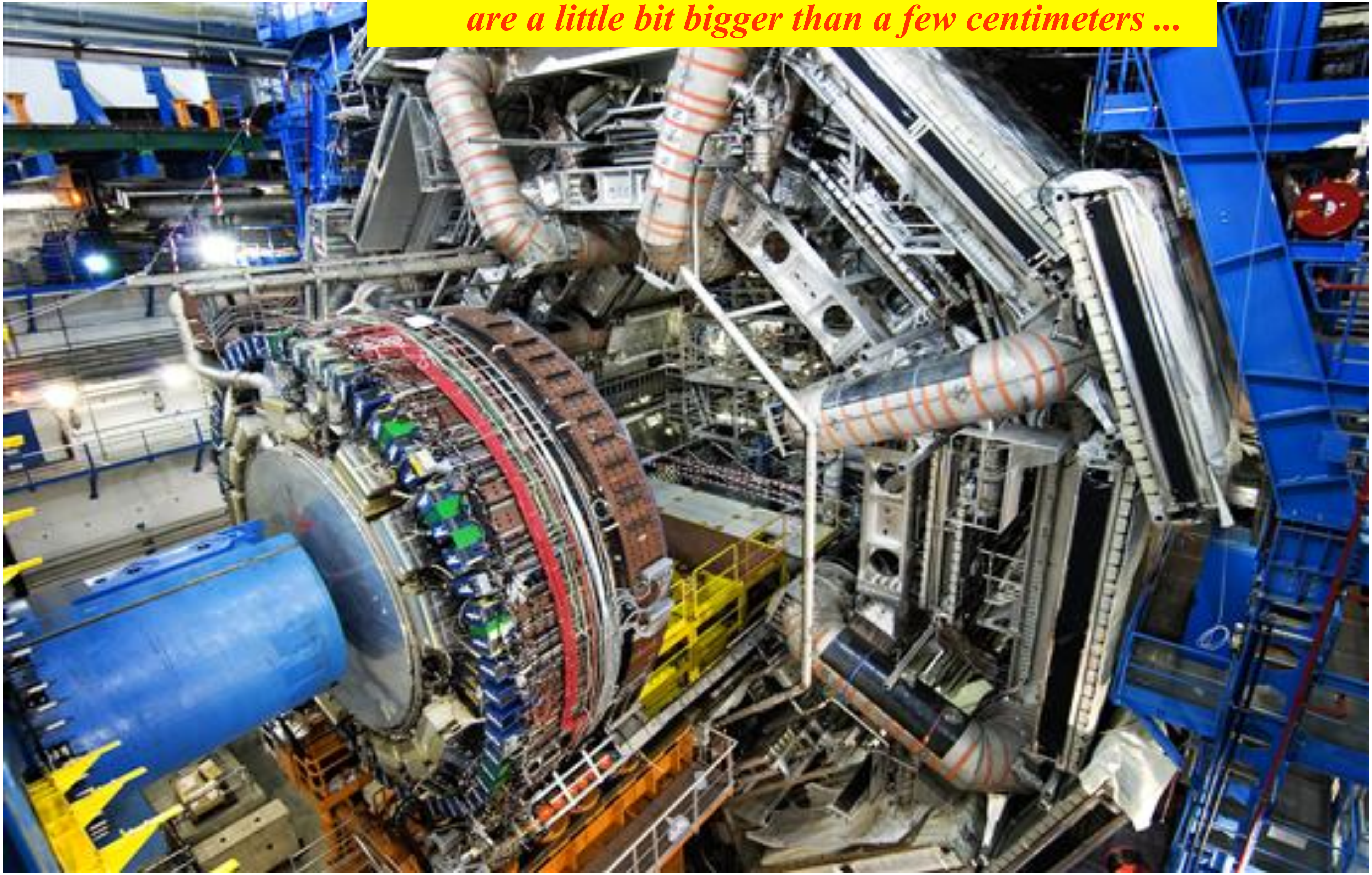


$$\alpha_0 = 0, \quad \rightarrow \quad \gamma_0 = \frac{1 + \alpha_0^2}{\beta_0} = \frac{1}{\beta_0}$$

