Measurements, Statistics and Errors

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1. Measurement
The Standard Signal Path

The documentation about the measurement device → significance, numbers + units.
Gnosis

Signal/Numbers → (? → proof of model: yes/no

Model → Hypothesis

Measurement means:
- Abstraction from raw data by calibration
- Significance of values by Error estimation
- Reduction of noise by optimization of device
- Model + simulate the signals
- proof or falsify model
- adapt model
Signals

Signal = Voltage as a function of time \( U(t) \)

<table>
<thead>
<tr>
<th>discrete</th>
<th>continous</th>
</tr>
</thead>
<tbody>
<tr>
<td>analog</td>
<td>digital</td>
</tr>
<tr>
<td>causal</td>
<td>non-causal</td>
</tr>
</tbody>
</table>

Definition:

- Time: \( t \in \mathbb{R} \) (sometimes \( \in \mathbb{R}_0^+ \))
- Amplitude: \( s(t) \in \mathbb{R} \)
- Power: \( s^2(t) \in \mathbb{R}_0^+ \) (constants are renormalized to 1, e.g. \( P = \frac{U^2}{R} \) or \( P = I^2R \))

Energy-Signal:

\[
\int_{-\infty}^{\infty} s^2(t) dt < \infty
\]

(Problems with \( \sin() \), \( \cos() \), \( \text{rect()} \) and stochastic/noise signals)

Power-Signal:

\[
\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} s^2(t) dt < \infty
\]
Errors

1. systematic (deterministic) error
2. unsystematic (statistic) error (noise)

**systematic error**
- due to characteristics of the measurement device. (ADC/DAC: *offset, gain, linearity-errors*)
- Improved by improvements of apparatus.
- Limits: Quantum mechanics

**statistic error**
- unforseen fluctuations, random, stochastic, noise
- it is possible to estimate the extend
- can be reduced by statistical methods (averaging), multiple measurements

**accuracy**
Definition is context depeptant: accuracy of 100 devices can be a matter of precision!
2. Noise
Noise

acoustic \rightarrow \text{electronics}

Noise sources:

- Brownian movement of charges (thermal noise)
- Variations of the number of charges involved in conduction (Shortky noise/shot noise, flicker noise)
- Quantum effects (zero point fluctuations)

Noise classes:

- \textbf{White} noise \rightarrow \textbf{flat} spectrum
- \textbf{Pink} noise \rightarrow \text{low-passed} spectrum
- \textbf{Blue} noise \rightarrow \text{high-passed spectrum}

Amplitude density distribution:

\begin{align*}
\text{probability density} & \rightarrow \text{Gauß} \\
\text{if many indep. sources contribute} & \rightarrow \text{stochastic signal}
\end{align*}
Sensor Noise

Sources:

- Thermal noise (Johnson-/Nyquist-noise)
- Shot noise (Shottky-Noise)
  (quants of the elementary charge)
- $1/f^\alpha$ noise
- (e.g. flicker noise)
**Thermal Noise**

Brownian motion of charge carriers

Noise Voltage: \( U_{\text{rms}} = \sqrt{4kT \cdot R \cdot B} \)

**Note:** Thermal noise is only emitted by structures with electromagnetic losses (resistors, \( R \)), which also absorb power!

\[ I_{\text{rms}} = \sqrt{4kT \frac{B}{R}} \]

Reileigh-Jeans approximation, high \( T \), small \( \nu \)

Noise Power: \( P_n = \frac{1}{4} U_{\text{rms}} I_{\text{rms}} = kTB \approx U^2 \approx I^2 \)

Received from \( R_1 \) (power matching)

CAS-Bohrm Diagnostics 2008 (c) Markus Hoffmann p.11/53
Amplitude Characteristics

\[ U_\sigma = U_{\text{rms}} \sim \sqrt{BW} \]

white noise  gaussian distribution

\[ \bar{U}_{\text{mean}} = 0 \]
\[ U_{\text{peak}} = \infty \]
The Physics behind

- **Power Spectral Density** $\rho(\nu)$:
  
  $$\rho = kT \quad ; \quad P = \int_{BW} \rho \, d\nu$$

- derived from **Planck's law**:
  
  $$\Rightarrow \rho = kT \frac{h\nu}{e^{h\nu/kT} - 1} \approx kT$$

  (+ zero point vacuum fluctuations $\frac{1}{2}h\nu$)

\[\begin{align*}
[\frac{W}{\text{Hz}} = J] & \quad T = 290K \\
& \quad 4 \cdot 10^{-21} \frac{W}{\text{Hz}}
\end{align*}\]

Problem: total power $= \infty$!
Thermal Noise Spectrum

For frequencies $>10$ GHz you can gain more than proportionally by cooling!

