Introduction to Transverse Beam Dynamics

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The Ideal World

I.) Magnetic Fields and Particle Trajectories
Largest storage ring: The Solar System

astronomical unit: average distance earth-sun
1AE ≈ 150 *10^6 km
Distance Pluto-Sun ≈ 40 AE
0.) Introduction and Basic Ideas

→ guide the particles on a well defined orbit („design orbit“)
→ focus the particles to keep each single particle trajectory
   within the vacuum chamber of the storage ring, i.e. close to the design orbit.

„... in the end and after all it should be a kind of circular machine“
→ need transverse deflecting force

\[ F = q \left( \vec{E} + \vec{v} \times \vec{B} \right) \]

Lorentz force

typical velocity in high energy machines: \( v \approx c \approx 3 \times 10^8 \frac{m}{s} \)

old greek dictum of wisdom:
if you are clever, you use magnetic fields in an accelerator wherever it is possible.
The ideal circular orbit

condition for circular orbit:

\[ F_L = e \nu B \]

\[ F_{\text{centr}} = \frac{\gamma m_0 v^2}{\rho} \]

\[ \frac{\gamma m_0 v}{\rho} = e \sqrt{B} \]

\[ \frac{p}{e} = B \rho \]

\[ B \rho = "\text{beam rigidity}" \]
1.) The Magnetic Guide Field

Dipole Magnets:

define the ideal orbit
homogeneous field created by two flat pole shoes

Normalise magnetic field to momentum:

\[ \frac{p}{e} = B \rho \quad \Rightarrow \quad \frac{1}{\rho} = \frac{eB}{p} \]

convenient units:

\[ B = [T] = \left[ \frac{Vs}{m^2} \right] \quad p = \left[ \frac{GeV}{c} \right] \]

Example LHC:

\[ B = 8.3T \]
\[ p = 7000 \frac{GeV}{c} \]
\[ \frac{1}{\rho} = e \frac{8.3 \frac{Vs}{m^2}}{7000 \times 10^9 \frac{eV}{c}} = \frac{8.3 \times 3 \times 10^8 \frac{m}{s}}{7000 \times 10^9 \frac{m^2}{c}} = 0.333 \frac{8.3}{7000} \frac{1}{m} \]
The Magnetic Guide Field

\[ \rho = 2.53 \text{ km} \quad \rightarrow \quad 2\pi \rho = 17.6 \text{ km} \approx 66\% \]

\[ B \approx 1\ldots 8 \text{ T} \]

rule of thumb:
\[ \frac{1}{\rho} \approx 0.3 \frac{B [T]}{p [\text{GeV}/c]} \]

„normalised bending strength“
2.) **Quadrupole Magnets:**

required: *focusing forces to keep trajectories in vicinity of the ideal orbit*

linear increasing Lorentz force

linear increasing magnetic field

\[ B_y = g \cdot x \quad B_x = g \cdot y \]

quadrupole field:

normalised gradient of a quadrupole magnet:

\[ k = \frac{g}{p/e} \]

simple rule:

\[ k \approx 0.3 \frac{g(T/m)}{p(GeV/c)} \]

LHC main quadrupole magnet

\[ g \approx 25 \ldots 220 \ T/m \]

what about the vertical plane:

... Maxwell

\[ \nabla \times B = \nabla \times \left( \frac{\delta F}{\delta t} \right) = 0 \]

\[ \Rightarrow \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y} \]
3.) The equation of motion:

**Linear approximation:**

* ideal particle $\Rightarrow$ design orbit

* any other particle $\Rightarrow$ coordinates $x, y$ small quantities $x, y \ll \rho$

$\Rightarrow$ magnetic guide field: only linear terms in $x$ & $y$ of $B$ are taken into account

in doing so, we can treat the particles as harmonic oscillators !!!