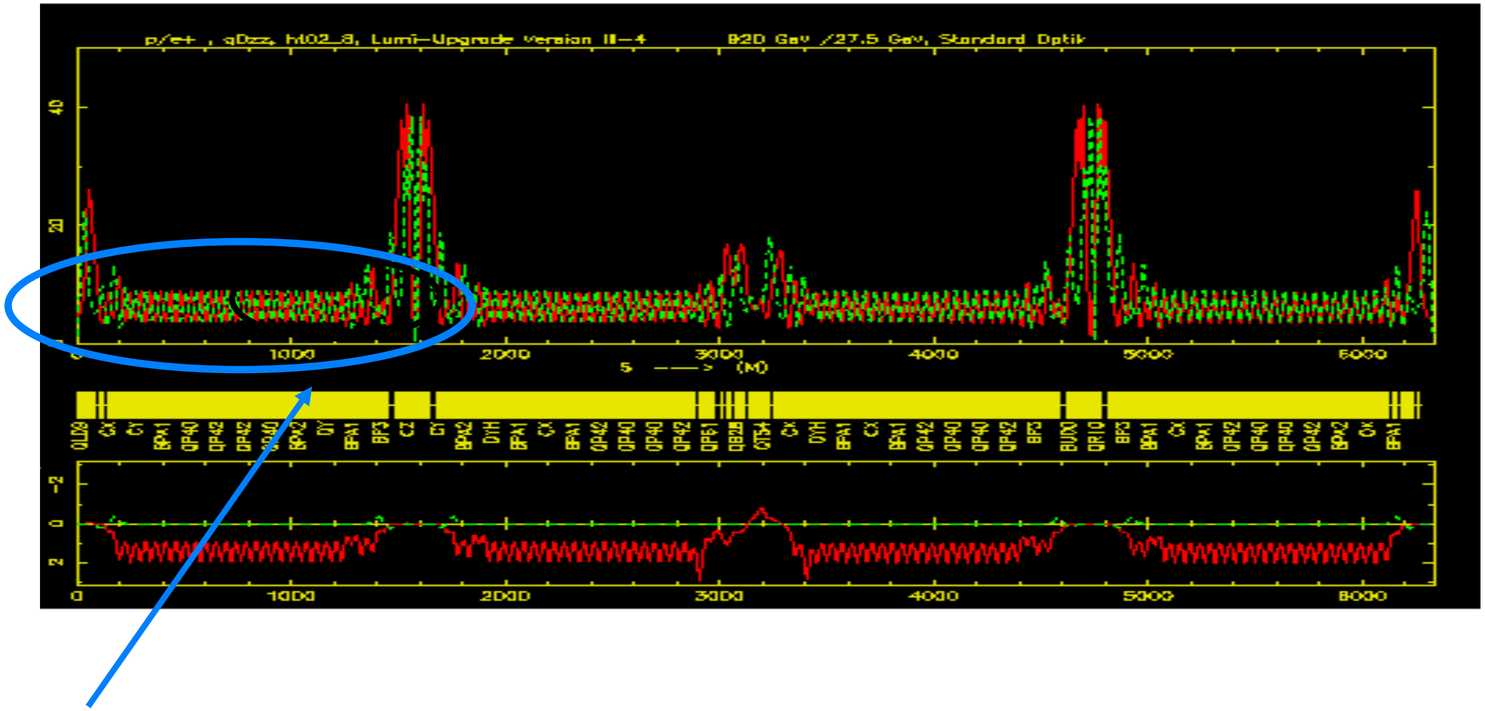
$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

... clearly there is an

... unfortunately ... in general
high energy detectors that are
installed in that drift spaces
are a little bit bigger than a few centimeters ...