The zero-point noise cannot deliver power to a load. But it can be amplified to a noticeable magnitude.
Cosmic Noise
Shot(tky) Noise

no current, no shot noise!

\[ P_s \sim (\langle I \rangle)^2 = 2eI \Delta f \]
\[ \delta I \sim \sqrt{2eI \Delta f} \]

\( \tau \): transit time
Flicker Noise

Markow Process
Lorentz spectrum

\[ \log(p) \sim \frac{1}{f} \]

spectra with different \( \tau \)
Noise Spectrum – Example
System Noise Measurement

Input Noise:

\[
\frac{P_{\text{eff}}}{g_{\text{ain}}} = P_{\text{input noise}} \quad \rightarrow \text{projected to the input}
\]

\[\text{SNR} = 1\] if input signal level = input noise level.

At the output

1. open input, measure \( P_{\text{eff,1}} \)
2. close input, inject additional noise \((P_r)\) so that \( P_{\text{eff,2}} = 2P_{\text{eff,1}} \)
3. read \( P_r \) from generator, \( \rightarrow P_r = \text{input noise.} \)
Signal to Noise Ratio

Power ratio, inside bandwidth!

\[ \text{SNR} := \frac{\hat{P}_{\text{signal}}}{\hat{P}_{\text{noise}}} = \left( \frac{\hat{A}_{\text{signal, rms}}}{\hat{A}_{\text{noise, rms}}} \right)^2 \]

Measurement:

\[ \frac{S + N}{N} = \frac{S}{N} + 1 \]

\[ \bar{P} := \int_{BW} \rho(v) \, dv \]

\[ \text{SNR(dB)} := 10 \log_{10} \left( \frac{\bar{P}_{\text{signal}}}{\bar{P}_{\text{noise}}} \right) = 20 \log_{10} \left( \frac{\hat{A}_{\text{signal, rms}}}{\hat{A}_{\text{noise, rms}}} \right) \]

\[ = P_{\text{signal}}[\text{dBm}] - P_{\text{noise}}[\text{dBm}] \]

Units: [SNR(dB)]=dBc (=“dB below carrier“)
Noise Figures

Noise factor:

\[
F = \frac{P_r}{kT_0B} + 1 \left( \frac{SNR_{in}}{SNR_{out}} \right)
\]

- Relation of input noise + noise of the source to noise of the source only
- Number of additional \( kTB \) units of noise necessary on the input to double the output noise

Noise figure:

\[
F' = 10\log F \quad [\text{dB}]
\]

- Is a measure of the system itself
- The figure is independent from \( T \) and \( B \)!

Examples

1. The system adds no additional noise: \( F = 1, F' = 0 \) dB (The input noise equals thermal noise on the input. If the real temperature is lower than 290 K, \( F \) can be smaller.)

2. A very noisy system reduces the SNR from 100 (20 dB) to 10 (10 dB):
   \[ F = 10, F' = 10 \text{ dB} \]
A Data Sheet

For NRA models that either have a serial prefix less than U14446 or are fitted with the Noise-figures RF board, N6379A-60091.

### Instrument’s own noise figure

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Noise figure</th>
<th>Noise figure over a limited temperature range of 25°C ± 3°C</th>
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<tbody>
<tr>
<td>10 MHz to &lt; 500 MHz</td>
<td>$&lt; 4.9 \pm 0.0025 \text{ dB in MHz} \pm 0.0005 \text{ dB in MHz}$</td>
<td>$&lt; 4.4 \text{ dB } \pm 0.0025 \text{ dB in MHz}$</td>
</tr>
<tr>
<td>600 MHz to &lt; 2.3 GHz</td>
<td>$&lt; 7.4 \pm 0.0015 \text{ dB in MHz} \pm 0.0005 \text{ dB in MHz}$</td>
<td>$&lt; 6.3 \text{ dB } \pm 0.0015 \text{ dB in MHz}$</td>
</tr>
<tr>
<td>2.3 GHz to 3.0 GHz</td>
<td>$&lt; 9.8 \pm 0.0015 \text{ dB in MHz} \pm 0.0005 \text{ dB in MHz}$</td>
<td>$&lt; 7.8 \text{ dB } \pm 0.0015 \text{ dB in MHz}$</td>
</tr>
<tr>
<td>&gt; 3.0 GHz to 13.2 GHz</td>
<td>$&lt; 12.0 \text{ dB}$</td>
<td>$&lt; 9.6 \text{ dB}$</td>
</tr>
<tr>
<td>&gt; 13.2 GHz to 26.5 GHz</td>
<td>$&lt; 18.0 \text{ dB}$</td>
<td>$&lt; 17.5 \text{ dB}$</td>
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<td>&gt; 13.2 GHz to 26.5 GHz</td>
<td>$&lt; 16.0 \text{ dB}$</td>
<td>$&lt; 12.5 \text{ dB}$</td>
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Noise Temperature