( a pendulum clock for example )

... as long as they are inside a storage ring element

well $10^{12}$ pendulum clocks might be a problem of its own.
Taylor Expansion of the B field:

\[ B_y(x) = B_{y0} + \frac{dB_y}{dx} x + \frac{1}{2!} \frac{d^2 B_y}{dx^2} x^2 + \frac{1}{3!} \frac{d^3 B_y}{dx^3} x^3 + \ldots \]

only terms linear in \( x, y \) taken into account  dipole fields
quadrupole fields

Separate Function Machines:

Split the magnets and optimise them according to their job:

bending, focusing etc

Example:

heavy ion storage ring TSR

* man sieht nur
dipole und quads \( \rightarrow \) linear
Equation of Motion:

Consider local segment of a particle trajectory ...
... and remember the old days: (Goldstein page 27)

radial acceleration:

\[ a_r = \frac{d^2 \rho}{dt^2} - \rho \left( \frac{d\theta}{dt} \right)^2 \]

Ideal orbit: \( \rho = \text{const}, \frac{d\rho}{dt} = 0 \)

Force: \( F = m \rho \left( \frac{d\theta}{dt} \right)^2 = m \rho \omega^2 \)

\( F = \frac{mv^2}{\rho} \)

→ centrifugal force

general trajectory: \( \rho \rightarrow \rho + x \)

\[ F = m \frac{d^2}{dt^2} (x + \rho) - \frac{mv^2}{x + \rho} = e B y \, v \]
\[ F = m \frac{d^2}{dt^2} (x + \rho) - \frac{mv^2}{x + \rho} = e B_y v \]

cooking recipe: develop for small \( x \)
constant / linear magnetic fields
normalize fields to momentum of particle
independent variable: \( t \to s \)

\[ x'' + x \left( \frac{1}{\rho^2} - k \right) = 0 \]

to make that very clear:

\( x = \text{horizontal distance of the particle from the design orbit} \)
\( s = \text{position of that particle in the ring (independent variable)} \)
\( x' = \text{angle of that particle with respect to the ideal orbit} \)
\( x'' = \text{change of this angle as a function of } ''s'' \)

... and then any particle behaves like Grandma's pendulum clock, running around at the speed of light.
Remarks:

* \[ x'' + \left( \frac{1}{\rho^2} - k \right) \cdot x = 0 \]

... there seems to be a focusing even without a quadrupole gradient

"weak focusing of dipole magnets"

\[ k = 0 \quad \Rightarrow \quad x'' = -\frac{1}{\rho^2} \cdot x \]

even without quadrupoles there is a retrieving force (i.e. focusing) in the bending plane of the dipole magnets

... in large machines it is weak. (!)

* Equation for the vertical motion:

\[ \frac{1}{\rho^2} = 0 \quad \text{no dipoles ... in general ...} \]

\[ k \leftrightarrow -k \quad \text{quadrupole field changes sign} \]

\[ y'' + k \ y = 0 \]
**Hard Edge Model:**

\[ x'' + \left( \frac{1}{\rho^2} - k(s) \right) x = 0 \]

... this equation is not correct !!!

\[ x''(s) + \left( \frac{1}{\rho^2(s)} - k(s) \right) x(s) = 0 \]

bending and focusing fields ... are functions of the independent variable „s“

Inside a magnet we assume constant focusing properties !

\[ \frac{1}{\rho} = \text{const} \quad k = \text{const} \]

\[ B \cdot l_{\text{eff}} = \int_0^{l_{\text{max}}} B \, ds \]
4.) Solution of Trajectory Equations

Define … hor. plane: $K = 1/\rho^2 - k$

… vert. Plane: $K = k$

\[
x'' + K x = 0
\]

Differential Equation of harmonic oscillator … with spring constant $K$

Ansatz: \[
x(s) = a_1 \cdot \cos(\omega s) + a_2 \cdot \sin(\omega s)
\]

general solution: linear combination of two independent solutions

\[
x'(s) = -a_1 \omega \sin(\omega s) + a_2 \omega \cos(\omega s)
\]

\[
x''(s) = -a_1 \omega^2 \cos(\omega s) - a_2 \omega^2 \sin(\omega s) = -\omega^2 x(s) \quad \rightarrow \quad \omega = \sqrt{K}
\]

general solution:

\[
x(s) = a_1 \cos(\sqrt{K} s) + a_2 \sin(\sqrt{K} s)
\]
**Hor. Focusing Quadrupole** $K > 0$:

\[
x(s) = x_0 \cdot \cos(\sqrt{|K|s}) + x'_0 \cdot \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|s})
\]

\[
x'(s) = -x_0 \cdot \sqrt{|K|} \cdot \sin(\sqrt{|K|s}) + x'_0 \cdot \cos(\sqrt{|K|s})
\]

