BEAM DIAGNOSTICS
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1) E.M. FIELDS USED FOR DIAGNOSTICS
2) TRANSVERSE EFFECTS
3) LONGITUDINAL EFFECTS
4) COLLECTIVE EFFECTS

Concentrate on:
circular machines, bunched beams, high energy,
relation between beam dynamics and diagnostics.
1) E.M. FIELDS USED FOR DIAGNOSTICS

Beam diagnostics uses electromagnetic fields $\vec{E}$ and $\vec{B}$ created by the beam. The near field: It is a Lorentz transformed Coulomb field, electric field concentrated in the transverse direction, induces wall currents in chamber which are used to measure the beam with a monitor. The radiation field: Fields above the cut-off frequency $\omega > \omega_{\text{cut-off}}$ can propagate in chamber. As synchrotron radiation they are used for diagnostics to measure beam at the source. We distinguish ‘near field’ attached to charges and ‘far’- or ‘' radiation field’ propagating.
The $E$-field of a point charge induces surface charges of opposite sign on a circular chamber of radius $a$ with Lorentz contracted distribution $dq_w/ds$ of RMS width

$$\sigma_w = \frac{a}{\sqrt{2\gamma}}$$

being very small for $\gamma \gg 1$.

Charges induced by a bunch current $I(t)$ have nearly the same distribution but wall current does not contain DC-part, an uniform beam induces an uniform static charge which does not move $I_w(t) = -(I(t) - \langle I \rangle)$. 

Induced charges by a moving proton

Wall current induced by a bunch
**Loop monitor:** The magnetic field of the beam induces a voltage in the loop. If the strip forming the loop is of finite width the electric field induces surface charges. The coupling to the beam is inductive for a thin loop and capacitive for a wide one. With the two balanced it is often called 'strip line monitor'.

**Cavity monitor:** Cavity gets excited by multiple bunch passages and oscillates in a monopole (left) as intensity or dipole (right) mode as position monitor. It has a high sensitivity and is used for low intensities.
Measure the average current
The wall current induced by the beam does not contain the DC-part. However, it can be estimated from the monitor reading between the widely spaced bunches.

\[ -I_w(t) \]

Position monitors and radiation field
Diffraction radiation from aperture changes can propagate and reach monitors if above cur-off frequency, \( \omega_{\text{cur-off}} \geq 2.405c/a \) with circular chamber radius \( a \). Monitor signals should be low-pass filtered.

Magnetic field created by DC-part of beam does not induce a wall current but penetrates and can be used to measure the average current outside.
Bandwidth and read-out

Intensity monitors, read-out and signal processor have limited bandwidth. They integrate/differentiate or distort signal. Might only give $I_b = \langle I(t) \rangle$ but over limited bandwidth original signal $I(t)$ may be restored.

A position monitor with low bandwidth measures just the average dipole moment of the bunch $y_b I_b = \langle y(t) I(t) \rangle$.

For larger bandwidth the bunch length can be resolved $y_b I(t)$ and for long bunches of a very large bandwidth also the position variation along the bunch can be observed (head-tail modes).
Fourier transform - spectrum and network analyzers

Fourier Transform converts $f(t) \rightarrow F(\omega)$

$$F(\omega) = \mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int f(t) [\cos(\omega t) - i \sin(\omega t)] \, dt$$

$$= F_r(\omega) + i F_i(\omega)$$

The cosine and sine transforms, called also real and imaginary part or resistive and reactive part, expressed also in amplitude and phase

$$A^2(\omega) = |F_r(\omega)|^2 + |F_i(\omega)|^2, \quad \tan \phi = \frac{F_i(\omega)}{F_r(\omega)}$$

$$A^2 \propto \text{power spectrum.}$$ Inverse transform

$$f(t) = \mathcal{F}^{-1}[F(\omega)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \, dt.$$

Periodic functions are developed in a series

$$f(t) = a_0 + 2 \sum_{1}^{\infty} \left( a_p \cos(p\omega_0 t) + b_p \sin(p\omega_0 t) \right)$$

$$a_p = (1/T_0) \int_{0}^{T_0} f(t) \cos(p\omega_0 t) \, dt$$

$$b_p = (1/T_0) \int_{0}^{T_0} f(t) \sin(p\omega_0 t) \, dt.$$ 

Spectrum analyzer Fourier transforms time signal $f(t)$ but integration range is limited and only amplitude is obtained, no phase since no absolute time involved. Most analyzers use swept frequency, i.e. at any moment only one frequency with finite bandwidth $\delta \omega$ is measured. Sweep speed is limited by the desired band-width. Some spectrum analyzers store signal $f(t)$ over certain time span make a Fast Fourier Transform (FFT) while the next time sample and a certain spectral range of interest is observed all the time.
Real time analyzer with parallel processor have been developed for RADAR

Search for Extra Terrestrial Intelligence
Unknown frequency, modulation, communication method, etc.
Network analyzer
A Network analyzer measures the beam response in amplitude $A$ and phase $\phi$, or in real $F_r(\omega)$ and imaginary $F_i(\omega)$ part, to harmonic excitation, transfer function. Most use a swept frequency, some a double channel FFT analyzer and noise excitation.

We can also compare relative amplitude and phase response of two monitors to a beam excitation and get information about beta function and phase.
## 2) TRANSVERSE EFFECTS

### Measurements

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Field gradients of quadrupoles focus horizontally in F, vertically in D-quad, quantified by $K = (e/p)(dB_y/dx) = d(1/\rho)/dx$ with $\ell K = 1/f$.

