Joint US-CERN-Japan-Russia School on Particle Accelerators

Course on Synchrotron Radiation and Free Electron Lasers
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Storage Ring Design

Part 4:

Beam Instabilities

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Lectures 1, 2 and 3: summary

In Lecture 1, we derived expressions for the radiation damping times and equilibrium emittances in an electron storage ring.

In Lecture 2, we derived expressions for the natural emittance in storage rings with different lattice styles, in terms of the number of cells and the beam energy.

In Lecture 3, we discussed the need for sextupoles to correct the natural chromaticity in a storage ring, and considered the impact of sextupoles on the dynamic aperture and beam lifetime.

In this final lecture on Storage Ring Design, we shall discuss the interaction of the beam with the vacuum chamber and the components that it contains.

We shall see that interactions between the beam and its surroundings can make the beam unstable.

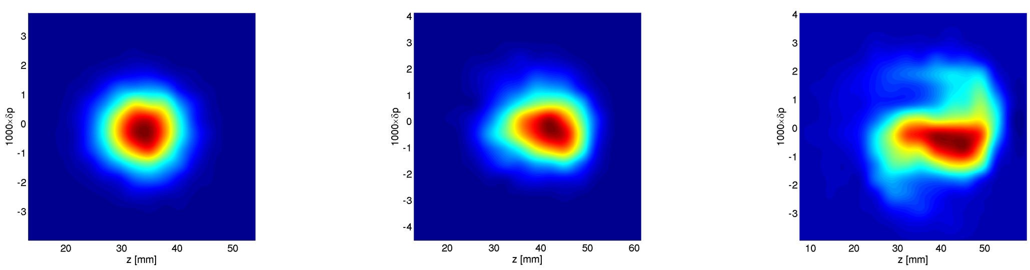
Instabilities of different kinds place important limits on the beam intensity that can be achieved (without degrading the beam quality) in a storage ring.

As well as having important implications for the design and operational performance of light sources, instabilities provide an introduction to wake fields and impedances, which can be used to model certain synchrotron radiation effects (e.g. CSR).

Multi-bunch and single-bunch instabilities

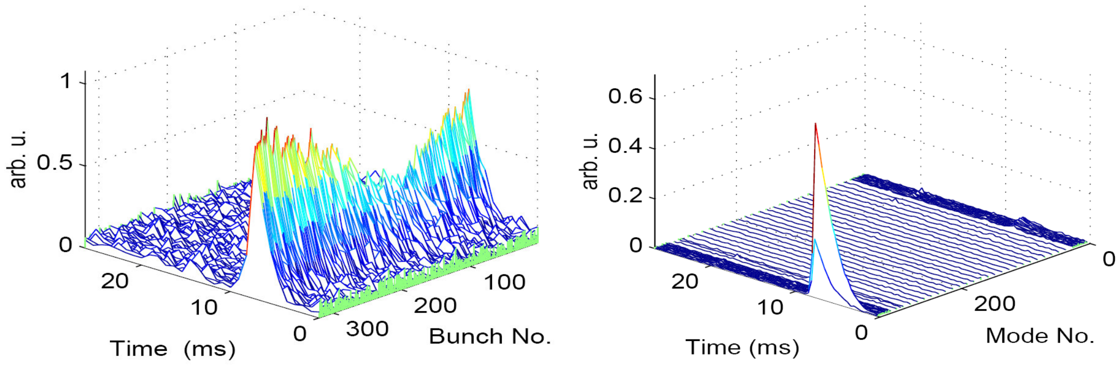
Broadly speaking, instabilities in electron storage rings can be classified as multi-bunch or single-bunch.

Single-bunch instabilities occur within individual bunches, irrespective of the presence (or absence) of other bunches in the ring. Time scales tend to be of order of a single turn or less; such instabilities are difficult to address using feedback systems.



Tracking simulation of longitudinal phase space distribution in a storage ring with wake fields, and (left to right) increasing bunch population.

We shall begin by considering multi-bunch instabilities. In this type of instability, coherent motion of one bunch affects other bunches in the storage ring. Time scales of the growth of such instabilities tend to be of order several turns or tens of turns; such instabilities can be mitigated using fast feedback systems.



"Grow-damp" measurements of multibunch instability in the ALS, from J. Fox et al, "Multi-bunch instability diagnostics via digital feedback systems at PEP-II, DAΦNE, ALS and SPEAR", Proceedings of PAC'99, New York.

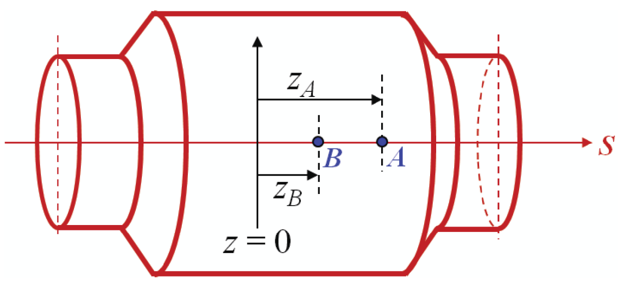
The electromagnetic fields generated by a charged particle (or bunch of particles) moving through an accelerator vacuum chamber are very complex.

However, the significant effects can generally be represented by a *wake function*, that describes the force on one particle from the fields generated by a particle some distance ahead of it.

For example, the change in the energy of a particle that follows a *point-like* bunch with population N_A through a particular component may be written:

$$\Delta E_B = -e^2 N_A W_{\parallel}(z_B - z_A), \quad (1)$$

where $W_{\parallel}(z)$ is the longitudinal wake function for the component.

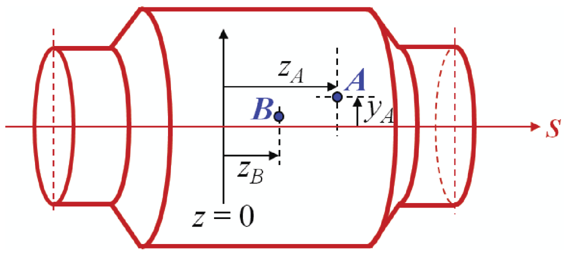


If we normalise both sides of Eq. (1) by the beam energy E_0, we obtain the change in the energy deviation:

$$\Delta\delta_B = -\frac{e^2 N_A}{E_0} W_\parallel(z_B - z_A), \tag{2}$$

Note that the wake function $W_\parallel(z)$ has units of *volts per coulomb* (in SI units).

As well as longitudinal forces, there are transverse forces arising from the electromagnetic fields around particles, that lead to transverse deflections.