For convenience expressed in matrix formalism:

\[
\begin{pmatrix}
    x \\
    x'
\end{pmatrix}_{s_1} = M_{foc} \ast 
\begin{pmatrix}
    x \\
    x'
\end{pmatrix}_{s_0}
\]

\[
M_{foc} = \begin{pmatrix}
    \cos(\sqrt{|K|s}) & \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|s}) \\
    -\sqrt{|K|} \sin(\sqrt{|K|s}) & \cos(\sqrt{|K|s})
\end{pmatrix}
\]

\[s = s_0 \quad \rightarrow \quad s = s_1\]

**determine** $a_1, a_2$ **by boundary conditions**:

\[
s = 0 \quad \rightarrow \quad \begin{cases}
    x(0) = x_0, \quad a_1 = x_0 \\
    x'(0) = x'_0, \quad a_2 = \frac{x'_0}{\sqrt{K}}
\end{cases}
\]
**hor. defocusing quadrupole:**

\[ x'' - K x = 0 \]

**Remember from school:**

\[ f(s) = \cosh(s), \quad f'(s) = \sinh(s) \]

**Ansatz:**

\[ x(s) = a_1 \cdot \cosh(\omega s) + a_2 \cdot \sinh(\omega s) \]

\[
M_{\text{defoc}} = \begin{pmatrix}
\cosh \sqrt{|K|} l & \frac{1}{\sqrt{|K|}} \sinh \sqrt{|K|} l \\
\sqrt{|K|} \sinh \sqrt{|K|} l & \cosh \sqrt{|K|} l
\end{pmatrix}
\]

**drift space:**

\[ K = 0 \]

\[
M_{\text{drift}} = \begin{pmatrix}
1 & l \\
0 & 1
\end{pmatrix}
\]

\[ ! \quad \text{with the assumptions made, the motion in the horizontal and vertical planes are independent ,, ... the particle motion in x & y is uncoupled} \]
Thin Lens Approximation:

matrix of a quadrupole lens

\[
M = \begin{pmatrix}
\cos \sqrt{|k|} l & \frac{1}{\sqrt{|k|}} \sin \sqrt{|k|} l \\
-\sqrt{|k|} \sin \sqrt{|k|} l & \cos \sqrt{|k|} l
\end{pmatrix}
\]

in many practical cases we have the situation:

\[
f = \frac{1}{k l_q} \gg l_q \quad \text{... focal length of the lens is much bigger than the length of the magnet}
\]

\[
limes: \quad l_q \rightarrow 0 \quad \text{while keeping} \quad k l_q = \text{const}
\]

\[
M_x = \begin{pmatrix} 1 & 0 \\ 1 & f \end{pmatrix} \quad \quad M_z = \begin{pmatrix} 1 & 0 \\ -1 & f \end{pmatrix}
\]

... useful for fast (and in large machines still quite accurate) „back on the envelope calculations“... and for the guided studies!
**Transformation through a system of lattice elements**

combine the single element solutions by multiplication of the matrices

\[
M_{\text{total}} = M_{QF} \times M_D \times M_{QD} \times M_{\text{Bend}} \times M_{D^*} \ldots
\]

\[
\begin{pmatrix}
  x \\
  x'
\end{pmatrix}_{s_2} = M(s_2,s_1) \times \begin{pmatrix}
  x \\
  x'
\end{pmatrix}_{s_1}
\]

**focusing lens**

**dipole magnet**

**defocusing lens**

---

typical values in a strong foc. machine:

\[
x \approx \text{mm}, \quad x' \leq \text{mrad}
\]
5.) *Orbit & Tune:*

**Tune:** number of oscillations per turn

- 64.31
- 59.32

*Relevant for beam stability:*

- **Non integer part**

**LHC revolution frequency:** 11.3 kHz

0.31 * 11.3 = 3.5 kHz
Question: what will happen, if the particle performs a second turn?

