

Introduction to Non-Linear Dynamics

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- Books on non-linear dynamical systems
 - M. Tabor, *Chaos and Integrability in Nonlinear Dynamics, An Introduction*, Willey, 1989.
 - A.J Lichtenberg and M.A. Lieberman, *Regular and Chaotic Dynamics*, 2nd edition, Springer 1992.
- Books on beam dynamics
 - E. Forest, *Beam Dynamics - A New Attitude and Framework*, Harwood Academic Publishers, 1998.
 - A. Wolski, *Beam Dynamics in High Energy Particle Accelerators*, Imperial College Press, 2014.
- Lectures on non-linear beam dynamics
 - A. Chao, *Advanced topics in Accelerator Physics*, USPAS, 2000.
 - W. Herr, *Mathematical and Numerical Methods for Non-linear Beam Dynamics*, CAS 2015.
 - L. Nadolski, *Lectures on Non-linear beam dynamics*, Master NPAC, LAL, Orsay 2013.
 - Y. Papaphilippou, *Lectures on Non-linear dynamics in particles accelerators*, Universita la Sapienza, Rome, Italy, June 2016.

- Introduce “historical” approaches of non-linear dynamics (i.e. classical perturbation theory)
 - Show their usefulness
 - Demonstrate their practical limitation especially in beam dynamics
 - Connect naturally with the lectures of W. Herr on “Non-linear dynamics methods and tools” and the lectures on “Non-linear dynamics phenomenology”

- Non-linear magnets, such as chromaticity sextupoles (especially in low emittance rings), octupoles,...
- Magnet imperfections and misalignments
- Insertion devices (wigglers, undulators) for synchrotron radiation storage rings
- Injection elements
- Magnet fringe fields (especially in high-intensity machines)
- Power supply ripple
- Ground motion (for e^+ / e^-)
- Electron (Ion) cloud
- Beam-beam effect (for colliders)
- Space-charge effect (especially in high-intensity machines)

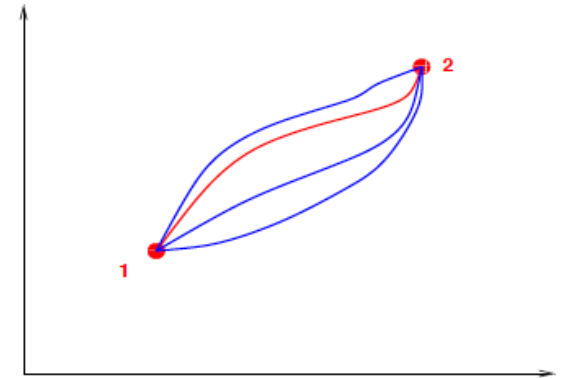
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- Beam-beam effect (for colliders)
- Space-charge effect (especially in high-intensity machines)
- Performance impact
 - Reduced injection efficiency (especially in low emittance rings)
 - Particle losses causing
 - Reduced intensity and/or beam lifetime
 - Radio-activation (hands-on maintenance, equipment lifetime, super-conducting magnet quench)
 - Reduced machine availability
 - Emittance increase
 - Reduced number of bunches and/or increased crossing angle, affecting luminosity (for colliders)
 - Allow to damp instabilities (see V. Kornilov lecture on “Landau damping”)
 - Can be used for beam extraction

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 - Can be used for beam extraction
- Cost issues
 - Magnetic field quality and alignment tolerances
 - Number of magnet correctors and families (power convertors)
 - Design of collimation system (for colliders and high-intensity machines)

Reminder of Lagrangian and Hamiltonian formalism

- Describe motion of particles in q_n coordinates (n degrees of freedom) from time t_1 to time t_2
- It can be achieved by the **Lagrangian function** $L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t)$ with (q_1, \dots, q_n) the **generalized coordinates** and $(\dot{q}_1, \dots, \dot{q}_n)$ the **generalized velocities**

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- ❑ The Lagrangian is defined as $L = T - V$, i.e. difference between **kinetic** and **potential** energy
- ❑ The integral $W = \int L(q_i, \dot{q}_i, t) dt$ defines the **action**
- ❑ **Hamilton's principle**: system evolves so as the action becomes extremum (principle of **stationary action**)



□ By using **Hamilton's principle**, i.e. $\delta W = 0$, over some time interval t_1 and t_2 for two stationary points $\delta q(t_1) = \delta q(t_2) = 0$ (see appendix), the following differential equations for each degree of freedom are obtained, the **Euler-Lagrange equations**

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

□ In other words, by knowing the form of the Lagrangian, the equations of motion can be derived

- For a simple **force law** contained in a potential function, governing motion among interacting particles, the Lagrangian is (or as Landau-Lifshitz put it “experience has shown that...”)

$$L = T - V = \sum_{i=1}^n \frac{1}{2} m_i \dot{q}_i^2 - V(q_1, \dots, q_n)$$

- For velocity independent potentials, Lagrange equations become

$$m_i \ddot{q}_i = - \frac{\partial V}{\partial q_i} ,$$

i.e. **Newton's equations.**

- ❑ Some **disadvantages** of the Lagrangian formalism:
 - ❑ **No uniqueness**: different Lagrangians can lead to same equations
 - ❑ **Physical significance** not straightforward (even its basic form given more by “experience” and the fact that it actually works that way!)
- ❑ Lagrangian function provides in general n second order differential equations (**coordinate space**)
- ❑ We already observed the advantage to move to a system of $2n$ first order differential equations, which are more straightforward to solve (**phase space**)
- ❑ These equations can be derived by the **Hamiltonian** of the system

- The **Hamiltonian** of the system is defined as the **Legendre transformation** of the Lagrangian

$$H(\mathbf{q}, \mathbf{p}, t) = \sum_i \dot{q}_i p_i - L(\mathbf{q}, \dot{\mathbf{q}}, t)$$

where the **generalised momenta** are $p_i = \frac{\partial L}{\partial \dot{q}_i}$

- The **generalised velocities** can be expressed as a function of the **generalised momenta** if the previous equation is invertible, and thereby define the Hamiltonian of the system

- **Example:** consider $L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} \sum_i m_i \dot{q}_i^2 - V(q_1, \dots, q_n)$

- From this, the momentum can be determined as $p_i = \frac{\partial L}{\partial \dot{q}_i} = m_i \dot{q}_i$

which can be trivially inverted to provide the Hamiltonian

$$H(\mathbf{q}, \mathbf{p}) = \sum_i \frac{p_i^2}{2m_i} + V(q_1, \dots, q_n)$$

- The **equations of motion** can be derived from the Hamiltonian following the same variational principle as for the Lagrangian (“least” action) but also by simply taking the differential of the Hamiltonian (see appendix)

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

- These are indeed $2n + 2$ equations describing the motion in the “**extended**” phase space $(q_1, \dots, q_n, p_1, \dots, p_n, t, -H)$

- ❑ The variables $(q_i, \dots, q_n, p_1, \dots, p_n, t, -H)$ are called **canonically conjugate** (or canonical) and define the evolution of the system in **phase space**
- ❑ These variables have the special property that they preserve volume in phase space, i.e. satisfy the well-known **Liouville's theorem**
- ❑ The variables used in the Lagrangian do not necessarily have this property
- ❑ Hamilton's equations can be written in **vector form**
 $\dot{\mathbf{z}} = \mathbf{J} \cdot \nabla H(\mathbf{z})$ with $\mathbf{z} = (q_i, \dots, q_n, p_1, \dots, p_n)$
and $\nabla = (\partial q_i, \dots, \partial q_n, \partial p_1, \dots, \partial p_n)$
- ❑ The $2n \times 2n$ matrix $\mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}$ is called the **symplectic matrix**

- ❑ Crucial step in study of Hamiltonian systems is identification of **integrals of motion**
- ❑ Consider a time dependent function of phase space. Its time evolution is given by

$$\begin{aligned} \frac{d}{dt} f(\mathbf{p}, \mathbf{q}, t) &= \sum_{i=1}^n \left(\frac{dq_i}{dt} \frac{\partial f}{\partial q_i} + \frac{dp_i}{dt} \frac{\partial f}{\partial p_i} \right) + \frac{\partial f}{\partial t} \\ &= \sum_{i=1}^n \left(\frac{\partial H}{\partial p_i} \frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial f}{\partial p_i} \right) + \frac{\partial f}{\partial t} = [H, f] + \frac{\partial f}{\partial t} \end{aligned}$$

where $[H, f]$ is the **Poisson bracket** of f with H

- ❑ If a quantity is explicitly **time-independent** and its Poisson bracket with the Hamiltonian vanishes (i.e. **commutes** with the H), it is a **constant** (or **integral**) of motion (as an **autonomous** Hamiltonian itself)

- The Poisson brackets between two functions of a set of canonical variables can be defined by the differential operator

$$[f, g] = \sum_{i=1}^n \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i} \frac{\partial f}{\partial q_i} \right)$$

- From this definition, and for any three given functions, the following properties can be shown

$$[af + bg, h] = a[f, h] + b[g, h], \quad a, b \in \mathbb{R} \quad \text{bilinearity}$$

$$[f, g] = -[g, f] \quad \text{anticommutativity}$$

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0 \quad \text{Jacobi's identity}$$

$$[f, gh] = [f, g]h + g[f, h]$$

- Poisson brackets operation satisfies a **Lie algebra**

Canonical transformations

- Find a **function** for transforming the Hamiltonian from variable (\mathbf{q}, \mathbf{p}) to (\mathbf{Q}, \mathbf{P}) so system becomes **simpler** to study
- This transformation should be **canonical** (or **symplectic**), so that the Hamiltonian properties of the system are preserved

- These “mixed variable” **generating** functions are derived by

$$F_1(\mathbf{q}, \mathbf{Q}) : p_i = \frac{\partial F_1}{\partial q_i}, \quad P_i = -\frac{\partial F_1}{\partial Q_i} \quad F_3(\mathbf{Q}, \mathbf{p}) : q_i = -\frac{\partial F_3}{\partial p_i}, \quad P_i = -\frac{\partial F_3}{\partial Q_i}$$

$$F_2(\mathbf{q}, \mathbf{P}) : p_i = \frac{\partial F_2}{\partial q_i}, \quad Q_i = \frac{\partial F_2}{\partial P_i} \quad F_4(\mathbf{p}, \mathbf{P}) : q_i = -\frac{\partial F_4}{\partial p_i}, \quad Q_i = \frac{\partial F_4}{\partial P_i}$$

