1.) the basic ideas

Lorentz force

\[ \vec{F} = q \left( \vec{E} + \vec{v} \times \vec{B} \right) \]

\text{typical velocity in high energy machines:}

\[ v \approx c \approx 3 \times 10^8 \text{ m/s} \]

Example:

\[ B = 1 \text{ T} \quad \rightarrow \quad F = q \times 3 \times 10^8 \frac{m}{s} \times 1 \frac{Vs}{m^2} \]

\[ F = q \times 300 \frac{MV}{m} \]

\text{technical limit for el. field}

\[ E \leq 1 \frac{MV}{m} \]

Bernhard Holzer, CERN
The ideal circular orbit

condition for circular orbit:

\[ F_L = e v B \]
\[ F_{\text{centr}} = \frac{\gamma m_0 v^2}{\rho} \]

\[ \frac{\gamma m_0 v^2}{\rho} = e v B \]

\[ \frac{p}{e} = B \rho \]

B \rho = "beam rigidity"

old greek dictum of wisdom:
if you are clever, you use magnetic fields in an accelerator wherever it is possible.
2.) The Magnetic Guide Field

**Dipole Magnets:**

define the ideal orbit
homogeneous field created by two flat pole shoes

Normalise magnetic field to momentum:

\[
\frac{p}{e} = B \rho \quad \rightarrow \quad \frac{1}{\rho} = \frac{e B}{p}
\]

**Example LHC:**

\[
\begin{align*}
B &= 8.3 \, \text{T} \\
p &= 7000 \frac{\text{GeV}}{c}
\end{align*}
\]

Convenient units:

\[
B = [T] = \left[ \frac{V_s}{m^2} \right] \quad p = \left[ \frac{\text{GeV}}{c} \right]
\]

\[
B = 1 \ldots 8 \, \text{T}
\]

\[
\frac{1}{\rho} = e \frac{8.3 \frac{V_s}{m^2}}{7000 \times 10^9 \frac{eV}{c}} = \frac{8.3 \times 3 \times 10^8 \frac{m}{s}}{7000 \times 10^9 \frac{m^2}{s}}
\]

\[
\rho = 2.81 \, \text{km}
\]
Focusing Properties and Quadrupole Magnets

classical mechanics: pendulum

there is a restoring force, proportional to the elongation x:

\[ m \frac{d^2 x}{dt^2} = -c \cdot x \]

general solution: free harmonic oscillation

\[ x(t) = A \cdot \cos(\omega t + \phi) \]

this is how grandma‘s Kuckuck‘s clock is working!!!

Storage Rings: linear increasing Lorentz force to keep trajectories in vicinity of the ideal orbit

linear increasing magnetic field

\[ B_y = g \cdot x \quad B_x = g \cdot y \]

\[ F(x) = q \cdot v \cdot B(x) \]

as in the dipole case we normalise to the beam rigidity

\[ k = \frac{g}{B \rho} = \frac{g}{p / q} \]

LHC main quadrupole magnet

\[ g \approx 25 \ldots 220 \ T/m \]

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4.) The equation of motion:

**Linear approximation:**

* ideal particle → design orbit

* any other particle → coordinates x, y small quantities
  \( x, y \ll \rho \)

→ magnetic guide field: only linear terms in x & y of B have to be taken into account

**Taylor Expansion of the B field ...** normalised to momentum \( p/e = B\rho \)
and only terms linear in x, y taken into account
dipole fields / quadrupole fields

\[
\frac{B(x)}{p/e} = \frac{B_0}{B_0\rho} + \frac{g \cdot x}{p/e} + \frac{1}{2!} \frac{e g'}{p/e} + \frac{1}{3!} \frac{e g''}{p/e} + \ldots
\]

= \frac{1}{\rho} + k \cdot x

**Separate Function Machines:**

Split the magnets and optimise them according to their job:

* bending, focusing etc

Example:

heavy ion storage ring TSR

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Equation of Motion:

Remember:

Hamiltonian for ideal particle, $\delta = 0$

$$H = \frac{p_x^2 + p_y^2}{2} - \frac{x^2}{2\rho(s)^2} + \frac{k_1(s)}{2}(x^2 - y^2)$$

with $k$ and $\rho$ representing the normalised quadrupole and dipole fields

putting into Hamiltonian equations

$$\frac{\partial H}{\partial x} = -\frac{dp_x}{ds}, \quad \frac{\partial H}{\partial p_x} = \frac{dx}{ds}$$

we get the equation of motion

$$\frac{d^2x}{ds^2} + \left\{ \frac{1}{\rho(s)^2} - k_1(s) \right\} * x = 0, \quad \frac{d^2y}{ds^2} + k_1(s) * y = 0$$

... see e.g. Goldstein p 241
Equation of Motion:

In linear approximation (x, y << ρ and only dipole & quadruple fields) we can derive a differential equation for the transverse motion of the particles.

* **Equation for the horizontal motion:**

\[ x'' + x \left( \frac{1}{\rho^2} - k \right) = 0 \]

Under the influence of the focusing fields from the quadrupoles „k“ and dipoles 1/ρ² the transverse movement of the particles inside looks like a harmonic oscillation.

* **Equation for the vertical motion:**

\[ \frac{1}{\rho^2} = 0 \quad \text{no dipoles ... in general} \ldots \]

\[ k \leftrightarrow -k \quad \text{quadrupole field changes sign} \]

\[ y'' + k \ y = 0 \]

... mmmppfff ... just another differential equation .... but it does not look sooo comfortable.

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5.) Solution of Trajectory Equations

Define ... hor. plane: \( K = 1/\rho^2 - k \)
... vert. Plane: \( K = k \)

\[ x'' + K \, x = 0 \]

Differential Equation of harmonic oscillator ... with spring constant \( K > 0 \) \( \rightarrow \) focusing case

Ansatz: \( x(s) = a_1 \cdot \cos(\omega s) + a_2 \cdot \sin(\omega s) \)

general solution: linear combination of two independent solutions

\[ x'(s) = -a_1 \omega \sin(\omega s) + a_2 \omega \cos(\omega s) \]

\[ x''(s) = -a_1 \omega^2 \cos(\omega s) - a_2 \omega^2 \sin(\omega s) = -\omega^2 x(s) \quad \omega = \sqrt{K} \]

general solution:

\[ x(s) = a_1 \cos(\sqrt{K} s) + a_2 \sin(\sqrt{K} s) \]
**Hor. Focusing Quadrupole $K > 0$:**

\[
x(s) = x_0 \cdot \cos(\sqrt{|K|} s) + x'_0 \cdot \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|} s)
\]

\[
x'(s) = -x_0 \cdot \sqrt{|K|} \cdot \sin(\sqrt{|K|} s) + x'_0 \cdot \cos(\sqrt{|K|} s)
\]

**For convenience expressed in matrix formalism:**

\[
\begin{pmatrix}
  x \\
  x'
\end{pmatrix}
_{s_1}
= 
M_{foc}
* 
\begin{pmatrix}
  x \\
  x'
\end{pmatrix}
_{s_0}
\]

\[
M_{foc} = \begin{pmatrix}
  \cos(\sqrt{|K|} s) & 1/\sqrt{|K|} \sin(\sqrt{|K|} s) \\
  -\sqrt{|K|} \sin(\sqrt{|K|} s) & \cos(\sqrt{|K|} s)
\end{pmatrix}
_{s_0}
\]

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hor. defocusing quadrupole:

\[ x'' - K \, x = 0 \]

Remember from school:

\[ f(s) = \cosh(s) \quad , \quad f'(s) = \sinh(s) \]

Ansatz:

\[ x(s) = a_1 \cdot \cosh(\omega s) + a_2 \cdot \sinh(\omega s) \]

drift space:

\[ K = 0 \]

\[ x_1 = x_0 + x'_0 \cdot l \]

Ansatz:

\[ \sinh(s) , \quad \cosh(s) \]

\[ \left( \begin{array}{cccc} \cosh \sqrt{|K|} & l & 1 & \sinh \sqrt{|K|} \\
 \sqrt{|K|} & \sinh \sqrt{|K|} & \sqrt{|K|} & \cosh \sqrt{|K|} \end{array} \right) \]

\[ M_{\text{defoc}} = \left( \begin{array}{cc} 1 & l \\
 0 & 1 \end{array} \right) \]

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Combining the two planes:

Clear enough (hopefully ... ?): a quadrupole magnet that is focussing in one plane acts as defocusing lens in the other plane ... et vice versa.

