BASICS OF RF ELECTRONICS
(… OR GETTING STARTED WITH LLRF BUILDING BLOCKS)

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Fixed attenuators are widely used in RF electronics to set the proper signal level in the various circuit branches. Proper level setting is crucial to fully exploit the instrumentation dynamic range and to avoid circuit overload and damaging.

Attenuators are also used as matching pads as they can be designed to connect lines of different impedances. Also, the insertion of attenuators in front of mismatched loads reduces the VSWR seen at the source side.

Fixed attenuators are mainly characterized by the following parameters:

- Attenuation ΔdB;
- Max average power rate;
- Max peak power rate;
- Frequency range;
- Attenuation flatness over the specified frequency range;
- VSWR, size and weight, performance over the given temperature range, ...
Fixed attenuators are passive, 2-ports devices generally made by a network of resistors with a very broadband frequency response (dc ÷ many GHz, typical). They are designed to provide both the required attenuation and matching of the input/output lines, which might have different characteristic impedances. The attenuation $\Delta dB$, expressed in dB units, and the linear transmission coefficient $\alpha$ are defined as:

$$\Delta dB = 10 \cdot \log\left(\frac{P_{in}}{P_{out}}\right); \quad \alpha = \sqrt{\frac{P_{out}}{P_{in}}} = 10^{-\left(\frac{\Delta dB}{20}\right)}$$

**T-type Attenuator**

For an unbalanced configuration:

- $R_A = \frac{1 + \alpha^2 - 2\alpha\sqrt{Z_{out}/Z_{in}}}{1 - \alpha^2} Z_{in}$
- $R_B = \frac{2\alpha}{1 - \alpha^2} \sqrt{Z_{in} \cdot Z_{out}}$
- $R_C = \frac{1 + \alpha^2 - 2\alpha\sqrt{Z_{in}/Z_{out}}}{1 - \alpha^2} Z_{out}$

For a balanced configuration:

- $R_A = R_C = \frac{1 - \alpha}{1 + \alpha} Z_0$
- $R_B = \frac{2\alpha}{1 - \alpha^2} Z_0$

If $Z_{in} = Z_{out} = Z_0$:

- $R_A = R_C$

A. Gallo, Basics of RF Electronics
It is important to notice that in order to match unequal input/output line impedances a minimum attenuation is required, according to (case $Z_{0_{\text{out}}} \geq Z_{0_{\text{in}}}$):

$$\alpha_{\text{max}} = \sqrt{Z_{0_{\text{out}}}/Z_{0_{\text{in}}}} - \sqrt{Z_{0_{\text{out}}}/Z_{0_{\text{in}}}} - 1 \quad \Rightarrow \quad \Delta dB_{\text{min}} = 20 \cdot \log(1/\alpha_{\text{max}})$$

Fixed attenuators are available in a huge variety of packages, power ratings (up to $\approx 1$ kW), frequency ranges (to $> 18$ GHz), any attenuation value and all standard impedances used in communication electronics.
Low-level RF amplifiers are used to increase the signal level whenever it is required for proper treatment and/or manipulation. A very wide subject, can only be mentioned here. Nowadays almost only solid state technology (silicon or GaAs semiconductors, BJT and FET technology) is used for low/medium power (< 10 W) applications, up to ≈ 10GHz.

Construction techniques are MIC (Microwave Integrated Circuits) and MMIC (Monolithic Microwave Integrated Circuits). In MIC realization the transistor and its capacitance and resistors are soldered on microstrip lines laying on a proper substrate; MMIC are completely integrated circuits where all components (the transistors and their ancillaries) are fabricated on a common substrate.

Concerning small-signal amplifier, the operating class is generally “A” since power efficiency is not an issue for these kind of applications.
**SIGNAL AMPLIFIERS: BASIC SPECIFICATIONS**

- **Frequency range:**
  - from DC to > 10 GHz, multi-decades covered by a single device.

- **Output level:**
  - Maximum power at the amp output.

- **Gain and gain flatness:**
  - Ratio between the output (non-saturated) and input levels, typically expressed in dB. The flatness is defined as half of the gain variation over the entire specified frequency band.