![Diagram of noise temperature](image)

Put a resistor $T_{\text{noise}}$ so that $P_{\text{eff}}$ doubles:

$$T_{\text{noise}} := (F - 1) \cdot 290 \text{K} \quad [\text{K}]$$

for white noise

- for an ideal noise-free system $\rightarrow T_{\text{noise}} = 0 \text{ K}$.
- is not necessarily identical with the real temperature the system is on.
- A low-noise amplifier can have $T_{\text{noise}} = 20 \text{ K} \rightarrow F = 1.06, F' = 0.28 \text{ dB}$.
  (This is really good!)
3. Statistics
Mean and Standard Deviation

\[ \hat{x} := \frac{1}{N} \sum_{i=0}^{N-1} x_i \]

mean over \(N\) samples.

\[ \sigma^2 := \frac{1}{N-1} \sum_{i=0}^{N-1} (x_i - \hat{x})^2 \]

variance = (standard deviation)\(^2\)

"power of fluctuations"
RMS and the Running Statistics

\[ \sigma_N^2 = \frac{1}{N-1} \left[ \frac{1}{N} \sum_{i=0}^{N-1} x_i^2 - \frac{1}{N} \left( \sum_{i=0}^{N-1} x_i \right)^2 \right] \]

Signal to Noise Ratio (SNR): \( \text{SNR} = \frac{x^2}{\sigma^2} \)

Coefficient of Variation (CV): \( \text{CV} = \frac{\sigma}{\hat{x}} \cdot 100\% \)

Root Mean Square (RMS): \( x_{\text{rms}} := \sqrt{\frac{1}{N} \sum_{i=0}^{N-1} x_i^2} \)

"Power of fluctuations plus power of DC component"
Standard Deviation of Common Waveforms

a. Square wave, $V_{pp} = 2\sigma$

b. Sine wave, $V_{pp} = 2\sqrt{2}\sigma$

c. Triangle wave, $V_{pp} = \sqrt{12}\sigma$

d. Random noise, $V_{pp} = 6 - 8\sigma$
Histograms

Snapshot of N samples

\[ N = \sum_{i=0}^{M-1} H_i \quad \hat{x} := \frac{1}{N} \sum_{i=0}^{M-1} i \cdot H_i \quad \sigma^2 := \frac{1}{N-1} \sum_{i=0}^{M-1} (i - \hat{x})^2 H_i \]

Histogram

Probability mass function

Probability density distribution

\[ N \rightarrow \infty \]
Probability Distribution Functions

a. Square wave

b. Sine wave

c. Triangle wave

d. Random noise
The Normal Distribution

\[ P(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\hat{x})^2}{2\sigma^2}} \]

Gauß function

\[ f(x) = e^{-x^2} \]

raw shape

normalized

\[ \hat{x} = 20 \]
\[ \sigma = 3 \]

\[ \int_{-\infty}^{+\infty} P(x) \, dx = 1 \]
Underlying Process

Typical error: \[ \Delta A = \frac{\sigma_N}{\sqrt{N}} \]

- \( \sigma_N \) is an estimate of the standard deviation of the underlying process by \( N \) samples (e.g. extracted from the histogram).