**hor foc. quadrupole lens**

matrix of the same magnet in the vert. plane:

\[
M_{\text{foc}} = \begin{pmatrix}
\cos(\sqrt{|K|}l) & \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}l) \\
-\sqrt{|K|} \sin(\sqrt{|K|}l) & \cos(\sqrt{|K|}l)
\end{pmatrix}
\]

\[
M_{\text{defoc}} = \begin{pmatrix}
\cosh(\sqrt{|K|}l) & \frac{1}{\sqrt{|K|}} \sinh(\sqrt{|K|}l) \\
\sqrt{|K|} \sinh(\sqrt{|K|}l) & \cosh(\sqrt{|K|}l)
\end{pmatrix}
\]

\[
\begin{pmatrix}
x \\
x'
\end{pmatrix}_{f} = \begin{pmatrix}
\cos(\sqrt{|k|}l) & \frac{1}{\sqrt{|k|}} \sin(\sqrt{|k|}l) \\
-\sqrt{|k|} \sin(\sqrt{|k|}l) & \cos(\sqrt{|k|}l)
\end{pmatrix} \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
x \\
x'
\end{pmatrix}_{i}
\]

! with the assumptions made, the motion in the horizontal and vertical planes are independent „... the particle motion in x & y is uncoupled“ !
Transformation through a system of lattice elements

combine the single element solutions by multiplication of the matrices

\[
M_{\text{total}} = M_{QF} \times M_{D} \times M_{QD} \times M_{\text{Bend}} \times M_{D^*} \ldots.
\]

\[
\begin{pmatrix}
    x \\
    x'
\end{pmatrix}_{s_2} = M(s_2, s_1) \times \begin{pmatrix}
    x \\
    x'
\end{pmatrix}_{s_1}
\]

in each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator „

typical values in a strong foc. machine:
\[x \approx \text{mm}, x' \leq \text{mrad}\]
6.) Orbit & Tune:

Tune: number of oscillations per turn

64.31  
59.32

Relevant for beam stability: non integer part

LHC revolution frequency: 11.3 kHz  

\[ f_q = 0.31 \times 11.3 = 3.5 \text{kHz} \]
First turn steering "by sector:"

- One beam at the time
- Beam through 1 sector (1/8 ring), correct trajectory, open collimator and move on.
LHC Operation: the First Beam

Beam 1 on OTR screen
1st and 2nd turn

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Question: what will happen, if the particle performs a second turn?

... or a third one or ... $10^{10}$ turns
7.) The Beta Function

General solution of Hill’s equation:

\[
\psi(s) = \int \sqrt{\epsilon} \sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi) \, ds
\]

\( \epsilon, \phi \) = integration constants determined by initial conditions

\( \beta(s) \) periodic function given by focusing properties of the lattice ⇔ quadrupoles

\( \beta(s + L) = \beta(s) \)

Inserting (i) into the equation of motion …

\[
\Psi(s) = \int_0^s \frac{ds}{\beta(s)}
\]

\( \Psi(s) = \text{“phase advance” of the oscillation between point “0” and “s” in the lattice.} \)

For one complete revolution: number of oscillations per turn \( \text{“Tune”} \)

\[
Q_y = \frac{1}{2\pi} \int_0^s \frac{ds}{\beta(s)}
\]
The Beta Function

Amplitude of a particle trajectory:

\[ x(s) = \sqrt{\varepsilon} \times \sqrt{\beta(s)} \times \cos(\psi(s) + \varphi) \]

Maximum size of a particle amplitude

\[ \hat{x}(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \]

\(\beta\) determines the beam size ( ... the envelope of all particle trajectories at a given position “s” in the storage ring.

It reflects the periodicity of the magnet structure.