- **1 dB compression point:**
  - Output level corresponding to a 1 dB reduced gain because of the incipient device saturation.

- **Noise figure:**
  - Ratio between the input and output signal-to-noise ratios assuming an input unilateral spectral noise power density $dP_{in}/df = kT$, ($k =$Boltzmann constant, $T = 290 \, K$). Being $G$ its power gain, the amp generates an extra output spectral noise $d(\Delta P_{out})/df = G(NF-1)kT$.

- **Dynamic range:**
  - Potential excursion of the output level, upper-limited by compression/saturation and lower-limited by the noise power integrated over the application frequency band.

- **Two-Tone Third-order intercept point:**
  - Measures the amp linearity. If two-tone fed (two equal-amplitude signals of frequencies $f_1$ and $f_2$), the amp generates intermodulation products at $m_1f_1 \pm m_2f_2$ frequencies. The amplitudes of 3rd order products ($2f_1-f_2$, $2f_2-f_1$) grow with the 3rd power of the input signals so that an input level corresponding to equal fundamental and 3rd order products amplitudes can be extrapolated (laying usually beyond the amp dynamic range).

- **Input/Output VSWR or Return Loss:**
  - Measures of the input/output matching characteristics of the amplifier.
Transformers are widely used in RF electronics. They are very effective to:

- **Match lines of different impedance** with negligible insertion loss;
- **De-couple ground** while transmitting RF signals;
- **Connect balanced and unbalanced** circuits (balun)

RF transformers are also embedded in a number of other devices (splitters/combiners, mixers, amplifiers, …)

### Ideal Transformer transfer functions

\[
\begin{align*}
V_s &= n V_p \\
I_s &= -I_p / n \\
Z_p &= V_p / I_p = -Z_s / n^2
\end{align*}
\]
Together with transform ratio $n$ and connection topology, real transformers are characterized by operating bandwidth, insertion loss, max power rating, ... The lower cutoff frequency is determined by the windings active inductance $L_{act}$, while the high frequency cutoff is dominated by the inter-windings and intra-windings capacitances $C_{p-p}$, $C_{s-s}$ and $C_{p-s}$.

In-band insertion loss is due to the magnetic core dissipation and to the windings ohmic losses, accounted by the resistances $R_{loss}$, $R_p$ and $R_s$. 

The graph shows the insertion loss and return loss over frequency for a real RF transformer.
Power splitters/combiners are used to divide a signal into $N$ equal copies ($N = \text{any number, preferably a power of } 2$), or to make a vector sum of $N$ different signals. Ideally, the power into any output channel is $P_{\text{out}} = \frac{P_{\text{in}}}{N}$.

Basic characteristics are:
- Number of channels $N$;
- Operating frequency range;
- Max power ratings;
- Splitting technique (reactive or resistive);
- Insertion loss (over the nominal $10 \log N$);
- Isolation between channels;
- Phase and amplitude unbalance among output channels;
- ...
Isolation between ports A and B is obtained by a proper choice of the $R_{\text{int}}$ value ($R_{\text{int}} = 2Z_0$) in the 2nd auto-transformer, while the 1st transformer is needed to match the characteristic input impedance.

**AT\(_2\)** transformer equations

\[
V_{p_2} = \frac{V_{s_2}}{2}; \quad I_{p_2} = 2I_{s_2}
\]

Kirchhoff current law at node B

\[
I_B = \frac{V_{s_2}}{R_{\text{int}}} - I_{s_2} = I_{s_2} \left( \frac{2Z_0}{R_{\text{int}}} - 1 \right) = 0
\]

\[
R_{\text{int}} = 2Z_0
\]

Micro-strip design of a 2-way splitter/combiner with similar characteristics of a transformer based one.
**POWER Splitters/Combiners**

Splitter/combiners can be also resistive, consisting in a “star” connection of equal resistors. The frequency response can be much wider (extending from DC) and flatter in this case, at the expense of a larger insertion loss and no isolation (all ports equally coupled).

