Introduction to Transverse Beam Dynamics

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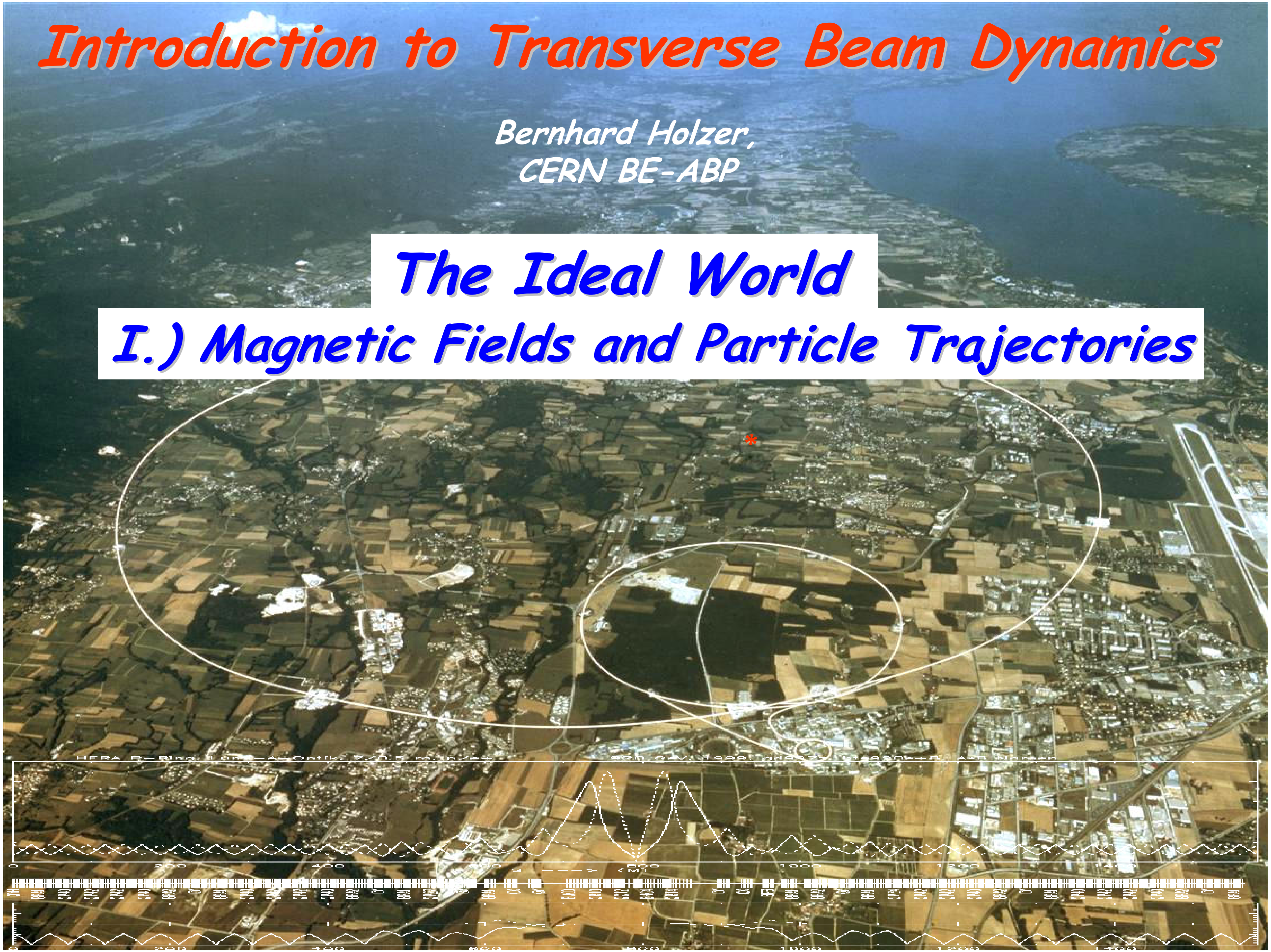
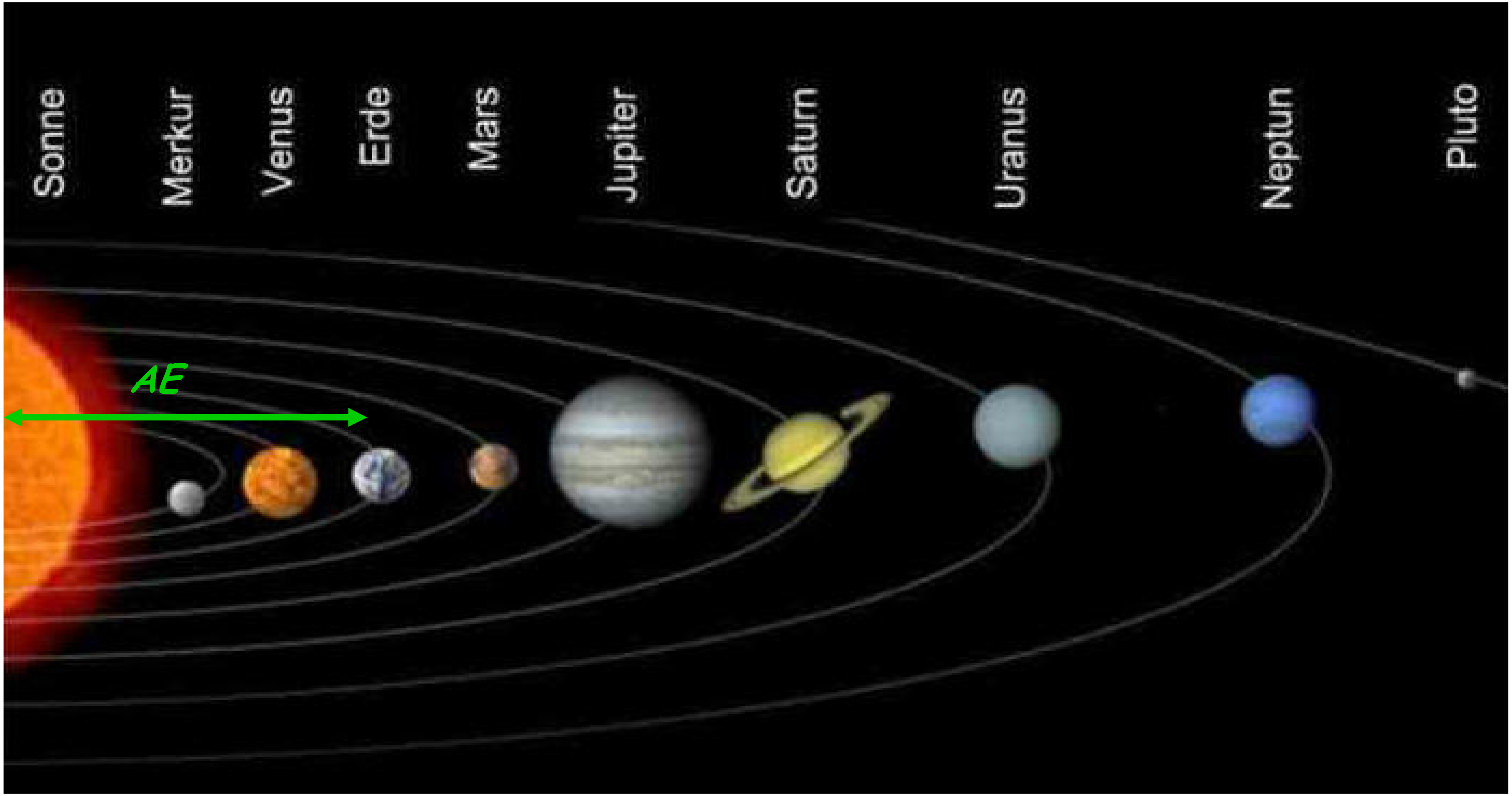
The Ideal World