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8.) Beam Emittance and Phase Space Ellipse

general solution of Hill equation

\[
\begin{align*}
(1) \quad x(s) &= \sqrt{\varepsilon} \sqrt{\beta(s)} \cos(\psi(s) + \phi) \\
(2) \quad x'(s) &= -\frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}} \left\{ \alpha(s) \cos(\psi(s) + \phi) + \sin(\psi(s) + \phi) \right\}
\end{align*}
\]

from (1) we get

\[
\cos(\psi(s) + \phi) = \frac{x(s)}{\sqrt{\varepsilon} \sqrt{\beta(s)}}
\]

Insert into (2) and solve for \(\varepsilon\)

\[
\varepsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'^2(s)
\]

* \(\varepsilon\) is a constant of the motion ... it is independent of ,,s“
* parametric representation of an ellipse in the \(x \times x'\) space
* shape and orientation of ellipse are given by \(\alpha, \beta, \gamma\)

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Beam Emittance and Phase Space Ellipse

\[ \varepsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s) x'^2(s) \]

Liouville: in reasonable storage rings area in phase space is constant.

\[ A = \pi \varepsilon = \text{const} \]

\( \varepsilon \) beam emittance = woozilycity of the particle ensemble, intrinsic beam parameter, cannot be changed by the foc. properties.

Scientifiquely speaking: it is the area covered in transverse x, x' phase space … and it is constant !!!
Particle Tracking in a Storage Ring

Calculate $x$, $x'$ for each linear accelerator element according to matrix formalism

plot $x$, $x'$ as a function of "s"
**Phase Space Ellipse**

**particle trajectory:** 
\[ x(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cos \{ \psi(s) + \phi \} \]

**max. Amplitude:** 
\[ \hat{x}(s) = \sqrt{\varepsilon \beta} \quad \text{determine } x' \text{ at that position …} \]

… put \( \hat{x}(s) \) into 
\[ \varepsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x''(s) \]

and solve for \( x' \)
\[ \varepsilon = \gamma \cdot \varepsilon \beta + 2\alpha \sqrt{\varepsilon \beta} \cdot x' + \beta x'' \]
\[ \rightarrow \quad x' = -\alpha \cdot \sqrt{\varepsilon / \beta} \]

\[ A \text{ high } \beta\text{-function means a large beam size and a small beam divergence.} \]
\[ \text{… et vice versa !!!} \]

\[ \star \text{ In the middle of a quadrupole } \beta = \text{maximum, } \alpha = \text{zero} \quad \left\{ \begin{array}{c} x' = 0 \end{array} \right\} \]

… and the ellipse is flat

Bernhard Holzer, CAS
**Phase Space Ellipse**

\[ \begin{align*}
\epsilon &= \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'^2(s) \\
\Rightarrow \quad \epsilon &= \frac{x^2}{\beta} + \frac{\alpha^2 x^2}{\beta} + 2\alpha \cdot xx' + \beta \cdot x'^2 \\
\end{align*} \]

... solve for \( x' \)

\[ x'_{1,2} = \frac{-\alpha \cdot x \pm \sqrt{\epsilon \beta - x^2}}{\beta} \]

... and determine \( x' \) via:

\[ \frac{dx'}{dx} = 0 \]

\[ x' = \sqrt{\epsilon \gamma} \]

\[ x = \pm \alpha \sqrt{\frac{\epsilon}{\gamma}} \]

*shape and orientation of the phase space ellipse depend on the Twiss parameters \( \beta, \alpha, \gamma \)*

\[ \begin{align*}
\alpha(s) &= -\frac{1}{2} \beta'(s) \\
\gamma(s) &= \frac{1 + \alpha(s)^2}{\beta(s)} \\
\end{align*} \]
**Emittance of the Particle Ensemble:**

\[ x(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cdot \cos(\Psi(s) + \phi) \]

\[ \hat{x}(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \]

*Gauß Particle Distribution:*

\[ \rho(x) = \frac{N \cdot e}{\sqrt{2\pi}\sigma_x} \cdot e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}} \]

particle at distance \(1\ \sigma\) from centre \(\leftrightarrow 68.3\%\) of all beam particles