- A general non-autonomous Hamiltonian is transformed to

$$H(\mathbf{Q}, \mathbf{P}, t) = H(\mathbf{q}, \mathbf{p}, t) + \frac{\partial F_j}{\partial t}, \quad j = 1, 2, 3, 4$$

- One generating function can be constructed by the other through **Legendre transformations**, e.g.

$$F_2(\mathbf{q}, \mathbf{P}) = F_1(\mathbf{q}, \mathbf{Q}) - \mathbf{Q} \cdot \mathbf{P}, \quad F_3(\mathbf{Q}, \mathbf{p}) = F_1(\mathbf{q}, \mathbf{Q}) - \mathbf{q} \cdot \mathbf{p}, \quad \dots$$

with the inner product define as $\mathbf{q} \cdot \mathbf{p} = \sum_i q_i p_i$

- A fundamental property of canonical transformations is the **preservation of phase space volume**
- This volume preservation in phase space can be represented in the old and new variables as

$$\int \prod_{i=1}^n dp_i dq_i = \int \prod_{i=1}^n dP_i dQ_i$$

- The volume element in old and new variables are related through the Jacobian

$$\prod_{i=1}^n dp_i dq_i = \frac{\partial(P_1, \dots, P_n, Q_1, \dots, Q_n)}{\partial(p_1, \dots, p_n, q_1, \dots, q_n)} \prod_{i=1}^n dP_i dQ_i$$

- These two relationships imply that the Jacobian of a canonical transformation should have determinant equal to 1

$$\left| \frac{\partial(P_1, \dots, P_n, Q_1, \dots, Q_n)}{\partial(p_1, \dots, p_n, q_1, \dots, q_n)} \right| = \left| \frac{\partial(p_1, \dots, p_n, q_1, \dots, q_n)}{\partial(P_1, \dots, P_n, Q_1, \dots, Q_n)} \right| = 1$$

- The transformation $Q = -p$, $P = q$, which **interchanges conjugate variables** is area preserving, as the Jacobian is

$$\frac{\partial(P,Q)}{\partial(p,q)} = \begin{vmatrix} \frac{\partial P}{\partial p} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial Q}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1$$

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- On the other hand, the transformation from **Cartesian to polar** coordinates $q = P \cos Q$, $p = P \sin Q$ is not, since

$$\frac{\partial(q,p)}{\partial(Q,P)} = \begin{vmatrix} -P \sin Q & P \cos Q \\ \cos Q & \sin Q \end{vmatrix} = -P$$

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- There are actually “**polar**” coordinates that are **canonical**, given by $q = -\sqrt{2P} \cos Q$, $p = \sqrt{2P} \sin Q$ for which

$$\frac{\partial(q,p)}{\partial(Q,P)} = \begin{vmatrix} \sqrt{2P} \sin Q & \sqrt{2P} \cos Q \\ -\frac{\cos Q}{\sqrt{2P}} & \frac{\sin Q}{\sqrt{2P}} \end{vmatrix} = 1$$

The Relativistic Hamiltonian for electromagnetic fields

- Neglecting self fields and radiation, motion can be described by a “single-particle” Hamiltonian

$$H(\mathbf{x}, \mathbf{p}, t) = c\sqrt{\left(\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{x}, t)\right)^2 + m^2c^2} + e\Phi(\mathbf{x}, t)$$

- $\mathbf{x} = (x, y, z)$ Cartesian positions
- $\mathbf{p} = (p_x, p_y, p_z)$ conjugate momenta
- $\mathbf{A} = (A_x, A_y, A_z)$ magnetic vector potential
- Φ electric scalar potential

- The ordinary kinetic momentum vector is written

$$\mathbf{P} = \gamma m \mathbf{v} = \mathbf{p} - \frac{e}{c} \mathbf{A}$$

with \mathbf{v} the velocity vector and $\gamma = (1 - v^2/c^2)^{-1/2}$ the relativistic factor

- It is generally a 3 degrees of freedom one plus time (i.e. 4 degrees of freedom)
- The Hamiltonian represents the total energy

$$H \equiv E = \gamma mc^2 + e\Phi$$

- The total kinetic momentum is

$$P = \left(\frac{H^2}{c^2} - m^2 c^2 \right)^{1/2}$$

- Using Hamilton's equations

$$(\dot{\mathbf{x}}, \dot{\mathbf{p}}) = [(\mathbf{x}, \mathbf{p}), H]$$

it can be shown that motion is governed by **Lorentz equations**

- ❑ Making a series of canonical transformations and approximations (see appendix)
 - ❑ From Cartesian to Frenet-Serret (rotating) coordinate system (bending in the horizontal plane)
 - ❑ Changing the independent variable from time to the path length s
 - ❑ Electric field set to zero, as longitudinal (synchrotron) motion is much slower than transverse (betatron) one
 - ❑ Consider static and transverse magnetic fields
 - ❑ Rescale the momentum and move the origin to the periodic orbit
 - ❑ For the ultra-relativistic limit $\beta_0 \rightarrow 1$, $\frac{1}{\beta_0^2 \gamma^2} \rightarrow 0$ the Hamiltonian becomes

$$\mathcal{H}(x, y, l, p_x, p_y, \delta) = (1 + \delta) - e\hat{A}_s - \left(1 + \frac{x}{\rho(s)}\right) \sqrt{(1 + \delta)^2 - p_x^2 - p_y^2}$$

with $\frac{P_t - P_0}{P_0} \equiv \delta$

- Note that the Hamiltonian is non-linear even in the absence of any field component (i.e. for a drift)!
- Last approximation: transverse momenta (rescaled to the reference momentum) are considered to be much smaller than 1, i.e. the square root can be expanded. Considering also the large machine approximation $x \ll \rho$, (dropping cubic terms), the Hamiltonian is simplified to

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2(1 + \delta)} - \frac{x(1 + \delta)}{\rho(s)} - e\hat{A}_s$$

- This expansion may not be a good idea, especially for low energy, small size rings

- Considering the **general expression** of the the **longitudinal component** of the **vector potential** is (see appendix)

- In curvilinear coordinates (curved elements)

$$A_s = \left(1 + \frac{x}{\rho(s)}\right) B_0 \Re e \sum_{n=0}^{\infty} \frac{b_n + ia_n}{n+1} (x + iy)^{n+1}$$

- In Cartesian coordinates $A_s = B_0 \Re e \sum_{n=0}^{\infty} \frac{b_n + ia_n}{n+1} (x + iy)^{n+1}$

with the **multipole coefficients** being written as

$$a_n = \frac{1}{B_0 n!} \left. \frac{\partial^n B_x}{\partial x^n} \right|_{x=y=0} \quad \text{and} \quad b_n = \frac{1}{B_0 n!} \left. \frac{\partial^n B_y}{\partial x^n} \right|_{x=y=0}$$

- The **general non-linear Hamiltonian** can be written as

$$\mathcal{H}(x, y, p_x, p_y, s) = \mathcal{H}_0(x, y, p_x, p_y, s) + \sum_{k_x, k_y} h_{k_x, k_y}(s) x^{k_x} y^{k_y}$$

with the **periodic functions** $h_{k_x, k_y}(s) = h_{k_x, k_y}(s + C)$

- Dipole:

$$H = \frac{x\delta}{\rho} + \frac{x^2}{2\rho^2} + \frac{p_x^2 + p_y^2}{2(1 + \delta)}$$

- Quadrupole:

$$H = \frac{1}{2}k_1(x^2 - y^2) + \frac{p_x^2 + p_y^2}{2(1 + \delta)}$$

- Sextupole:

$$H = \frac{1}{3}k_2(x^3 - 3xy^2) + \frac{p_x^2 + p_y^2}{2(1 + \delta)}$$

- Octupole:

$$H = \frac{1}{4}k_3(x^4 - 6x^2y^2 + y^4) + \frac{p_x^2 + p_y^2}{2(1 + \delta)}$$

■ Assume a simple case of **linear transverse magnetic fields**,

$$B_x = b_1(s)y$$

$$B_y = -b_0(s) + b_1(s)x \quad ,$$

□ main bending field

$$-B_0 \equiv b_0(s) = \frac{P_0 c}{e \rho(s)} \quad [\text{T}]$$

□ normalized
quadrupole gradient

$$K(s) = b_1(s) \frac{e}{c P_0} = \frac{b_1(s)}{B \rho} \quad [1/\text{m}^2]$$

□ magnetic rigidity

$$B \rho = \frac{P_0 c}{e} \quad [\text{T} \cdot \text{m}]$$

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$$B \rho = \frac{P_0 c}{e} \quad [\text{T} \cdot \text{m}]$$

- The vector potential has only a longitudinal component which in curvilinear coordinates is

$$B_x = -\frac{1}{1 + \frac{x}{\rho(s)}} \frac{\partial A_s}{\partial y} \quad , \quad B_y = \frac{1}{1 + \frac{x}{\rho(s)}} \frac{\partial A_s}{\partial x}$$

- The previous expressions can be integrated to give

$$A_s(x, y, s) = \frac{P_0 c}{e} \left[-\frac{x}{\rho(s)} - \left(\frac{1}{\rho(s)^2} + K(s) \right) \frac{x^2}{2} + K(s) \frac{y^2}{2} \right] = P_0 c \hat{A}_s(x, y, s)$$

- The Hamiltonian for linear fields can be finally written as

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2(1+\delta)} - \frac{x\delta}{\rho(s)} + \frac{x^2}{2\rho(s)^2} + \frac{K(s)}{2} (x^2 - y^2)$$