15.) Lattice Design:



Arc: ***regular (periodic) magnet structure:***

bending magnets → define the energy of the ring

main focusing & tune control, chromaticity correction,

multipoles for higher order corrections

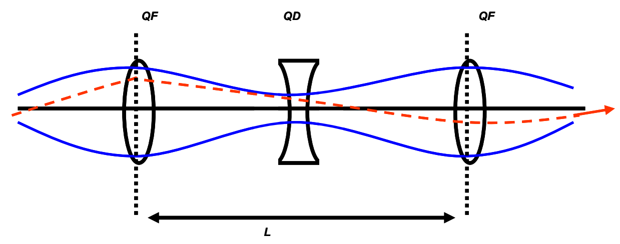
Straight sections: ***drift spaces for injection, dispersion suppressors,***

low beta insertions, RF cavities, etc....

... and the high energy experiments if they cannot be avoided

Once more unto the breach dear friends: The Matrices

The FoDo-Lattice



Matrix of a focusing quadrupole magnet:

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K}*l) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}*l) \\ -\sqrt{K}\sin(\sqrt{K}*l) & \cos(\sqrt{K}*l) \end{pmatrix}$$

... Can we do a bit easier ???
If the focal length f is much larger than the length of the quadrupole magnet,

$$f = 1/kl_Q >> l_Q$$

the matrix can be simplified by $kl_q = const,\ l_q \to 0$

$$M = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$

and then we can show that

$$\hat{\beta} = \frac{(1+\sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}} \quad !$$

$$\breve{\beta} = \frac{(1-\sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}} \quad !$$

proof see appendix

16.) Dipole Errors / Quadrupole Misalignment

The Design Orbit is defined by the strength and arrangement of the dipoles.
Under the influence of dipole imperfections and quadrupole misalignments we obtain a "Closed Orbit" which is hopefully still closed and not too far away from the design.

Dipole field error:

$$\theta = \frac{dl}{\rho} = \frac{\int B\,dl}{B\rho}$$

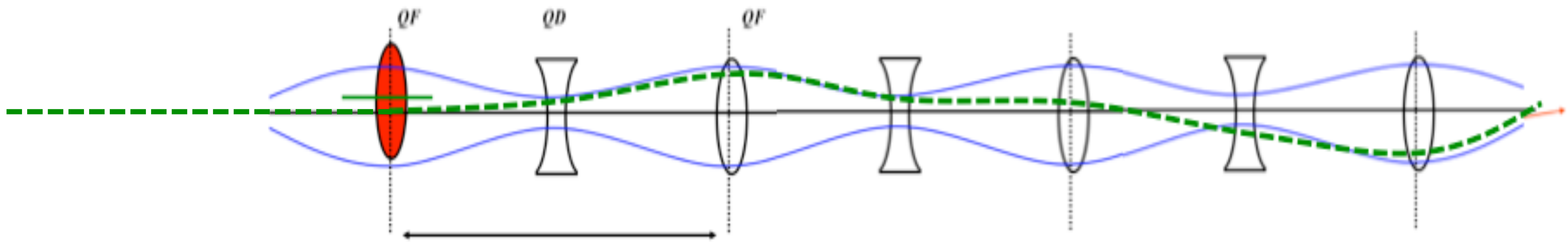
Quadrupole offset:

$$g = \frac{dB}{dx} \rightarrow \quad \Delta x \cdot g \quad = \Delta x \frac{dB}{dx} = \Delta B$$

misaligned quadrupoles (or orbit offsets in quadrupoles) create dipole effects that lead to a distorted "closed orbit"

normalised to p/e:

$$\Delta x \cdot k = \Delta x \cdot \frac{g}{B\rho} = \frac{1}{\rho} \qquad \begin{pmatrix} x \\ x' \end{pmatrix}_i = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ x' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{l}{\rho} \end{pmatrix}$$