The transverse forces in a particular component are described by a transverse wake function $W_\perp(z)$, defined so that:

$$\Delta p_{y,B} = -\frac{e^2 N_A}{E_0} y_A W_\perp(z_B - z_A). \tag{3}$$

Note that the transverse wake function $W_\perp(z)$ has units of *volts per coulomb per meter* (in SI units).

For some very simple cases, expressions for the wake functions may be derived by solving Maxwell's equations analytically.

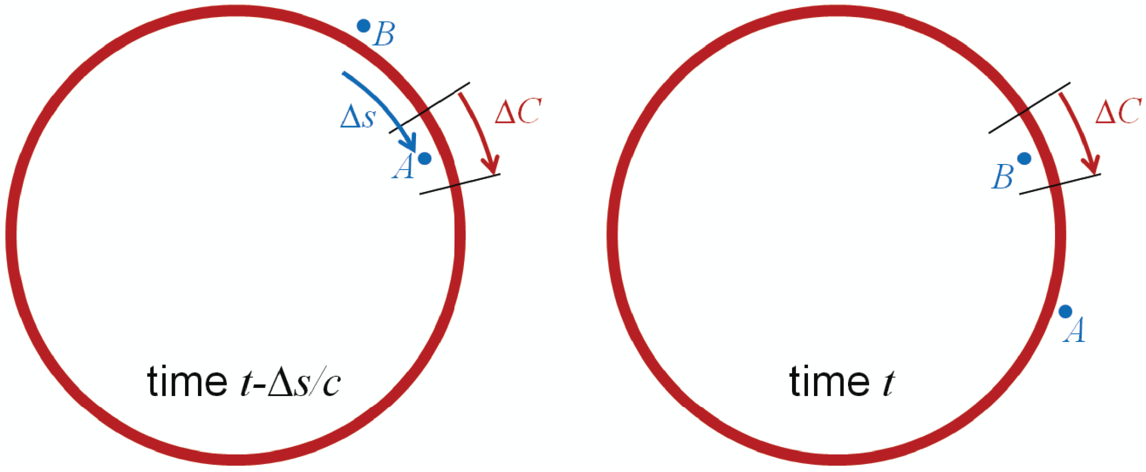
For example, the "resistive wall" wake functions for a straight beam pipe of length L, circular cross section of radius b, and conductivity σ are:

$$W_{\parallel}(z) = \frac{1}{2\pi b}\sqrt{\frac{Z_0 c}{4\pi}\frac{c}{\sigma}} \cdot \frac{L}{\sqrt{-z^3}}, \quad W_{\perp}(z) = -\frac{2}{\pi b^3}\sqrt{\frac{Z_0 c}{4\pi}\frac{c}{\sigma}} \cdot \frac{L}{\sqrt{-z}}. \quad (4)$$

However, more generally, one needs to use a modelling code ("Maxwell solver") to compute the wake functions numerically for a given component.

We will not discuss the details of computing wake functions: we shall assume they are known, and will just consider the effect that they have on the beam dynamics.

As an example, consider two bunches in a storage ring, with bunch B a distance $\Delta s \ll C_0$ behind bunch A (where C_0 is the ring circumference).



Since bunch A is a long way behind bunch B, we assume we can neglect the effect of the wake fields induced by bunch B.

Bunch A then performs betatron oscillations with equation of motion:

$$\ddot{y}_A + \omega_\beta^2 y_A = 0. \tag{5}$$

Note that a dot indicates a derivative with respect to *time*.

For bunch B, there are two contributions to the transverse force: one from the magnets giving the usual betatron oscillations, and the other from the wake field from bunch A.

Using Eq. (3), the wake function $W_\perp$ (for the full ring) gives a transverse deflection over a section of length ΔC:

$$\Delta p_{y,B} = -\frac{e^2 N_A}{E_0} y_A W_\perp(-\Delta s) \frac{\Delta C}{C_0}. \tag{6}$$

Motion with wake forces: two-bunch model

Taking into account the focusing effects of the magnets (which lead to the betatron oscillations) and the force from the wake field generated by bunch A, the equation of motion for bunch B is:

$$\ddot{y}_B + \omega_\beta^2 y_B = -\frac{e^2 N_A c}{E_0 T_0} y_A\!\left(t - \frac{\Delta s}{c}\right) W_\perp(-\Delta s), \tag{7}$$

where T_0 is the revolution period of the ring.

Note that the position of bunch A has to be evaluated at the time that it generated the wake field experienced by bunch B.

Assuming that bunch A only performs betatron motion (and does not experience any wake field effects), and neglecting damping effects from synchrotron radiation etc., the motion of bunch A can be written:

$$y_A(t) = y_{A,0} \cos(\omega_\beta t). \tag{8}$$

Substituting into Eq. (7), the motion of bunch B is given by:

$$y_B(t) = -y_{A,0} \frac{t}{\tau} \sin(\omega_\beta t), \tag{9}$$

where:

$$\frac{1}{\tau} = \frac{1}{2\omega_\beta} \frac{e^2 N_A c}{E_0 T_0} W_\perp(-\Delta s). \tag{10}$$

We see that the amplitude of the oscillation of bunch B grows linearly with time: the motion is unstable, and the bunch will eventually be lost from the beam pipe.

A more interesting case is one where the ring contains many bunches. To simplify things, we shall assume the bunches are equally spaced, and all have the same charge.

We should also take into account the fact that wake fields can persist over many revolution periods.

The equation of motion of bunch number n is then:

$$\ddot{y}_n + \omega_\beta^2 y_n =$$
$$-\frac{e^2 N_0 c}{E_0 T_0} \sum_k \sum_{m=0}^{M-1} y_m\!\left(t - kT_0 - \frac{m-n}{M} T_0\right) W_\perp(-kC_0 - \frac{m-n}{M} C_0), \tag{11}$$

where M is the total number of bunches, and $C_0 = cT_0$ is the ring circumference.

The summation over m accounts for the multiple bunches, and the summation over k accounts for multiple turns.

Solving the equation of motion (11) is not as bad as we might fear, at least if we are prepared to make some approximations.