- Hamilton's equations are

$$\frac{dx}{ds} = \frac{p_x}{1+\delta}, \quad \frac{dp_x}{ds} = \frac{\delta}{\rho(s)} - \left(\frac{1}{\rho^2(s)} + K(s) \right) x$$

$$\frac{dy}{ds} = \frac{p_y}{1+\delta}, \quad \frac{dp_y}{ds} = K(s)y$$

and they can be written as two second order uncoupled differential equations, i.e. **Hill's equations**

$$x'' + \frac{1}{1+\delta} \overbrace{\left(\frac{1}{\rho(s)^2} + K(s) \right)}^{K_x} x = \frac{\delta}{\rho(s)}$$

$$y'' - \frac{1}{1+\delta} \underbrace{K(s)}_{K_y} y = 0$$

with the usual solution for $\delta = 0$ and $u = x, y$

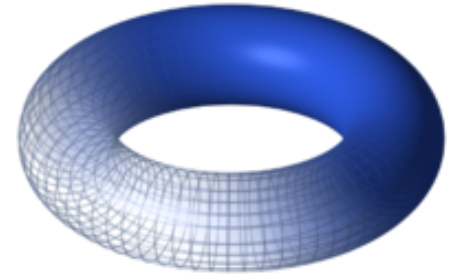
$$u(s) = \sqrt{\epsilon\beta(s)} \cos(\psi(s) + \psi_0)$$

$$u'(s) = \sqrt{\frac{\epsilon}{\beta(s)}} (\sin(\psi(s) + \psi_0) + \alpha(s) \cos(\psi(s) + \psi_0))$$

- There is a canonical transformation to some **optimal set** of variables which can simplify the phase-space motion
- This set of variables are the **action-angle** variables
- The action vector is defined as the integral $\mathbf{J} = \oint \mathbf{p}d\mathbf{q}$ over closed paths in phase space.
- An **integrable Hamiltonian** is written as a function of only the actions, i.e. $H_0 = H_0(\mathbf{J})$. Hamilton's equations give

$$\dot{\phi}_i = \frac{\partial H_0(\mathbf{J})}{\partial J_i} = \omega_i(\mathbf{J}) \Rightarrow \phi_i = \omega_i(\mathbf{J})t + \phi_{i0}$$

$$\dot{J}_i = -\frac{\partial H_0(\mathbf{J})}{\partial \phi_i} = 0 \Rightarrow J_i = \text{const.}$$



i.e. the **actions are integrals of motion** and the **angles are evolving linearly with time**, with **constant frequencies** which depend on the actions

- The actions define the surface of an **invariant torus**, topologically equivalent to the product of n circles

- Considering **on-momentum** motion, the Hamiltonian can be written as

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2} + \frac{K_x(s)x^2 - K_y(s)y^2}{2}$$

- The generating function from the original to action angle variables is

$$F_1(x, y, \phi_x, \phi_y; s) = -\frac{x^2}{2\beta_x(s)} [\tan \phi_x(s) + a_x(s)] - \frac{y^2}{2\beta_y(s)} [\tan \phi_y(s) + a_y(s)]$$

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- The **old variables** with respect to **actions and angles** are

$$u(s) = \sqrt{2\beta_u(s)J_u} \cos \phi_u(s), \quad p_u(s) = -\sqrt{\frac{2J_u}{\beta_u(s)}} (\sin \phi_u(s) + \alpha_u(s) \cos \phi_u(s))$$

and the Hamiltonian takes the form

$$\mathcal{H}_0(J_x, J_y, s) = \frac{J_x}{\beta_x(s)} + \frac{J_y}{\beta_y(s)}$$

- The “time” (longitudinal position) dependence can be eliminated by the transformation to **normalized coordinate**

$$\begin{pmatrix} \mathcal{U} \\ \mathcal{U}' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \mathcal{U} \\ \mathcal{U}' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \cos(\nu\phi) \\ \sin(\nu\phi) \end{pmatrix} \quad \text{with} \quad \nu = \frac{1}{2\pi} \oint \frac{du}{\beta(s)}$$

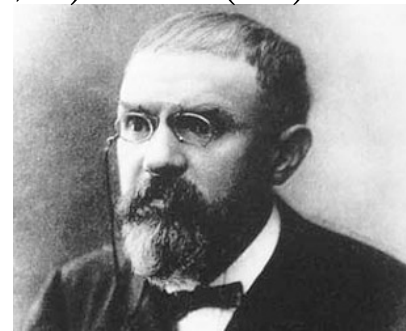
Canonical perturbation theory

- Consider a general Hamiltonian with n degrees of freedom

$$H(\mathbf{J}, \varphi, \theta) = H_0(\mathbf{J}) + \epsilon H_1(\mathbf{J}, \varphi, \theta) + \mathcal{O}(\epsilon^2)$$

where the non-integrable part $H_1(\mathbf{J}, \varphi, \theta)$ is 2π -periodic on the angles φ and the “time” θ

- Provided that ϵ is sufficiently small, tori should still exist but they are distorted
- We seek a canonical transformation that could “straighten up” the tori, i.e. it could transform the non-integrable part of the Hamiltonian (at first order in ϵ) to a function only of some new actions $\bar{H}(\bar{\mathbf{J}})$ plus higher orders in ϵ
- This can be performed by a mixed variable close to identity generating function $S(\bar{\mathbf{J}}, \varphi, \theta) = \bar{\mathbf{J}} \cdot \varphi + \epsilon S_1(\bar{\mathbf{J}}, \varphi, \theta) + \mathcal{O}(\epsilon^2)$ for transforming old variables to new ones $(\bar{\mathbf{J}}, \bar{\varphi})$
- In principle, this procedure can be carried to arbitrary powers of the perturbation



- By the canonical transformation equations (slide 19), the **old action** and **new angle** can be also represented by a power series in ϵ

$$\begin{aligned}
 \mathbf{J} &= \bar{\mathbf{J}} + \epsilon \frac{\partial S_1(\bar{\mathbf{J}}, \varphi, \theta)}{\partial \varphi} + \mathcal{O}(\epsilon^2) & \mathbf{J} &= \bar{\mathbf{J}} + \epsilon \frac{\partial S_1(\bar{\mathbf{J}}, \bar{\varphi}, \theta)}{\partial \bar{\varphi}} + \mathcal{O}(\epsilon^2) \\
 \bar{\varphi} &= \varphi + \epsilon \frac{\partial S_1(\bar{\mathbf{J}}, \varphi, \theta)}{\partial \bar{\mathbf{J}}} + \mathcal{O}(\epsilon^2) & \text{or} & & \varphi &= \bar{\varphi} - \epsilon \frac{\partial S_1(\bar{\mathbf{J}}, \bar{\varphi}, \theta)}{\partial \bar{\mathbf{J}}} + \mathcal{O}(\epsilon^2)
 \end{aligned}$$

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 \end{aligned}$$

- The previous equations expressing the old as a function of the new variables assume that there is possibility to **invert** the equation on the left, so that $S_1(\bar{\mathbf{J}}, \bar{\varphi}, \theta)$ becomes a function of the new variables
- The **new Hamiltonian** is then

$$\bar{H}(\bar{\mathbf{J}}, \bar{\varphi}, \theta) = H(\mathbf{J}(\bar{\mathbf{J}}, \bar{\varphi}), \varphi(\bar{\mathbf{J}}, \bar{\varphi}), \theta) + \epsilon \frac{\partial S_1(\bar{\mathbf{J}}, \bar{\varphi}, \theta)}{\partial \theta} + \mathcal{O}(\epsilon^2)$$

- The second term is appearing because of the “time” dependence through θ

- The question is what is the form of the **generating function** that eliminates the angle dependence
- The procedure is cumbersome (see appendix for details), but here is the final result,

$$S(\bar{\mathbf{J}}, \bar{\varphi}) = \bar{\mathbf{J}} \cdot \bar{\varphi} + \epsilon i \sum_{\mathbf{k} \neq \mathbf{0}} \frac{H_{1\mathbf{k}}(\bar{\mathbf{J}})}{\mathbf{k} \cdot \boldsymbol{\omega}(\bar{\mathbf{J}}) + p} e^{i(\mathbf{k} \cdot \bar{\varphi} + p\theta)} + \mathcal{O}(\epsilon^2)$$

with the frequency vector $\boldsymbol{\omega}(\bar{\mathbf{J}}) = \frac{\partial H_0(\bar{\mathbf{J}})}{\partial \bar{\mathbf{J}}}$
and the integers $\mathbf{k}, p \neq \mathbf{0}$

- If the denominator vanishes, i.e. for the **resonance condition** $\mathbf{k} \cdot \boldsymbol{\omega}(\bar{\mathbf{J}}) + p = 0$, the Fourier series coefficients (**driving terms**) become **infinite**
- It actually implies that even at **first order** in the perturbation parameter and in the vicinity of a resonance, it is **impossible** to construct a **generating function** for seeking some **approximate integrals of motion**

- In principle, the **technique works for arbitrary order**, but the **disentangling of variables** becomes difficult even to 2nd order!!!
- The solution was given in the late 60s by introducing the **Lie transforms** (e.g. see Deprit 1969), which are **algorithmic for constructing generating functions** and were adapted to beam dynamics by Dragt and Finn (1976)
- On the other hand, the problem of **small denominators** due to **resonances** is not just a mathematical one. The inability to construct solutions close to a **resonance** has to do with the **unpredictable nature of motion** and the **onset of chaos**
- **KAM theory** (see appendix) developed the mathematical framework into which local solutions could be constructed provided some general conditions on the size of the perturbation and the distance of the system from resonances are satisfied
- Very difficult though to apply **directly** this theorem to realistic physical systems, such as a particle accelerator

Example: Perturbation treatment of a sextupole

- Consider the simple case of a **periodic sextupole perturbation** and restrict the study only to one plane. The **Hamiltonian** is written as,

$$H(x, p_x, s) = \frac{p_x^2 + K(s)x^2}{2} + \frac{K_s(s)x^3}{3}$$

where $K(s)$ and $K_s(s)$ are periodic functions of time.

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where $K(s)$ and $K_s(s)$ are periodic functions of time.