Deviations $x/y$ and $x'/y'$ from nominal are focused, betatron oscillations around orbit. Their maximum amplitude is a measure for local focusing strength, given by so-called beta functions $\beta_{x/y}$. Number of oscillations per revolution $Q_{x/y}$, called tunes, measure global focusing strength.

Particle with momentum deviation $\Delta p$, bent differently by dipoles, different orbit, displaced $\Delta x = D_x \Delta p/p_0$ ($D$=dispersion), circumference $C_0$, revolution frequency $\omega_0$

$$\frac{\Delta C'}{C_0} = \alpha_c \frac{\Delta p}{p} \quad \frac{\Delta \omega_0}{\omega_0} = \eta_c \frac{\Delta p}{p_0} \quad \eta_c = \alpha_c - \frac{1}{\gamma^2} \quad \alpha_c = \text{momentum compaction, computed.}$$

Storage ring has bending dipole and focusing quadrupole magnets, arranged in lattice, equilibrium orbit for nominal position, angle, momentum $p_0$, revolution frequency $\omega_0$. 
Single particle trajectory observed turn by turn

In storage ring with linear focusing by F and D-quads observe position, angle \( x_k, x'_k \) at location \( s \) of single particle each turn \( k \). Plotting \( x'_k \) vs \( x_k \) gives phase-space ellipse having everywhere same area and emittance \( A = \pi \epsilon \) but different shape. In F-quad \( x_k \) are large but angles \( x'_k \) small giving low, flat ellipse, in D-quads high and narrow ellipse. Their heights and widths are maxima \( x' \) and \( \hat{x} \) in many turns, product gives \( \epsilon_x = x' \hat{x} = \text{const.} \), ratio \( \beta(s) = \hat{x} / x' \). Between quads, ellipses tilted, \( x'_k / x_k \)-correlation \( < 0 \) after F, \( > 0 \) after D-quad, complicated \( \epsilon, \beta(s) \) expression.

Trajectory represents betatron oscillation, with number \( Q/\text{turn}=\text{tune} \). Not harmonic in \( s \), but expressed by computed \( \beta(s) \), phase \( \phi(s) \)

\[
 x(s) = \sqrt{\epsilon \beta(s)} \cos(\phi(s) - \phi_0)
\]

Points \( x_k \) at fixed location \( s \) plotted against \( k \) can be fitted with harmonic functions \( \sin((p \pm q) \omega_0 t) \), fractional tune \( q = Q \)-integer, revolution frequency \( \omega_0 \).
Many particles — distribution
Many particles oscillating with different phase, amplitude, emittance $\epsilon_i$ but same ratio $\hat{x}_i/\hat{x}'_i = \beta_x$. Distribution of emittances with average $\epsilon = \langle \epsilon_i \rangle$

The phase space distribution $\psi(x, x')$ is not measured directly but its projected spatial $f(x)$ and angular $g(x')$ distributions with center-of-mass, variances or RMS values.

$$f(x) = \int \psi(x, x')dx', \ g(x') = \int \psi(x, x')dx$$

$$\langle x \rangle = \frac{\int f(x)dx}{\int f(x)dx}, \ \langle x' \rangle = \frac{\int g(x')dx'}{\int g(x')dx'}$$

$$\sigma_x^2 = \frac{\int f(x)x^2dx}{\int f(x)dx}, \ \sigma_{x'}^2 = \frac{\int g(x')x'^2dx'}{\int g(x')dx'}$$

$$\epsilon_x = \langle \epsilon_x \rangle \approx \sigma_x \sigma_{x'}$$
Distribution measurements

\[ f(x) = \int \psi(x, x') dx', \quad g(x') = \int \psi(x, x') dx \]

With beam position monitors
Center-of-mass position \( \langle x \rangle \), with a close pair of monitors also center-of-mass angle \( \langle x' \rangle \).
In measurements of whole beam the center-of-mass behaves like a single particle.

Imaging with SR \( \rightarrow f(x) \) and \( \sigma_x \)

Direct SR observation \( \rightarrow g(x') \) and \( \sigma'_x \)

Emittance
\[ \epsilon = \frac{\langle x^2 \rangle}{\beta_x} = \frac{\sigma^2_x}{\beta_x} \approx \sigma_x \sigma'_x \]
**Closed orbit perturbation by a dipole**

With a short dipole magnet we make a deflection $\Delta \theta$ and calculate the new closed orbit.

\[
x(s) = \sqrt{\epsilon \beta(s)} \cos(\phi(s) - \phi_0)
\]

\[
x(0) = x(2\pi R)
\]

\[
x'(0) = x'(2\pi R) + \Delta \theta
\]

\[
x(0) = \frac{\Delta \theta \beta_x(0) \cos(\pi Q)}{2 \sin(\pi Q)}
\]

\[Q = \phi(2\pi R) - \phi(0) = \text{tune}\]

If $Q$ is close to integer the distortion is large - integer stop band.

Raw measurement of distorted orbit (top) is scaled with root of beta function (middle) and plotted against phase advance $\phi$ (bottom). Optics check, deflection, measure orbit distortion, compare with calculated $\beta(s)$ and $\phi(s)$. Phase separation of two points by $n\pi$ changes only if error is inbetween.
Cusp of distortion for different tunes

Qualitative cusp tells if tune is above or below half integer, quantitative $x_0$ gives either $Q$ or $\beta(0)$, not accurate, needs minimal instrumentation, works in presence of noise and coupling, likes monitor and deflector close by.

$$x_0 = \frac{\Delta \theta \beta(0) \cos(\pi Q)}{2 \sin(\pi Q)}.$$

Response matrix

Many correctors and monitors give 'response matrix' and measures beta function at many locations and check calibrations.