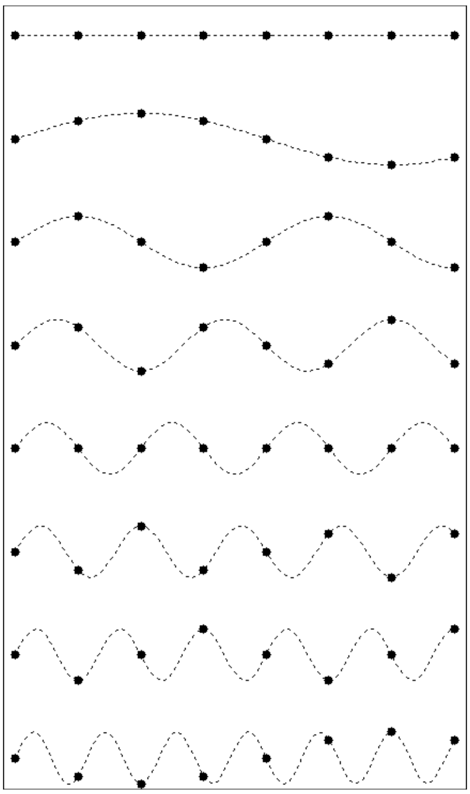
We try a solution of the form:

$$y_n^\mu(t) \propto \exp\left(2\pi i \frac{\mu n}{M}\right) \exp(-i\Omega_\mu t)\,. \qquad (12)$$

Actually, equation (12) represents a set of solutions, with the members of the set indexed by μ.

From the right hand side of equation (12), μ gives the phase advance in the transverse displacement between successive bunches.

Each bunch performs oscillations with frequency Ω_μ as it moves around the ring.

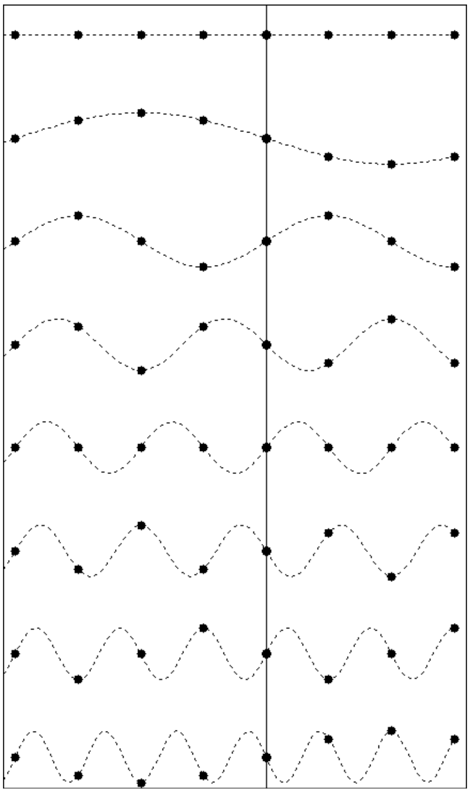


The mode number μ gives the phase advance in the transverse position between successive bunches.

The figure on the left shows the 8 (transverse) modes possible with 8 bunches, from $\mu = 0$ (all bunches in phase) at the top, through to $\mu = 7$ (phase advance $2\pi \times 7/8$ from one bunch to the next) at the bottom.

The real part of Ω_μ gives the oscillation frequency of each bunch as it moves around the ring. Because of the wake fields, the frequency may be different from the nominal betatron frequency ω_β.

The imaginary part of Ω_μ gives the growth or damping rate of the oscillations for the specified mode.



If we observe the bunches coming past at a fixed point in the ring, we see that each mode has associated with it a different oscillation frequency.

If a component in the ring has an electromagnetic mode with a frequency close to that of a particular beam mode, then the electromagnetic mode may resonate with the beam mode.

The effect of the resonance can be to drive both the electromagnetic mode and the beam mode to very large amplitudes.

If we assume that the frequency shift from the wake fields is small, i.e.:

$$|\Omega_\mu - \omega_\beta| \ll \omega_\beta, \tag{13}$$

then substituting the trial solution (12) into the equation of motion (11) gives:

$$\Omega_\mu - \omega_\beta \approx -i \frac{Me^2 N_0 c}{4\pi E_0 T_0 \nu_\beta} \sum_{p=-\infty}^{\infty} Z_\perp \Big([pM + \mu]\omega_0 + \omega_\beta \Big) \,, \tag{14}$$

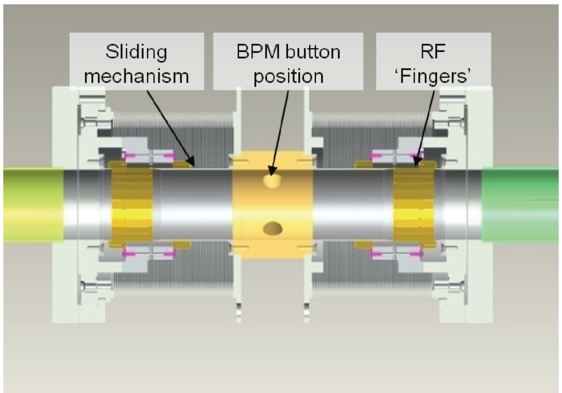
where ν_β is the betatron tune.

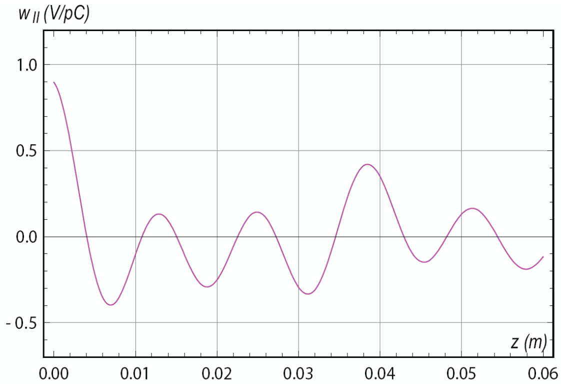
This is written in terms of the *impedance*, which is defined as the Fourier transform of the wake function:

$$Z_\perp(\omega) = \frac{i}{c} \int_{-\infty}^{\infty} e^{-i\frac{\omega z}{c}} W_\perp(z)\, dz. \tag{15}$$

(Note that the units of $Z_\perp(\omega)$ are *ohms per metre*, in SI units).

Calculating the impedance in a storage ring requires knowledge of the detailed design of all components in the vacuum chamber (including the chamber itself).





A simple impedance model: the broad-band resonator

Usually, only an approximate *impedance model* can be developed.