Matching \( \Rightarrow R = \frac{N - 2}{N} Z_0 \)

Transmission \( \Rightarrow \frac{P_{\text{out}}}{P_{\text{in}}} = \left(\frac{V_{\text{out}}}{V_{\text{in}}}\right)^2 = \frac{1}{(N - 1)^2} \)

As the insertion loss grows linearly with the number of ports, practical use is restricted to 3-ports devices.
Hybrid junctions and Directional couplers are 4-ports passive devices based on the same operational principles but with different coupling levels between ports. The two class of devices are used for different purposes.

\[
\begin{pmatrix}
0 & 0 & -j\sqrt{1-c^2} & c \\
0 & 0 & c & -j\sqrt{1-c^2} \\
-j\sqrt{1-c^2} & c & 0 & 0 \\
c & -j\sqrt{1-c^2} & 0 & 0
\end{pmatrix}
\]

**Directional Coupler**

### 90° Hybrid

\[
\begin{pmatrix}
0 & 0 & -j & 1 \\
0 & 0 & 1 & -j \\
-j & 1 & 0 & 0 \\
1 & -j & 0 & 0
\end{pmatrix}
\]

### 180° Hybrid

\[
\begin{pmatrix}
0 & 0 & 1 & -1 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
-1 & 1 & 0 & 0
\end{pmatrix}
\]
Distributed coupling between 2 lines travelling close each other is one of the possible hybrid/coupler configuration. The lines have a characteristic impedance $Z_0$ when travelling separately, while the 3-conductors system of the 2 coupled lines has even and odd excitation impedances $Z_0^+$ and $Z_0^-$, respectively.

The scattering matrix can be worked out by exploiting the 4-fold symmetry of the network. Being $\beta^\pm$ the propagation constants of the even and odd modes, we get:

$$\beta^+ = \beta^- \equiv \beta; \quad Z_0^+Z_0^- = Z_0^2$$

$$c = \frac{c_{\text{max}} \sin(2\beta d)}{\sqrt{1-c_{\text{max}}^2 \cos^2(2\beta d)}}$$

with: $c_{\text{max}} = \frac{Z_0^+ - Z_0^-}{Z_0^+ + Z_0^-}$

$$\tan \phi_c = \frac{\sqrt{1-c_{\text{max}}^2}}{\tan(2\beta d)}$$

$$\|S\| = \begin{pmatrix} 0 & 0 & -j\sqrt{1-c^2} e^{j\phi_c} & c e^{j\phi_c} \\ 0 & 0 & c e^{j\phi_c} & -j\sqrt{1-c^2} e^{j\phi_c} \\ -j\sqrt{1-c^2} e^{j\phi_c} & c e^{j\phi_c} & 0 & 0 \\ c e^{j\phi_c} & -j\sqrt{1-c^2} e^{j\phi_c} & 0 & 0 \end{pmatrix}$$
If the length of the coupled line is exactly \( 2d = \lambda/4 = \pi/2\beta \), the scattering matrix has its simplest form:

\[
\begin{pmatrix}
0 & 0 & -j\sqrt{1-c_{\text{max}}^2} & c_{\text{max}} \\
0 & 0 & c_{\text{max}} & -j\sqrt{1-c_{\text{max}}^2}
\end{pmatrix}
\]

If coupling factors lower than 10 dB \((c_{\text{max}} > 0.3)\) are required, broadside coupled strips can be used. At \( c_{\text{max}} = .707 \) (i.e. \( Z_0^+ = 5.8 Z_0^- \)) the device scattering matrix is that of a 90° hybrid junction. Coupled wound coils on ferrite cores are used at low frequencies.