In a Linac or Transfer Line – starting with a perfect orbit – the misaligned quadrupole creates an oscillation that is transformed from now on downstream via

$$\begin{pmatrix} x \\ x' \end{pmatrix}_f = M \begin{pmatrix} x \\ x' \end{pmatrix}_i$$

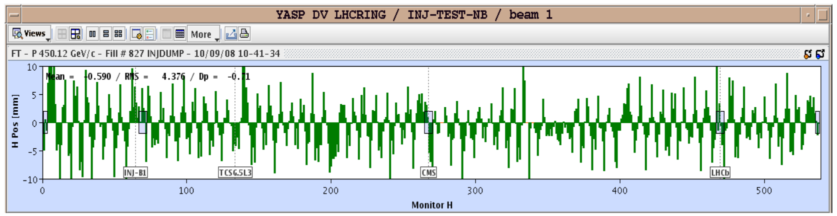
... and in a circular machine ??

we have to obey the periodicity condition.
The orbit is closed !! ... even under the influence of a orbit kick.

Calculation of the new closed orbit:
the general orbit will always be a solution of Hill, so ...

$$x(s) = a \cdot \sqrt{\beta}\cos(\psi(s) + \varphi$$

We set at the location of the error s=0, Ψ(s)=0
and require as 1st boundary condition:
periodic amplitude



$$x(s+L) = x(s)$$

$$a \cdot \sqrt{\beta(s+L)} \cdot \cos(\psi(s) + 2\pi Q - \varphi) = a \cdot \sqrt{\beta(s)} \cdot \cos(\psi(s) - \varphi)$$

$$\cos(2\pi Q - \varphi) = \cos(-\varphi) = \cos(\varphi)$$

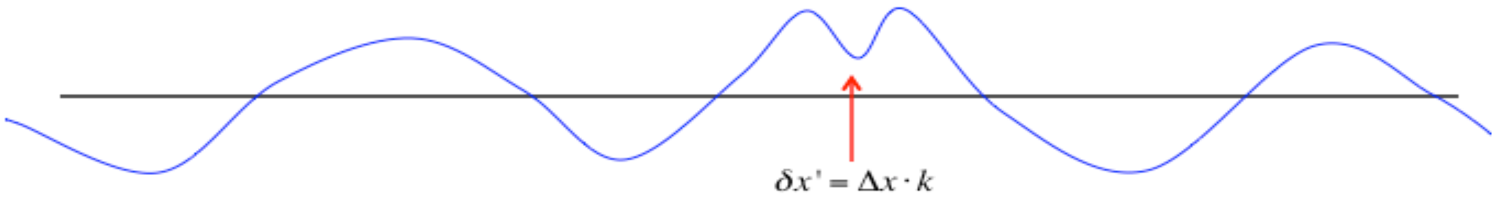
$$\rightarrow \varphi = \pi Q$$

$$\beta(s+L) = \beta(s)$$
$$\psi(s=0) = 0$$
$$\psi(s+L) = 2\pi Q$$

Misalignment error in a circular machine

2nd boundary condition: x' (s+L) + δx'= x'(s)

we have to close the orbit



$$x(s) = a \cdot \sqrt{\beta} \cos(\psi(s) - \varphi)$$

$$x'(s) = a \cdot \sqrt{\beta} \left(-\sin(\psi(s) - \varphi)\right) \psi' + \frac{\beta'(s)}{2\sqrt{\beta}} a \cdot \cos(\psi(s) - \varphi)$$

$$x'(s) = -a \cdot \frac{1}{\sqrt{\beta}} \left(\sin(\psi(s) - \varphi)\right) + \frac{\beta'(s)}{2\sqrt{\beta}} a \cdot \cos(\psi(s) - \varphi)$$

$$\psi(s) = \int \frac{1}{\beta(s)} ds$$

$$\psi'(s) = \frac{1}{\beta(s)}$$

boundary condition: x' (s+L) + δx'= x'(s)