ALS: Robin, Safranek, Portmann, Nishimura
Beam bumps
Few correctors are powered to make local bump without effect outside. To probe physical aperture, scrape beam tails, centering special magnets, find coupling sources, etc.

Most important for position monitors and correctors is closed orbit measurement and correction. Based on figure about 2 monitors/correctors per betatron wavelength are needed and at strategic points, interaction region, undulators, a few more including a pair without magnets in between. Different strategies are used, global correction based on orbit harmonics, local one with beam-bumps, find most effective corrector and many more. Ideal orbit goes through quadrupole centers. Check monitor alignment by observing beam position or tune motion vs. quadrupole or sextupole strength, 'beam based alignment'.

\[ \Delta x' = -x_0/f = -x_0L\Delta K, \]
with \( K = \) quad strength parameter.
**Measuring betatron frequency — tune**

Oscillating bunch in time and frequency.

\[ x_k = \hat{x} \cos(2\pi q k), \quad \omega_\beta = (p \pm q)\omega_0, \quad q = \frac{1}{6} \]

Harmonic excitation: swept frequency, width = damping

Harmonic excitation, sweep both ways

**Green function, transfer function; equivalent, but non-linearities.**

Pulse excite fit vs. amplitude (A. Müller)

Harmonic excitation, sweep both ways

**Applications:**

Measure \( \omega_\beta \) fractional tune \( q \), test optics.

\[ q = q(\Delta K) \quad \text{tune vs. quad, } \beta \]

Dynamic aperture

Vertical kick, horizontal response

\[ q = q(\Delta E) \rightarrow Q' \text{ chromaticity} \]
**Measuring $\beta$-function at a quadrupole**

Variation $L\Delta K = \Delta(1/f)$ of single quadrupole changes tune $\Delta Q$ proportional to local $\beta$ which offers an important optics check

$$\Delta Q = \frac{\beta}{4\pi} L\Delta K \rightarrow \beta = \frac{4\pi \Delta Q}{L \Delta K}.$$ 

This assumes **small** $\Delta Q$ otherwise $\beta$ itself changes giving quadratic expression in $\Delta Q$

$$\beta = \frac{4\pi \Delta Q}{L \Delta K} \left(1 + \frac{\Delta Q}{\tan Q}\right).$$

Still short lens approximation, longer quadrupole involve neighborhood optics. Best simulate quadrupole change on computer code and compare.

Hysteresis makes $\Delta K/K \neq \Delta I/I$ and limit accuracy demanding small changes or recycling. Incorporate a loop in quad and measure induced integrated voltage to get directly the flux change.

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**Measure dynamic acceptance**

Dynamic acceptance gives normalized maximum betatron oscillation amplitude of beam optics, i.e. the maximum beam emittance. Limited by non-linear elements giving tune changes with amplitudes making oscillations unstable. Measured by exciting oscillation and increasing amplitude until life-time is short. To calibrate, scraper is moved into beam to a distance $x_a$ where gets even shorter, acceptance $A = x_a^2/\beta$.

At life-time limit check tune in case of resonances. To avoid orbit distortion effects move scraper from both sides, window scraper.
**Coupling measurement**

Horizontal and vertical betatron oscillations are usually treated as independent. Some elements, rotated quads, solenoids, couple them 

normal quadrupole    rotated quadrupole

\[ \ddot{x} + Q_x^2 \omega_0^2 x = k y, \quad \ddot{y} + Q_y^2 \omega_0^2 y = k x \]

**Excitation response in both planes**

Takinawa, ISR: Beam is excited horizontally, response in this plane (bottom) and vertical (top) are shown. Energy is exchanged, same modulation in both planes if \( \beta_x \approx \beta_y \).

**Closest tune approach:**

Increasing F-quad approaches tunes to minimum value \( \Delta \lambda \) and separate them again

\[ \Delta \lambda \approx \frac{k}{(Q \omega_0)} \]
**Beta function and phase advance**

An exited oscillation is measured in monitors $i$ each revolution $k$

$$x_{ik} = \frac{\hat{x}}{\sqrt{\beta_{x0}}} \sqrt{\beta_{xi}} \cos(2\pi Q_x k + \mu_{xi})$$

get $\beta_{xi}$, $\mu_{xi}$ phase advance $\Delta \phi = \mu_{i+1} - \mu_i$, has small systematic errors, bunch signal gives cable delays. $\beta_{x,i+1}/\beta_{x,i} = (\hat{x}_{i+1,k}/\hat{x}_{i,k})^2$ needs calibration.

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**Check optics by phase measurement**

Measuring betatron phase around ring and comparing calculation we check beam optics and locate errors. Local focusing error creates a $\beta(s)$ beating around the ring at twice the betatron frequency, i.e. points being separated by $n\pi$ will keep this separation unless the error is located in between them.