- Extract the most information about the underlying process out of the sampled signal.
Propagation of Error

in general:

\[ f = f(\alpha_1, \alpha_2, \ldots, \alpha_n) \]

a function of (model) parameters \( \alpha_i \) with corresponding errors \( \Delta \alpha_i \).

\[ \Rightarrow \Delta f = \sqrt{\sum_i \left( \frac{\partial f}{\partial \alpha_i} \Delta \alpha_i \right)^2} \]

**Example:**

\[ x(t) = v \cdot t + x_0 \]

\[ \Rightarrow \Delta x = \sqrt{\left( \frac{\partial x}{\partial v} \Delta v \right)^2 + \left( \frac{\partial x}{\partial x_0} \Delta x_0 \right)^2 + \left( \frac{\partial x}{\partial t} \Delta t \right)^2} \]

\[ = \sqrt{(\Delta v \cdot t)^2 + (\Delta x_0)^2} \]
Accuracy and Precision

- measure, estimate, predict
- "true value"
- measured value
- systematic error
- random noise

- accuracy is a measure of calibration
- precision is a measure of statistics

Question: measuring with 100 different devices. ——> accuracy or precision?
The Central Limit Theorem

- generation of random numbers (noise)
  - white noise $\text{RND} := [0; 1]$
  - gaussian noise:
    "the sum of random numbers (any distribution) becomes gaussian distributed" better:

  $x = \sqrt{-2 \log_{10}(\text{RND}_1) \cdot \cos(2\pi \text{RND}_2)}$

  $\bar{x} = 0, \quad \sigma = 1$

- pseudo-random:

  $\text{RND} = (as + b) \mod c$

  seed

$\begin{align*}
  x &= \text{RND} \\
  \bar{x} &= 0.5 \\
  \sigma &= \frac{1}{\sqrt{12}} \approx 0.29
\end{align*}$

$\begin{align*}
  x &= \text{RND} + \text{RND} \\
  \bar{x} &= 1 \\
  \sigma &= \frac{1}{\sqrt{6}} \approx 0.4
\end{align*}$

$\begin{align*}
  x &= \text{RND} + \cdots + \text{RND} (12 \times) \\
  \bar{x} &= 6 \\
  \sigma &= 1
\end{align*}$
Chi Square Distribution

$X_i$ are $k$ independent, normally distributed random variables with $\mu_i = 0$, $\sigma_{x_i}^2 = 1$:

$$Q := \sum_{i=1}^{k} X_i^2$$

is distributed according to „Chi Square“

$$Q \sim \chi^2_k$$

$k$: number of degrees of freedom

For $k \gg 1$: $Q$ becomes Gaussian distributed.

$$f(x; k) = \begin{cases} 
\frac{x^{k/2 - 1}}{2^{k/2} \Gamma(k/2)} e^{-x/2} & \text{for } x > 0, \\
0 & \text{for } x \leq 0.
\end{cases}$$
Chi Square Test

Null-hypothesis: is true if your alternative hypothesis cannot be supported.

Statistical Test: falsify the Null-hypothesis!

χ²-Test: the statistics of samples of data (derived from model) has a χ²-distribution if the null-hypothesis is true.

Or gauss or any, if the proportionality distribution approximates large # of samples

Theory: \( n_{j0} = F_0(x)_j \cdot N \)

\[ \chi^2 = \sum_{j=1}^{m} \frac{(n_j - n_{j0})^2}{n_{j0}} \]

⇒ χ² distributed, \( m - 1 \) degrees of freedom if \( N \) large

If χ² is larger than a significance level \( \alpha \), the hypothesis will be rejected.
4. Advanced Concepts
Power Spectral Density

averaged power levels related to intervals of noise frequency

\[
\rho_x(v) := \lim_{T \to \infty} \frac{1}{T} \left\langle \left| \int_{-T/2}^{T/2} X(t) e^{2\pi i vt} dt \right|^2 \right\rangle \quad \text{power spectrum of } X(t)
\]

Unit: \( \left[ \frac{W}{Hz} \right] \), \( \left[ \frac{dBm}{Hz} \right] \), \( \left[ \frac{dBc}{Hz} \right] \), or \( \left[ \frac{rad^2}{Hz} \right] \) (phase noise). Amplitude: \( \left[ \frac{rad}{\sqrt{Hz}} \right] \)

Wiener-Khinchin theorem:
\[
\rho_x(v) = \text{Fourier Transformation of Autocorrelation function of } X(t)
\]

\[
\rho_x(v) = \int_{-\infty}^{\infty} g_x(\tau) e^{2\pi i vt} d\tau \quad \text{fourier transform of}
\]

\[
g_x(t) := \left\langle \int_{-\infty}^{\infty} x(\tau)x(t+\tau) d\tau \right\rangle \quad \text{average autocorrelation function}
The Fourier Transformation

time domain  frequency domain

\[ s(t) \quad \rightarrow \quad S(\omega) \]

