13.) The "\( \Delta p / p \neq 0 \)" Problem

**ideal accelerator:** all particles will see the same accelerating voltage. 

\[ \Delta p / p = 0 \]

"nearly ideal" accelerator: van de Graaf

\[ \Delta p / p \approx 10^{-5} \]

"not-at-all ideal" accelerator: RF structures

\[ \Delta p / p \approx 10^{-3} \]
14.) Gradient Errors

Remember: Matrix in Twiss Form

Transfer Matrix from point „0“ in the lattice to point „s“:

\[
M(s) = \begin{pmatrix}
\sqrt{\frac{\beta_s}{\beta_0}} (\cos \psi_s + \alpha_0 \sin \psi_s) & \sqrt{\beta_s \beta_0} \sin \psi_s \\
(\alpha_0 - \alpha_s) \cos(\psi_s - (1 + \alpha_0 \alpha_s) \sin \psi_s) & \sqrt{\beta_s \beta_0} (\cos(\psi_s - \alpha_0 \sin \psi_s))
\end{pmatrix}
\]

For one complete turn the Twiss parameters have to obey periodic boundary conditions:

\[
\beta(s + L) = \beta(s) \\
\alpha(s + L) = \alpha(s) \\
\gamma(s + L) = \gamma(s)
\]

\[
M(s) = \begin{pmatrix}
\cos \psi_{\text{turn}} + \alpha_s \sin \psi_{\text{turn}} & \beta_s \sin \psi_{\text{turn}} \\
- \gamma_s \sin \psi_s & \cos \psi_{\text{turn}} - \alpha_s \sin \psi_{\text{turn}}
\end{pmatrix}
\]
Introduce Quadrupole Error in the Lattice

optics perturbation described by thin lens quadrupole

\[
M_{\text{dist}} = M_{\Delta k} \cdot M_0 = \left( \begin{array}{cc}
1 & 0 \\
\Delta kds & 1 \\
\end{array} \right) \cdot \left( \begin{array}{cc}
\cos \psi_{\text{turn}} + \alpha \sin \psi_{\text{turn}} & \beta \sin \psi_{\text{turn}} \\
-\gamma \sin \psi_{\text{turn}} & \cos \psi_{\text{turn}} - \alpha \sin \psi_{\text{turn}} \\
\end{array} \right)
\]

\text{quad error} \quad \text{ideal storage ring}

\[
M_{\text{dist}} = \left( \begin{array}{cc}
\cos \psi_0 + \alpha \sin \psi_0 & \beta \sin \psi_0 \\
\Delta kds (\cos \psi_0 + \alpha \sin \psi_0) - \gamma \sin \psi_0 & \Delta kds \beta \sin \psi_0 + \cos \psi_0 - \alpha \sin \psi_0 \\
\end{array} \right)
\]

rule for getting the tune

\[
\text{Trace}(M) = 2 \cos \psi = 2 \cos \psi_0 + \Delta kds \beta \sin \psi_0
\]

Bernhard Holzer, CAS
Quadrupole error $\rightarrow$ Tune Shift

\[
\psi = \psi_0 + \Delta \psi \quad \Rightarrow \quad \cos(\psi_0 + \Delta \psi) = \cos \psi_0 + \frac{\Delta k ds \beta \sin \psi_0}{2}
\]

Remember the old-fashioned trigonometric stuff and assume that the error is small !!!

\[
\cos \psi_0 \cos \Delta \psi - \sin \psi_0 \sin \Delta \psi = \cos \psi_0 + \frac{k ds \beta \sin \psi_0}{2}
\]

\[
\Delta \psi = \frac{k ds \beta}{2}
\]

And referring to $Q$ instead of $\psi$:

\[
\psi = 2\pi Q
\]

\[
\Delta Q = \int_{s_0}^{s_0+1} \frac{\Delta k(s) \beta(s) ds}{4\pi}
\]

$\beta$ is a measure for the sensitivity of the beam.

Field quality, power supply tolerances etc are much tighter at places where $\beta$ is large.

$\beta$ is proportional to the $\beta$-function at the quadrupole.

Mini beta quads: $\beta \approx 1900$ m
Arc quads: $\beta \approx 80$ m
A quadrupol error leads to a shift of the tune
... and this can be used to measure the \( \beta \)-function

\[
\Delta Q = \int_{s_0}^{s_0+l} \frac{\Delta k \beta(s)}{4\pi} ds \approx \frac{\Delta k l_{quad} \bar{\beta}}{4\pi}
\]

Example: measurement of \( \beta \) in a storage ring: tune spectrum

Beyond that: without proof (e.g. CERN-94-01)
A quadrupole error will always lead to a tune shift, but in addition to a change of the beta–function.

\[
\Delta \beta(s) = \frac{\beta(s)}{2\sin(2\pi Q)} \int \beta(\tilde{s}) \Delta k(\tilde{s}) \cos(2|\psi(s) - \psi(\tilde{s})| - \pi Q) d\tilde{s}
\]

As before, the effect of the error depends on the \( \beta \)-function at the observation point as well as at the place of the error itself, on the error strength and there is again a resonance denominator \( \rightarrow \) half integer tunes are forbidden.

Bernhard Holzer, CAS
15.) Chromaticity:
A Quadrupole Error for \( \Delta p/p \neq 0 \)

Influence of external fields on the beam: prop. to magn. field & prop. zu 1/p

\[
dipole\ magnet \quad \alpha = \frac{\int B \, dl}{p/e}
\]

\[
focusing\ lens \quad k = \frac{g}{p/e}
\]

\[
x_D(s) = D(s) \frac{\Delta p}{p}
\]

Bernhard Holzer, CAS
Chromaticity: $Q'$

\[ k = \frac{g}{p/e} \quad p = p_0 + \Delta p \]

in case of a momentum spread:

\[ k = \frac{eg}{p_0 + \Delta p} \approx \frac{e}{p_0} (1 - \frac{\Delta p}{p_0}) g = k_0 + \Delta k \]

\[ \Delta k = -\frac{\Delta p}{p_0} k_0 \]

… which acts like a quadrupole error in the machine and leads to a tune spread:

\[ \Delta Q = -\frac{1}{4\pi} \frac{\Delta p}{p_0} k_0 \beta(s) ds \]

definition of chromaticity:

\[ \Delta Q = Q' \frac{\Delta p}{p} \quad ; \quad Q' = -\frac{1}{4\pi} \int k(s) \beta(s) ds \]
... what is wrong about Chromaticity:

$Q'$ is a number indicating the size of the tune spot in the working diagram, $Q'$ is always created if the beam is focussed

→ it is determined by the focusing strength $k$ of all quadrupoles

$$Q' = -\frac{1}{4\pi} \int \beta(s)k(s)ds$$

$k =$ quadrupole strength

$\beta =$ betafunction indicates the beam size … and even more the sensitivity of the beam to external fields

Example: LHC

$Q' = -250$

$\Delta p/p = +/- 0.2 \times 10^{-3}$

$\Delta Q = 0.256 \ldots 0.36$

→Some particles get very close to resonances and are lost

in other words: the tune is not a point

it is a pancake

Ideal situation: cromaticity well corrected ( $Q' \approx 1$ )

Tune signal for a nearly uncompensated cromaticity ( $Q' \approx 20$ )

Bernhard Holzer, CAS
Tune and Resonances

\[ m \times Q_x + n \times Q_y + l \times Q_s = \text{integer} \]

Tune diagram up to 3rd order

... and up to 7th order

Homework for the operateurs:
find a nice place for the tune
where against all probability
the beam will survive

Bernhard Holzer, CAS
**Sextupole Magnets:**

1. sort the particles according to their momentum

\[ x_D(s) = D(s) \frac{\Delta p}{p} \]

2. apply a magnetic field that rises quadratically with \( x \) (sextupole field)

\[
B_x = \tilde{g} x y \\
B_y = \frac{1}{2} \tilde{g}(x^2 - y^2)
\]

\[
\frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x} = \tilde{g} x
\]

Correlation of \( Q' \)

\[ Q'_{\text{cell}_{-x}} = \frac{-1}{4\pi} \left( k_{qf} \tilde{\beta}_x l_{qf} - k_{qd} \tilde{\beta}_x l_{qd} \right) + \frac{1}{4\pi} \sum_{F_{\text{sext}}} k_2^F l_{\text{sext}} D_x^F \beta_x^F - \frac{1}{4\pi} \sum_{D_{\text{sext}}} k_2^D l_{\text{sext}} D_x^D \beta_x^D \]

\[ Q'_{\text{cell}_{-y}} = \frac{-1}{4\pi} \left( -k_{qf} \tilde{\beta}_y l_{qf} + k_{qd} \tilde{\beta}_y l_{qd} \right) - \frac{1}{4\pi} \sum_{F_{\text{sext}}} k_2^F l_{\text{sext}} D_x^F \beta_y^F + \frac{1}{4\pi} \sum_{D_{\text{sext}}} k_2^D l_{\text{sext}} D_x^D \beta_y^D \]

**Sextupole Magnets:**

normalised quadrupole strength:

\[ k_{\text{sext}} = \frac{\tilde{g} x}{p / e} = m_{\text{sext}} \cdot x \]

\[ k_{\text{sext}} = m_{\text{sext}} \cdot D \frac{\Delta p}{p} \]

Bernhard Holzer, CAS
Chromaticity in a FODO Cell

\[ Q' = \frac{-1}{4\pi} \int k(s) \beta(s) \, ds \]

**β-Function in a single FoDo**

\[ \hat{\beta} = \frac{(1 + \sin \frac{\psi_{cell}}{2})L}{\sin \psi_{cell}} \quad \bar{\beta} = \frac{(1 - \sin \frac{\psi_{cell}}{2})L}{\sin \psi_{cell}} \]

\[ Q' = \frac{-1}{4\pi} \int k(s) \beta(s) \, ds \]

\[ Q' = \frac{-1}{4\pi} \int \frac{\hat{\beta} - \bar{\beta}}{f_Q} \]

\[ Q' = \frac{-1}{4\pi} \int \frac{1}{f_Q} \left\{ \frac{L(1 + \sin \frac{\psi_{cell}}{2}) - L(1 - \sin \frac{\psi_{cell}}{2})}{\sin \psi_{cell}} \right\} \]
using some TLC transformations ... $Q'$ can be expressed in a very simple form:

$$Q' = \frac{-1}{4\pi} \cdot \frac{1}{f_Q} \cdot \frac{2L \sin \frac{\psi_{cell}}{2}}{\sin \psi_{cell}}$$

$$Q' = \frac{-1}{4\pi} \cdot \frac{1}{f_Q} \cdot \frac{L \sin \frac{\psi_{cell}}{2}}{\sin \frac{\psi_{cell}}{2} \cos \frac{\psi_{cell}}{2}}$$

$$Q'_{cell} = \frac{-1}{4\pi f_Q} \cdot \frac{L \tan \frac{\psi_{cell}}{2}}{\sin \frac{\psi_{cell}}{2}}$$

and so we have to power the sextupoles properly ...

$$\Delta Q'_x = \frac{-1}{4\pi} \left\{ k^F_x l_s D^F_x \beta^F_x - k^D_x l_s D^D_x \beta^D_x \right\}$$

$$\Delta Q'_y = \frac{-1}{4\pi} \left\{ -k^F_x l_s D^F_y \beta^F_y + k^D_x l_s D^D_y \beta^D_y \right\}$$

Bernhard Holzer, CAS
5.) Lattice Design: Insertions

... the most complicated one: the drift space

Question to the auditorium: what will happen to the beam parameters $\alpha$, $\beta$, $\gamma$ if we stop focusing for a while ...?

$$
\begin{pmatrix}
\beta \\
\alpha \\
\gamma
\end{pmatrix}_s =
\begin{pmatrix}
C^2 & -2SC & S^2 \\
-CC' & SC' + S'C & -SS' \\
C'^2 & -2S'C' & S'^2
\end{pmatrix}
\begin{pmatrix}
\beta \\
\alpha \\
\gamma
\end{pmatrix}_0
$$

transfer matrix for a drift: $M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$

$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$ 
$\alpha(s) = \alpha_0 - \gamma_0 s$ 
$\gamma(s) = \gamma_0$

„0“ refers to the position of the last lattice element 
„s“ refers to the position in the drift

Bernhard Holzer, CAS
**location of the waist:**

given the initial conditions $\alpha_0, \beta_0, \gamma_0$: where is the point of smallest beam dimension in the drift ... or at which location occurs the beam waist?

beam waist:

$$\alpha(s) = 0 \quad \Rightarrow \quad \alpha_0 = \gamma_0 \cdot s$$

beam size at that position:

$$\gamma(l) = \gamma_0$$

$$\alpha(l) = 0$$

$$\rightarrow \quad \gamma(l) = \frac{1 + \alpha^2(l)}{\beta(l)} = \frac{1}{\beta(l)}$$

$$\beta(l) = \frac{1}{\gamma_0}$$

Bernhard Holzer, CAS
\textbf{β-Function in a Drift:}

let’s assume we are at a symmetry point in the center of a drift.