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Beta beating caused by a focusing error

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Experimental tracking to check a simulation, P. Morton et al.
### 3) LONGITUDINAL EFFECTS

**Measurements**

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**Longitudinal dynamics**

Particle $\Delta p > 0$ is bent less in dipoles, its closed orbit is outside from nominal by $\Delta x(s)$ and gets extra bending in quadupoles. Circumference of this off-momentum orbit is changed by $\Delta C$. In linear approximation

$$\Delta x = D_x \frac{\Delta p}{p_0}, \quad \frac{\Delta C}{C_0} = \alpha_c \frac{\Delta p}{p_0}, \quad \alpha_c = \left\langle \frac{D_x(s)}{C_0 \rho} \right\rangle.$$  

$D_x$ and $\alpha_c$, are computed with optics codes. Different $C$ and $\beta_c$ of off-momentum particles change revolution time $T$, frequency $\omega_0$.

A cavity is oscillating with peak voltage $\hat{V}$ at frequency $\omega_{RF} = \hbar \omega_r$, being a harmonic $\hbar$ of the nominal revolution frequency, replaces the energy $U_s$ lost per particle and turn and provides energy focusing.

$$\Delta T = -\frac{\Delta \omega_0}{\omega_0} = \left( \alpha_c - \frac{1}{\gamma^2} \right) \frac{\Delta p}{p_0} = \eta_c \frac{\Delta p}{p_0}.$$  

Usually $\alpha_x > 0$, however $\eta_c$ vanishes at transition energy defined by $\gamma_T^2 = 1/\alpha_c$ and is positive above and negative below this.
RF-system — longitudinal focusing

nominal: \( p_0 \approx E_0/c, \ T_0 = 2\pi/\omega_0 \)

synchr.: \( t_s, \ \phi_s = h\omega_0 t_s, \ e\hat{V}\sin\phi_s = U_s \)

deviation: \( \tau = t - t_s, \ \Delta E = E - E_0 \ll E_0 \)

dynamics: \( \frac{\Delta \tau}{T_0} = -\frac{\Delta \omega_0}{\omega_0} = \eta_c \frac{\Delta p}{p_0} \)

Particle arriving at \( t_s, \ \tau = 0 \) receives energy \( U_s \) just compensate loss by S.R.

Late particle, \( \tau > 0 \), gets less voltage, its energy will be lower, goes around faster, arrives earlier next turn and gains more energy. Result, particle oscillates around \( t_s \) and \( E_0 \) with frequency \( \omega_s \), making synchrotron oscillation.

\[ \tau = \hat{\tau} \sin(\omega_s t), \ \epsilon = \hat{\epsilon} \cos(\omega_s t), \ \hat{\tau} = \eta_c \hat{\epsilon} / \omega_s \]

\[ \omega_s^2 = -\omega_0^2 \eta_c h e \hat{V} \cos\phi_s / 2\pi E_0, \ Q_s = \frac{\omega_s}{\omega_0}, \ \epsilon = \frac{\Delta E}{E} \]

\[ \frac{\eta_c \epsilon^2}{2} + \frac{\omega_s^2 \tau^2}{2\eta_c} = \text{Hamiltonian} = \text{const.} \]

RF-system replaces in average energy lost by synchrotron radiation and focuses in time and energy. Particles oscillated with \( \omega_s \) around \( t_s \) and \( E_0 \) and form bunches with distribution in energy and time of RMS values \( \sigma_\tau \) and \( \sigma_\epsilon = \sigma_E/E_0 \)

\[ \sigma_\tau = \frac{\eta_c}{\omega_s} \sigma_\epsilon = 1 / \omega_0 \sqrt{\frac{2\pi E_0 \eta_c}{h e \hat{V} \cos\phi_s}} \sigma_\epsilon \]
**Bunch signals**

Measure bunch length and filling pattern.

\[ \tilde{I}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} I(t) \cos(\omega t) dt \quad \text{(symmetric)} \]

1 bunch single traversal, \( \sigma_t \sigma_\omega = 1 \)

\[ I_p = \frac{1}{T_0} \int_0^{T_0} I(t) \cos(p\omega_0 t) dt = \frac{\omega_0}{\sqrt{2\pi}} \tilde{I}(p\omega_0) \]

Time/frequency signal of 1 bunches

Time/frequency signal of 4 bunches
Timing signals of colliding bunches

For colliding beams it is important that the bunches meet in the interaction point where the experiment is located and where the beta functions and beam dimensions have a minimum. This is checked with intensity monitors located symmetrically on both sides of the interaction region.

Synchronous phase and energy loss

The energy loss $U_s$ per turn due synchrotron radiation is compensated by the RF-system with a bunch traversing cavity at synchronous phase $\phi_s$ satisfying $eV_{RF} \sin \phi_s = U_s$. Cable delays make absolute phase measurement difficult but we get its change versus $V_{RF}$ by comparing cavity voltage with the one induced by bunch at $\omega_{RF}$ in intensity monitor using vector voltmeter.

$U_s \propto \int B^2 ds$ but deflection angle $\propto \int B ds$, fringe fields affect the two differently.
Energy change

A beam momentum change is made by varying RF-frequency with constant magnets

\[
\frac{\Delta \omega_{RF}}{\omega_{RF}} = \frac{\Delta \omega}{\omega_0} = -\eta_c \frac{\Delta p}{p_0}, \quad \Delta x = D_x \frac{\Delta p}{p_0}
\]

Since \( C = h\lambda_{RF} \) a lower frequency forces beam on a longer orbit where energy is lower. This is used to measure energy dependent beam parameters, dispersion or chromaticity.

Dispersion measurement

Measure orbit difference for energy change, get dispersion from \( \Delta x = D_x \Delta p/p_0 \). Main interest \( D_x \) but there might be residual \( D_y \).