To investigate the effects of impedances, simplified models can be used. One such model represents the wake field as a damped oscillation:

$$W_{\perp}(z) = \frac{cR_s}{Q}\frac{\omega_r}{\bar{\omega}_r} e^{\frac{\alpha z}{c}} \sin\left(\frac{\bar{\omega}_r z}{c}\right), \tag{16}$$

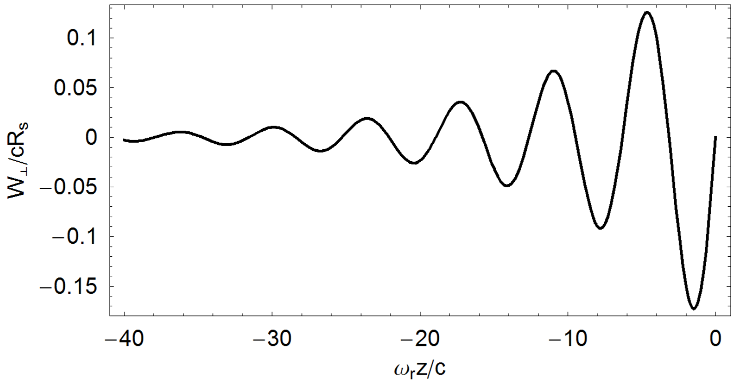
where:

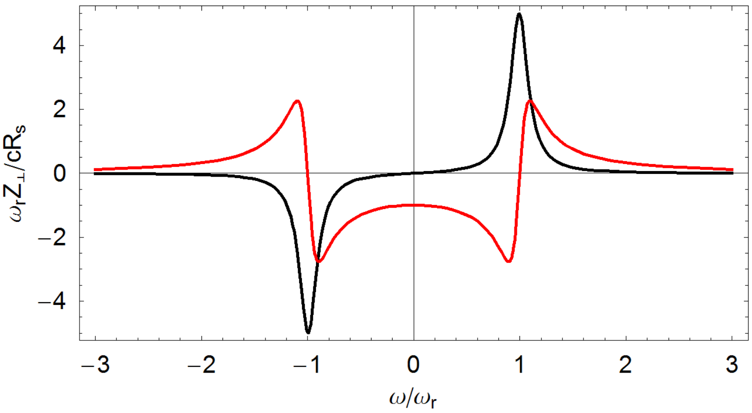
$$\alpha = \frac{\omega_r}{2Q}, \quad \text{and} \quad \bar{\omega}_r = \sqrt{|\omega_r^2 - \alpha^2|}. \tag{17}$$

This leads to the *resonator* impedance:

$$Z_{\perp}(\omega) = \frac{c}{\omega}\frac{R_s}{1 + iQ(\omega_r/\omega - \omega/\omega_r)}. \tag{18}$$

Example of a broad-band resonator, with $Q = 5$





Multibunch instability growth rate

From equation (14):

$$\Omega_\mu - \omega_\beta \approx -i\frac{Me^2N_0c}{4\pi E_0T_0\nu_\beta}\sum_{p=-\infty}^{\infty} Z_\perp\Big([pM+\mu]\omega_0+\omega_\beta\Big),$$

the imaginary part of $Z_\perp$ leads to a tune shift, while the real part leads to growth (or damping) of the oscillations.

For a resonator, the impedance is largest at frequencies close to the resonant frequency. We expect to see the largest growth rate for:

$$(p + M\mu)\omega_0 + \omega_\beta \approx -\omega_r. \tag{19}$$

The dominant term in the summation in equation (14) has:

$$p \approx -\frac{(\omega_r+\omega_\beta)}{M\omega_0} - \frac{\mu}{M}. \tag{20}$$

Since $0 \leq \mu < M$, for large M there is likely to be a mode for which the resonance condition (19) is closely satisfied for some integer value of p.

In this case (for a resonator impedance):

$$Z_\perp\Big([pM + \mu]\omega_0 + \omega_\beta\Big) \approx Z_\perp(-\omega_r) = \frac{cR_s}{\omega_r}. \tag{21}$$

Then, the exponential growth rate of the amplitude of the resonant mode is:

$$\frac{1}{\tau_\mu} = \mathrm{Im}(\Omega_\mu) \approx \frac{Me^2N_0c}{4\pi E_0 T_0 \nu_\beta}\frac{cR_s}{\omega_r}. \tag{22}$$

Note that the growth rate is proportional to the total current in the ring ($I_0 = MeN_0/T_0$). As current is injected into the ring, at some point the growth rate will exceed the damping rate (from radiation, feedbacks...), and the beam will become unstable.

The physical significance of the impedance is probably best understood in terms of the longitudinal dynamics.

Recall that the longitudinal wake function $W_\parallel(-z)$ determines the change in energy of particle B that is a distance z behind particle A (which has charge eN_a):

$$\Delta E_B = -e^2 N_A W_\parallel(z_B - z_A). \tag{23}$$

If we consider the wake field generated by a longitudinal charge distribution, with line density $e\lambda(z)$, then the change in energy of a particle at position z within the charge distribution will be given by:

$$\Delta E(z) = -e^2 \int_{-\infty}^{\infty} \lambda(z')\, W_\parallel(z - z')\, dz'. \tag{24}$$

The energy change associated with the different frequency components in the charge distribution is obtained by a Fourier transform:

$$\begin{aligned}
\Delta\tilde{E}(\omega) &= \int e^{-i\frac{\omega z}{c}}\,\Delta E(z)\,\frac{dz}{c} && (25)\\
&= -e^2\int\int e^{-i\frac{\omega z}{c}}\,\lambda(z')\,W_{\|}(z-z')\,\frac{dz'}{c}\,\frac{dz}{c} && (26)\\
&= -e^2 c\int\int e^{-i\frac{\omega z'}{c}}\,\lambda(z')\,e^{-i\frac{\omega z''}{c}}\,W_{\|}(z'')\,\frac{dz'}{c}\,\frac{dz''}{c}. && (27)
\end{aligned}$$

The last line follows by making the substitution $z = z' + z''$.

Notice that this can be written:

$$\Delta\tilde{E}(\omega) = -e^2 c\,\tilde{\lambda}(\omega)\,Z_{\|}(\omega), \tag{28}$$

where $\tilde{\lambda}$ and $Z_{\|}$ are the Fourier transforms of the charge distribution λ and the wake function $W_{\|}$, respectively.