I.) Magnetic Fields and Particle Trajectories



Largest storage ring: The Solar System

astronomical unit: average distance earth-sun
$1AE \approx 150 * 10^6$ km
Distance Pluto-Sun $\approx$ 40 AE



0.) Introduction and Basic Ideas

→ guide the particles on a well defined orbit („design orbit“)
→ focus the particles to keep each single particle trajectory within the vacuum chamber of the storage ring, i.e. close to the design orbit.

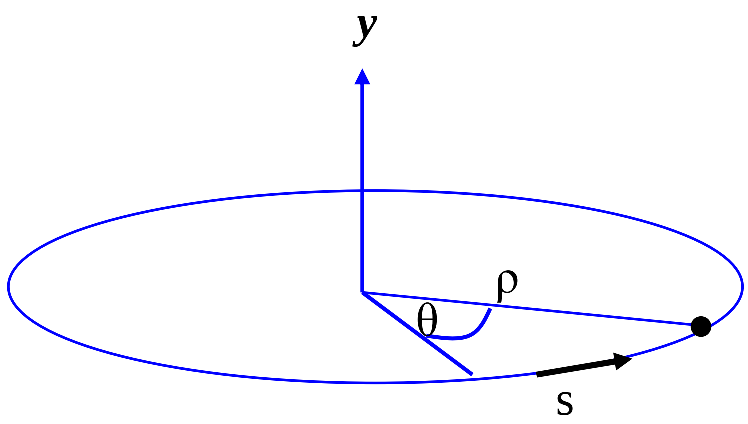
„ ... in the end and after all it should be a kind of circular machine“
→ need transverse deflecting force

Lorentz force $$F = q\,(\cancel{\vec{E}} + \vec{v} \times \vec{B})$$

typical velocity in high energy machines: $$v \approx c \approx 3*10^8\ \frac{m}{s}$$

old greek dictum of wisdom:
if you are clever, you use magnetic fields in an accelerator wherever it is possible.

The ideal circular orbit



circular coordinate system

condition for circular orbit:

Lorentz force $F_L = e\,v\,B$

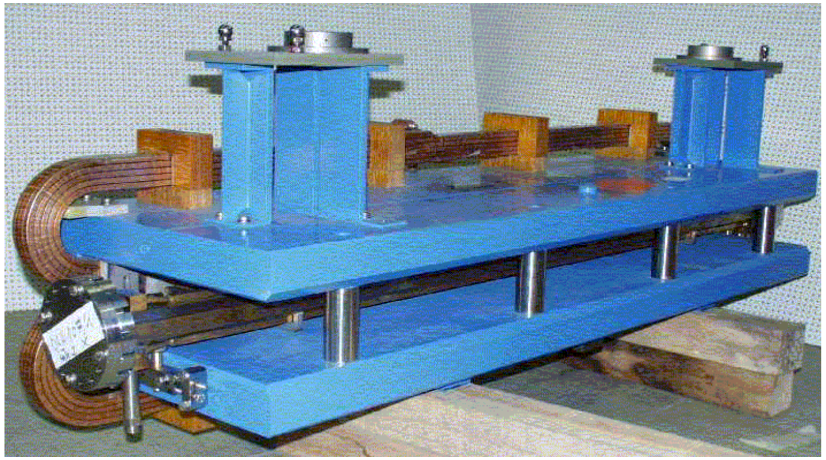
centrifugal force $F_{centr} = \dfrac{\gamma m_0 v^2}{\rho}$

$$\frac{\gamma m_0 \cancel{v^2}}{\rho} = e\,\cancel{v}\,B$$

$$\boxed{\frac{p}{e} = B\,\rho}$$

$B\rho$ = "beam rigidity"

1.) The Magnetic Guide Field



Dipole Magnets:

define the ideal orbit
homogeneous field created by two flat pole shoes

Normalise magnetic field to momentum:

$$\frac{p}{e} = B\rho \quad \longrightarrow \quad \frac{1}{\rho} = \frac{e\,B}{p}$$

convenient units:

$$B = [T] = \left[\frac{Vs}{m^2}\right] \qquad p = \left[\frac{GeV}{c}\right]$$

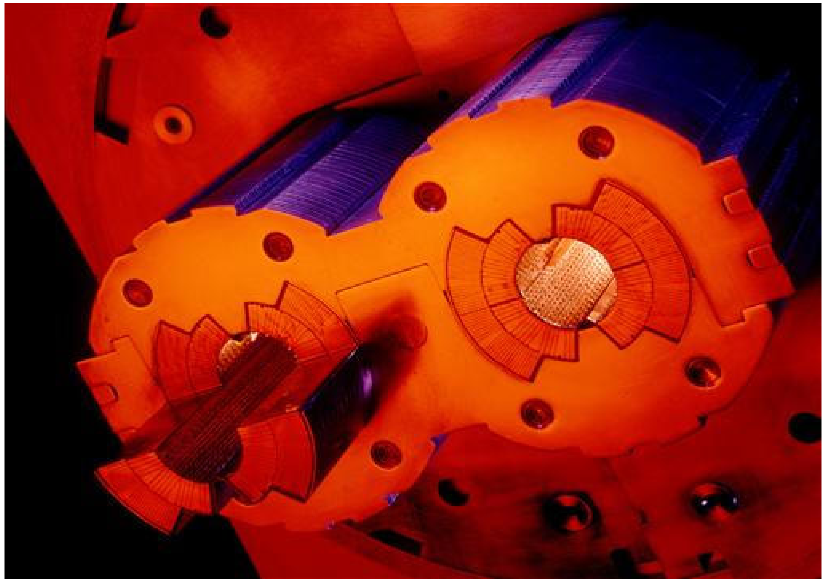
Example LHC:

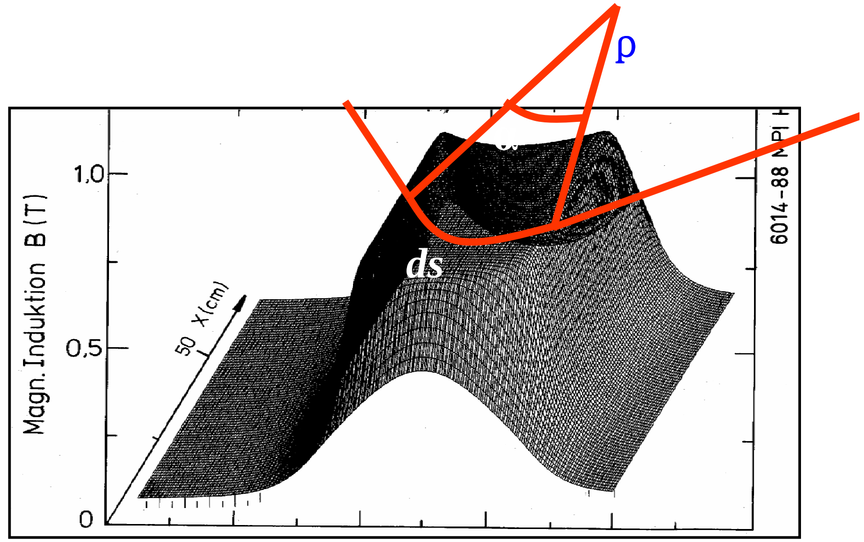
$$\left.\begin{aligned} B &= 8.3\,T \\ p &= 7000\,\frac{GeV}{c} \end{aligned}\right\}$$

$$\frac{1}{\rho} = e\,\frac{8.3\,{}^{Vs}\!/_{m^2}}{7000*10^9\,{}^{eV}\!/_{c}} = \frac{8.3\,s\,3*10^8\,{}^{m}\!/_{s}}{7000*10^9\,m^2}$$

$$\frac{1}{\rho} = 0.333\,\frac{8.3}{7000}\,{}^{1}\!/_{m}$$

The Magnetic Guide Field





field map of a storage ring dipole magnet

$$\rho = 2.53\,\boldsymbol{km} \longrightarrow \boldsymbol{2\pi\rho = 17.6\ km}$$

$$\boldsymbol{\approx 66\%}$$

$$\boldsymbol{B} \approx 1 \ldots 8\ \boldsymbol{T}$$

rule of thumb:

$$\frac{1}{\rho} \approx 0.3\,\frac{\boldsymbol{B}[\boldsymbol{T}]}{\boldsymbol{p}[\boldsymbol{GeV/c}]}$$