Energy loss distribution due to S.R.
Collider \( e^- \) and \( e^+ \) lose energy in dipoles, replaced by RF. At \( D_x > 0 \), opposite orbit shift, shown as difference. (Anke Müller).
Synchrotron oscillation frequencies
Single particle incoherent oscillation with frequency $\omega_s$, not seen by intensity monitor.

$$\omega_s = \omega_0 \sqrt{-\frac{e\eta_c h V_{RF} \cos \phi_s}{2\pi \beta^2 E_0}}, \quad \eta_c = \alpha_c - \frac{1}{\gamma^2}.$$ 

Monitor sees coherent motion, like the center-of-mass dipole mode with frequency $\omega_{s1}$ or quadrupole oscillation between, ‘short time, large energy spread’, vice versa, with $\omega_{s2}$. They represent a phase/amplitude modulation with sidebands around $p\omega_0$. Without other forces $\omega_{s1} = \omega_s$, $\omega_{s2} = 2\omega_s$ and check RF-voltage. Collective effects change these frequencies.

Injection phase error excites dipole, mismatch (wrong bunch length/energy-spread ratio) quadrupole mode; check injection. RF-voltage modulation in phase excites dipole, in amplitude, quadrupole mode.
Off-energy focusing — chromaticity
Curvature in a magnet is inversely proportional to momentum $1/\rho \propto 1/p$ giving for a higher momentum particle less focusing in quadrupoles and a tune dependence on momentum, called chromaticity which is always negative without correction.

$$Q' = \frac{\Delta Q}{\Delta p/p_0}, \quad \xi = \frac{\Delta Q/Q}{\Delta p/p_0}$$

Corrected with sextupoles at finite dispersion.

The local sextupole field is in the horizontal plane a quadrupole of strength $\propto x$, in the vertical plane it is a rotated quadrupole.

Sextupoles are installed at places of finite dispersion. A particle with excess energy is focused less by the quad but, since it is displaced to the outside, gets extra focusing by the sextupole as compensation. Without chromatic correction the energy oscillation of a particle makes a tune modulation and can cross resonances. Furthermore an instability, called 'head-tail' can occur.

Sextupoles are non-linear elements which can create resonances and limit dynamic aperture and have to be distributed such as to limit these effects.
Measure chromaticity
Chromaticity and change of momentum

\[ Q' = \frac{dQ}{dp/p} \quad \frac{dp}{p} = -\frac{1}{\eta_c \omega_{RF}} \frac{d\omega_{RF}}{\eta_c \omega_{RF}} \]

\[ p = p_0 + \Delta p \]

\[ p = p_0 \]

\[ \rho \]

\[ \Delta x \]

\[ \text{F-quad} \]

\[ \text{sextupole} \]

To get the chromaticity we measure the tunes as a function of \( f_{RF} \). This is done with the sextupoles on for the corrected and with them turned off for the natural chromaticity. The latter is also obtained by varying momentum through a dipole field change but keeping the beam on the nominal orbit going through the sextupole centers where they have no influence, this is based on \( dp/p_0 = dB/B_0 \).
Large rings, specially colliders, have long dispersion-free straight section with strong focusing. The chromaticity created there is only corrected where $D_x > 0$ and builds-up before reaching this. Measuring betatron phase advance for different energy deviations gives chromatic phase advance and checks the chromatic correction. LEP example shows saw-tooth of local chromaticity but vanishing over the whole ring. Mismatched off-energy orbit shows fine structure by beta beating.
3) COLLECTIVE EFFECTS

Longitudinal Overview
The single particle motion is given by external guide fields, dipoles, quadrupoles, RF, etc. Beam with many particles induces currents in vacuum chamber **impedance** and creates **self fields** acting back on it. This collective action can: give **synchrotron frequency shift** by modified focusing, increase initial disturbance, **instability**, **change particle distribution**.

**Multi-turn effects** driven by narrow-band cavity with memory build up instability in many turns with small self-fields treated as **perturbation**. Start a small disturbance from a stationary beam, calculate fields this produces through impedance and check if they increase/decrease the initial amplitude, gives growth/damping rate. Check this for orthogonal (independent) modes of disturbances.

Bunch induces fields in passive cavity which oscillate and act back next turn, in/decreasing original disturbance depending on phase.

**Single traversal effects** driven by strong self-fields from broad impedances change distribution, modify oscillation modes and can couple them. Self consistent solutions are difficult to get, **bunch lengthening**.
Wake function and impedance

Resonator

Beam induces wall current $I_w = -(I_b - \langle I_b \rangle)$

Cavities have narrow oscillation modes which drive coupled bunch instabilities. Each resembles an **RCL - circuit** and is treated as such. This circuit has shunt impedance $R_s$, inductance $L$, capacity $C$. Since they cannot be identified related parameters are used and measured directly: **resonance frequency** $\omega_r$, **quality factor** $Q$, **damping rate** $\alpha_s$:

$$\omega_r = \frac{1}{\sqrt{LC}}, \quad Q = R_s\sqrt{\frac{C}{L}} = \frac{R_s}{L\omega_r} = R_s C \omega_r$$

$$\alpha_s = \frac{\omega_r}{2Q}, \quad L = \frac{R_s}{Q\omega_r}, \quad C = \frac{Q}{\omega_r R_s}.$$
Impedance
Driving circuit with current $I = \hat{I} \cos(\omega t)$

\begin{align*}
V_R &= I_R R_s \\
V_C &= \frac{1}{C} \int I_C dt \\
V_L &= L \frac{dI_L}{dt}
\end{align*}