\[ \beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2 \]

as \( \alpha_0 = 0, \rightarrow \gamma_0 = \frac{1 + \alpha_0^2}{\beta_0} = \frac{1}{\beta_0} \)

and we get for the \( \beta \) function in the neighborhood of the symmetry point

\[ \beta(s) = \beta_0 + \frac{s^2}{\beta_0} \]

\[ !!! \]

\textit{Nota bene:}
1.) this is very bad !!!
2.) this is a direct consequence of the conservation of phase space density (... in our words: \( \varepsilon = \text{const} \) ... and there is no way out.
3.) Thank you, Mr. Liouville !!!
A bit more in detail: $\beta$-Function in a Drift

If we cannot fight against Liouville theorem ... at least we can optimise

Optimisation of the beam dimension:

$$\beta(l) = \beta_0 + \frac{1^2}{\beta_0}$$

Find the $\beta$ at the center of the drift that leads to the lowest maximum $\beta$ at the end:

$$\frac{d \hat{\beta}}{d \beta_0} = 1 - \frac{1^2}{\beta_0^2} = 0$$

$\Rightarrow \beta_0 = 1$

$\Rightarrow \hat{\beta} = 2\beta_0$

If we choose $\beta_0 = \ell$ we get the smallest $\beta$ at the end of the drift and the maximum $\beta$ is just twice the distance $\ell$
... and why all that ??

High Light of the HEP-Year 2012 / 13 naturally the HIGGS

ATLAS event display: Higgs => two electrons & two muons

Bernhard Holzer, CAS
Problem: Our particles are VERY small!!

Overall cross section of the Higgs:

\[ \Sigma_{\text{react}} \approx 1 \text{ pb} \]

The particles are indeed “very small”

\[ 1 \text{b} = 10^{-24} \text{cm}^2 \]

\[ 1 \text{pb} = 10^{-12} \times 10^{-24} \text{cm}^2 = 1/\text{mio} \times 1/\text{mio} \times 1/\text{mio} \times 1/\text{mio} \times 1/\text{mio} \times 1/10000 \text{mm}^2 \]

The only chance we have: compress the transverse beam size … at the IP

During collider run we had in Run 1 …

1400 bunches circulating, with 800 Mio proton collisions per second in the experiments and collected only 450 Higgs particles in three years.

LHC typical:

\[ \sigma = 0.1 \text{ mm} \Rightarrow 16 \mu\text{m} \]

Bernhard Holzer, CAS
6.) **Luminosity & Minibeta Insertion**

\[
R = L \times \Sigma_{\text{react}}
\]

\[
L = \frac{1}{4\pi e^2 f_0 n_b} \times \frac{I_{p1} I_{p2}}{\sigma_x \sigma_y}
\]

**Example: Luminosity run at LHC**

- \(\beta_{x,y} = 0.55 \text{ m}\)
- \(f_0 = 11.245 \text{ kHz}\)
- \(\epsilon_{x,y} = 5 \times 10^{-10} \text{ rad m}\)
- \(n_b = 2808\)
- \(\sigma_{x,y} = 17 \mu m\)
- \(I_p = 584 \text{ mA}\)

\[
L = 1.0 \times 10^{34} \text{ \frac{1}{cm^2 s}}
\]

The production rate of events is determined by the cross section \(\Sigma_{\text{react}}\) and the luminosity, which is given by the design of the accelerator.

Bernhard Holzer, CAS
The LHC Mini-Beta-Insertions

\[ \beta(s) = \beta_0 + \frac{s^2}{\beta_0} \]

LHC mini β optics
**Mini-β Insertions: A look into phase space**

A mini-β insertion is always a kind of special symmetric drift space.

\[ \frac{\alpha^*}{\epsilon} = 0 \]

\[ \gamma^* = 1 + \frac{\alpha^2}{\beta^*} = \frac{1}{\beta^*} \]

\[ \sigma^* = \sqrt{\frac{\epsilon}{\beta^*}} \]

\[ \beta^* = \frac{\sigma^*}{\sigma_r^*} \]

at a symmetry point \( \beta \) is just the ratio of beam dimension and beam divergence.

Bernhard Holzer, CAS
**Mini-β Insertions: Phase advance**

By definition the *phase advance* is given by:

\[ \Phi(s) = \int \frac{1}{\beta(s)} \, ds \]

**Now in a mini β insertion:**

\[ \beta(s) = \beta_0 \left( 1 + \frac{s^2}{\beta_0^2} \right) \]

\[ \rightarrow \Phi(s) = \frac{1}{\beta_0} \int_0^L \frac{1}{\frac{1}{\beta_0^2} + \frac{s^2}{\beta_0^2}} \, ds = \arctan \frac{L}{\beta_0} \]

---

**Consider the length of the drift spaces on both sides of the IP:**

*the phase advance of a mini β insertion is always close to π*

*in other words: the tune will increase by half an integer.*
Are there any problems?

Sure there are...

* large $\beta$ values at the doublet quadrupoles $\Rightarrow$ large contribution to chromaticity $Q'$ ... and no local correction (... why not ???)

$$Q' = \frac{-1}{4\pi} \int K(s)\beta(s) ds$$

* aperture of mini $\beta$ quadrupoles limit the luminosity

beam envelope at the first mini $\beta$ quadrupole lens in the HERA proton storage ring

* field quality and magnet stability most critical at the high $\beta$ sections

effect of a quad error:

$$\Delta Q = \int_{s_0}^{s_0+L} \frac{\Delta K(s)\beta(s) ds}{4\pi}$$

$$\Delta \beta(s) = \frac{\beta(s)}{2\sin(2\pi Q)} \int \beta(\bar{s}) \Delta k(\bar{s}) \cos(2|\psi(s) - \psi(\bar{s})| - \pi Q) d\bar{s}$$

$\Rightarrow$ keep distance ,,s“ to the first mini $\beta$ quadrupole as small as possible
**Mini-β Insertions: some guide lines**

* calculate the periodic solution in the arc
* introduce the drift space needed for the insertion device (detector ...)
* put a quadrupole doublet (or triplet ??) as close as possible
* introduce additional quadrupole lenses to match the beam parameters to the values at the beginning of the arc structure

parameters to be optimised & matched to the periodic solution:

\[
\begin{align*}
\alpha_x, \beta_x, & \quad D_x, D_x' \\
\alpha_y, \beta_y, & \quad Q_x, Q_y
\end{align*}
\]

8 individually powered quad magnets are needed to match the insertion (... at least)

Bernhard Holzer, CAS
7.) **Dispersion Suppressors**

There are two rules of paramount importance about dispersion:

- It is nasty
- It is not easy to get rid of it.