Now we define $\tilde{V}$ to be the energy loss (in frequency space) per unit charge:

$$\tilde{V}(\omega) = -\frac{\Delta\tilde{E}(\omega)}{e}, \tag{29}$$

and $\tilde{I}(\omega)$ to be the current spectrum:

$$\tilde{I}(\omega) = ec\tilde{\lambda}(\omega). \tag{30}$$

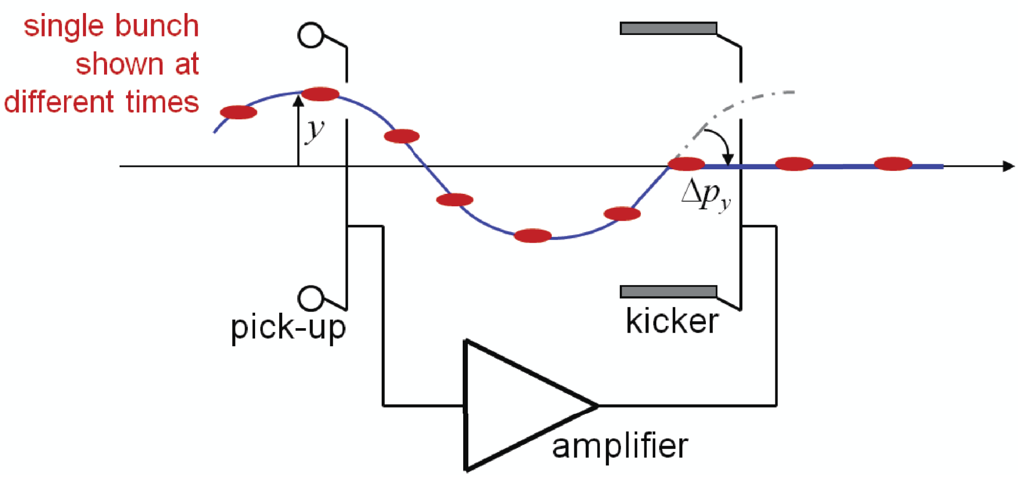
Then, equation (28) becomes:

$$\tilde{V}(\omega) = \tilde{I}(\omega)\,Z_{\|}(\omega). \tag{31}$$

In other words, $Z_{\|}$ performs the function we would expect of an impedance, in relating the voltage to the current in frequency space.

At low currents, radiation damping can be sufficient to prevent the growth of multibunch instabilities.

But at the currents required by third-generation synchrotron light sources, feedback systems are commonly required.



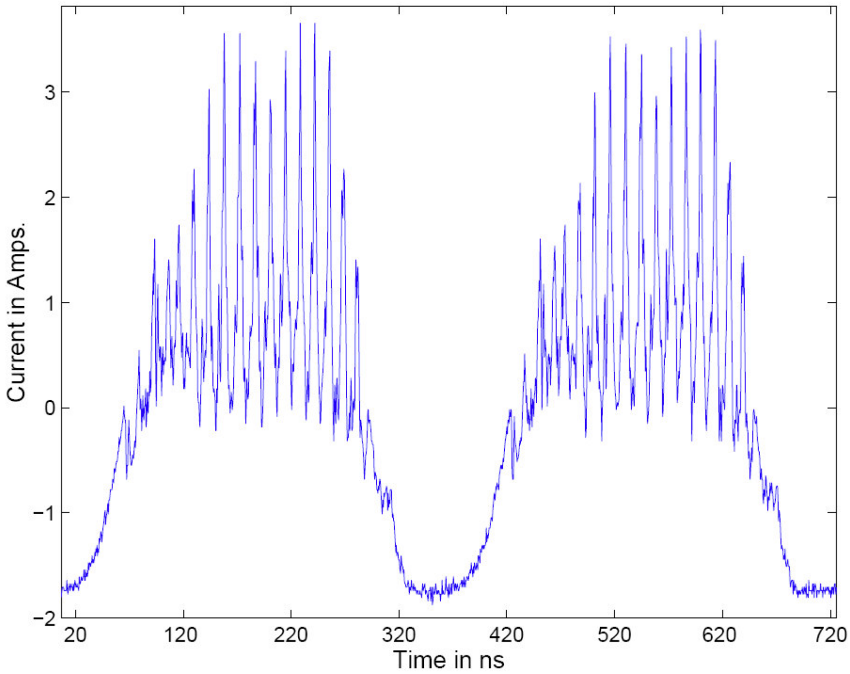
As well as coupling the motion of different bunches, wake fields can couple the motion of different particles within individual bunches.

As a result, the charge distribution within individual bunches can be modified, and with sufficient numbers of particles, can become unstable.

We can use simplified models to understand how short-range wake fields can lead to single-bunch instabilities.

We shall discuss the microwave instability, for which we only need to consider the longitudinal dynamics.

Example of a longitudinal instability in the Los Alamos PSR



From C. Beltran, A.A. Browman and R.J. Macek, "Calculations and observations of the longitudinal instability caused by the ferrite inductors at the Los Alamos Proton Storage Ring", Proceedings of the 2003 Particle Accelerator Conference, Portland, Oregon (2003).

The Vlasov equation

Neglecting damping effects (such as synchrotron radiation), Liouville's theorem tells us that the particle density Ψ in phase space is conserved:

$$\frac{d\Psi}{dt} = 0. \tag{32}$$

In the longitudinal phase space, this is expressed by the *Vlasov equation*:

$$\frac{\partial\Psi}{\partial t} + \dot{\theta}\frac{\partial\Psi}{\partial\theta} + \dot{\delta}\frac{\partial\Psi}{\partial\delta} = 0, \tag{33}$$

where $\Psi = \Psi(\theta, \delta; t)$ is the particle density in phase space, and θ and δ are dynamical variables.

$\theta = 2\pi s/C_0$ is an angle describing the position of a particle around the ring, and $\delta = E/E_0 - 1$ is the energy deviation of a particle.

If we know $\dot{\theta}$ and $\dot{\delta}$, then we can (in principle) solve the Vlasov equation to find how the phase space distribution of particles within a bunch evolves over time.

This will tell us, for example, whether the distribution is stable or not.

Generally, solving the Vlasov equation is difficult, and requires (computationally expensive) numerical techniques. But by making some approximations, we can obtain some useful analytical results.

First of all we write down an expression for $\dot{\theta}$, which is just the rate at which particles move around the ring:

$$\dot{\theta} = (1 - \alpha_p \delta)\omega_0. \tag{34}$$

Here, α_p is the momentum compaction factor, and ω_0 is the (nominal) revolution frequency.

Finding $\dot{\delta}$ in the presence of wake fields is more complicated. To simplify things, we will ignore energy changes from synchrotron radiation and RF cavities.

The energy changes from wake fields then depend on the charge distribution around the ring, which determines the current spectrum.

Let us assume that the charge distribution takes the form:

$$\Psi(\theta, \delta; t) = \Psi_0(\delta) + \Delta\Psi(\delta)\, e^{i(n\theta - \omega_n t)}, \qquad (35)$$

where $\Delta\Psi(\delta)$ represents a *perturbation* to a distribution that is otherwise uniform around the ring.

Note that the perturbation takes the form of a wave: n is an integer that gives the number of wavelengths around the ring.

Our goal is to find the frequency ω_n for a given n: the imaginary part of ω_n will indicate the stability of the perturbation.