$\omega_r = \frac{1}{\sqrt{LC}}$

$V_R = V_C = V_L = V$, $I_R + I_C + I_L = I$

$\hat{I} = \dot{I}_R + \dot{I}_C + \dot{I}_L = \frac{\dot{V}}{R_s} + C\ddot{V} + \frac{V}{L}$

using $L = R_s/\omega_r Q$ and $C = Q/(\omega_r R_s)$

$$\dot{V} + \frac{\omega_r}{Q} \dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q} \hat{I}$$

Seeking harmonic solution

$$V(t) = \hat{I} R_s \frac{\cos(\omega t) + \frac{Q}{\omega_r \omega} \frac{\omega^2 - \omega_r^2}{\omega_r \omega} \sin(\omega t)}{1 + \frac{Q^2 (\omega^2 - \omega_r^2)^2}{\omega_r^2 \omega}}$$

Has cosine term in phase with exciting current, absorbs energy, resistive. Sine term is out of phase, does not absorb energy, reactive. Ratio between voltage and current is impedance as function of frequency $\omega$

$$Z_r = \frac{R_s}{1 + Q^2 \left(\frac{\omega^2 - \omega_r^2}{\omega_r \omega}\right)^2}, \ Z_i = \frac{-R_s Q \frac{\omega^2 - \omega_r^2}{\omega_r \omega}}{1 + Q^2 \left(\frac{\omega^2 - \omega_r^2}{\omega_r \omega}\right)^2}.$$
Induced voltage and energy loss by a stationary bunch

Circulating symmetric bunch ($N_b$ particles) has current

$$I(t) = I_0 + 2 \sum I_p \cos(p \omega_0 t), \quad I_p = \int_0^{T_0} I(t) \cos(p \omega_0 t) dt$$

In impedance $Z(\omega)$ it induces voltage

$$V(t) = \sum I_p [Z_r(p \omega_0) \cos(p \omega_0 t) - Z_i(p \omega_0) \sin(p \omega_0 t)]$$

Energy lost per particles and turn $U = \int_0^{T_0} I(t)V(t)dt/N_b$

$$U = \frac{2T_0}{N_b} \sum_1^\infty I_p^2 Z_r(p \omega_0) = \frac{2e}{I_0} \sum_1^\infty I_p^2 Z_r(p \omega_0)$$

using $\int_0^{T_0} \cos(p' \omega_0 t) \sin(p \omega_0 t) dt = 0$, $I_0 = eN_b/T_0$

$\int_0^{T_0} \cos(p' \omega_0 t) \cos(p \omega_0 t) dt = \frac{T_0}{2}$ for $p' = p$

$0$ for $p' \neq p$
Measuring energy loss

\[ U = \frac{2T_0}{N_b} \sum_{1}^{\infty} I_p Z_r(p\omega_0) \approx \frac{2}{N_b} \int_0^{\infty} |\tilde{I}(\omega)| Z_r(\omega) d\omega \]

Comparing the phase of the bunch signal with the one of the RF is difficult if there are many cavities. Also, at higher currents signals are larger, warm up the cable and expand it, leading to a current dependent delay.

A method developed at DESY avoids this by comparing the distance between two adjacent bunches as a function of the current in the second.

Measuring the energy loss for different bunch length helps to get information of the frequency dependence of the impedance.
Typical ring impedance

Aperture changes form cavity-like objects with $\omega_r$, $R_s$ and $Q$ and impedance $Z(\omega)$ developed for $\omega < \omega_r$, where it is inductive

$$Z(\omega) = \frac{R_s}{1 + \left(\frac{Q(\omega^2 - \omega_r^2)}{\omega_r}\right)^2} \approx j\frac{R_s\omega}{Q\omega_r} + \ldots$$

Sum impedance at $\omega \ll \omega_{rk}$ divided by mode number $n = \omega/\omega_0$ is with inductance $L$

$$\left|\frac{Z}{n}\right|_0 = \sum_k \frac{R_{sk}\omega_0}{Q_k\omega_{rk}} = L\omega_0 = L\frac{\beta c}{R}.$$
Potential well bunch lengthening

We take a parabolic bunch form

\[
I_b(\tau) = \hat{I} \left(1 - \frac{\tau^2}{\tau^2} \right) = \frac{3\pi I_0}{2\omega_0 \hat{\tau}} \left(1 - \frac{\tau^2}{\hat{\tau}^2} \right)
\]

\[
\frac{dI_b}{d\tau} = -\frac{3\pi I_0 \tau}{\omega_0 \hat{\tau}^3}, \quad I_0 = \langle I_b \rangle,
\]

\[
V = \hat{V} \left(\sin \phi_s + h\omega_0 \cos \phi_s \tau\right) + \frac{3\pi I_0 L \tau}{\omega_0 \hat{\tau}^3}
\]

\[
V = \hat{V} \left[\sin \phi_s + \cos \phi_s h\omega_0 \left(1 + \frac{3\pi |Z/n| I_0}{h\hat{V} \cos \phi_s (\omega_0 \hat{\tau})^3} \right) \right]
\]

\[
\omega_s^2 = -\frac{\omega_0^2 h\eta_c e \hat{V} \cos \phi_s}{2\pi E}
\]