**signal**  **spectrum**

Given \( f : D \rightarrow \mathbb{C} \), where \( D \subseteq \mathbb{R} \), the **Fourier transformation** of \( f \) is:

\[ F(\omega) := \int_D f(t) e^{-i\omega t} dt \]

and the **back transformation**

\[ f(t) := \int_{-\infty}^{\infty} F(\omega) e^{+i\omega t} d\omega \ . \]

**Note:** We use \( \omega = 2\pi v \) to get rid of the constants.
## Fourier Transformation Examples

| $s(t)$   | time domain | $S(f)$       | frequency domain | $|S|$ |
|----------|-------------|--------------|-----------------|------|
| 1        | ![1](image) | $\delta(f)$  | ![1](image)     |      |
| $\delta(t)$ | ![delta](image) | 1            | ![delta](image) |      |
| $\omega(t)$ | ![omega](image) | $\omega(f)$ | ![omega](image) |      |
| $e^{-\pi t^2}$ | ![Gaussian](image) | $e^{-\pi f^2}$ | ![Gaussian](image) |      |
| $2 \cos(2\pi F t)$ | ![2cos](image) | $\delta(f + F) + \delta(f - F)$ | ![2cos](image) |      |
Calculation with Fourier Transforms

\[ x(t) \xrightarrow{\text{FT}} X(\omega) \xrightarrow{\text{FT}} x(-t) \]

Symmetry:

\[ \text{FT}^2 \{x(t)\} = x(-t) \]

Linearity:

\[ \text{FT}\{c_1x_1(t) + c_2x_2(t)\} = c_1 X_1(\omega) + c_2 X_2(\omega) \]

Scaling:

\[ \text{FT}\{x(\lambda t)\} = \frac{1}{|\lambda|} X\left(\frac{\omega}{\lambda}\right) \]

Convolution:

\[ \text{FT}\{x_1(t) * x_2(t)\} = X_1(\omega) \cdot X_2(\omega) \quad ; \quad \text{FT}\{x_1(t) \cdot x_2(t)\} = X_1(\omega) * X_2(\omega) \]

For a real input, the transformation produces a complex spectrum which is symmetric:

\[ X(\omega) = X^*(-\omega) \]

Complex conjugate

\[ X(\text{cos-like}) = \text{real} \]
\[ X(\text{sin-like}) = \text{imaginary} \]

Time-Shift:

\[ \text{FT}\{x(t + t_0)\} = e^{i\omega t_0} X(\omega) \]
Autocorrelation

Given two functions $f, g : D \rightarrow \mathbb{C}$, where $D \subseteq \mathbb{R}$, the **cross correlation** of $f$ with $g$:

$$(f \circ g)(t) := K \int_D f(\tau)g(t + \tau) d\tau$$

**Autocorrelation**

$A_g(t) := g \circ g = K \int_D g(\tau)g(t + \tau) d\tau$

Detect a known waveform in a noisy background, e.g. echoes.

**Symmetry:**

$$A_g(-t) = A_g^*(t)$$

**Convolution:**

$$f[n] \circ g[n] = f[n] * g[-n]$$

**Peak:**

$$|A_g(t)| \leq A_g(0)$$

**Periodicity:**

$$g(t) \text{ periodic } \iff A_g(t) \text{ periodic}$$
Power Spectral Density (2)

Spectrum of the autocorrelation function:

\[ s(t) \ast s(-t) \xrightarrow{\text{FT}} S(\omega) \cdot S^*(\omega) = |S(\omega)|^2 = \rho(\omega) \quad [\text{W/Hz} = J] \]

- we lose information about phase (or time/shift/location), Time invariance
- white noise part is contained in \(g_x(0)\),
- relation to the variance and RMS of \(x(t)\):

\[ \langle x^2 \rangle = \int_{v_1}^{v_2} \rho_x(v) dv = P_{BW} = (\text{RMS}_x)^2 = x_{\text{eff}}^2 = \sigma_x^2 \mid \text{[v_1,v_2]} \]

Wiener-Khinchin:
mean square value in time = mean square value in frequency
- analog and digital measurement techniques possible.
Complex Noise
(Amplitude & Phase Noise)