- We proceed to the **transformation in action angle variables** to write the Hamiltonian in the form

$$\begin{aligned} H = H_0(J) + H_1(\phi, J) &= \frac{J}{\beta(s)} + \frac{2\sqrt{2}K_s(s)}{3} (J\beta(s))^{3/2} \cos^3 \phi \\ &= \frac{J}{\beta(s)} + \frac{K_s(s)}{3\sqrt{2}} (J\beta(s))^{3/2} (\cos 3\phi + 3 \cos \phi) \end{aligned}$$

- The perturbation procedure implies to split the perturbation in an **average part** over the angles and an **oscillating part**

$$H_1 = \langle H_1 \rangle_\phi + \{H_1\} = \frac{\sqrt{2}k_2(s)}{12} (J\beta(s))^{3/2} (\cos 3\phi + 3 \cos \phi)$$

where $\langle H_1 \rangle_\phi = \left(\frac{1}{2\pi} \right) \oint H_1(J, \varphi) d\varphi$

and $\{H_1\} = H_1 - \langle H_1 \rangle_\phi$

$$= \sum_{k,p} H_{1k}(J) e^{i(k \cdot \varphi + p\theta)}$$

- The **average part** should be only a **function of the action**
- Its derivative with respect to the action should provide the **frequency shift (tune-shift)** due to the non-linearity
- It can be shown that this quantity vanishes for a sextupole perturbation

$$\left\langle \frac{\partial H_1(\phi, J)}{\partial J} \right\rangle_\phi = \frac{k_2(s)\beta(s)}{8\sqrt{2}\pi} (J\beta(s))^{1/2} \int_0^{2\pi} (\cos 3\phi + 3 \cos \phi) d\phi = 0$$

- Sextupoles do not provide any tune-shift at **first order**
- But we know by experience that this is not true, i.e. first order perturbation theory fails to give the correct answer
- One has to go to **higher order** (see appendix)

- The **oscillating part** is then the same as the original Hamiltonian

$$\{H_1\} = H_1 - \langle H_1 \rangle_{\bar{\phi}} = H_1 = \frac{K_s(s)}{3\sqrt{2}} (\bar{J}\beta(s))^{3/2} (\cos 3\phi + 3 \cos \phi)$$

- Following the canonical perturbation procedure the **generating function** is

$$S(\bar{J}, \bar{\phi}) = \bar{J} \cdot \bar{\phi} + i \sum_{k,p \neq 0} \frac{H_{1k}(\bar{J})}{k \cdot \nu(\bar{J}) + p} e^{i(k \cdot \bar{\phi} + p\theta)} + \dots$$

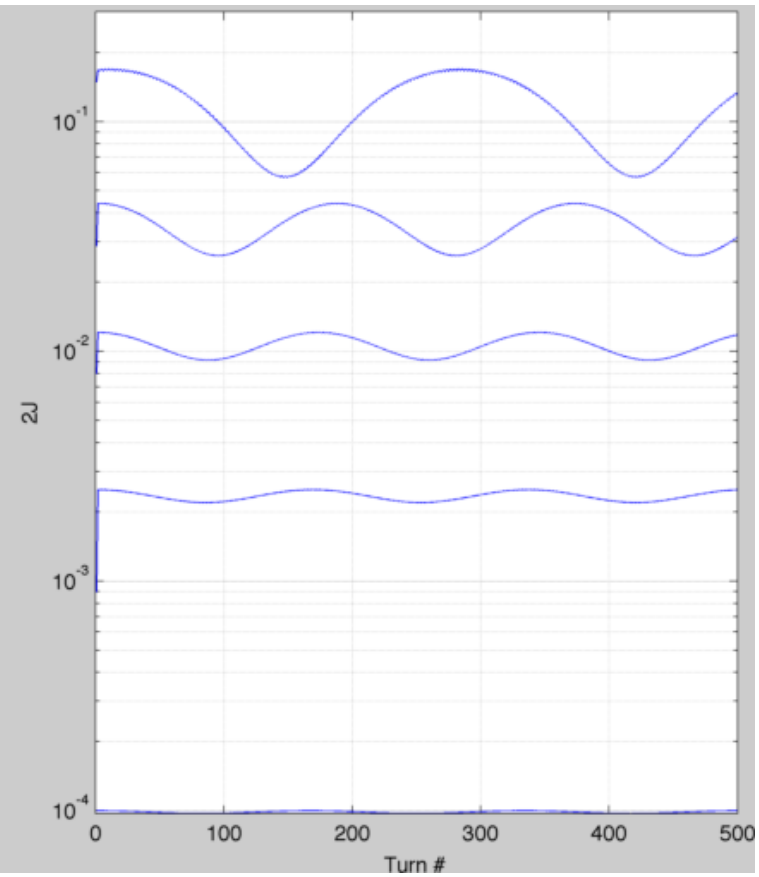
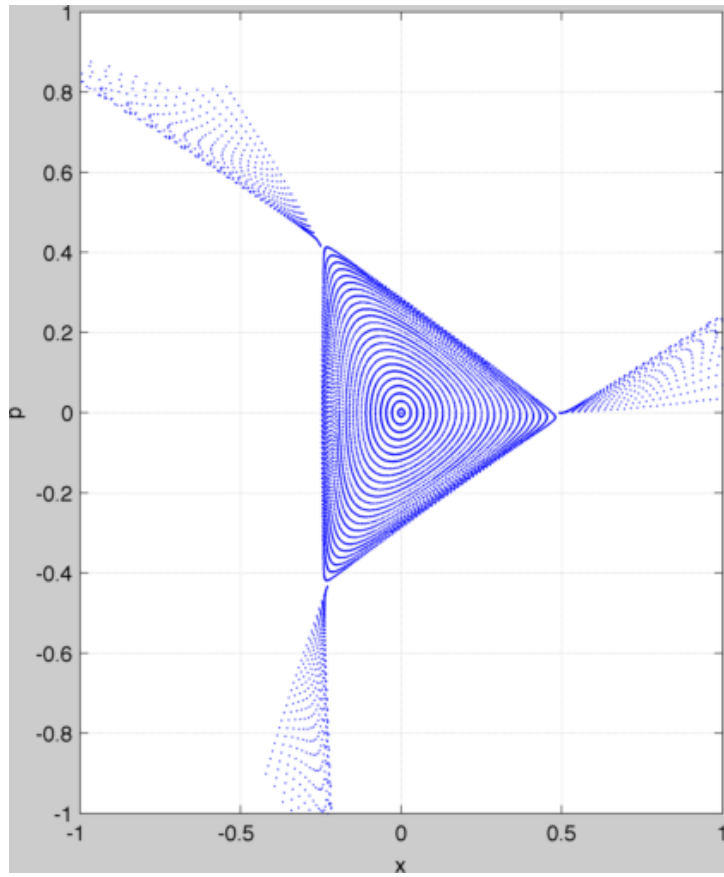
- The **only non-zero Fourier terms** are for $k = 1, 3$ and

$$S(\bar{J}, \bar{\phi}) = \bar{J} \cdot \bar{\phi} + i \frac{K_s(s)}{6\sqrt{2}} (\bar{J}\beta(s))^{3/2} \sum_{p=-\infty}^{\infty} \left(\frac{e^{i(3\bar{\phi} + p\theta)}}{3\nu + p} + \frac{3e^{i(\bar{\phi} + p\theta)}}{\nu + p} \right)$$

- We derived (with a lot of effort) the common result that sextupoles **at first order** excite **integer** and **third integer** resonances
- Again this is not generally true! It is known that sextupoles can drive **any resonance** (either if they are large enough, or if the particle is far away from the closed orbit)
- This can be shown again by pursuing the perturbation approach to **second order** (as for the tune-shift)
- A useful application is to use the generating function for computing the correction to the original invariant, as the new one should be an integral of motion (at first order)

$$J \approx \bar{J} + \frac{\partial S_1(\bar{J}, \varphi, \theta)}{\partial \varphi}$$

- For small perturbations, the new action variable is almost an invariant but for larger ones phase space gets deformed
- Close to the integer or third integer resonance, canonical perturbation theory cannot be applied
- The solution is provided by **secular perturbation theory**



- The general accelerator Hamiltonian is written as

$$\mathcal{H}(x, y, p_x, p_y, s) = \mathcal{H}_0(x, y, p_x, p_y, s) + \sum_{k_x, k_y} h_{k_x, k_y}(s) x^{k_x} y^{k_y}$$

- The transverse coordinates can be expressed in action-angle variables as

$$u(s) = \sqrt{\frac{J_u \beta_u(s)}{2}} \left(e^{i(\phi_u(s) + \theta_u(s))} + e^{-i(\phi_u(s) + \theta_u(s))} \right)$$

- The Hamiltonian in action-angle variables is

$$\mathcal{H}'(J_x, J_y, \phi_x, \phi_y) = H_0(J_x, J_y) + H_1(J_x, J_y, \phi_x, \phi_y)$$

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- The integrable part $H_0(J_x, J_y) = \frac{1}{R} (\nu_x J_x + \nu_y J_y)$

- The perturbation

$$H_1(J_x, J_y, \phi_x, \phi_y; s) = \sum_{k_x, k_y} J_x^{k_x/2} J_y^{k_y/2} \sum_j \sum_l^{k_x, k_y} g_{j, k, l, m}(s) e^{i[(j-k)\phi_x + (l-m)\phi_y]}$$

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$$\mathcal{H}(x, y, p_x, p_y, s) = \mathcal{H}_0(x, y, p_x, p_y, s) + \sum_{k_x, k_y} h_{k_x, k_y}(s) x^{k_x} y^{k_y}$$

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$$H_1(J_x, J_y, \phi_x, \phi_y; s) = \sum_{k_x, k_y} J_x^{k_x/2} J_y^{k_y/2} \sum_j \sum_l g_{j, k, l, m}(s) e^{i[(j-k)\phi_x + (l-m)\phi_y]}$$