\[
\omega_s^2 = \omega_s^2 \left[1 + \frac{3\pi |Z/n| I_0}{h\hat{V}_{RF} \cos \phi_s (\omega_0 \hat{\tau})^3} \right]
\]

\[
\Delta \omega_s = \frac{\omega_s - \omega_{s0}}{\omega_{s0}} \approx \frac{3\pi |Z/n| I_0}{2h\hat{V}_{RF} \cos \phi_s (\omega_0 \hat{\tau}_0)^3}
\]
Decreasing $\omega_s$ reduces longitudinal focusing, increases bunch length $\hat{\tau}$. Relative energy spread $\hat{\epsilon} = \hat{\tau}\omega_s/\eta_c$ is given for electrons by synchrotron radiation, for protons the product (emittance) $\hat{\tau}\hat{\epsilon}=\text{const.}$

\[
\frac{\omega_s^2}{\omega_s^2} = 1 + \frac{3\pi|Z/n|_0 I_0}{h\hat{V}_{RF} \cos \phi_s(\omega_0 \hat{\tau})^3}
\]

\[
\frac{\omega_s - \omega_{s0}}{\omega_{s0}} \approx \frac{3\pi|Z/n|_0 I_0}{2h\hat{V} \cos \phi_s(\omega_0 \hat{\tau})^3}
\]

Only incoherent frequency $\omega_s$ of single particles is changed (reduced $\gamma > \gamma_T$, increased $\gamma < \gamma_T$), not coherent dipole (rigid bunch) frequency $\omega_{s1}$. The two get separated.

From observed bunch lengthening impedance is estimated.

Frequency measurement would be better, but $\omega_s$ is invisible and $\omega_{s1}$ do not move, however, quadrupole mode can be used

\[
\frac{\omega_{s2} - 2\omega_{s0}}{2\omega_{s0}} = \frac{\Delta\omega_{s2}}{\omega_{s2}} \approx \frac{1}{4} \frac{\Delta\omega_s}{\omega_{s0}}.
\]
Measurements of $\omega_{s1}$ and $\omega_{s2}$

The measurement of the dipole and quadrupole synchrotron frequencies as a function of current gives an easy estimate of the reactive longitudinal impedance. Since the RF-voltage might change a little with current due to beam loading the observation of $\omega_1$ might serve as calibration. This is evident from the measurement at the SLAC damping ring for two different bunch lengths.
Robinson instability

Qualitative treatment

Important longitudinal instability of a bunch interacting with an narrow impedance, called Robinson instability. In a qualitative approach we take single bunch and a narrowband cavity of resonance frequency $\omega_r$ and impedance $Z(\omega)$ taking only its resistive part $Z_r$. The revolution frequency $\omega_0$ depends on energy deviation $\Delta E$

$$\frac{\Delta \omega_0}{\omega_0} = -\eta_c \frac{\Delta p}{p}.$$ 

While the bunch is executing a coherent dipole mode oscillation $\epsilon(t) = \hat{\epsilon} \cos(\omega_s t)$ its energy and revolution frequency are modulated. **Above transition** $\omega_0$ is small when the energy is high and $\omega_0$ is large when the energy is small. If the cavity is tuned to a resonant frequency slightly smaller than the RF-frequency $\omega_r < p\omega_0$ the bunch sees a higher impedance and loses more energy when it has an energy excess and it loses less energy when it has a lack of energy. This leads to a damping of the oscillation. If $\omega_r > p\omega_0$ this is reversed and leads to an instability. Below transition energy the dependence of the revolution frequency is reversed which changes the stability criterion.
Frequency domain, only one harmonic $p$

\[ \epsilon = \hat{e} e^{-\alpha_s t} \sin(\omega_s t), \text{ damping if } \alpha_s > 0 \]

\[ \alpha_s = \frac{\omega_s p I_0^2 (Z_r(\omega_{p+}) - Z_r(\omega_{p-}))}{2I_0 \hbar V \cos \phi_s} \]

\[ \gamma > \gamma_T, \cos \phi_s < 0, \text{ stable } Z_r(\omega_{p-}) > Z_r(\omega_{p+}) \]

Damping rate \( \propto Z_r \) difference at side-bands.

RF-cavity: \( p = \hbar, \ I_0 \approx I_0 \).

\[ \alpha_s \approx \frac{\omega_s I_0 (Z_r(\omega_{p+}) - Z_r(\omega_{p-}))}{2V \cos \phi_s} \]

General \[ \alpha_s = \sum_p \frac{\omega_s p I_0^2 (Z_r(\omega_{p+}) - Z_r(\omega_{p-}))}{2I_0 \hbar V \cos \phi_s} \]

Qualitative understanding

- Oscillating bunch \( (Q_s = 0.25) \)
- Stationary bunch
- Perturbation
- Cavity field induced by the two sidebands
- Phase motion of the bunch center

Narrow band \( \rightarrow \) long memory, vice-versa
Transverse collective effects

Transverse impedance

Field excited by \( Ix = D = \hat{D}\cos(\omega t) \)

\[
\frac{\partial E_z}{\partial x} = -kIx, \quad E_z(x) = -kIx^2
\]

\( Z_L(x) = -\int E_zdz/I = -E_z\ell/I = k\ell x^2 \)

\[
\int B\,d\vec{a} = -\int \vec{E}\,ds, \quad B_yx\ell = E_z\ell = -k\ell Dx
\]

\( \hat{B}_y = -k\hat{D}\cos(\omega t), \quad B = -k\hat{D}\sin(\omega t)/\omega \)

field \( B \) out of phase with \( D = Ix \)

\( \hat{B}_y = -k\hat{D}/\omega, \) Lorentz force \( \hat{F} \approx -ec\hat{B}_y \)

\[
Z_T = -\frac{Fx\ell}{e\hat{D}} = \frac{ck\ell}{\omega} = \frac{cZ_L}{x^2\omega} = \frac{c}{2\omega} \frac{d^2Z_L}{d\omega^2}.
\]

Used special case to define transverse impedance and its relation to second derivative of the longitudinal impedance of same mode. In General we have the impedances long.:: integrated field/current; trans.:: integrated defl. field/ dipole moment On resonance, \( E_z \) is in, \( B_y \) out of phase of \( I \).