**single particle trajectories, \(N \approx 10^{11}\) per bunch**

**LHC:**

\[ \beta = 180\ m \]

\[ \varepsilon = 5 \times 10^{-10}\ m\ rad \]

\[ \sigma = \sqrt{\varepsilon \beta} = \sqrt{5 \times 10^{-10}\ m \times 180\ m} = 0.3\ mm \]

**aperture requirements:** \(r_0 = 12 \times \sigma\)

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13.) Liouville during Acceleration

\[ \varepsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'^2(s) \]

Beam Emittance corresponds to the area covered in the \( x, x' \) Phase Space Ellipse

Liouville: Area in phase space is constant.

**But so sorry ... \( \varepsilon \neq \text{const} \) !**

Classical Mechanics:

**phase space = diagram of the two canonical variables**

**position & momentum**

\[ x \quad \quad p_x \]
According to Hamiltonian mechanics:
phase space diagram relates the variables $x$ and $p_x$

**Liouville's Theorem:**
$$\int p_x \, dx = \text{const}$$

for convenience (i.e. because we are lazy bones) we use in accelerator theory
$x'$ instead of $p_x$

$$x' = \frac{dx}{ds} = \frac{dx}{dt} \frac{dt}{ds} = \frac{\beta_x}{\beta} = \frac{p_x}{p}$$

where $p \sim p_s$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad ; \quad \beta_x = \frac{\dot{x}}{c}$$

$$\int x' \, dx = \int \frac{p_x}{p} \, dx \propto \text{const} \frac{1}{m_0 c \cdot \gamma \beta}$$

$$\Rightarrow \quad \varepsilon = \int x' \, dx \propto \frac{1}{\beta \gamma}$$

the beam emittance shrinks during acceleration
$$\varepsilon \sim 1 / \gamma$$

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Nota bene:

1.) A proton machine ... or an electron linac ... needs the highest aperture at injection energy !!!
    as soon as we start to accelerate the beam size shrinks as $\gamma^{-1/2}$ in both planes.

$$\sigma = \sqrt{\varepsilon \beta}$$

2.) To confuse the students we introduce often a "normalized" emittance $\varepsilon_n$
    ... which is energy independent

$\varepsilon_n = \varepsilon_0 \cdot \beta \gamma$

Example: HERA proton ring

injection energy: 40 GeV $\gamma = 43$

flat top energy: 920 GeV $\gamma = 980$

$\varepsilon_n = 5.0 \times 10^{-6}$ mrad
$\varepsilon_0 (40\text{GeV}) = 1.2 \times 10^{-7}$ mrad
$\varepsilon_0 (920\text{GeV}) = 5.1 \times 10^{-9}$ mrad

7 $\sigma$ beam envelope at $E = 40$ GeV

... and at $E = 920$ GeV

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11.) Résumé

1.) Beam rigidity

\[ \frac{P}{e} = B \rho \]

2.) Equation of motion

\[ x'' + x \left( \frac{1}{\rho^2} - k \right) = 0 \]
\[ y'' + k \ y = 0 \]

3.) Transfer matrix foc. quadrupole

\[ M_{\text{foc}} = \begin{pmatrix} \cos(\sqrt{K} s) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} s) \\ -\sqrt{K} \sin(\sqrt{K} s) & \cos(\sqrt{K} s) \end{pmatrix} \]

defoc. quadrupole

\[ M_{\text{defoc}} = \begin{pmatrix} \cosh \sqrt{|K| l} & \frac{1}{\sqrt{|K|}} \sinh \sqrt{|K| l} \\ \sqrt{|K|} \sinh \sqrt{|K| l} & \cosh \sqrt{|K| l} \end{pmatrix} \]

drift

\[ M_{\text{drift}} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \]

4.) general solution of Hill’s equation

\[ x(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi) \]

5.) Tune

\[ Q_s = \frac{1}{2\pi} \int \frac{ds}{\beta(s)} \]

6.) Emittance as phase space ellipse

\[ \varepsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s) x'^2(s) \]
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