- The coefficients $g_{j, k, l, m}(s) = \frac{h_{k_x, k_y}(s)}{2^{\frac{j+k+l+m}{2}}} \binom{k_x}{j} \binom{k_y}{l} \beta_x^{k_x/2}(s) \beta_y^{k_y/2}(s) e^{i[(j-k)\theta_x(s) + (l-m)\theta_y(s)]}$

depend on the optics, with the indexes $k_x = j + k$, $k_y = l + m$

- As the coefficients $h_{k_x, k_y}(s)$ are periodic, the perturbation can be expanded in Fourier series

$$H_1(J_x, J_y, \phi_x, \phi_y; \theta) = \sum_{k_x, k_y} J_x^{k_x/2} J_y^{k_y/2} \sum_j^{k_x} \sum_l^{k_y} \sum_{p=-\infty}^{\infty} g_{j,k,l,m;p} e^{i[(j-k)\phi_x + (l-m)\phi_y - p\theta]}$$

with the **resonance driving terms**

$$g_{j,k,l,m;p} = \binom{k_x}{j} \binom{k_y}{l} \frac{1}{2^{\frac{j+k+l+m}{2}}} \frac{1}{2\pi} \oint h_{k_x, k_y}(s) \beta_x^{k_x/2}(s) \beta_y^{k_y/2}(s) e^{i[(j-k)\phi_x(s) + (l-m)\phi_y(s) + p\theta]}$$

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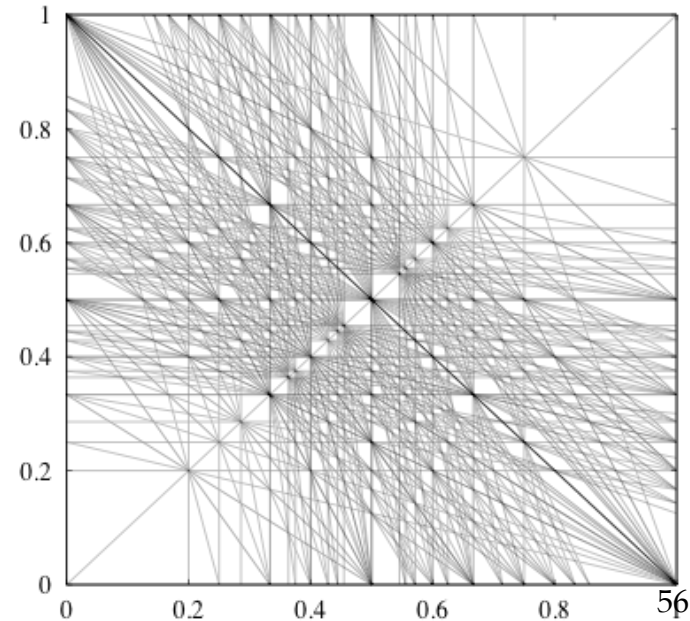
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- For $n_x = j - k$, $n_y = l - m$, resonance conditions appear for $n_x \nu_x + n_y \nu_y = p$
- Goal of accelerator design and correction systems is to minimize the resonance driving terms
 - ❑ Change magnet design so that $h_{k_x, k_y}(s)$ become smaller
 - ❑ Introduce magnetic elements capable of creating a cancelling effect
 - ❑ Sort magnets or non-linear elements in a way that phase terms are minimised

- The general resonance conditions is $n_x \nu_x + n_y \nu_y = p$ with order $n_x + n_y$
- For all the polynomial field terms of a $2m$ -pole, the excited resonances (**at first order**) satisfy the condition $n_x + n_y = m$ but there are also **sub-resonances** for which $n_x + n_y < m$
- For **normal** (erect) multi-poles, the resonances (**at first order**) are $(n_x, n_y) = (m, 0), (m - 2, \pm 2), \dots$ whereas for **skew** multi-poles $(n_x, n_y) = (m - 1, \pm 1), (m - 3, \pm 3), \dots$
- If perturbation is large, **all** resonances can be potentially excited
- The **resonance conditions form lines** in frequency space and fill it up as the **order grows** (the rational numbers form a dense set inside the real numbers), but Fourier amplitudes should also decrease

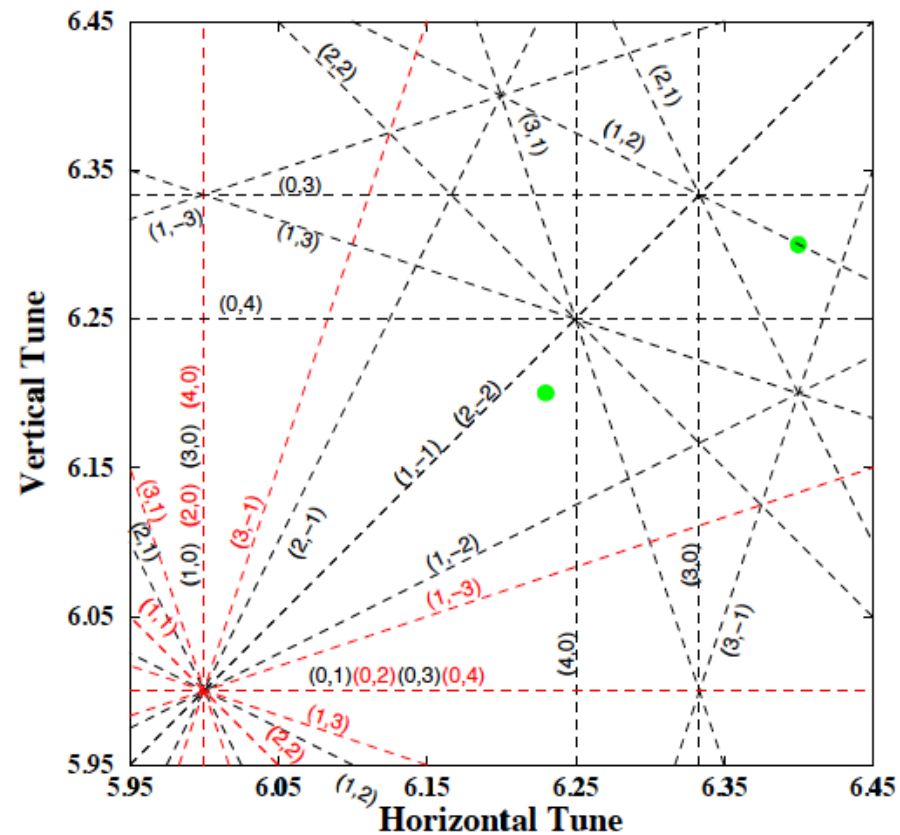


- If lattice is made out of N identical cells, and the perturbation follows the same periodicity, resulting in a reduction of the resonance conditions to the ones satisfying $n_x \nu_x + n_y \nu_y = jN$

- These are called **systematic** resonances

- Practically, any (linear) lattice perturbation breaks super-periodicity and any **random** resonance can be excited

- Careful choice of the working point is necessary



- First order correction to the tunes is computed by the derivatives with respect to the action of the average part of perturbation. For a given term, $h_{k_x, k_y}(s) x^{k_x} y^{k_y}$ the leading order correction to the tunes are

$$\delta\nu_x = \frac{J_x^{k_x/2-1} J_y^{k_y/2}}{4\pi^2} \sum_j^{k_x} \sum_l^{k_y} \bar{g}_{j,k,l,m} \oint e^{i[(j-k)\phi_x + (l-m)\phi_y]}$$

$$\delta\nu_y = \frac{J_x^{k_x/2} J_y^{k_y/2-1}}{4\pi^2} \sum_j^{k_x} \sum_l^{k_y} \bar{g}_{j,k,l,m} \oint e^{i[(j-k)\phi_x + (l-m)\phi_y]}$$

where $\bar{g}_{j,k,l,m}$ is the average of $g_{j,k,l,m}(s)$ around the ring.

- In the accelerator jargon if $\delta\nu_{x,y}$ is independent of the action, it is referred to as **tune-shift**, whereas, if it depends on the action, it is called **tune-spread** (or amplitude detuning)
- At first order, $\delta\nu_{x,y} = 0$, for **odd multi-poles** $k_x = j + k$, $k_y = l + m$ (trigonometric functions give zero averages).

- **Hamiltonian formalism** provides the **natural framework** to analyse (linear and non-linear) beam dynamics
- **Canonical (symplectic) transformations** enable to move from variables describing a distorted phase space to something simpler (ideally circles)
- The **generating functions** passing from the old to the new variables are bounded to **diverge** in the vicinity of **resonances** (emergence of chaos, see next lectures)
- Calculating this generating function with **canonical perturbation theory** becomes **hopeless** for higher orders
- Need clearly a new approach (through **Lie transformations of accelerator maps**) enabling derivation of the generating functions in an **algorithmic way**, in principle to **arbitrary order** (see Lectures of W. Herr)
- For real accelerator models, we have to rely on numerical integration of the equations of motion, i.e. **particle tracking** and **methods** to analyse it (see Lectures of NLD Phenomenology)

- The variation of the action can be written as

$$\delta W = \int_{t_1}^{t_2} (L(q + \delta q, \dot{q} + \delta \dot{q}, t) - L(q, \dot{q}, t)) dt = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt$$

- Taking into account that $\delta \dot{q} = \frac{d\delta q}{dt}$, the 2nd part of the integral can be integrated by parts giving

$$\delta W = \left. \frac{\partial L}{\partial \dot{q}} \delta q \right|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right) \delta q dt = 0$$

- The first term is zero because $\delta q(t_1) = \delta q(t_2) = 0$ so the second integrant should also vanish, providing the following differential equations for each degree of freedom, the **Lagrange equations**

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

- The **equations of motion** can be derived from the Hamiltonian following the same variational principle as for the Lagrangian (“least” action) but also by simply taking the differential of the Hamiltonian

$$dH = \sum_i p_i d\dot{q}_i + \dot{q}_i dp_i - \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{p_i} d\dot{q}_i - \underbrace{\frac{\partial L}{\partial q_i}}_{\dot{p}_i} dq_i - \frac{\partial L}{\partial t} dt$$