Solving the Vlasov equation: perturbation approach

Let us normalise the particle distribution so that:

$$\int_{-\infty}^{\infty} \Psi_0(\delta)\, d\delta = 1. \qquad (36)$$

Since the perturbation contains a single frequency ω_n, we can immediately write down the current spectrum (ignoring the DC component):

$$\tilde{I}(\omega) = 2\pi I_0\, \bar{\delta}(\omega - \omega_n) \int \Delta\Psi(\delta)\, d\delta, \qquad (37)$$

where $\bar{\delta}(\omega - \omega_n)$ is a Dirac delta function.

Using Eq. (31), the energy change over one turn is given by:

$$\frac{\Delta E(z)}{e} = -\frac{1}{2\pi}\int \tilde{I}(\omega)\, Z_{\parallel}(\omega)\, e^{i\frac{\omega z}{c}}\, d\omega, \tag{38}$$

where z is the longitudinal coordinate of a particle with respect to a reference particle moving around the ring at the speed of light.

Substituting for $\tilde{I}(\omega)$ from Eq. (37), and using the fact that for particles moving with speed c:

$$z \approx \frac{\theta}{2\pi}C_0 - ct, \tag{39}$$

we find:

$$\dot{\delta} \approx -\frac{\omega_0}{2\pi}\frac{I_0}{E_0/e}\left(\int \Delta\Psi(\delta)\, d\delta\right) Z_{\parallel}(\omega_n)\, e^{i(n\theta-\omega_n t)}. \tag{40}$$

We now have explicit expressions for $\dot{\theta}$ [Eq. (34)] and $\dot{\delta}$ [Eq. (40)].

If we substitute these expressions into the Vlasov equation (33), and keep terms only up to first order in the perturbation $\Delta\Psi(\delta)$, we find:

$$iZ_{\parallel}(\omega_n)\frac{\omega_0}{2\pi}\frac{I_0}{E_0/e}\left(\int \Delta\Psi(\delta)\, d\delta\right)\frac{\partial\Psi_0(\delta)/\partial\delta}{(n\omega-\omega_n)} = \Delta\Psi(\delta), \tag{41}$$

where $\omega = (1-\alpha_p\delta)\omega_0$.

We cannot solve this directly for ω_n, because the perturbation $\Delta\Psi(\delta)$ is unknown. However, we notice that if we integrate both sides over δ, then we can cancel the integral over the perturbation...

The resulting equation is known as the *dispersion relation*:

$$iZ_{\|}(\omega_n)\frac{\omega_0}{2\pi}\frac{I_0}{E_0/e}\int\frac{\partial\Psi_0(\delta)/\partial\delta}{(n\omega-\omega_n)}\,d\delta = 1. \qquad (42)$$

Although we have not "solved" the Vlasov equation, we have derived from it an integral equation (the dispersion relation) for the frequency of a perturbation with a given wavelength.

The imaginary part of the frequency will tell us if the perturbation is damped, or anti-damped.

The relationship between ω_n and n (which is found from the dispersion relation) depends on the energy distribution $\Psi_0(\delta)$.

As a simple example, let us consider the case of a "cold" beam, in which all particles have zero energy deviation. The energy distribution is a delta function:

$$\Psi_0(\delta) = bar\delta(\delta). \qquad (43)$$

Integrating by parts, we find that:

$$\int\frac{\partial\Psi_0(\delta)/\partial\delta}{(n\omega-\omega_n)}\,d\delta = \int\frac{\Psi_0}{(n\omega-\omega_n)^2}n\frac{\partial\omega}{\partial\delta}\,d\delta = \frac{n\omega_0\alpha_p}{(n\omega_0-\omega_n)^2}. \qquad (44)$$

We then find from the dispersion relation (42):

$$\omega_n = n\omega_0 \pm \sqrt{iZ_\parallel(\omega_n)\frac{I_0}{E_0/e}\frac{n\omega_0^2\alpha_p}{2\pi}}. \tag{45}$$

Except for the case that the impedance $Z_\parallel(\omega_n)$ has complex phase $3\pi/2$, there is always a solution for the frequency that has a positive imaginary part.

From equation (35):

$$\Psi(\theta,\delta;t) = \Psi_0(\delta) + \Delta\Psi(\delta)\, e^{i(n\theta-\omega_n t)},$$

we see that the beam is always unstable. Physically, this is because there is no process in our model that will damp a mode that is driven by the impedance.

A more realistic situation is one in which the beam has a Gaussian energy spread:

$$\Psi_0(\delta) = \frac{1}{\sqrt{2\pi}\sigma_\delta}\, e^{-\frac{\delta^2}{2\sigma_\delta^2}}. \tag{46}$$

Substituting this into the dispersion relation gives:

$$i\frac{Z_\parallel(\omega_n)}{n}\frac{1}{(2\pi)^{3/2}}\frac{I_0}{E_0/e}\frac{1}{\alpha_p\sigma_\delta^2}\int \frac{\zeta\, e^{-\zeta^2/2}}{\zeta+\Delta_n}\, d\zeta = 1, \tag{47}$$

where:

$$\Delta_n = \frac{\omega_n - n\omega_0}{n\omega_0\alpha_p\sigma_\delta}. \tag{48}$$

Let us write equation (47) as:

$$\frac{1}{(2\pi)^{3/2}} \frac{I_0}{E_0/e} \frac{1}{\alpha_p \sigma_\delta^2} \frac{Z_\parallel(\omega_n)}{n} = \left[i \int \frac{\zeta\, e^{-\zeta^2/2}}{\zeta + \Delta_n} d\zeta \right]^{-1}. \qquad (49)$$

Note that an instability occurs if ω_n, and hence Δ_n, has a positive imaginary part.

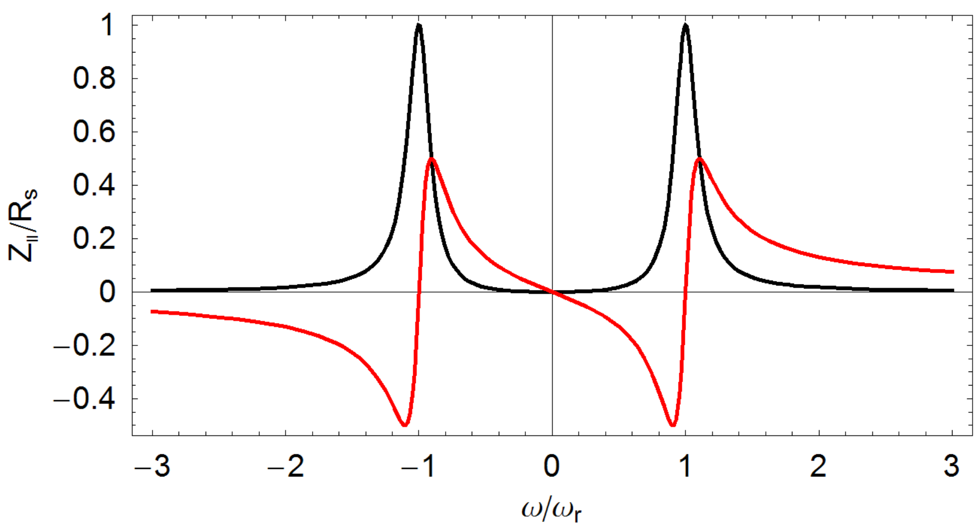
The right hand side, evaluated for all *real* values of Δ_n, defines a boundary in the complex plane.