Another possible directional coupler structure is represented by 2 parallel lines connected through coupling holes. Two equal holes separated by \( \lambda/4 \) are sufficient to produce the proper scattering matrix at a given frequency. Device bandwidth can be enlarged with multiple holes with different optimal dimensions.
The branch line coupler is another geometry giving the desired port-to-port coupling. The scattering matrix can be still worked out by exploiting the 4-fold symmetry of the network. Under the assumptions:

I) $\beta d_1 = \beta d_2 = \pi / 2$;

II) $(1/Z_1)^2 - (1/Z_2)^2 = (1/Z_0)^2$

the scattering matrix has the following form:

$$
\begin{pmatrix}
0 & 0 & -jZ_1/Z_0 & -Z_1/Z_2 \\
0 & 0 & -Z_1/Z_2 & -jZ_1/Z_0 \\
-jZ_1/Z_0 & -Z_1/Z_2 & 0 & 0 \\
-Z_1/Z_2 & -jZ_1/Z_0 & 0 & 0
\end{pmatrix}
$$

$$
Z_2 = Z_0 \\
Z_1 = Z_0 / \sqrt{2}
$$

$$
= \frac{1}{\sqrt{2}}
\begin{pmatrix}
0 & 0 & j & 1 \\
0 & 0 & 1 & j \\
1 & j & 0 & 0
\end{pmatrix}
$$
The hybrid ring (also called “rat-race”) coupler is a suitable geometry to get 180° hybrids.

Under the assumptions:

\[ d_{1,2} = d_{2,3} = d_{3,4} = \pi / 2\beta = \lambda / 4; \quad d_{1,4} = 3\pi / 2\beta = 3\lambda / 4 \]

the scattering matrix has the following form:

\[
\begin{bmatrix}
Z_1^2 - 2Z_0^2 & -2jZ_1Z_0 & 0 & 2jZ_1Z_0 \\
\frac{Z_1^2 - 2Z_0^2}{Z_1^2 + 2Z_0^2} & \frac{-2jZ_1Z_0}{Z_1^2 + 2Z_0^2} & 0 & \frac{2jZ_1Z_0}{Z_1^2 + 2Z_0^2} \\
\frac{-2jZ_1Z_0}{Z_1^2 + 2Z_0^2} & \frac{Z_1^2 - 2Z_0^2}{Z_1^2 + 2Z_0^2} & \frac{-2jZ_1Z_0}{Z_1^2 + 2Z_0^2} & 0 \\
\frac{2jZ_1Z_0}{Z_1^2 + 2Z_0^2} & 0 & \frac{-2jZ_1Z_0}{Z_1^2 + 2Z_0^2} & \frac{Z_1^2 - 2Z_0^2}{Z_1^2 + 2Z_0^2}
\end{bmatrix}
\]

\[ Z_1 = \sqrt{2} Z_0 \]

\[
\|S\| = \begin{bmatrix}
0 & 1 & 0 & -1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
-1 & 0 & 1 & 0
\end{bmatrix}
\]

\[
\|S\| \otimes \begin{bmatrix}
V_1 \\
0 \\
V_3 \\
0
\end{bmatrix} = \frac{-j}{\sqrt{2}} \begin{bmatrix}
0 \\
V_3 + V_1 \\
0 \\
V_3 - V_1
\end{bmatrix}
\]
Directional couplers are used to sample or to unequal split/sum RF signals.

Hybrids are used whenever splitting/combination of RF signals out-of-phase (90°) or counter-phase (180°) are required (I&Q modulators/detectors, differential combination, ...).

Basic characteristics are:
- Coupling coefficient (3 dB for hybrids);
- Directivity / Isolation between uncoupled ports;
- Operating frequency range;
- Max power ratings;
- Coupling type (holes, distributed, rings, ...);
- Insertion loss (over the nominal coupling factor);
- Phase and amplitude unbalance among output channels;
- Phase and amplitude flatness over frequency;
- ...

Coupler Directivity: \(-20 \log |d|\) measurement of the imperfect isolation between ideally uncoupled port.
Circulators are non-reciprocal 3-ports (typically) devices whose scattering matrix is ideally given by:

\[
S = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

Their basic structure is a symmetrical, 120° Y junction with a ferrite disk placed at the center biased by an axial magnetic field. Biased ferrites show a tensor magnetic permeability, i.e. an anisotropic behavior. The incident wave on one port excites 2 unbalanced (because of the anisotropy) waves rotating in the 2 opposite directions, so that the coupling to the output ports is also unbalanced. By proper design the junction it is possible to have almost 100% transmission in one port and no transmission in the other in a given frequency band (> 1 octave).