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or

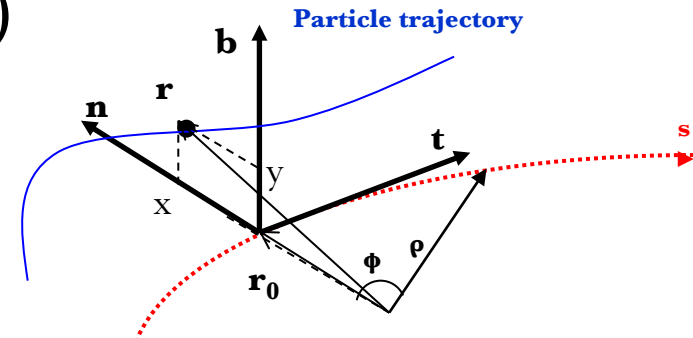
$$dH(q, p, t) = \sum_i \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial L}{\partial t} dt = \sum_i \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial t} dt$$

- By equating terms, **Hamilton's equations** are derived

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

- These are indeed $2n + 2$ equations describing the motion in the “**extended**” phase space $(q_1, \dots, q_n, p_1, \dots, p_n, t, -H)$

- It is useful (especially for rings) to transform the Cartesian coordinate system to the **Frenet-Serret system** moving to a closed curve, with path length s



- The position coordinates in the two systems are connected by $\mathbf{r} = \mathbf{r}_0(s) + X\mathbf{n}(s) + Y\mathbf{b}(s) = x\mathbf{u}_x + y\mathbf{u}_y + z\mathbf{u}_z$
- The **Frenet-Serret unit vectors** and their derivatives are defined as $(\mathbf{t}, \mathbf{n}, \mathbf{b}) = \left(\frac{d}{ds}\mathbf{r}_0(s), -\rho(s)\frac{d^2}{ds^2}\mathbf{r}_0(s), \mathbf{t} \times \mathbf{n} \right)$

$$\frac{d}{ds} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{\rho(s)} & 0 \\ 0 & 0 & \tau(s) \\ \frac{1}{\rho(s)} & 0 & -\tau(s) \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}$$

with $\rho(s)$ the radius of curvature and $\tau(s)$ the torsion which vanishes in case of planar motion

- We are seeking a canonical transformation between

$$(\mathbf{q}, \mathbf{p}) \mapsto (\mathbf{Q}, \mathbf{P}) \text{ or}$$

$$(x, y, z, p_x, p_y, p_z) \mapsto (X, Y, s, P_x, P_y, P_s)$$

- The generating function is

$$(\mathbf{q}, \mathbf{P}) = - \left(\frac{\partial F_3(\mathbf{p}, \mathbf{Q})}{\partial \mathbf{p}}, \frac{\partial F_3(\mathbf{p}, \mathbf{Q})}{\partial \mathbf{Q}} \right)$$

- By using the relationship between the positions, the generating function is

$$F_3(\mathbf{p}, \mathbf{Q}) = -\mathbf{p} \cdot \mathbf{r} + \overline{F_3}(\mathbf{Q}) = -\mathbf{p} \cdot \mathbf{r}$$

□ for planar motion, the momenta are

$$\mathbf{P} = (P_X, P_Y, P_s) = \mathbf{p} \cdot (\mathbf{n}, \mathbf{b}, (1 + \frac{X}{\rho})\mathbf{t})$$

□ Taking into account that the **vector potential** is also transformed in the same way

$$(A_X, A_Y, A_s) = \mathbf{A} \cdot (\mathbf{n}, \mathbf{b}, (1 + \frac{X}{\rho})\mathbf{t})$$

the new **Hamiltonian** is given by

$$\mathcal{H}(\mathbf{Q}, \mathbf{P}, t) = c \sqrt{(P_X - \frac{e}{c}A_X)^2 + (P_Y - \frac{e}{c}A_Y)^2 + \frac{(P_s - \frac{e}{c}A_s)^2}{(1 + \frac{X}{\rho(s)})^2} + m^2c^2} + e\Phi$$

- It is more convenient to use the **path length** s , instead of the time as **independent variable**
- The Hamiltonian can be considered as having **4 degrees of freedom**, where the 4th “**position**” is **time** and its conjugate momentum is $P_t = -\mathcal{H}$

- ❑ It is more convenient to use the **path length** s , instead of the time as **independent variable**
- ❑ The Hamiltonian can be considered as having **4 degrees of freedom**, where the 4th “**position**” is **time** and its conjugate momentum is $P_t = -\mathcal{H}$
- ❑ In the same way, the new Hamiltonian with the path length as the independent variable is just $P_s = -\tilde{\mathcal{H}}(X, Y, t, P_X, P_Y, P_t, s)$ with

$$\tilde{\mathcal{H}} = -\frac{e}{c}A_s - \left(1 + \frac{X}{\rho(s)}\right) \sqrt{\left(\frac{P_t + e\Phi}{c}\right)^2 - m^2c^2 - (P_x - \frac{e}{c}A_X)^2 - (P_Y - \frac{e}{c}A_Y)^2}$$

- ❑ It can be proved that this is indeed a **canonical transformation**
- ❑ Note the existence of the **reference orbit** for zero vector potential, for which $(X, Y, P_X, P_Y, P_s) = (0, 0, 0, 0, P_0)$

- Due to the fact that **longitudinal** (synchrotron) motion is **much slower** than the **transverse** (betatron) one, the electric field can be set to zero and the Hamiltonian is written as

$$\tilde{\mathcal{H}} = -\frac{e}{c}A_s - \left(1 + \frac{X}{\rho(s)}\right) \sqrt{\underbrace{\left(\frac{\mathcal{H}}{c}\right)^2 - m^2c^2}_{P^2} - (P_x - \frac{e}{c}A_X)^2 - (P_Y - \frac{e}{c}A_Y)^2}$$

- The Hamiltonian is then written as

$$\tilde{\mathcal{H}} = -\frac{e}{c}A_s - \left(1 + \frac{X}{\rho(s)}\right) \sqrt{(P^2 - (P_x - \frac{e}{c}A_X)^2 - (P_Y - \frac{e}{c}A_Y)^2)}$$

- If **static** magnetic fields are considered, the time dependence is also dropped, and the system is having **2 degrees of freedom + “time”** (path length)

- Due to the fact that **total momentum is much larger** than the transverse ones, another transformation may be considered, where the transverse momenta are rescaled

$$(\mathbf{Q}, \mathbf{P}) \mapsto (\bar{\mathbf{q}}, \bar{\mathbf{p}}) \text{ or}$$

$$(X, Y, t, P_X, P_Y, P_t) \mapsto (\bar{x}, \bar{y}, \bar{t}, \bar{p}_x, \bar{p}_y, \bar{p}_t) = \left(X, Y, -c t, \frac{P_X}{P_0}, \frac{P_Y}{P_0}, -\frac{P_t}{P_0 c} \right)$$

- The new variables are indeed canonical if the Hamiltonian is also rescaled and written as

$$\bar{\mathcal{H}}(\bar{x}, \bar{y}, \bar{t}, \bar{p}_x, \bar{p}_y, \bar{p}_t) = \frac{\tilde{\mathcal{H}}}{P_0} = -e\bar{A}_s - \left(1 + \frac{\bar{x}}{\rho(s)} \right) \sqrt{\bar{p}_t^2 - \frac{m^2 c^2}{P_0} - (\bar{p}_x - e\bar{A}_x)^2 - (\bar{p}_y - e\bar{A}_y)^2}$$

with $(\bar{A}_x, \bar{A}_y, \bar{A}_z) = \frac{1}{P_0 c} (A_x, A_y, A_s)$

and $\frac{m^2 c^2}{P_0} = \frac{1}{\beta_0^2 \gamma_0^2}$

- Along the reference trajectory $\bar{p}_{t0} = \frac{1}{\beta_0}$ and

$$\left. \frac{d\bar{t}}{ds} \right|_{P=P_0} = \left. \frac{\partial \bar{H}}{\partial \bar{p}_t} \right|_{P=P_0} = -\bar{p}_{t0} = -\frac{1}{\beta_0}$$

- It is thus useful to **move the reference frame to the reference trajectory** for which another canonical transformation is performed

$$(\bar{\mathbf{q}}, \bar{\mathbf{p}}) \mapsto (\hat{\mathbf{q}}, \hat{\mathbf{p}}) \text{ or}$$

$$(\bar{x}, \bar{y}, \bar{t}, \bar{p}_x, \bar{p}_y, \bar{p}_t) \mapsto (\hat{x}, \hat{y}, \hat{t}, \hat{p}_x, \hat{p}_y, \hat{p}_t) = \left(\hat{x}, \hat{y}, \bar{t} + \frac{s - s_0}{\beta_0}, \hat{p}_x, \hat{p}_y, \bar{p}_t - \frac{1}{\beta_0} \right)$$

□ Along the reference trajectory $\bar{p}_{t0} = \frac{1}{\beta_0}$ and

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□ The mixed variable generating function is

$$(\hat{\mathbf{q}}, \bar{\mathbf{p}}) = \left(\frac{\partial F_2(\bar{\mathbf{q}}, \hat{\mathbf{p}})}{\partial \hat{\mathbf{p}}}, \frac{\partial F_2(\bar{\mathbf{q}}, \hat{\mathbf{p}})}{\partial \bar{\mathbf{q}}} \right) \text{ providing}$$

$$F_2(\bar{\mathbf{q}}, \hat{\mathbf{p}}) = \bar{x}\hat{p}_x + \bar{y}\hat{p}_y + \left(\bar{t} + \frac{s - s_0}{\beta_0} \right) \left(\hat{p}_t + \frac{1}{\beta_0} \right)$$

□ The Hamiltonian is then

$$\hat{\mathcal{H}}(\hat{x}, \hat{y}, \hat{t}, \hat{p}_x, \hat{p}_y, \hat{p}_t) = \frac{1}{\beta_0} \left(\frac{1}{\beta_0} + \hat{p}_t \right) - e\hat{A}_s - \left(1 + \frac{\hat{x}}{\rho(s)} \right) \sqrt{\left(\hat{p}_t + \frac{1}{\beta_0} \right)^2 - \frac{1}{\beta_0^2 \gamma_0^2} - (\hat{p}_x - e\hat{A}_x)^2 - (\hat{p}_y - e\bar{A}_y)^2}$$

□ First note that $\hat{p}_t = \bar{p}_t - \frac{1}{\beta_0} = \bar{p}_t - \bar{p}_{t0} = \frac{P_t - P_0}{P_0} \equiv \delta$
and $l = \hat{t}$

□ In the **ultra-relativistic limit** $\beta_0 \rightarrow 1$, $\frac{1}{\beta_0^2 \gamma^2} \rightarrow 0$
and the Hamiltonian is written as

$$\mathcal{H}(x, y, l, p_x, p_y, \delta) = (1 + \delta) - e\hat{A}_s - \left(1 + \frac{x}{\rho(s)}\right) \sqrt{(1 + \delta)^2 - (p_x - e\hat{A}_x)^2 - (p_y - e\hat{A}_y)^2}$$

where the “hats” are dropped for simplicity

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where the “hats” are dropped for simplicity

□ If we consider **only transverse field** components, the **vector potential** has **only a longitudinal** component and the Hamiltonian is written as

$$\mathcal{H}(x, y, l, p_x, p_y, \delta) = (1 + \delta) - e\hat{A}_s - \left(1 + \frac{x}{\rho(s)}\right) \sqrt{(1 + \delta)^2 - p_x^2 - p_y^2}$$

□ Note that the Hamiltonian is non-linear even in the absence of any field component (i.e. for a drift)!