**remember:** oscillation amplitude for a particle with momentum deviation

\[
x(s) = x_p(s) + D(s) \frac{\Delta p}{p}
\]

**beam size at the IP**

\[
\sigma^* = 17 \mu m
\]

**dispersion trajectory**

\[
D = 1.5 m \quad \frac{\Delta p}{p} \approx 1.1 \times 10^{-4} \quad x_D = 165 \mu m
\]

**Dispersion in a FoDo cell with Dipoles:**

(proof see appendix)

\[
D(s) = S(s) * \int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(s) * \int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}
\]

\[
\hat{D} = \frac{\ell^2}{\rho} \left( \frac{1 + \frac{1}{2} \sin \psi_{cell}}{\sin^2 \frac{\psi_{cell}}{2}} \right) \quad \tilde{D} = \frac{\ell^2}{\rho} \left( \frac{1 - \frac{1}{2} \sin \psi_{cell}}{\sin^2 \frac{\psi_{cell}}{2}} \right)
\]

Bernhard Holzer, CAS
**Dispersion Suppressor Schemes**

There are some locations in the ring, where the dispersion has to vanish,

* at the **IP**, to avoid unnecessary increase of the beam size
* at the **RF** to avoid unwanted coupling between transv. and long. oscillations
* at the injection / extraction points etc

The way the trick goes: ... we turn it the other way round

Starting from \( D = D' = 0 \), we create dispersion in combining the dispersive effect of the dipoles and the phase advance at their location (defined by the quadrupoles) in such a way to get the \( D, D' \) values of the periodic arc

**Three major schemes:**

1.) The straight forward one: **Quadrupole based Dispersion suppressor**

use additional quadrupole lenses to match the optical parameters ... including the \( D(s), D'(s) \) terms

* Dispersion **suppressed** by 2 quadrupole lenses,
* \( \beta \) and \( \alpha \) **restored** to the values of the periodic solution by 4 additional quadrupoles

\[
D(s), \ D'(s), \quad \begin{align*}
\beta_x(s), \alpha_x(s) \\
\beta_y(s), \alpha_y(s)
\end{align*}
\rightarrow \quad 6 \text{ additional quadrupole lenses required}
\]
Dispersion Suppressor Quadrupole Scheme

Advantage:

- easy,
- flexible: it works for any phase advance per cell
- does not change the geometry of the storage ring,
- can be used to match between different lattice structures (i.e. phase advances)

Disadvantage:

- additional power supplies needed (→ expensive)
- requires stronger quadrupoles
- due to higher $\beta$ values: more aperture required
8.) The Missing Bend Dispersion Suppressor

… turn it the other way round:

Start with \( D(s) = 0, \quad D'(s) = 0 \)

and create dispersion – using dipoles - in such a way, that it fits exactly the conditions at the centre of the first regular arc cells:

\[
D(s) = S(s) \int \frac{1}{\rho(\bar{s})} C(\bar{s}) \, d\bar{s} - C(s) \int \frac{1}{\rho(\bar{s})} S(\bar{s}) \, d\bar{s} \quad \rightarrow \quad \hat{D} = \frac{\ell^2}{\rho} \left( \frac{1 + \frac{1}{2} \sin^2 \psi_{cell}}{\sin^2 \frac{\psi_{cell}}{2}} \right), \quad D' = 0
\]

Depending on the phase advance, add at the end of the arc:

\( m \) cells without dipoles

followed by

\( n \) regular arc cells.
The Missing Bend Dispersion Suppressor

conditions for the (missing) dipole field scheme:

\[ \frac{2m + n}{2} \Phi_C = (2k + 1) \frac{\pi}{2} \]

\[ \sin \frac{n\Phi_C}{2} = \frac{1}{2}, \ k = 0, 2 \ \ldots \ \text{or} \]

\[ \sin \frac{n\Phi_C}{2} = \frac{-1}{2}, \ k = 1, 3 \ \ldots \]

Cooking Recipe:

At the end of the arc we add \( m \) cells without dipoles followed by \( n \) regular arc cells.

Example:

phase advance in the arc \( \Phi_C = 60^\circ \)

number of suppr. cells \( m = 1 \)

number of regular cells \( n = 1 \)
9.) The **Half Bend Dispersion Suppressor**

at the end of the arc cells we add a number of “*n*” additional cells, with different dipole strength.

depending on the phase advance per cell different possibilities exist with the general condition for vanishing dispersion

\[ 2 \delta_{\text{supr}} \sin^2 \left( \frac{n \Phi_c}{2} \right) = \delta_{\text{arc}} \]

\( \delta_{\text{arc}} = \) dipole strength in the arc
\( \delta_{\text{supr}} = \) dipole strength in the suppressor cells
\( n = \) number of suppressor cells
\( \Phi_c = \) phase advance of the cells

(proof see appendix)

so if we require

\[ \delta_{\text{supr}} = \frac{1}{2} \delta_{\text{arc}} \]

which means we install dipoles of half the arc strength

we get

\[ \sin^2 \left( \frac{n \Phi_c}{2} \right) = 1 \]

and equivalent for \( D' = 0 \)

\[ \sin(n \Phi_c) = 0 \quad n \Phi_c = k \pi, \quad k = 1, 3, ... \]
**The Half Bend Dispersion Suppressor**

Combining these two conditions

\[
\sin^2 \left( \frac{n \Phi_c}{2} \right) = 1, \quad \sin(n \Phi_c) = 0
\]

The phase advance in the \( n \) suppressor cells has to accumulate to an odd multiple of \( \pi \)

\[
n \Phi_c = k \pi, \quad k = 1, 3, ...
\]

**Example:**

- Phase advance in the arc \( \Phi_c = 60^\circ \)
- Number of suppressor cells \( n = 3 \)

- Phase advance in the arc \( \Phi_c = 90^\circ \)
- Number of suppressor cells \( n = 2 \)

**Strength of suppressor dipoles is half as strong as that of arc dipoles,**

\[
\delta_{\text{suppr}} = \frac{1}{2} \delta_{\text{arc}}
\]
**Dispersion Suppressor: dipole based schemes**

**Advantage:**
- elegant
- does not affect the beam optics
- no additional quadrupoles required
- no impact on aperture
- lower field dipoles are often required anyway