$$-a \cdot \frac{1}{\sqrt{\beta(\tilde{s}+L)}} \left(\sin(2\pi Q - \varphi)\right) + \frac{\beta'(\tilde{s}+L)}{2\beta(\tilde{s}+L)} \sqrt{\beta(\tilde{s}+L)}\, a \cdot \cos(2\pi Q - \varphi) + \frac{\Delta\tilde{s}}{\rho} =$$

$$= -a \cdot \frac{1}{\sqrt{\beta(\tilde{s})}} \left(\sin(-\varphi)\right) + \frac{\beta'(\tilde{s})}{2\beta(\tilde{s})} \sqrt{\beta(\tilde{s})}\, a \cdot \cos(-\varphi)$$

Nota bene: $\tilde{s}$ *refers to the location of the kick*

Misalignment error in a circular machine

Now we use: $\beta(s+L) = \beta(s)$, $\varphi=\pi Q$

$$\frac{-a}{\sqrt{\beta(\tilde{s})}}\left(\sin(\pi Q)+\frac{\beta'(\tilde{s})}{2\beta(\tilde{s})}\sqrt{\beta(\tilde{s})}\,a\cdot\cos(\pi Q)+\frac{\Delta\tilde{s}}{\rho} = \frac{a}{\sqrt{\beta(\tilde{s})}}\left(\sin(\pi Q)+\frac{\beta'(\tilde{s})}{2\beta(\tilde{s})}\sqrt{\beta(\tilde{s})}\,a\cdot\cos(\pi Q)\right.\right.$$

$$\Rightarrow 2\,a\cdot\frac{\sin(\pi Q)}{\sqrt{\beta(\tilde{s})}} = \frac{\Delta\tilde{s}}{\rho} \quad \Rightarrow$$

$$a = \frac{\Delta\tilde{s}}{\rho}\cdot\sqrt{\beta(\tilde{s})}\,\frac{1}{2\sin(\pi Q)}$$

! this is the amplitude of the orbit oscillation resulting from a single kick

inserting in the equation of motion

$$x(s) = a\cdot\sqrt{\beta}\cos(\psi(s)+\varphi$$

$$x(s) = \frac{\Delta\tilde{s}}{\rho}\cdot\frac{\sqrt{\beta(\tilde{s})}\sqrt{\beta(s)}\cos(\psi(s)-\varphi)}{2\sin(\pi Q)}$$

! the distorted orbit depends on the kick strength,
! the local β function
! the β function at the observation point

!!! there is a resoncance denominator
→ watch your tune !!!

Misalignment error in a circular machine

For completeness:

if we do not set $\psi(s=0)=0$ *we have to write a bit more, but finally we get:*

$$x(s)=\frac{\sqrt{\beta(s)}}{2\sin(\pi Q)}*\oint\sqrt{\beta(\tilde{s})}\,\frac{1}{\rho(\tilde{s})}\cos\left(\left|\psi(\tilde{s})-\psi(s)\right|-\pi Q\right)d\tilde{s}$$