„normalised bending strength“

2.) Quadrupole Magnets:

required: *focusing forces to keep trajectories in vicinity of the ideal orbit*

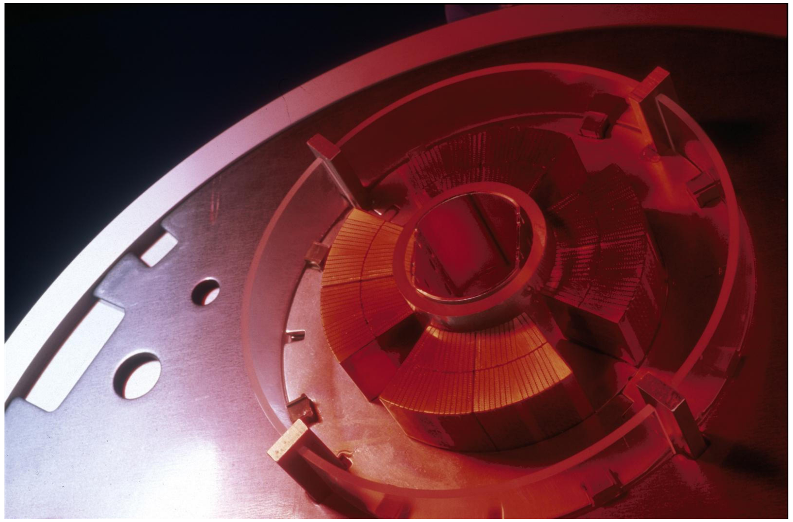
linear increasing Lorentz force

linear increasing magnetic field $B_y = g\ x \qquad B_x = g\ y$

quadrupole field:

normalised gradient of a quadrupole magnet:

$$\longrightarrow \qquad k = \frac{g}{p/e}$$



LHC main quadrupole magnet

$$g \approx 25 \ldots 220\ \ T/m$$

simple rule:

$$\boxed{k = 0.3 \frac{g(T/m)}{p(GeV/c)}}$$

what about the vertical plane:
... Maxwell

$$\vec{\nabla} \times \vec{B} = \cancel{\vec{j}} + \cancel{\frac{\partial \vec{E}}{\partial t}} = 0 \qquad \Rightarrow \qquad \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$$

3.) The equation of motion:

Linear approximation:

** ideal particle → design orbit*

** any other particle → coordinates x, y small quantities*
$x,y \ll \rho$

→ magnetic guide field: only linear terms in x & y of B are taken into account

in doing so, we can treat the particles as harmonic oscillators !!!
(a pendulum clock for example)

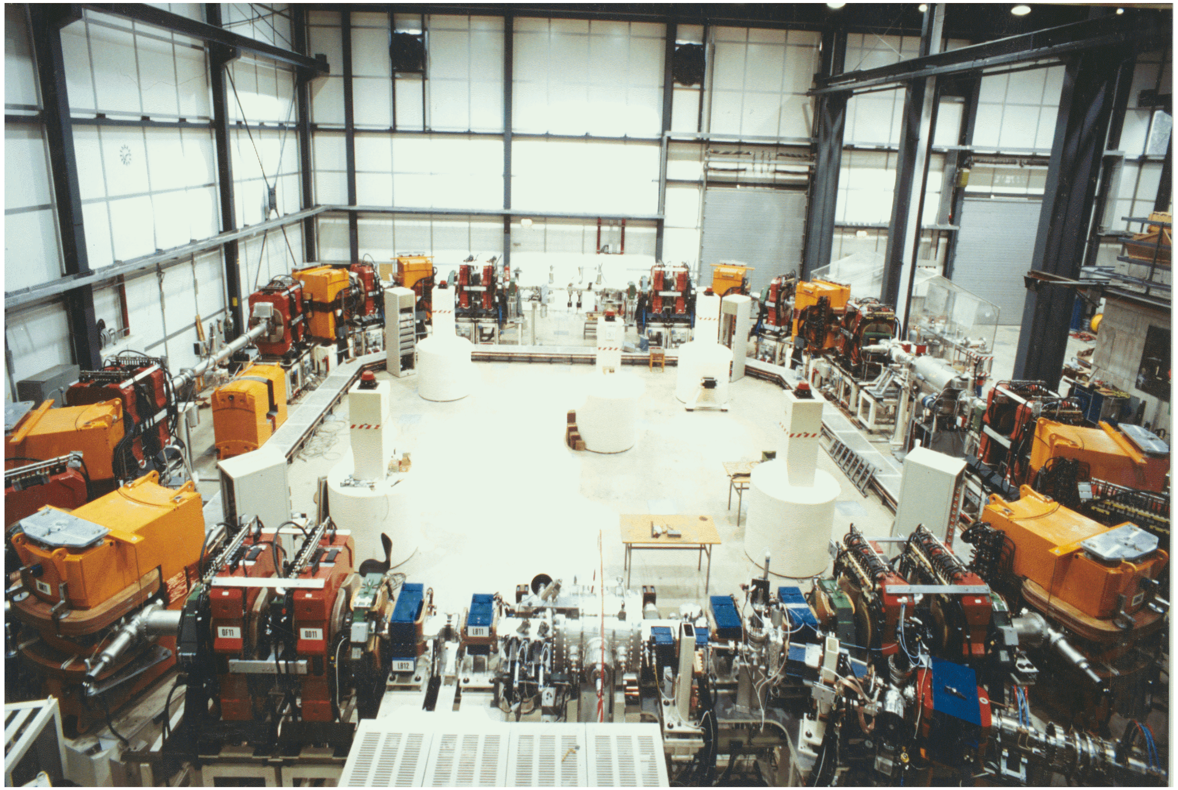
... as long as they are inside a storage ring element