**Disadvantage:**
- changes the geometry of the machine
  - think first, digg later
- requires different dipoles
  - (e.g. shorter, or weaker)
- limits the achievable beam momentum
  \[ B^*p = p/q \]
- works for specific phase advances per cell

Bernhard Holzer, CAS
Resume ‘

1.) Dispersion in a FoDo cell:
small dispersion ↔ large bending radius
short cells
strong focusing

\[ \hat{D} = \frac{\ell^2}{\rho} \left( 1 + \frac{1}{2} \sin \frac{\psi_{cell}}{2} \right) \]
\[ \tilde{D} = \frac{\ell^2}{\rho} \left( 1 - \frac{1}{2} \sin \frac{\psi_{cell}}{2} \right) \]

2.) Chromaticity of a cell:
small Q’ ↔ weak focusing
small \( \beta \)

\[ Q_{total}' = \frac{-1}{4\pi} \oint \left\{ K(s) - mD(s) \right\} \beta(s) ds \]

3.) Position of a waist at the cell end:
\( \alpha_0, \beta_0 = \text{values at the end of the cell} \)

\[ l = \frac{\alpha_0}{\gamma_0} \]
\[ \beta(1) = \frac{1}{\gamma_0} \]

4.) \( \beta \) function in a drift

\[ \beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2 \]

5.) Mini \( \beta \) insertion

\[ \beta(s) = \beta_0 + \frac{s^2}{\hat{\beta}_0} \]

Bernhard Holzer, CAS
Appendix I: The Beam Matrix and the Mini-Beta

„Once more unto the breach dear friends:“

Transformation of Twiss parameters

just because it is mathematical more elegant ...

let ‘s define a beam matrix: \( B_0 = \begin{pmatrix} \beta_0 & -\alpha_0 \\ \alpha_0 & \gamma_0 \end{pmatrix} \)

and a orbit vector: \( X_0 = \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}, \; X^T = (x_0, x'_0) \)

the product \( {X_0}^T \cdot B_0^{-1} \cdot X_0 = (x_0, x'_0) \cdot \begin{pmatrix} \gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0 \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} = \gamma_0 x_0^2 + 2\alpha_0 x_0 x'_0 + \beta_0 x'_0^2 = \epsilon \)

transformation of the orbit vector: \( X_1 = M \cdot X_0 \)

... is constant

and so we get: \( \epsilon = {X_0}^T \cdot B_0^{-1} \cdot X_0 = {X_0}^T \cdot M^T (M^T)^{-1} \cdot B_0^{-1} \cdot M^{-1} M \cdot X_0 \)
\( = {X_0}^T \cdot M^T \left\{ (M^T)^{-1} \cdot B_0^{-1} \cdot M^{-1} \right\} \cdot M \cdot X_0 \)

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Transformation of Twiss parameters

and using \( A^T B^T = (BA)^T \) and \( A^{-1} B^{-1} = (BA)^{-1} \)

\[
\begin{align*}
\varepsilon &= X_0^T M^T \left\{ (M^T)^{-1} (MB_0)^{-1} \right\} M X_0 \\
&= X_0^T M^T \left\{ MB_0 M^T \right\}^{-1} M X_0 \\
&= (MX_0)^T \left\{ MB_0 M^T \right\}^{-1} MX_0 \\
&= X_1^T \left\{ MB_0 M^T \right\}^{-1} X_1
\end{align*}
\]

but we know already that

\[
\varepsilon = \text{const} = X_0^T B_0^{-1} X_0 = X_1^T B_1^{-1} X_1
\]

and in the end and after all we learn that ...

\[
B_1 = \begin{pmatrix} \beta_1 & -\alpha_1 \\ \alpha_1 & \gamma_1 \end{pmatrix} = M * B_0 * M^T
\]

\[
\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{12}m_{21} + m_{22}m_{11} & -m_{12}m_{22} \\ m_{12}^2 & -2m_{22}m_{21} & m_{22}^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}
\]

in full equivalence to ...

Bernhard Holzer, CAS
Transformation of Twiss parameters

Example again the drift space

... starting from \( \alpha_0 = 0 \)

\[
M = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}
\]

\[
B_0 = \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} = \begin{pmatrix} \beta_0 & 0 \\ 0 & 1/\beta_0 \end{pmatrix}
\]

Beam parameters after the drift: \( B_1 = MB_0M^T = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \ast \begin{pmatrix} \beta_0 & 0 \\ 0 & 1/\beta_0 \end{pmatrix} \ast \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \)

\[
= \begin{pmatrix} \beta_0 + \frac{s^2}{\beta_0} & s \\ \frac{s}{\beta_0} & \frac{1}{\beta_0} \end{pmatrix}
\]

\( \beta_1 = \beta_0 + \frac{s^2}{\beta_0} \)
Appendix II: Dispersion

... solution of the inhomogenous equation of motion

The dispersion function is given by

\[ D(s) = S(s) * \int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(s) * \int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s} \]

Proof:

\[ D'(s) = S'(s) * \int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} + S(s) * \frac{C(s)}{\rho(s)} - C'(s) * \int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s} - C(s) \frac{S(\tilde{s})}{\rho(\tilde{s})} \]

\[ D'(s) = S'(s) * \int \frac{C}{\rho} d\tilde{s} - C(s) * \int \frac{S}{\rho} d\tilde{s} \]

\[ D''(s) = S''(s) * \int \frac{C}{\rho} d\tilde{s} + S'(s) * \frac{C'(s)}{\rho} - C''(s) * \frac{S'}{\rho} \]

\[ D''(s) = S''(s) * \int \frac{C}{\rho} d\tilde{s} - C''(s) * \frac{1}{\rho} (CS' - SC') \]

\[ = \text{det}(M) = 1 \]

\[ D''(s) = S''(s) * \int \frac{C}{\rho} d\tilde{s} - C''(s) * \int \frac{S}{\rho} d\tilde{s} + \frac{1}{\rho} \]

Now the principal trajectories \( S \) and \( C \) fulfill the homogeneous equation

\[ S''(s) = -K * S, \quad C''(s) = -K * C \]
and so we get:

\[ D''(s) = -K \cdot S(s) \cdot \int \frac{C}{\rho} \, d\bar{s} + K \cdot C(s) \cdot \int \frac{S}{\rho} \, d\bar{s} + \frac{1}{\rho} \]

\[ D''(s) = -K \cdot D(s) + \frac{1}{\rho} \]

\[ D''(s) + K \cdot D(s) = \frac{1}{\rho} \]

\[ \text{qed.} \]
Appendix III: Dispersion Suppressors

... the calculation of the half bend scheme in full detail (for purists only)