... or a third one or ... $10^{10}$ turns
19th century:

Ludwig van Beethoven: „Mondschein Sonate“

Sonate Nr. 14 in cis-Moll (op. 27/II, 1801)
Astronomer Hill:

differential equation for motions with periodic focusing properties
„Hill‘s equation“

Example: particle motion with periodic coefficient

\[ x''(s) - k(s)x(s) = 0 \]

restoring force \( \neq \) const, \( k(s) = \) depending on the position \( s \)
\( k(s+L) = k(s), \) periodic function

we expect a kind of quasi harmonic oscillation: \( \text{amplitude & phase will depend on the position } s \ \text{in the ring.} \)
6.) The Beta Function

General solution of Hill’s equation:

\[ x(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi) \]

\( \varepsilon, \Phi \) = integration constants determined by initial conditions

\( \beta(s) \) periodic function given by focusing properties of the lattice \( \leftrightarrow \) quadrupoles

\[ \beta(s + L) = \beta(s) \]

Inserting (i) into the equation of motion …

\[ \psi(s) = \int_{0}^{s} \frac{ds}{\beta(s)} \]

\( \Psi(s) = \text{„phase advance“ of the oscillation between point „0“ and „s“ in the lattice. For one complete revolution: number of oscillations per turn „Tune“} \)

\[ Q_{y} = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)} \]
7.) Beam Emittance and Phase Space Ellipse

General solution of Hill equation

\begin{align}
    x(s) &= \sqrt{\varepsilon} \sqrt{\beta(s)} \cos(\psi(s) + \phi) \\
    x'(s) &= -\frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}} \left\{ \alpha(s) \cos(\psi(s) + \phi) + \sin(\psi(s) + \phi) \right\}
\end{align}

From (1) we get

\[ \cos(\psi(s) + \phi) = \frac{x(s)}{\sqrt{\varepsilon} \sqrt{\beta(s)}} \]

Insert into (2) and solve for ε

\[ \varepsilon = \gamma(s) x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s) x'^2(s) \]

* ε is a constant of the motion ... it is independent of „s“
* parametric representation of an ellipse in the x x' space
* shape and orientation of ellipse are given by α, β, γ
Beam Emittance and Phase Space Ellipse

\[ \varepsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s) x'^2(s) \]

**Liouville:** in reasonable storage rings area in phase space is constant.

\[ A = \pi \varepsilon = \text{const} \]

\( \varepsilon \) beam emittance = wozilycity of the particle ensemble, intrinsic beam parameter, cannot be changed by the foc. properties.

Scientifiquely speaking: area covered in transverse \( x, x' \) phase space … and it is constant !!!
Particle Tracking in a Storage Ring

Calculate $x, x'$ for each linear accelerator element according to matrix formalism

plot $x, x'$ as a function of "s"
... and now the ellipse:

Note for each turn $x$, $x'$ at a given position "$s_1" and plot in the phase space diagram.
Résumé:

**beam rigidity:**  \[ B \cdot \rho = \frac{p}{q} \]

**bending strength of a dipole:**  \[ \frac{1}{\rho} \left[ m^{-1} \right] = \frac{0.2998 \cdot B_0(T)}{p(\text{GeV}/c)} \]

**focusing strength of a quadrupole:**  \[ k \left[ m^{-2} \right] = \frac{0.2998 \cdot g}{p(\text{GeV}/c)} \]

**focal length of a quadrupole:**  \[ f = \frac{1}{k \cdot l_q} \]

**equation of motion:**  \[ x'' + Kx = \frac{1}{\rho} \frac{\Delta p}{p} \]

**matrix of a foc. quadrupole:**  \[ x_{s2} = M \cdot x_{s1} \]

\[ M = \begin{pmatrix} \cos \sqrt{|K|} l & \frac{1}{\sqrt{|K|}} \sin \sqrt{|K|} l \\ -\sqrt{|K|} \sin \sqrt{|K|} l & \cos \sqrt{|K|} l \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \]
6.) **Bibliography:**

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6.) Mathew Sands: *The Physics of e+ e- Storage Rings*, SLAC report 121, 1970

7.) D. Edwards, M. Syphers: *An Introduction to the Physics of Particle Accelerators*, SSC Lab 1990
7.) Appendix: The equation of motion

Linear approximation:

* ideal particle → design orbit

* any other particle → coordinates \( x, y \) small quantities

\( x, y \ll \rho \)

→ magnetic guide field: only linear terms in \( x \) & \( y \) of \( B \)
have to be taken into account