Isolators are circulators with one port internally terminated. Circulators and isolators are used for a number of tasks, such as matching sources and loads, protecting sources against backward power, capturing and draining reflected power from a device to another device.
Filters are 2-ports devices “tailored” to obtain a specific required frequency response (the $s_{21}$ of the network). Typically the response is maximized at some bands of interest, and minimized at other frequency bands that have to be rejected.

Filters are classified on the base of their nature, topology, dissipation, ...

- **Analog/Digital**, depending on the nature (continuous or sampled & digitized) of the input signal;
- **Lumped/Distributed**, depending on the nature of the internal components (L-C cells, DR cavities, $\mu$-strip cells, ...);
- **Reflective/Absorbing** depending on the path of the stopbands (reflected or internally dissipated);
- **LowPass/HighPass/BandPass/Notch/Comb**, depending on the profile of the frequency response.

![Circuit Frequency Response](image)
Filters with different response around transition between pass and stop bands are available for different applications. They implement different rational complex polynomials in their transfer functions, the most popular ones being:

- Bessel, for a maximally flat group delay;
- Butterworth, for a maximally flat frequency response in the pass-band;
- Gaussian, providing a gaussian response to a Dirac pulse and no overshoot for an input step function. The Gaussian filter also minimizes the group delay;
- Chebyshev, providing a steep transition with some passband (type I) or stopband (type II) ripples. They provide the closest possible response w.r.t. an ideal rectangular filter;
- Elliptic, providing the steeper possible transition between the pass-band and the stop-band by equalizing the ripple in both.
Design and construction of filters is becoming more and more a specialized activity, so that “home made” devices are seldom used, and mainly for very specific task. Design phase make use of dedicated software packages (such as Touchstone) through various iterative steps. In addition to the typologies already listed, other important characteristics defining filter performances are:

- **Insertion Loss**, defined as the in-band signal attenuation;

- **Phase linearity/Group delay**: figures of the quality of the filter phase response across the pass-band, that should present a constant negative slope to avoid distortion of time-profile of in-band pulses. The slope of the response phase is the filter group delay, equal to the signal latency while travelling across the device.

- **Input/output impedance** and **VSWR**: characteristic impedance of the device and reflectivity of signals transmitted (within pass-band) and rejected (within stop-band).
Digital filters act on sampled and digitized input signals. There are 2 basic filter architectures:

- **Finite Impulsive Response** (FIR), where the output $y$ is a linear combination of the last $N$ sampled values of the input $x$. The coefficients $h_i$ of the expansion represent the discretization of the filter Green’s function.

- **Infinite Impulsive Response** (IIR), where the output $y$ is a linear combination of the last $N$ and $M$ sampled values of the input $x$ and output $y$, respectively. IIR filters directly implement a feedback architecture, which may generate sharp frequency responses with a limited number of samples.
Differences between analog and digital filtering are quite evident. Digital filtering is a complex operation requiring many steps such as down-conversion (necessary in most cases), A-to-D conversion, digital data manipulation, D-to-A conversion and final frequency up-conversion. However, powerful ICs nowadays available (DSP, FPGA, ...) are capable to perform various tasks in a single chip.

On the other hand, digital filtering provide incomparable flexibility and operational adaptivity, since the transfer function can be modified and optimized in real time by simply changing the weighting coefficients. Beam feedbacks can greatly benefit this feature.

The z-domain transfer function $H_z(z)$ gives direct information on the filter frequency response being related to the Laplace $H_L(s)$ and Fourier $H_F(j\omega)$ transfer functions by the mathematical expressions:

$$H_L(s) = H_z(z) \big|_{z=e^{sT}} \quad H_F(j\omega) = H_z(z) \big|_{z=e^{j\omega T}}$$

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$$H_z(z) = \frac{1}{1-p_k z^{-k}}$$