1.) the lattice is split into 3 parts: \textit{(Gallia divisa est in partes tres)}

* periodic solution of the arc \hspace{1cm} \text{periodic } \beta, \text{ periodic dispersion } D
* section of the dispersion suppressor \hspace{1cm} \text{periodic } \beta, \text{ dispersion vanishes}
* FoDo cells without dispersion \hspace{1cm} \text{periodic } \beta, D = D' = 0
2.) calculate the dispersion D in the periodic part of the lattice

transfer matrix of a periodic cell:

$$M_{0\rightarrow S} = \begin{pmatrix}
\sqrt{\frac{\beta_s}{\beta_0}} (\cos \phi + \alpha_0 \sin \phi) & \sqrt{\beta_s \beta_0} \sin \phi \\
(\alpha_0 - \alpha_s) \cos \phi - (1 + \alpha_0 \alpha_s) \sin \phi & \sqrt{\beta_s \beta_0} (\cos \phi - \alpha_s \sin \phi)
\end{pmatrix}$$

for the transformation from one symmetry point to the next (i.e. one cell) we have:

$$\Phi_C = \text{phase advance of the cell, } \alpha = 0 \text{ at a symmetry point. The index } "c" \text{ refers to the periodic solution of one cell.}$$

$$M_{cell} = \begin{pmatrix}
C & S & D \\
C' & S' & D' \\
0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
\cos \Phi_C & \beta_C \sin \Phi_C & D(l) \\
\frac{-1}{\beta_C} \sin \Phi_C & \cos \Phi_C & D'(l) \\
0 & 0 & 1
\end{pmatrix}$$

The matrix elements D and D' are given by the C and S elements in the usual way:

$$D(l) = S(l) * \int_0^l \frac{1}{\rho(s)} C(s) ds - C(l) * \int_0^l \frac{1}{\rho(s)} S(s) ds$$

$$D'(l) = S'(l) * \int_0^l \frac{1}{\rho(s)} C(s) ds - C'(l) * \int_0^l \frac{1}{\rho(s)} S(s) ds$$
here the values $C(l)$ and $S(l)$ refer to the symmetry point of the cell (middle of the quadrupole) and the integral is to be taken over the dipole magnet where $\rho \neq 0$. For $\rho = \text{const}$ the integral over $C(s)$ and $S(s)$ is approximated by the values in the middle of the dipole magnet.

Transformation of $C(s)$ from the symmetry point to the center of the dipole:

$$C_m = \sqrt{\frac{\beta_m}{\beta_c}} \cos \Delta \Phi = \sqrt{\frac{\beta_m}{\beta_c}} \cos \left( \frac{\Phi_c}{2} \pm \varphi_m \right) \quad S_m = \sqrt{\beta_m \beta_c} \sin \left( \frac{\Phi_c}{2} \pm \varphi_m \right)$$

where $\beta_c$ is the periodic $\beta$ function at the beginning and end of the cell, $\beta_m$ its value at the middle of the dipole and $\varphi_m$ the phase advance from the quadrupole lens to the dipole center.

Now we can solve the integral for $D$ and $D'$:

$$D(l) = S(l) \ast \int_0^l \frac{1}{\rho(\bar{s})} C(\bar{s}) \, d\bar{s} - C(l) \ast \int_0^l \frac{1}{\rho(\bar{s})} S(\bar{s}) \, d\bar{s}$$

$$D(l) = \beta_c \sin \Phi_c \ast \frac{L}{\rho} \sqrt{\frac{\beta_m}{\beta_c}} \ast \cos \left( \frac{\Phi_c}{2} \pm \varphi_m \right) - \cos \Phi_c \ast \frac{L}{\rho} \sqrt{\beta_m \beta_c} \ast \sin \left( \frac{\Phi_c}{2} \pm \varphi_m \right)$$
\[ D(l) = \delta \sqrt{\beta_m \beta_c} \left\{ \sin \Phi_c \left[ \cos \left( \frac{\Phi_c}{2} + \varphi_m \right) + \cos \left( \frac{\Phi_c}{2} - \varphi_m \right) \right] - \right. \\
\left. - \cos \Phi_c \left[ \sin \left( \frac{\Phi_c}{2} + \varphi_m \right) + \sin \left( \frac{\Phi_c}{2} - \varphi_m \right) \right] \right\} \]

I have put \( \delta = \frac{L}{\rho} \) for the strength of the dipole.

**remember the relations**

\[
\cos x + \cos y = 2 \cos \frac{x + y}{2} \cos \frac{x - y}{2} \\
\sin x + \sin y = 2 \sin \frac{x + y}{2} \cos \frac{x - y}{2}
\]

\[ D(l) = \delta \sqrt{\beta_m \beta_c} \left\{ \sin \Phi_c \left[ 2 \cos \frac{\Phi_c}{2} \cos \varphi_m - \cos \Phi_c \right] \left[ 2 \sin \frac{\Phi_c}{2} \cos \varphi_m \right] \right\} \]

\[ D(l) = 2\delta \sqrt{\beta_m \beta_c} \cos \varphi_m \left\{ \sin \Phi_c \cos \frac{\Phi_c}{2} - \cos \Phi_c \sin \frac{\Phi_c}{2} \right\} \]

**remember:**

\[
\sin 2x = 2 \sin x \cdot \cos x \\
\cos 2x = \cos^2 x - \sin^2 x
\]

\[ D(l) = 2\delta \sqrt{\beta_m \beta_c} \cos \varphi_m \left\{ 2 \sin \frac{\Phi_c}{2} \cos^2 \frac{\Phi_c}{2} - \left( \cos^2 \frac{\Phi_c}{2} - \sin^2 \frac{\Phi_c}{2} \right) \sin \frac{\Phi_c}{2} \right\} \]

Bernhard Holzer, CAS
\[ D(l) = 2\delta \sqrt{\beta_m \beta_C} \cos \varphi_m \sin \Phi_C \times \left\{ \frac{2 \cos^2 \frac{\Phi_C}{2} - \cos^2 \frac{\Phi_C}{2} + \sin^2 \frac{\Phi_C}{2}}{2} \right\} \]

\[ D'(l) = 2\delta \sqrt{\beta_m / \beta_C} \cos \varphi_m \cos \Phi_C \]

in full analogy one derives the expression for \( D' \):

\[ D'(l) = 2\delta \sqrt{\beta_m / \beta_C} \cos \varphi_m \cos \frac{\Phi_C}{2} \]

As we refer the expression for \( D \) and \( D' \) to a periodic structure, namely a FoDo cell we require periodicity conditions:

\[
\begin{pmatrix}
D_C \\
D'_C \\
1
\end{pmatrix} = M_C \times \begin{pmatrix}
D_C \\
D'_C \\
1
\end{pmatrix}
\]

and by symmetry: \( D'_C = 0 \)

With these boundary conditions the Dispersion in the FoDo is determined:

\[ D_C \cos \Phi_C + \delta \sqrt{\beta_m \beta_C} \cos \varphi_m \times 2 \sin \frac{\Phi_C}{2} = D_C \]
(A1) \[ D_C = \delta \sqrt{\beta_m \beta_c} \cos \phi_m / \sin \frac{\Phi_c}{2} \]

This is the value of the periodic dispersion in the cell evaluated at the position of the dipole magnets.