- ❑ It is useful for study purposes (especially for finding an “integrable” version of the Hamiltonian) to make an extra approximation
- ❑ For this, **transverse momenta** (rescaled to the reference momentum) are considered to be **much smaller than 1**, i.e. the square root can be expanded.
- ❑ Considering also the large machine approximation $x \ll \rho$, (dropping cubic terms), the Hamiltonian is simplified to

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2(1 + \delta)} - \frac{x(1 + \delta)}{\rho(s)} - e\hat{A}_s$$

- ❑ This expansion may **not** be a **good idea**, especially for **low energy, small size rings**

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□ Note that the Hamiltonian is non-linear even in the absence of any field component (i.e. for a drift)!

- From Gauss law of magnetostatics, a vector potential exist

$$\nabla \cdot \mathbf{B} = 0 \quad \rightarrow \quad \exists \mathbf{A} : \quad \mathbf{B} = \nabla \times \mathbf{A}$$

- Assuming transverse 2D field, vector potential has only one component A_s . The Ampere's law in vacuum (inside the beam pipe) $\nabla \times \mathbf{B} = 0 \quad \rightarrow \quad \exists V : \quad \mathbf{B} = -\nabla V$

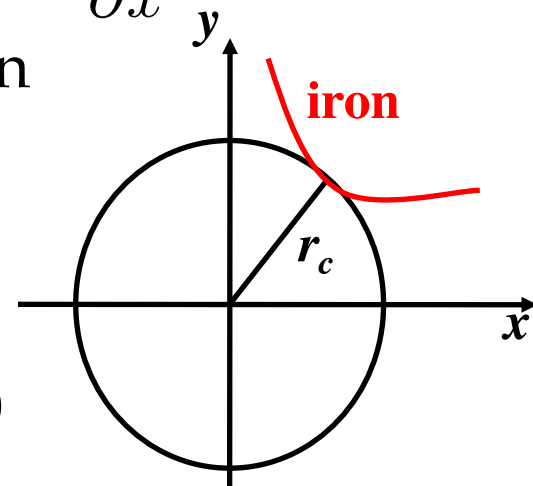
- Using the previous equations, the relations between field components and potentials are

$$B_x = -\frac{\partial V}{\partial x} = \frac{\partial A_s}{\partial y}, \quad B_y = -\frac{\partial V}{\partial y} = -\frac{\partial A_s}{\partial x}$$

i.e. Riemann conditions of an analytic function



Exists complex potential of $z = x + iy$ with power series expansion convergent in a circle with radius $|z| = r_c$ (distance from iron yoke)



$$\mathcal{A}(x + iy) = A_s(x, y) + iV(x, y) = \sum_{n=1}^{\infty} \kappa_n z^n = \sum_{n=1}^{\infty} (\lambda_n + i\mu_n)(x + iy)^n$$

- From the complex potential we can derive the fields

$$B_y + iB_x = -\frac{\partial}{\partial x}(A_s(x, y) + iV(x, y)) = -\sum_{n=1}^{\infty} n(\lambda_n + i\mu_n)(x + iy)^{n-1}$$

- Setting $b_n = -n\lambda_n$, $a_n = n\mu_n$

$$B_y + iB_x = \sum_{n=1}^{\infty} (b_n - ia_n)(x + iy)^{n-1}$$

- Define normalized coefficients

$$b'_n = \frac{b_n}{10^{-4}B_0} r_0^{n-1}, \quad a'_n = \frac{a_n}{10^{-4}B_0} r_0^{n-1}$$

on a reference radius r_0 , 10^{-4} of the main field to get

$$B_y + iB_x = 10^{-4}B_0 \sum_{n=1}^{\infty} (b'_n - ia'_n) \left(\frac{x + iy}{r_0}\right)^{n-1}$$

- **Note:** $n' = n - 1$ is the US convention

- Expand term by term the Hamiltonian $H(\mathbf{J}(\bar{\mathbf{J}}, \bar{\varphi}), \varphi(\bar{\mathbf{J}}, \bar{\varphi}), \theta)$ to leading order in ϵ

$$H_0(\mathbf{J}(\bar{\mathbf{J}}, \bar{\varphi})) = H_0(\bar{\mathbf{J}}) + \epsilon \frac{\partial H_0(\bar{\mathbf{J}})}{\partial \bar{\mathbf{J}}} \frac{\partial S_1(\bar{\mathbf{J}}, \bar{\varphi}, \theta)}{\partial \bar{\varphi}} + \mathcal{O}(\epsilon^2)$$

$$\epsilon H_1(\mathbf{J}(\bar{\mathbf{J}}, \bar{\varphi}), \varphi(\bar{\mathbf{J}}, \bar{\varphi}), \theta) = \epsilon H_1(\bar{\mathbf{J}}, \bar{\varphi}) + \mathcal{O}(\epsilon^2)$$

- The new Hamiltonian can also be expanded in orders of ϵ

$$\bar{H} = \bar{H}_0 + \epsilon \bar{H}_1 + \dots$$

- Equating the terms of equal orders in ϵ , we obtain

- Zero order $\bar{H}_0 = H_0(\bar{\mathbf{J}})$

- First order $\bar{H}_1 = \frac{\partial S_1(\bar{\mathbf{J}}, \bar{\varphi}, \theta)}{\partial \theta} + \omega(\bar{\mathbf{J}}) \cdot \frac{\partial S_1(\bar{\mathbf{J}}, \bar{\varphi}, \theta)}{\partial \bar{\varphi}} + H_1(\bar{\mathbf{J}}, \bar{\varphi})$

where the frequency vector is $\omega(\bar{\mathbf{J}}) = \frac{\partial H_0(\bar{\mathbf{J}})}{\partial \bar{\mathbf{J}}}$

- From the first order Hamiltonian, the angles have to be eliminated. For this purpose, it can be split in two parts:

- Average part: $\langle H_1 \rangle_{\bar{\varphi}} = \left(\frac{1}{2\pi} \right)^n \oint H_1(\bar{\mathbf{J}}, \bar{\varphi}) d\bar{\varphi}$

- Oscillating part: $\{H_1\} = H_1 - \langle H_1 \rangle_{\bar{\varphi}}$

- The 1st order perturbation part of the Hamiltonian then becomes

$$\bar{H}_1 = \frac{\partial S_1(\bar{\mathbf{J}}, \bar{\varphi}, \theta)}{\partial \theta} + \omega(\bar{\mathbf{J}}) \cdot \frac{\partial S_1(\bar{\mathbf{J}}, \bar{\varphi}, \theta)}{\partial \bar{\varphi}} + \langle H_1(\bar{\mathbf{J}}, \bar{\varphi}) \rangle_{\bar{\varphi}} + \{H_1(\bar{\mathbf{J}}, \bar{\varphi})\}$$

- Thus, the generating function should be chosen such that the angle dependence is eliminated, for which

$$\bar{H}_1(\bar{\mathbf{J}}) = \langle H_1(\bar{\mathbf{J}}, \bar{\varphi}) \rangle_{\bar{\varphi}} \quad \text{and} \quad \frac{\partial S_1(\bar{\mathbf{J}}, \bar{\varphi}, \theta)}{\partial \theta} + \omega(\bar{\mathbf{J}}) \cdot \frac{\partial S_1(\bar{\mathbf{J}}, \bar{\varphi}, \theta)}{\partial \bar{\varphi}} = -\{H_1(\bar{\mathbf{J}}, \bar{\varphi})\}$$

- The new Hamiltonian is a function of the new actions

$$\bar{H}(\bar{\mathbf{J}}) = H_0(\bar{\mathbf{J}}) + \epsilon \langle H_1(\bar{\mathbf{J}}, \bar{\varphi}) \rangle_{\bar{\varphi}} + \mathcal{O}(\epsilon^2) \quad \text{with the new frequency vector}$$

$$\bar{\omega}(\bar{\mathbf{J}}) = \frac{\partial \bar{H}(\bar{\mathbf{J}})}{\partial \bar{\mathbf{J}}} = \omega(\bar{\mathbf{J}}) + \epsilon \frac{\partial \langle H_1(\bar{\mathbf{J}}, \bar{\varphi}) \rangle_{\bar{\varphi}}}{\partial \bar{\mathbf{J}}} + \mathcal{O}(\epsilon^2)$$