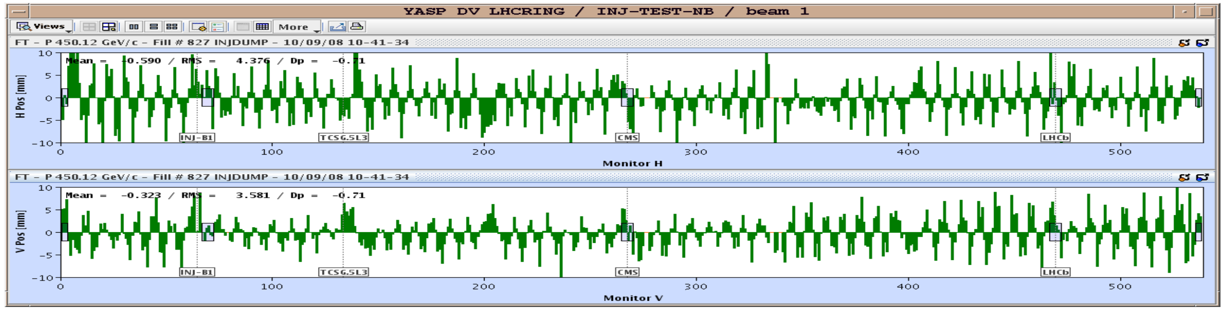
Reminder: LHC

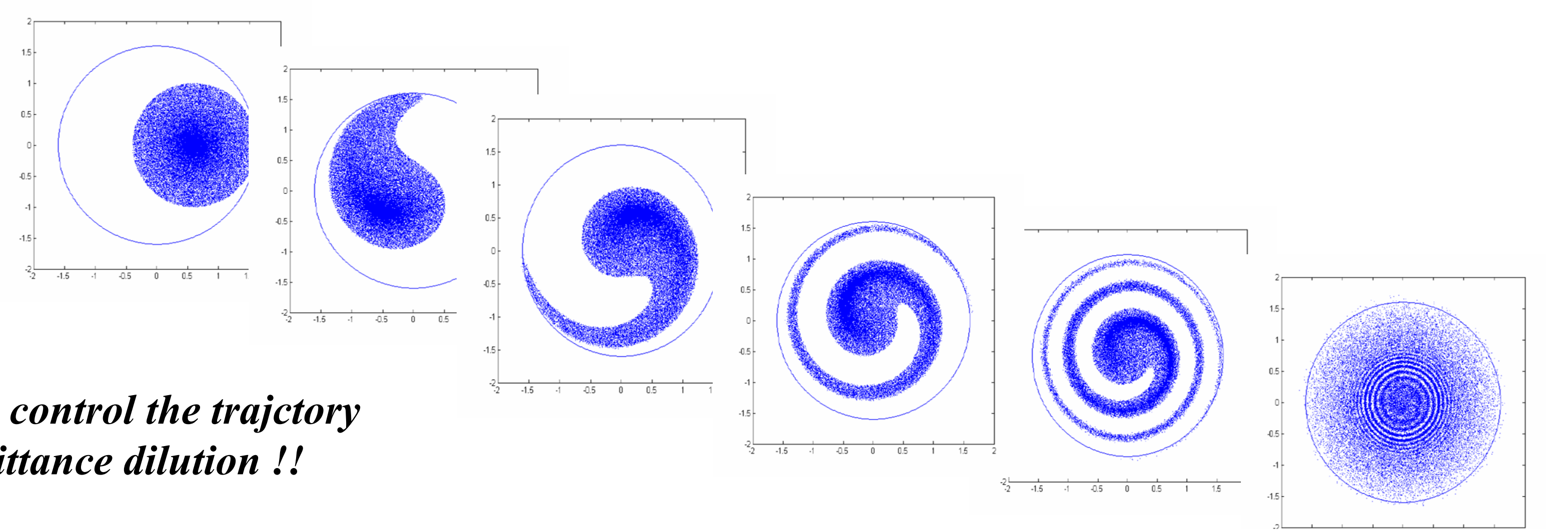
Tune: $Q_x = 64.31$, $Q_y = 59.32$

Relevant for beam stability:

non integer part

avoid integer tunes





Remember ...

... we have to control the trajctory

top avoid emittance dilution !!

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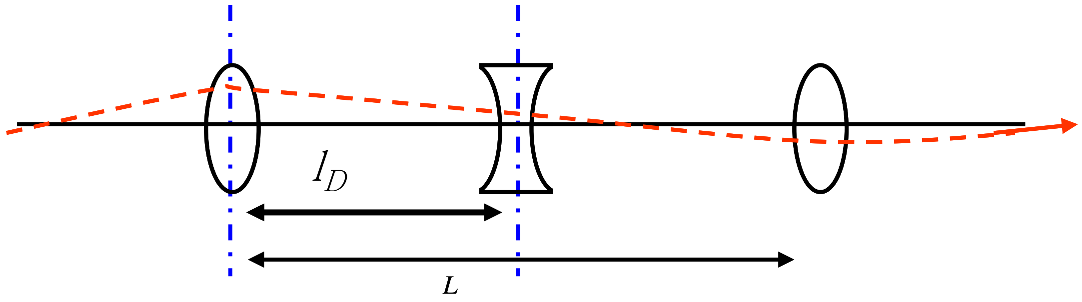
Appendix