The left hand side may be evaluated for some mode n and over a range of assumed values for ω_n, to generate a line in the complex plane.

If part of this line lies on the unstable side of the boundary defined by the right hand side of (49), then it is possible that the mode may be unstable.

As an example, consider a resonator impedance. In the longitudinal case, this is given by:

$$Z_\parallel(\omega) = \frac{R_s}{1 + iQ(\omega_r/\omega - \omega/\omega_r)}. \qquad (50)$$



In the complex plane, we draw the line (stability boundary) defined by:

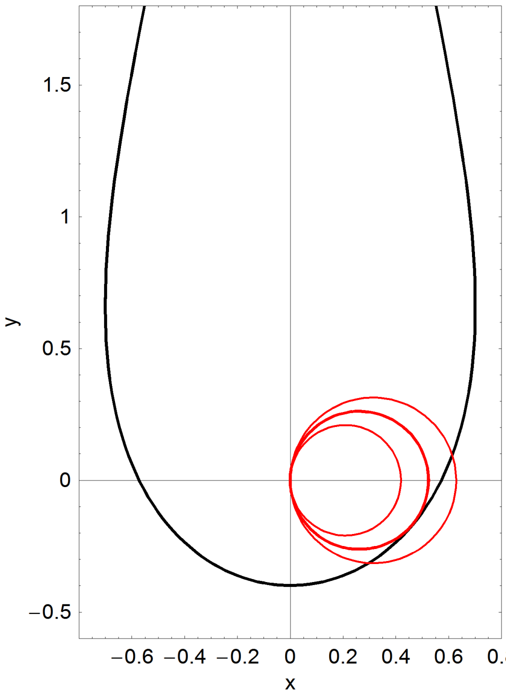
$$x + iy = \left[i \int \frac{\zeta\, e^{-\zeta^2/2}}{\zeta + \Delta_n}\, d\zeta \right]^{-1} \tag{51}$$

for $\mathrm{Im}(\Delta_n) = 0$, in black.

We draw the line defined by:

$$x+iy = \frac{1}{(2\pi)^{3/2}} \frac{I_0}{E_0/e} \frac{1}{\alpha_p \sigma_\delta^2} \frac{Z_{\|}(\omega)}{n} \tag{52}$$

for all ω, (and selected values of the various parameters) in red.



For:

$$\frac{1}{(2\pi)^{3/2}} \frac{I_0}{E_0/e} \frac{1}{\alpha_p \sigma_\delta^2} \frac{R_s}{n} \approx \frac{\pi}{6}, \tag{53}$$

the lines just touch: all the modes have frequencies with negative (or zero) imaginary part.

This implies that if:

$$\frac{1}{(2\pi)^{3/2}} \frac{I_0}{E_0/e} \frac{1}{\alpha_p \sigma_\delta^2} \frac{R_s}{n} < \frac{\pi}{6}, \tag{54}$$

then all the modes will be stable.

For a given magnitude of the impedance (i.e. value of R_s), it appears that modes with a small value of n are more likely to be unstable.

The broad-band resonator model is not always a good approximation for the impedance in storage rings.

Since an accurate model for the impedance is not usually known until late in the design stage (if at all!), the quantity R_s/n is often replaced by an indicative value, denoted Z/n.

Also, the stability boundary can be replaced by a circle with radius $1/\sqrt{2\pi}$: this is a conservative approximation.

The stability condition is then written:

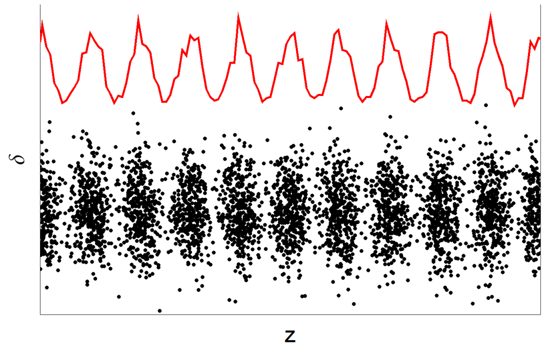
$$\frac{Z}{n} < 2\pi \frac{E_0/e}{I_0} \alpha_p \sigma_\delta^2. \tag{55}$$

This is known as the Keil-Schnell criterion. It ought not be taken as a strict quantitative limit on the beam stability: rather, it provides a convenient way for comparing impedances (characterised by a single number, Z/n) between different rings.

Recall that a "cold" beam (with zero energy spread) was *always* unstable in the presence of an impedance.

Now we have found that a beam with a non-zero energy spread can be stable, as long as the impedance is not too large.

The energy spread improves stability, because of the momentum compaction: particles with different energies move around the ring at different rates.



The combination of energy spread and momentum compaction means that any modulation on the charge density tends to get smeared out: the larger the energy spread, or the larger the momentum compaction factor, the faster the smearing.

An instability will only occur if the modulation grows faster than it is smeared out. The momentum compaction provides a damping mechanism, sometimes referred to as Landau damping.

Our model is based on a "coasting" beam, i.e. a beam in which charge is uniformly distributed around the ring (neglecting a small modulation).

In electron storage rings, the beam will be bunched because of the RF cavities. However, our model can still be applied if we consider modes with:

$$n > \frac{C_0}{\sigma_z}, \tag{56}$$

where C_0 is the ring circumference, and σ_z is the bunch length.

Modes with smaller n cannot appear, because the wavelength of the density modulation would be larger than the length of a bunch.

If the mode number n is very large, so that $n \gg C_0/\sigma_z$, then from point of view of the development of an instability, there is little difference between a bunched beam and a coasting beam.