- The question that remains to be answered is whether a generating function can be found that eliminates the angle dependence
- The oscillating part of the perturbation and the generating function can be expanded in Fourier series

$$\{H_1(\bar{\mathbf{J}}, \bar{\boldsymbol{\varphi}})\} = \sum_{\mathbf{k}, p} H_{1\mathbf{k}}(\bar{\mathbf{J}}) e^{i(\mathbf{k} \cdot \bar{\boldsymbol{\varphi}} + p\theta)} \quad S_1(\bar{\mathbf{J}}, \bar{\boldsymbol{\varphi}}, \theta) = \sum_{\mathbf{k}, p} S_{1\mathbf{k}}(\bar{\mathbf{J}}) e^{i(\mathbf{k} \cdot \bar{\boldsymbol{\varphi}} + p\theta)}$$

with
$$\mathbf{k} \cdot \bar{\boldsymbol{\varphi}} = k_1 \bar{\varphi}_1 + \dots + k_n \bar{\varphi}_n$$

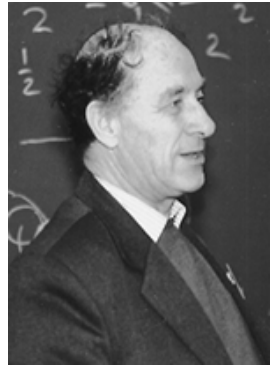
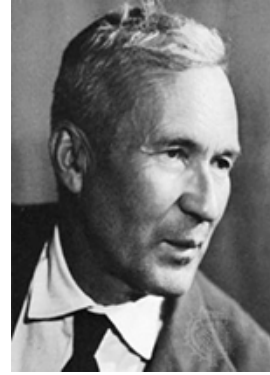
- Following the relationship for the angle elimination, the Fourier coefficients of the generating function should satisfy

$$S_{1\mathbf{k}}(\bar{\mathbf{J}}) = i \frac{H_{1\mathbf{k}}(\bar{\mathbf{J}})}{\mathbf{k} \cdot \boldsymbol{\omega}(\bar{\mathbf{J}}) + p} \quad \text{with} \quad \mathbf{k}, p \neq 0$$

- Then, the generating function can be written as

$$S(\bar{\mathbf{J}}, \bar{\boldsymbol{\varphi}}) = \bar{\mathbf{J}} \cdot \bar{\boldsymbol{\varphi}} + \epsilon i \sum_{\mathbf{k} \neq 0} \frac{H_{1\mathbf{k}}(\bar{\mathbf{J}})}{\mathbf{k} \cdot \boldsymbol{\omega}(\bar{\mathbf{J}}) + p} e^{i(\mathbf{k} \cdot \bar{\boldsymbol{\varphi}} + p\theta)} + \mathcal{O}(\epsilon^2)$$

- Original idea of **Kolmogorov** (1954) (super-convergent series expansion) later proved by **Arnold** (1963) and **Moser** (1962)
- If a Hamiltonian system is subjected to weak nonlinear perturbation, some invariant tori are deformed and survive
- Trajectories starting on one of these tori remain on it thereafter, executing quasi-periodic motion with a fixed frequency vector depending only on the torus.
- The family of tori is parameterized over a Cantor set of frequency vectors, while in the gaps of the Cantor set chaotic behavior can occur
- The **KAM theorem** specifies quantitatively the size of the perturbation for this to be true.
- The **KAM tori** that survive are those that have “sufficiently irrational” frequencies
- The conditions of the KAM theorem become increasingly difficult to satisfy for systems with more degrees of freedom. As the number of dimensions of the system increases, the volume occupied by the tori decreases
- A complement of KAM theory for the stability of dynamical systems were given by **Nekhoroshev** (1971) who proved that if the density of tori is large all solutions will stay close to the tori for exponentially long times showing practical stability of motion



- It can be shown that at second order in perturbation theory the Hamiltonian depending only on the actions can be written

$$\bar{H}_2(\bar{J}) = \left\langle \frac{1}{2} \frac{\partial^2 H_0}{\partial \bar{J}^2} \left(\frac{\partial S_1}{\partial \phi} \right)^2 + \frac{\partial H_1}{\partial \bar{J}} \frac{\partial S_1}{\partial \phi} \right\rangle_{\phi}$$

- This can be simplified to $\bar{H}_2(\bar{J}) = \left\langle \frac{\partial H_1}{\partial \bar{J}} \frac{\partial S_1}{\partial \phi} \right\rangle_{\phi}$

- The two terms are $\frac{\partial H_1}{\partial \bar{J}} = \frac{K_s(s)}{2\sqrt{2}} \bar{J}^{1/2} \beta(s)^{3/2} (\cos 3\phi + 3 \cos \phi)$

$$\frac{\partial S_1}{\partial \phi} = -\frac{\bar{J}^{3/2}}{2\sqrt{2}} \int_s^{s+C} K_s(s') \beta(s')^{3/2} \left[\frac{\cos(\phi + \psi(s') - \psi(s) - \pi\nu)}{\sin(\pi\nu)} + \frac{\cos 3(\phi + \psi(s') - \psi(s) - \pi\nu)}{\sin(3\pi\nu)} \right] ds'$$

- The 2nd order Hamiltonian is given by the angle-averaged product of the last two terms.
- It is quadratic in the sextupole strength and the new action.

The 2nd order tune-shift is the derivative in the action

$$\nu(\bar{J}) = \left\langle \frac{\partial H_2}{\partial \bar{J}} \right\rangle_{\phi,s} = -\frac{\bar{J}}{16\pi} \int_0^C ds K_s(s) \beta(s)^{3/2} \int_s^{s+C} K_s(s') \beta(s')^{3/2} \times \left[\frac{\cos(\phi + \psi(s') - \psi(s) - \pi\nu)}{\sin(\pi\nu)} + \frac{\cos 3(\phi + \psi(s') - \psi(s) - \pi\nu)}{\sin(3\pi\nu)} \right] ds'_{83}$$

- Expand both the perturbation and generating function in Fourier series of the form

$$S_1(\bar{J}, \bar{\phi}, \theta) = \sum_k S_{1k}(\bar{J}, \theta) e^{ik\bar{\phi}} \quad \text{and} \quad \{H_1(\bar{J}, \bar{\phi}, \theta)\} = \sum_k H_{1k}(\bar{J}, \theta) e^{ik\bar{\phi}}$$

- The equation relating the amplitudes is

$$i k \nu S_{1k} + \frac{\partial S_{1k}}{\partial \theta} = -H_{1k}$$

which can be solved yielding

$$S_{1k} = \frac{i}{2 \sin(\pi k \nu)} \int_{\theta}^{\theta+2\pi} H_{1k} e^{ik\nu(\theta' - \theta - \pi)} d\theta'$$

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- Following the canonical perturbation procedure the generating function is

$$S_1 = \sum_k \frac{i}{2 \sin(\pi k \nu)} \int_{\theta}^{\theta+2\pi} H_{1k} e^{ik[\phi + \nu(\theta' - \theta - \pi)]} d\theta'$$

- For the sextupole, and letting $\psi(s) = \int_0^s \frac{ds'}{\beta(s')}$ we have

$$S_1 = -\frac{\bar{J}^{3/2}}{2\sqrt{2}} \int_s^{s+C} K_s(s') \beta(s')^{3/2} \left[\frac{\sin(\phi + \psi(s') - \psi(s) - \pi\nu)}{\sin(\pi\nu)} + \frac{\sin 3(\phi + \psi(s') - \psi(s) - \pi\nu)}{3 \sin(3\pi\nu)} \right] ds'$$

- The single resonance accelerator Hamiltonian (Hagedorn (1957), Schoch (1957), Guignard (1976, 1978))

$$H(J_x, J_y, \phi_x, \phi_y, s) = \frac{1}{R}(\nu_x J_x + \nu_y J_y) + g_{n_x, n_y} \frac{2}{R} J_x^{\frac{k_x}{2}} J_y^{\frac{k_y}{2}} \cos(n_x \phi_x + n_y \phi_y + \phi_0 - p\theta)$$

with $g_{n_x, n_y} e^{i\phi_0} = g_{j, k, l, m; p}$

- From the generating function

$$F_r(\phi_x, \phi_y, \hat{J}_x, \hat{J}_y, s) = (n_x \phi_x + n_y \phi_y - p\theta) \hat{J}_x + \phi_y \hat{J}_y$$

the relationships between old and new variables are

$$\hat{\phi}_x = (n_x \phi_x + n_y \phi_y - p\theta), \quad J_x = n_x \hat{J}_x$$

$$\hat{\phi}_y = \phi_y, \quad J_y = n_y \hat{J}_x + \hat{J}_y$$

- The following Hamiltonian is obtained

$$\hat{H}(\hat{J}_x, \hat{J}_y, \hat{\phi}_x) = \frac{(n_x \nu_x + n_y \nu_y - p) \hat{J}_x + \hat{J}_y}{R} + g_{n_x, n_y} \frac{2}{R} (n_x \hat{J}_x)^{\frac{k_x}{2}} (n_y \hat{J}_x + \hat{J}_y)^{\frac{k_y}{2}} \cos(\hat{\phi}_x + \phi_0) \quad 86$$

- There are two integrals of motion
 - The Hamiltonian, as it is independent on “time”
 - The new action \hat{J}_y as the Hamiltonian is independent on $\hat{\phi}_y$
- The two invariants in the old variables are written as:

$$c_1 = \frac{J_x}{n_x} - \frac{J_y}{n_y}$$

$$c_2 = \left(\nu_x - \frac{p}{n_x + n_y}\right)J_x + \left(\nu_y - \frac{p}{n_x + n_y}\right)J_y + 2g_{n_x, n_y} J_x^{\frac{k_x}{2}} J_y^{\frac{k_y}{2}} \cos(n_x \phi_x + n_y \phi_y + \phi_0 - p\theta)$$

- Two cases can be distinguished
 - n_x, n_y have **opposite** sign, i.e. **difference** resonance, the motion is the one of an ellipse, so bounded
 - n_x, n_y have the **same** sign, i.e. **sum** resonance, the motion is the one of an hyperbola, so **not** bounded
- These are **first order** perturbation theory considerations
- The distance from the resonance is obtained as

$$\Delta = \frac{g_{n_x, n_y}}{R} J_x^{\frac{k_x-2}{2}} J_y^{\frac{k_y-2}{2}} (k_x n_x J_x + k_y n_y J_y)$$