FoDo in thin lens approximation



$$l_D = L/2$$
$$\tilde{f} = 2f$$

Calculate the matrix for a half cell, starting in the middle of a foc. quadrupole:

$$M_{halfCell} = M_{QD/2} * M_{lD} * M_{QF/2}$$

$$M_{halfCell} = \begin{pmatrix} 1 & 0 \\ 1/\tilde{f} & 1 \end{pmatrix} * \begin{pmatrix} 1 & l_D \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ -1/\tilde{f} & 1 \end{pmatrix}$$

note: $\tilde{f}$ denotes the focusing strength of half a quadrupole, so $\tilde{f} = 2f$

$$M_{halfCell} = \begin{pmatrix} 1 - l_D/\tilde{f} & l_D \\ -l_D/\tilde{f}^2 & 1 + l_D/\tilde{f} \end{pmatrix}$$

for the second half cell set f → -f

FoDo in thin lens approximation

Matrix for the complete FoDo cell

$$M = \begin{pmatrix} 1 + l_D/\tilde{f} & l_D \\ -l_D/\tilde{f}^2 & 1 - l_D/\tilde{f} \end{pmatrix} * \begin{pmatrix} 1 - l_D/\tilde{f} & l_D \\ -l_D/\tilde{f}^2 & 1 + l_D/\tilde{f} \end{pmatrix}$$

$$M = \begin{pmatrix} 1 - \frac{2l_D^2}{\tilde{f}^2} & 2l_D(1 + \frac{l_D}{\tilde{f}}) \\ 2(\frac{l_D^2}{\tilde{f}^3} - \frac{l_D}{\tilde{f}^2}) & 1 - 2\frac{l_D^2}{\tilde{f}^2} \end{pmatrix}$$

Now we know, that the phase advance is related to the transfer matrix by

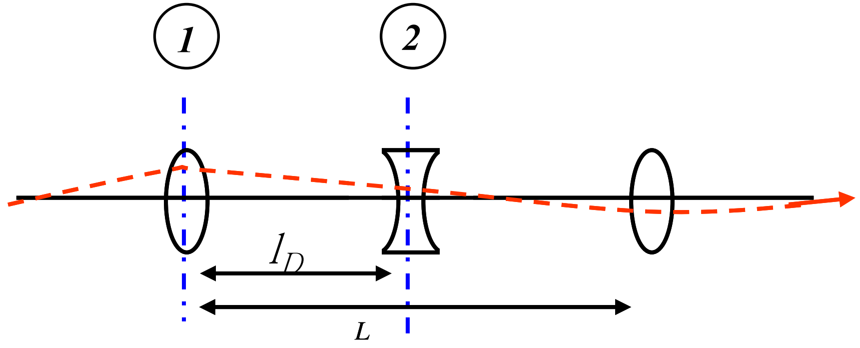
$$\cos\psi_{cell} = \frac{1}{2} trace(M) = \frac{1}{2} * (2 - \frac{4 l_d^2}{\tilde{f}^2}) = 1 - \frac{2 l_d^2}{\tilde{f}^2}$$

After some beer and with a little bit of trigonometric gymnastics

$$\cos(x) = \cos^2(x/2) - \sin^2(x/2) = 1 - 2\sin^2(x/2)$$

Transfer Matrix for half a FoDo cell:

$$M_{halfcell} = \begin{pmatrix} 1 - l_D/\tilde{f} & l_D \\ -l_D/\tilde{f}^2 & 1 + l_D/\tilde{f} \end{pmatrix}$$