well 10^{12} pendulum clocks might be a problem of its own.

Taylor Expansion of the B field:

$$B_y(x) = B_{y0} + \frac{dB_y}{dx}\,x + \cancel{\frac{1}{2!}\frac{d^2B_y}{dx^2}}\,x^2 + \cancel{\frac{1}{3!}\frac{eg''}{dx^3}} + \ldots$$

only terms linear in x, y taken into account ***dipole fields***
quadrupole fields



Separate Function Machines:

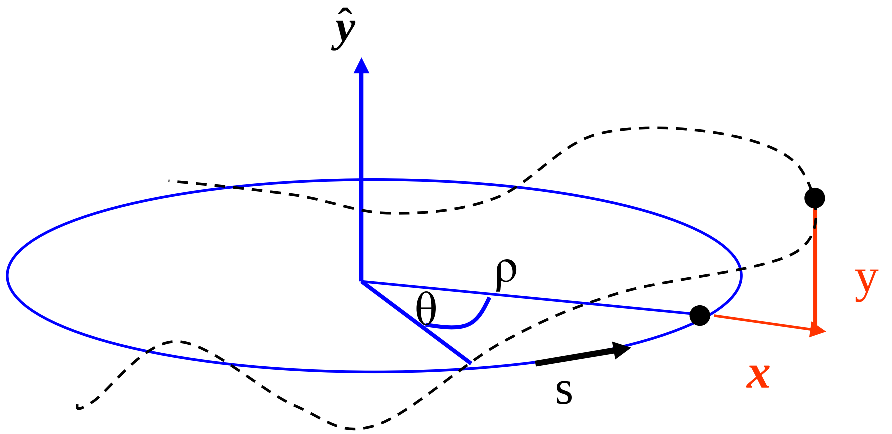
Split the magnets and optimise them according to their job:

bending, focusing etc

Example:
heavy ion storage ring TSR

* *man sieht nur*
dipole und quads → linear

Equation of Motion:



Consider local segment of a particle trajectory
... and remember the old days:
(Goldstein page 27)

radial acceleration:

$$a_r = \frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2$$

general trajectory: $\rho \rightarrow \rho + x$

$$\boldsymbol{F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v}$$

Ideal orbit: $\rho = const, \quad \frac{d\rho}{dt} = 0$

Force: $F = m\rho\left(\frac{d\theta}{dt}\right)^2 = m\rho\omega^2$

$$F = mv^2/\rho$$

→ centrifugal force

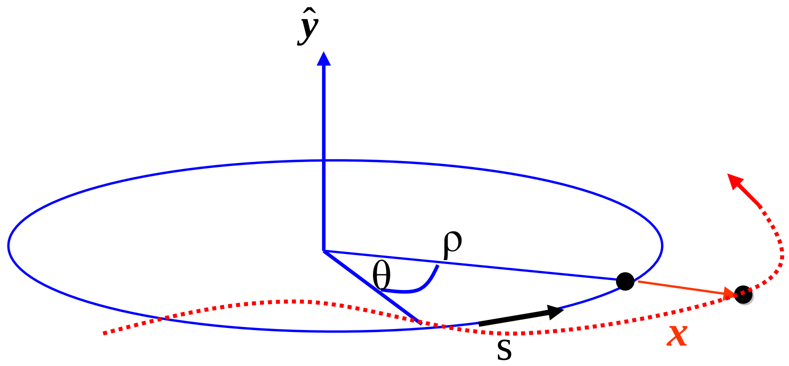
$$F = m \frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v$$