General deflecting mode, using \( x = \hat{x}e^{j\omega t} \)

\[
Z_T(\omega) = \frac{\int \left( \vec{E}(\omega) + [\vec{v} \times \vec{B}(\omega)] \right) ds}{Ix(\omega)}
\]

Relation \( Z_L \) to \( Z_T \) of different modes:

In ring of global and vacuum chamber radii \( R \) and \( b \) the impedances, averaged for different modes, have semi-empirical ratio

\[
Z_T(\omega) \approx \frac{2RZ_L(\omega)}{b^2 \omega/\omega_0}
\]

From area available for the wall current we expect \( Z_L \propto 1/b \), therefore \( Z_T \propto 1/b^3 \).
Transverse multi-traversal instability of a single bunch

A bunch $p$ traverses a cavity with off-set $x$, excites a field $-E_z$ which converts after $T_r/4$ into field $-B_y$, then into $E_z$ and after into $B_y$. The bunch oscillates with tune $Q$ having a fractional part $q = 1/4$ seen as sidebands at $\omega_0(\text{integer } \pm q)$ by a stationary observer.

A) Cavity is tuned to upper sideband. Next turn bunch traverses in situation 'A', $t = T_r/4$ with velocity in $-x$-direction and gets by $B_y$ force in $+x$-direction which damps oscillation.

B) Cavity is tuned to lower sideband, bunch traverses next in situation 'B', $t = T_r3/4 = T_r(1 - 1/4)$ with negative velocity and force in same direction, increases velocity, instability.

\[
damping\ rate\ a = \frac{e\omega_0\beta_x}{4\pi m_0 c^2 \gamma I_0} \sum_{\omega > 0} \left( I_{p+}^2 Z_{Tr}(\omega_{p+}) - I_{p-}^2 Z_{Tr}(\omega_{p-}) \right), \ \omega_{p\pm} = \omega_0(p \pm q).
\]
Transverse instability of many bunches

$M$ bunches can oscillate in $M$ independent modes $n = M\Delta \phi / 2\pi$, phase $\Delta \phi$ between them seen in global view. Locally, bunches pass with increasing time delay shown as bullets fitted by upper (solid) and lower (dashed) side-band frequency. Higher frequencies can be fitted and spectrum repeats every $4\omega_0$. 

$\omega_p = \omega_0(pM \pm (n + q))$

<table>
<thead>
<tr>
<th>Spectrum $n = 3$, $q = 1/4$</th>
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</thead>
<tbody>
<tr>
<td>$\Delta \phi = \pi$, $n = 2$</td>
</tr>
<tr>
<td>$\Delta \phi = 3\pi/2$, $n = 3$</td>
</tr>
</tbody>
</table>

![Diagram of bunches and spectra](image-url)
Head-tail mode oscillation

Synchrotron motion in $\Delta E$ and $\tau$ affect transverse motion via chromaticity $Q' = dQ/(dp/p)$. For $\gamma > \gamma_T$ has excess energy moving from head to tail and lack going from tail to head. For $Q' > 0$, phase advances in first and lags in second step; vice versa for $Q' < 0$ or $\gamma < \gamma_T$. Figure shows betatron motion in steps of its period $T_\beta = T_0/q$.

$Q' = 0 \quad Q' > 0$

CERN booster; Gareyte, Sacherer.
Model of head-tail instability

Above transition energy:

$Q' = 0$: Going from head to tail or from tail to head has same phase change. Phase lag and advance between head an tail interchange, neither damping nor growth.

$Q' < 0$: Going from head to tail there is a loss in phase, going from tail to head a gain (picture), giving a systematic phase advance between head and tail and in average growth.

$Q' > 0$: Going from head to tail there is a gain in phase, going from tail to head a loss, giving a systematic phase lag between head and tail and in average damping.

Below transition this situation is reversed.

Head tail spectrum:

$$y I_p$$

Diagram: 

- Head has phase lag, amplitude is increased
- Tail has phase advance, amplitude is decreased

Diagram showing energy transfer from head to tail with $Q' < 0$.

Diagram showing energy transfer from tail to head with $Q' > 0$.

Diagram showing energy transfer from head to tail with $Q' = 0$.
Summary
The instability treatment used here was invented by K. Robinson. This was generalized to nearly all longitudinal and transverse bunched beam instabilities by Frank Sacherer.

This demands for resistive impedance at upper, $Z^+$, and lower, $Z^-$, side-band to fulfill a stability condition

<table>
<thead>
<tr>
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<th>above transition</th>
<th>below transition</th>
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<tr>
<td>longitudinal, stability</td>
<td>$Z^+_r &lt; Z^-_r$</td>
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<tr>
<td>transverse $Q' = 0$, stability</td>
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Ken Robinson

Frank Sacherer