To make an estimate for the stability criterion in this case, we simply modify the Keil-Schnell criterion by replacing the average current in the ring, I_0, by the peak current in a bunch:

$$\hat{I} = \frac{eNc}{\sqrt{2\pi}\sigma_z}, \tag{57}$$

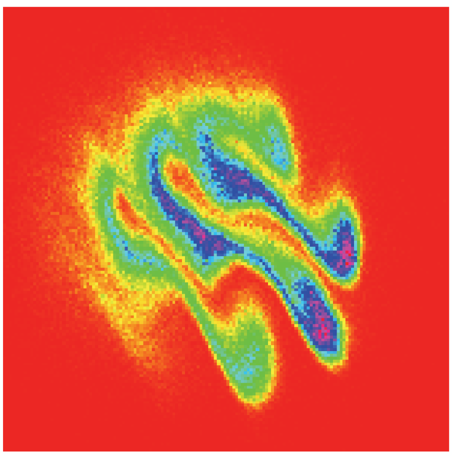
where N is the bunch population.

Making this modification gives the Keil-Schnell-Boussard criterion for stability in a bunched beam:

$$\frac{Z}{n} < (2\pi)^{\frac{3}{2}} \frac{E_0/e}{eNc} \alpha_p \sigma_z \sigma_\delta^2. \tag{58}$$

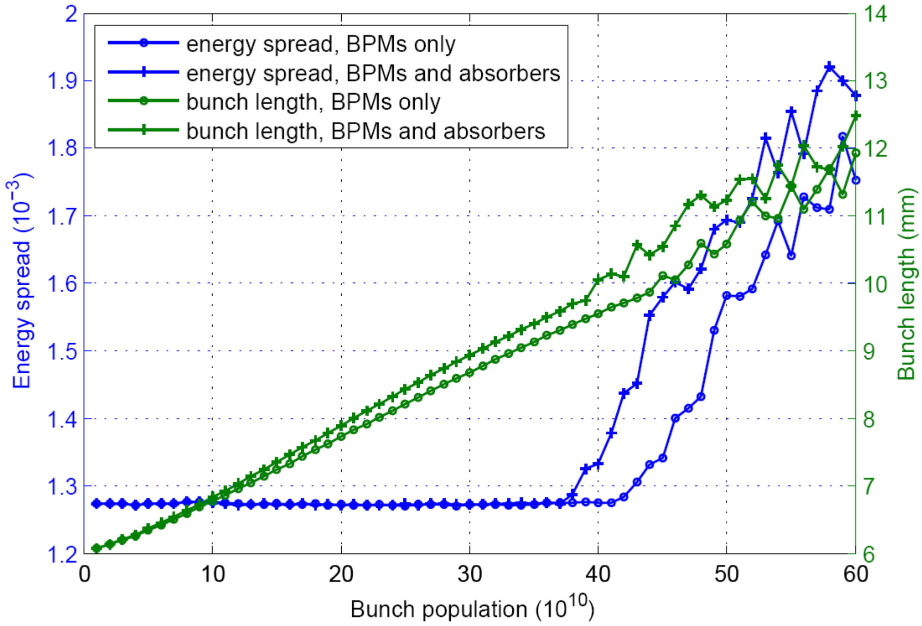
The model we have developed is based on perturbation theory. This means we can tell (in principle) whether an initial (small) density modulation will be damped or anti-damped by the impedance of the ring; but we cannot describe the dynamics in any more detail.

We do, however, expect the instability to have a clear threshold. Below a particular bunch charge, the longitudinal emittance should damp to (approximately) the value determined by the synchrotron radiation.



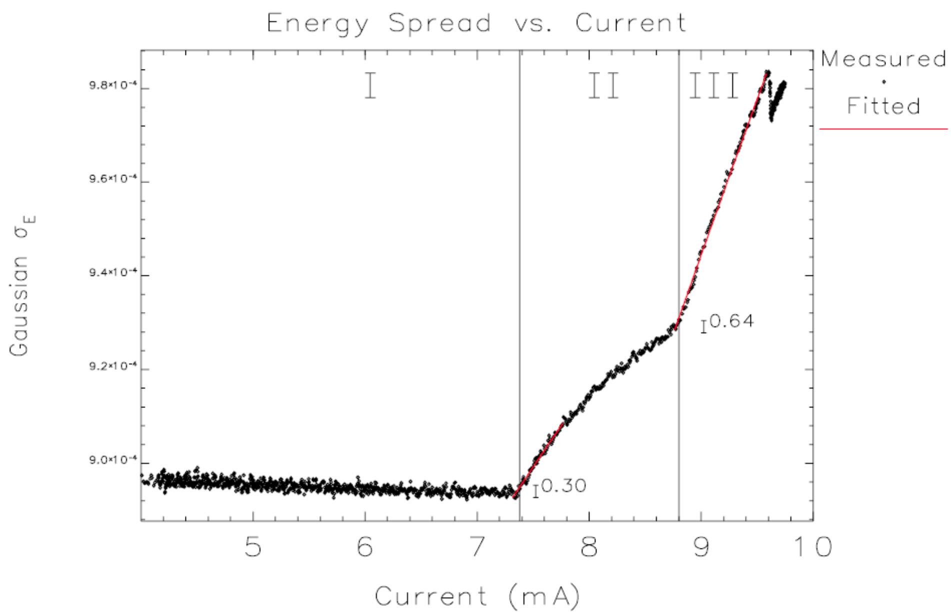
Above the threshold bunch charge, we expect to see an increase in the longitudinal emittance; in fact, the longitudinal distribution may never properly reach equilibrium.

The instability threshold is visible in tracking simulations...



Energy spread and bunch length as functions of bunch population in the ILC damping rings, with two different impedance models.

...and in observations in storage rings:



Y.-C. Chae et al, "Measurement of the longitudinal microwave instability in the APS storage ring," Proceedings of PAC 2001, Chicago.

The main impact of the microwave instability is to increase the energy spread.

For third generation synchrotron light sources, this may not be too serious. But it could make it difficult to operate an FEL.

The timescale of the microwave instability generally puts it beyond the reach of feedback systems. To stay below the instability threshold it is necessary to:

- design the storage ring with appropriate values for the energy, bunch charge, momentum compaction factor, bunch length and (natural) energy spread;
- stay within a specified "impedance budget" in the design and manufacture of components within the vacuum chamber.

Final remarks

There are many different varieties of beam instabilities and (more generally) collective effects.

In this lecture, we have considered just two of them, as illustrative examples:

- (transverse) coupled bunch instability;
- (longitudinal) single bunch instability (microwave instability).

Understanding and control of instabilities is critical part of storage ring design and operation, especially where beam quality is important (for example, where the storage ring includes an FEL).

Our understanding of instabilities continues to develop with new observations, new theories, and new modelling techniques.