3.) Calculate the dispersion in the suppressor part:

We will now move to the second part of the dispersion suppressor: The section where \( D = D' = 0 \) the dispersion is generated ... or turning it around where the Dispersion of the arc is reduced to zero.

The goal will be to generate the dispersion in this section in a way that the values of the periodic cell that have been calculated above are obtained.

\[ D(l) = S(l) \int_0^l \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(l) \int_0^l \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s} \]

Bernhard Holzer, CAS
as the dispersion is generated in a number of \( n \) cells the matrix for these \( n \) cells is

\[
M_n = M^\ast_n = \begin{pmatrix}
\cos n\Phi_c & \beta_c \sin n\Phi_c & D_n \\
-\frac{1}{\beta_c} \sin n\Phi_c & \cos n\Phi_c & D' \\
0 & 0 & 1
\end{pmatrix}
\]

\[
D_n = \beta_c \sin n\Phi_c \ast \delta_{\text{supr}} \ast \sum_{i=1}^{n} \cos(i\Phi_c - \frac{1}{2}\Phi_c \pm \varphi_m) \ast \frac{\sqrt{\beta_m}}{\beta_c} - \\
- \cos n\Phi_c \ast \delta_{\text{supr}} \ast \sum_{i=1}^{n} \sqrt{\beta_m} \beta_c \ast \sin(i\Phi_c - \frac{1}{2}\Phi_c \pm \varphi_m)
\]

\[
D_n = \sqrt{\beta_m} \beta_c \ast \sin n\Phi_c \ast \delta_{\text{supr}} \ast \sum_{i=1}^{n} \cos((2i - 1)\Phi_c \pm \varphi_m) - \sqrt{\beta_m} \beta_c \ast \delta_{\text{supr}} \ast \cos n\Phi_c \sum_{i=1}^{n} \sin((2i - 1)\Phi_c \pm \varphi_m)
\]

**remember:**

\[
\sin x + \sin y = 2 \sin \frac{x+y}{2} \ast \cos \frac{x-y}{2} \quad \cos x + \cos y = 2 \cos \frac{x+y}{2} \ast \cos \frac{x-y}{2}
\]

\[
D_n = \delta_{\text{supr}} \ast \sqrt{\beta_m} \beta_c \ast \sin n\Phi_c \ast \sum_{i=1}^{n} \cos((2i - 1)\Phi_c) \ast 2 \cos \varphi_m - \\
- \delta_{\text{supr}} \ast \sqrt{\beta_m} \beta_c \ast \cos n\Phi_c \sum_{i=1}^{n} \sin((2i - 1)\Phi_c) \ast 2 \cos \varphi_m
\]

Bernhard Holzer, CAS
\[ D_n = 2\delta_{\text{supr}} \sqrt{\beta_m \beta_c} \cos \Phi_m \left\{ \sum_{i=1}^{n} \cos(2(i-1)\frac{\Phi_c}{2}) \sin n\Phi_c - \sum_{i=1}^{n} \sin(2(i-1)\frac{\Phi_c}{2}) \cos n\Phi_c \right\} \]

\[ D_n = 2\delta_{\text{supr}} \sqrt{\beta_m \beta_c} \cos \Phi_m \left\{ \sin n\Phi_c \left[ \frac{\sin \frac{n\Phi_c}{2} \cos \frac{n\Phi_c}{2}}{\sin \frac{\Phi_c}{2}} \right] - \cos n\Phi_c \left[ \frac{\sin \frac{n\Phi_c}{2} \sin \frac{n\Phi_c}{2}}{\sin \frac{\Phi_c}{2}} \right] \right\} \]

\[ D_n = 2\delta_{\text{supr}} \sqrt{\beta_m \beta_c} \cos \Phi_m \left\{ \sin n\Phi_c \left[ \frac{\sin \frac{n\Phi_c}{2} \cos \frac{n\Phi_c}{2}}{\sin \frac{\Phi_c}{2}} \right] - \cos n\Phi_c \left[ \frac{\sin \frac{n\Phi_c}{2} \sin \frac{n\Phi_c}{2}}{\sin \frac{\Phi_c}{2}} \right] \right\} \]

set for more convenience \( x = n\Phi_c/2 \)

\[ D_n = \frac{2\delta_{\text{supr}} \sqrt{\beta_m \beta_c} \cos \Phi_m}{\sin \frac{\Phi_c}{2}} \left\{ \sin 2x \sin x \cos x - \cos 2x \sin^2 x \right\} \]

\[ D_n = \frac{2\delta_{\text{supr}} \sqrt{\beta_m \beta_c} \cos \Phi_m}{\sin \frac{\Phi_c}{2}} \left\{ 2 \sin x \cos x \sin x - (\cos^2 x - \sin^2 x) \sin^2 x \right\} \]
This expression gives the dispersion generated in a certain number of \( n \) cells as a function of the dipole kick \( \delta \) in these cells.
At the end of the dispersion generating section the value obtained for \( D(s) \) and \( D'(s) \) has to be equal to the value of the periodic solution:

\[
\text{equating (A1) and (A2) gives the conditions for the matching of the periodic dispersion in the arc to the values } D = D' = 0 \text{ after the suppressor.}
\]
\[ \rightarrow 2\delta_{\text{sup}} \sin^2\left(\frac{n\Phi_C}{2}\right) = \delta_{\text{arc}} \]
\[ \rightarrow \sin(n\Phi_C) = 0 \]
\[ \begin{align*}
\rightarrow \delta_{\text{sup}} &= \frac{1}{2} \delta_{\text{arc}} \\
\delta_{\text{sup}} &= \frac{1}{2} \delta_{\text{arc}}
\end{align*} \]