Compare to the twiss parameter form of M

$$M_{1\to 2} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\psi_{12} + \alpha_1 \sin\psi_{12}) & \sqrt{\beta_1\beta_2}\sin\psi_{12} \\ \frac{(\alpha_1 - \alpha_2)\cos\psi_{12} - (1 + \alpha_1\alpha_2)\sin\psi_{12}}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\psi_{12} - \alpha_2 \sin\psi_{12}) \end{pmatrix}$$

In the middle of a foc (defoc) quadrupole of the FoDo we allways have $\alpha = 0$, and the half cell will lead us from β_{max} to β_{min}

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\check{\beta}}{\hat{\beta}}}\cos\frac{\psi_{cell}}{2} & \sqrt{\check{\beta}\hat{\beta}}\sin\frac{\psi_{cell}}{2} \\ \frac{-1}{\sqrt{\hat{\beta}\check{\beta}}}\sin\frac{\psi_{cell}}{2} & \sqrt{\frac{\hat{\beta}}{\check{\beta}}}\cos\frac{\psi_{cell}}{2} \end{pmatrix}$$

Solving for β_{max} and β_{min} and remembering that …. $\sin\frac{\psi_{cell}}{2} = \frac{l_d}{\tilde{f}} = \frac{L}{4f}$

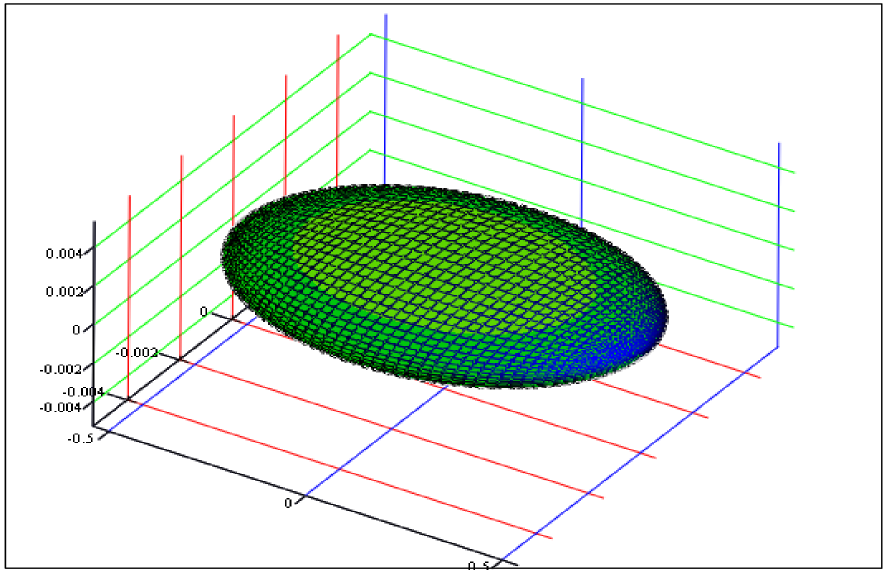
$$\frac{m_{22}}{m_{11}} = \frac{\hat{\beta}}{\breve{\beta}} = \frac{1 + l_d / \tilde{f}}{1 - l_d / \tilde{f}} = \frac{1 + \sin(\psi_{cell}/2)}{1 - \sin(\psi_{cell}/2)}$$

$$\frac{m_{12}}{m_{21}} = \hat{\beta}\breve{\beta} = \tilde{f}^2 = \frac{l_d^2}{\sin^2(\psi_{cell}/2)}$$

$\rightarrow$

$$\hat{\beta} = \frac{(1 + \sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}} \quad !$$

$$\breve{\beta} = \frac{(1 - \sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}} \quad !$$



(Z, X, Y)

The maximum and minimum values of the β-function are solely determined by the phase advance and the length of the cell.

Longer cells lead to larger β

typical shape of a proton bunch in a FoDo Cell