Harmonic Signal with Noise:
\[ x(t) = A_0 (1 + \delta \alpha(t)) e^{2\pi i v_0 t + \delta \phi(t)} \]

\[ \rho(v) = A_0^2 (\delta(v - v_0) + \rho_\alpha(v - v_0) + \rho_\phi(v - v_0) + \Theta^R(\rho_\alpha, \rho_\phi)) \]

\[ \delta(v - v_0) \]

\[ \rho_\phi \sim \frac{1}{f} \]

\[ \rho_\alpha \]

carrier frequency
amplitude noise
phase noise

carrier \( \delta \) peak

DESY
Phase- and Time-Jitter

\[ \langle \Delta t^2 \rangle_{v_1,v_2} := \frac{1}{2\pi v_0} \langle \Delta \phi^2 \rangle_{v_1,v_2} \]

\[ \langle \Delta \phi^2 \rangle_{v_1,v_2} := \int_{v_1}^{v_2} \rho_{\phi}(v) \, dv \]
Noise Propagation

Adding noise sources:

\[ P = P_1 + P_2 \]

\[ U_{\text{eff}} = \sqrt{\delta U_1^2 + \delta U_2^2 + 2\gamma \cdot \delta U_1 \delta U_2} \]

\[ \rho(v) = \rho_1(v) + \rho_2(v) + 2\gamma(v) \sqrt{\rho_1(v)} \sqrt{\rho_2(v)} \]

\( \delta \leq \gamma \leq 1 \) correlation coefficient

Time domain

Frequency domain
Noise Propagation (2)

**Amplifier/Attenuator:**

\[ y(t) = g x(t) \]

\[ P_y = g^2 P_x \]

\[ U_{\text{eff},y} = g \cdot \delta U_x \]

\[ \rho_y(v) = g^2 \rho_x(v) \]

\[ \rho_{\alpha,y}(v) = \rho_{\alpha,x}(v) \]

\[ \rho_{\phi,y}(v) = \rho_{\phi,x}(v) \]

**LTI-Filter:**

\[ y(t) = h(t) * x(t) \]

Transfer function:

\[ \rho_y(v) = |H(v)|^2 \rho_x(v) \]

\[ \rho_{\alpha,y}(v) = \left| \frac{H(v + v_0)}{H(v_0)} \right|^2 \rho_{\alpha,x}(v) \]

\[ \rho_{\phi,y}(v) = \left| \frac{H(v + v_0)}{H(v_0)} \right|^2 \rho_{\phi,x}(v) \]

The power of the phase noise is not(!) amplified!
Quantisation Noise

Transfer-function of ADC:

- Quantisation-Error
  $|A| < 0.5$ LSB
- $\text{RMS}(\Delta A) \approx \sqrt{12}$ LSB
- missing codes
- code transition noise

For a full scale $\sin(\cdot)$-signal:

$$\text{SNR} = 6.02n + 1.76\text{dB} + 10\log\left(\frac{f_s}{2\text{BW}}\right)$$

increases with lower BW.

$\rightarrow$ doubling the sampling frequency increases SNR by 3dB (same signal BW)

$\rightarrow$ "oversampling"
Quantisation Noise Spectra

\[ \text{SNR} = 6.02n + 1.76\text{dB} + 10 \log \left( \frac{f_s}{2 \text{BW}} \right) \]

it is assumed that the noise is equally distributed over the full BW.

\[ \text{! This is often not the case!} \]

Mostly the noise is correlated with the input signal!

- The lower the signal, the more correlation!
- In case of strong correlation the noise is concentrated at the various harmonics of the input signal, just where you don't want them.
- Dithering and broad input signal spectrum randomizes the quantisation noise.
5. Applications
Stochastic Signals in Accelerator Diagnostics

Apply additional and/or artificial (pseudo random) noise

Use the noise which is there anyway

- **Tune measurements** from shotky noise of the beam itself (without increasing the emittances), esp. hadron beams
- Synchrotron light emmission → noise source, beam exitation

- dithering methods
- beam size blow ups (nbunch lengthening)
- decorrelate signals to avoid systematic errors, interference and resonant excitations.

- **transfer function** measurements

- **Stochastic cooling**

in many other cases the stochastic part of the signal is unwanted (noise)

→ Noise Filtering Techniques
Fighting the Noise

How can we reconstruct a signal to which noise has been added?

One idea is to use a Low-Pass Filter:

Problems are: latency, dissipation, high frequency cut-off and still noise in the low frequencies.

→ solution to this: the Kalman Filter

→ other solution: (non-causal)

noisy Signal  →  Transformation  →  remove noise  →

inverse Transformation  →  Signal
Measurements, Statistics and Errors

The art of measurement is always also the art of error treatment!

The End