and at the same time the phase advance in the arc cell has to obey the relation:

\[ n\Phi_C = k \pi, \quad k = 1, 3, \ldots \]
Appendix IV: Dispersion in a FoDo Cell

equation of motion

\[ x'' + K(s) \cdot x = 0 \]

\[ K = -k + \sqrt{\rho^2} \]

single particle trajectory considering both planes

\[
\begin{pmatrix}
  x(s) \\
  x'(s) \\
  y(s) \\
  y'(s)
\end{pmatrix}
= M \cdot
\begin{pmatrix}
  x(s_0) \\
  x'(s_0) \\
  y(s_0) \\
  y'(s_0)
\end{pmatrix}
\]

e.g. matrix for a quadrupole lens:

\[
M_{foc} =
\begin{pmatrix}
  \cos(\sqrt{|k|}s) & \frac{1}{\sqrt{|k|}} \sin(\sqrt{|k|}s) & 0 & 0 \\
  -\sqrt{|k|} \sin(\sqrt{|k|}s) & \cos(\sqrt{|k|}s) & 0 & 0 \\
  0 & 0 & \cosh(\sqrt{|k|}s) & \frac{1}{\sqrt{|k|}} \sinh(\sqrt{|k|}s) \\
  0 & 0 & \sqrt{|k|} \sinh(\sqrt{|k|}s) & \cosh(\sqrt{|k|}s)
\end{pmatrix}
\]

\[
\begin{pmatrix}
  C_x & S_x & 0 & 0 \\
  C'_x & S'_x & 0 & 0 \\
  0 & 0 & C_y & S_y \\
  0 & 0 & C'_y & S'_y
\end{pmatrix}
\]
Dispersion

Momentum error:

\( \frac{\Delta p}{p} \neq 0 \)

\[ x'' + x \left( \frac{1}{\rho^2} - k \right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho} \]

General solution:

\[ x(s) = x_h(s) + x_i(s) \]

\[ D(s) = \frac{x_i(s)}{\Delta p/p} \]

\[ \begin{pmatrix} x \\ x' \\ \frac{\Delta p}{p} \end{pmatrix} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \frac{\Delta p}{p} \end{pmatrix}_s \]

\[ x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p} \]

Bernhard Holzer, CAS
**Dispersion**

the dispersion function $D(s)$ is (...obviously) defined by the focusing properties of the lattice and is given by:

$$D(s) = S(s) * \int \frac{1}{\rho(\vec{s})} C(\vec{s}) d\vec{s} - C(s) * \int \frac{1}{\rho(\vec{s})} S(\vec{s}) d\vec{s}$$

! weak dipoles $\rightarrow$ large bending radius $\rightarrow$ small dispersion

Example: Drift

$$M_D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad D(s) = S(s) * \int \frac{1}{\rho(\vec{s})} C(\vec{s}) d\vec{s} - C(s) * \int \frac{1}{\rho(\vec{s})} S(\vec{s}) d\vec{s}$$

$$= 0 \quad = 0$$

$\rightarrow M_D = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

...in similar way for quadrupole matrices,

!!! in a quite different way for dipole matrix (see appendix)

Bernhard Holzer, CAS
**Dispersion in a FoDo Cell:**

!! we have now introduced dipole magnets in the FoDo:
- we still neglect the weak focusing contribution $1/\rho^2$
- but take into account $1/\rho$ for the dispersion effect

assume: length of the dipole $= l_D$

Calculate the matrix of the FoDo half cell in thin lens approximation:

- in analogy to the derivations of $\hat{\beta}$, $\beta$
  - *thin lens approximation*: 
    
    $f = \frac{1}{kl_Q} \gg 1_Q$
  
  - *length of quad negligible* 
    
    $1_Q \approx 0$, \( \rightarrow 1_D = \frac{1}{2} L \)
  
  - *start at half quadrupole* 
    
    $\frac{1}{f'} = \frac{1}{2f'}$

Bernhard Holzer, CAS
Matrix of the half cell

\[ M_{\text{HalfCell}} = M_{QD} \frac{1}{2} * M_B * M_{QF} \frac{1}{2} \]

\[ M_{\text{HalfCell}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} * \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \]

\[ M_{\text{HalfCell}} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\frac{1}{f} & 1+\frac{1}{f} \end{pmatrix} \]

calculate the dispersion terms \( D, D' \) from the matrix elements

\[ D(s) = S(s) \int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(s) \int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s} \]
\[
D(\ell) = \ell \star \frac{1}{\rho} \int_{\ell}^0 (1 - \frac{s}{f}) \, ds - \left(1 - \frac{\ell}{f}\right) \frac{1}{\rho} \int_0^\ell s \, ds
\]

\[D(\ell) \frac{\ell^2}{2\rho} \]

and we get the complete matrix including the dispersion terms \(D, D'\)

\[
M_{\text{halfCell}} = \begin{pmatrix}
C & S & D \\
C' & S' & D' \\
0 & 0 & 1
\end{pmatrix}
\]
Dispersion in a FoDo Cell:

boundary conditions for the transfer from the center of the foc. to the center of the defoc. quadrupole

\[
\begin{pmatrix}
\sqrt{\beta} \\
0 \\
1
\end{pmatrix} = M_{1/2} \begin{pmatrix}
\hat{D} \\
0 \\
1
\end{pmatrix}
\]

\[
\hat{D} = \hat{D}(1 - \frac{\ell}{f^2}) + \frac{\ell^2}{2\rho}
\]

\[
0 = -\frac{\ell}{f^2} \hat{D} + \frac{\ell}{\rho} (1 + \frac{\ell}{2f})
\]

Bernhard Holzer, CAS

where $\psi_{cell}$ denotes the phase advance of the full cell and $l/f = \sin(\psi/2)$
Dispersion in a FoDo Cell

Nota bene:

! small dispersion needs strong focusing → large phase advance
!! ↔ there is an optimum phase for small β
!!! ...do you remember the stability criterion?
   ½ trace = cos ψ ↔ ψ < 180°
!!!! ...life is not easy

latest news: TLEP

E=175 GeV
C_0 = 100km, ψ=90°
l = 50m, ρ = 9400m, D = 14cm

\[
\hat{D} = \frac{l^2}{\rho} \frac{(1 + \frac{1}{2} \sin \frac{\psi_{cell}}{2})}{\sin^2 \frac{\psi_{cell}}{2}}
\]

Bernhard Holzer, CAS