Comb filter transfer function
LLRF servo-loops and feedback loops often need to apply **AM** and **PM** modulation to the **RF drive** signal. The response of a **resonant cavity** to **AM** and **PM** excitations depends on its **bandwidth** and **tuning** relative to the carrier:

\[
v_i(t) = A_i[1 + a_i(t)]\cos(\omega_c t) \quad \Rightarrow \quad v_o(t) = A_o[1 + a_o(t)]\cos(\omega_c t)
\]

\[
v_i(t) = A_i \cos[\omega_c t + \phi_i(t)] \quad \Rightarrow \quad v_o(t) = A_o \cos[\omega_c t + \Delta\phi_o + \phi_o(t)]
\]

\[
L\text{-transform}
\]

\[
x(t) \quad \Rightarrow \quad \hat{x}(s) \quad \Rightarrow \quad G(s) = \frac{\hat{a}_o(s)}{\hat{a}_i(s)} = \frac{\hat{\phi}_o(s)}{\hat{\phi}_i(s)} = \frac{1}{1 + s/\sigma} \quad \text{with} \quad \sigma = \frac{\omega_r}{2Q_L}
\]

\[
v_i(t) = A_i[1 + a_i(t)]\cos(\omega_c t) \quad \Rightarrow \quad v_o(t) = A_o[1 + a_{o,a}(t)] \cos[\omega_c t + \Delta\phi_o + \phi_{o,a}(t)]
\]

\[
v_i(t) = A_i \cos[\omega_c t + \phi_i(t)] \quad \Rightarrow \quad v_o(t) = A_o[1 + a_{o,p}(t)] \cos[\omega_c t + \Delta\phi_o + \phi_{o,p}(t)]
\]

\[
G_{aa}(s) = \frac{\hat{a}_{o,a}(s)}{\hat{a}_i(s)}; \quad G_{pp}(s) = \frac{\hat{\phi}_{o,p}(s)}{\hat{\phi}_i(s)}; \quad G_{ap}(s) = \frac{\hat{\phi}_{o,a}(s)}{\hat{a}_i(s)}; \quad G_{pa}(s) = \frac{\hat{a}_{o,p}(s)}{\hat{\phi}_i(s)}
\]
It may be demonstrated that **direct** and **cross** modulation transfer functions are given by:

\[
G_{pp}(s) = G_{aa}(s) = \frac{1}{2} \left[ \frac{A(s + j\omega_c)}{A(j\omega_c)} + \frac{A(s - j\omega_c)}{A(-j\omega_c)} \right]; \quad G_{ap}(s) = -G_{pa}(s) = \frac{1}{2j} \left[ \frac{A(s + j\omega_c)}{A(j\omega_c)} - \frac{A(s - j\omega_c)}{A(-j\omega_c)} \right]
\]

with \( A(s) \) = transfer function in Laplace domain of the filter applied to the modulated signal. If the signal is filtered by a resonant cavity, one has to consider \( A(s) = A_{cav}(s) \) given by:

\[
A_{cav}(s) = A_0 \frac{2\sigma s}{s^2 + 2\sigma s + \omega_r^2}
\]

with \( \omega_r \approx \omega_c + \sigma \tan \phi_z \)

where \( \phi_z \) is the **cavity tuning angle**, i.e. the phase of the cavity transfer function at the carrier frequency \( \omega_c \). Finally one gets:

\[
G_{pp}(s) = G_{aa}(s) = \frac{\sigma s + \sigma^2 (1 + \tan^2 \phi_z)}{s^2 + 2\sigma s + \sigma^2 (1 + \tan^2 \phi_z)}; \quad G_{ap}(s) = -G_{pa}(s) = -\frac{\sigma \tan \phi_z s}{s^2 + 2\sigma s + \sigma^2 (1 + \tan^2 \phi_z)}
\]
The general form of the modulation transfer functions features 2 poles (possibly a complex conjugate pair) and 1 zero, and degenerates to a single pole LPF response if the cavity is perfectly tuned (cross modulation terms vanish in this case).
In circular accelerators the beam phase depends on the cavity RF phase through the beam transfer function, while the cavity RF amplitude and phase depend on the beam phase through the beam loading mechanism. The whole generator-cavity-beam linear system can be graphically represented in a diagram called Pedersen Model.

The modulation transfer functions vary with the stored current and definitely couple the servo-loops and the beam loops implemented around the system.
Thank you for the moment and ...

See you tomorrow at 15.30