Taylor Expansion of the \( B \) field:

\[
B_y(x) = B_{y0} + \frac{dB_y}{dx} x + \frac{1}{2!} \frac{d^2 B_y}{dx^2} x^2 + \frac{1}{3!} \frac{eg}{dx^3} + \ldots
\]

normalise to momentum \( p/e = B\rho \)

\[
\frac{B(x)}{p/e} = \frac{B_0}{B_0 \rho} + \frac{g \ast x}{p/e} + \frac{1}{2!} \frac{eg}{p/e} + \frac{1}{3!} \frac{eg}{p/e} + \ldots
\]
**Equation of Motion:**

Consider local segment of a particle trajectory...

... and remember the old days:

(Goldstein page 27)

radial acceleration:

\[
a_r = \frac{d^2 \rho}{dt^2} - \rho \left(\frac{d\theta}{dt}\right)^2
\]

Ideal orbit: \( \rho = \text{const}, \quad \frac{d\rho}{dt} = 0 \)

Force: \( F = m\rho \left(\frac{d\theta}{dt}\right)^2 = m\rho \omega^2 \)

\[ F = m\nu^2 / \rho \]

general trajectory: \( \rho \rightarrow \rho + x \)

\[
F = m \frac{d^2}{dt^2} (x + \rho) - \frac{mv^2}{x + \rho} = e B_y \nu
\]
\[ F = m \frac{d^2}{dt^2}(x + \rho) - \frac{mv^2}{x + \rho} = e B_y v \]

1. \[ \frac{d^2}{dt^2}(x + \rho) = \frac{d^2}{dt^2} x \quad \text{... as } \rho = \text{const} \]

2. remember: \( x \approx \text{mm}, \rho \approx \text{m} \ldots \Rightarrow \text{develop for small } x \)

\[ \frac{1}{x + \rho} \approx \frac{1}{\rho} (1 - \frac{x}{\rho}) \]

Taylor Expansion

\[ f(x) = f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \ldots \]

\[ m \frac{d^2 x}{dt^2} - \frac{mv^2}{\rho} (1 - \frac{x}{\rho}) = eB_y v \]
guide field in linear approx.

\[ B_y = B_0 + x \frac{\partial B_y}{\partial x} \]

\[ m \frac{d^2 x}{dt^2} - \frac{mv^2}{\rho} (1 - \frac{x}{\rho}) = ev \left\{ B_0 + x \frac{\partial B_y}{\partial x} \right\} \]

\[ \frac{d^2 x}{dt^2} - \frac{v^2}{\rho} (1 - \frac{x}{\rho}) = \frac{e v B_0}{m} + \frac{e v x g}{m} \]

independent variable: \( t \to s \)

\[ \frac{dx}{dt} = \frac{dx}{ds} \frac{ds}{dt} \]

\[ \frac{d^2 x}{dt^2} = \frac{d}{dt} \left( \frac{dx}{ds} \right) \frac{ds}{dt} = \frac{d}{ds} \left( \frac{dx}{ds} \right) \frac{ds}{dt} \]\n
\[ \frac{d^2 x}{dt^2} = x'' v^2 + \frac{dx}{ds} \frac{dv}{ds} v \]

\[ x'' v^2 - \frac{v^2}{\rho} (1 - \frac{x}{\rho}) = \frac{e v B_0}{m} + \frac{e v x g}{m} \]

\[ : v^2 \]
\[
x'' - \frac{1}{\rho} \left(1 - \frac{x}{\rho}\right) = \frac{e}{mv} B_0 \frac{x}{mv} + \frac{e x g}{mv}
\]

\[
x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = \frac{B_0}{p/e} + \frac{x g}{p/e}
\]

\[
x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = -\frac{1}{\rho} + k x
\]

\[
x'' + x \left(\frac{1}{\rho^2} - k\right) = 0
\]

\[
m v = p
\]

**normalize to momentum of particle**

\[
\frac{B_0}{p/e} = -\frac{1}{\rho}
\]

\[
\frac{g}{p/e} = k
\]

**Equation for the vertical motion:**

\[
\frac{1}{\rho^2} = 0 \quad \text{no dipoles … in general …}
\]

\[
k \leftrightarrow -k \quad \text{quadrupole field changes sign}
\]

\[
y'' + k y = 0
\]