cooking recipe: ***develop for small x***
constant / linear magnetic fields
normalize fields to momentum of particle
independent variable: $t \rightarrow s$

$$\frac{B_0}{p/e} = -\frac{1}{\rho}$$

$$\frac{g}{p/e} = k$$

$$\boxed{x'' + x\left(\frac{1}{\rho^2} - k\right) = 0}$$



to make that very clear:

x = horizontal distance of the particle from the design orbit
s = position of that particle in the ring (independent variable)
x' = angle of that particle with respect to the ideal orbit
x'' = change of this angle as a function of "s"

$$x' = \frac{\Delta x}{\Delta s}$$

... and then any particle behaves like Grandma's pendulum clock, running around at the speed of light.

Remarks:

* $$x'' + \left(\frac{1}{\rho^2} - k\right) \cdot x = 0$$

… there seems to be a focusing even without a quadrupole gradient

„weak focusing of dipole magnets“

$$\boldsymbol{k} = 0 \quad \Rightarrow \quad \boldsymbol{x}'' = -\frac{1}{\rho^2}\,\boldsymbol{x}$$

even without quadrupoles there is a retriving force (i.e. focusing) in the bending plane of the dipole magnets

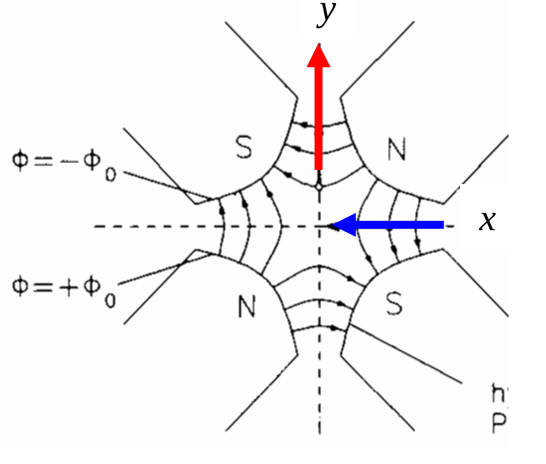
… in large machines it is weak. *(!)*

* *Equation for the vertical motion:*

$$\frac{1}{\rho^2} = 0$$ *no dipoles … in general …*

$$\boldsymbol{k} \quad \leftrightarrow \quad -\boldsymbol{k}$$ *quadrupole field changes sign*

$$\boldsymbol{y}'' + \boldsymbol{k}\;\boldsymbol{y} = 0$$



* ***Hard Edge Model:***

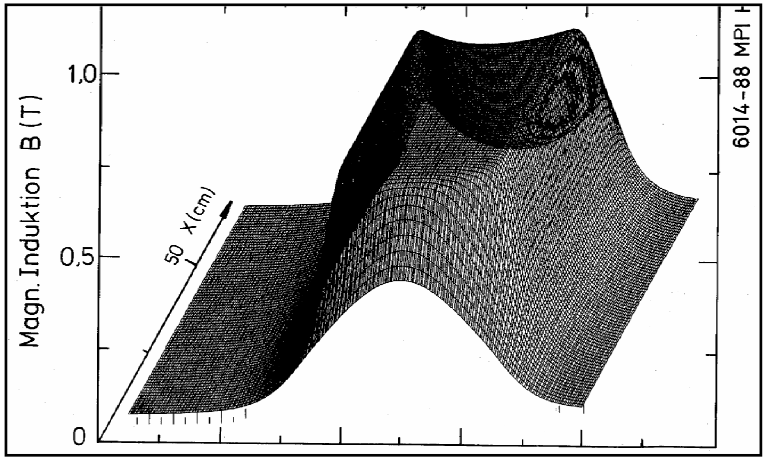
$$x'' + \left\{ \frac{1}{\rho^2} - k \right\} x = 0$$

… this equation is not correct !!!

$$x''(s) + \left\{ \frac{1}{\rho^2(s)} - k(s) \right\} x(s) = 0$$

bending and focusing fields … are functions of the independent variable „s“

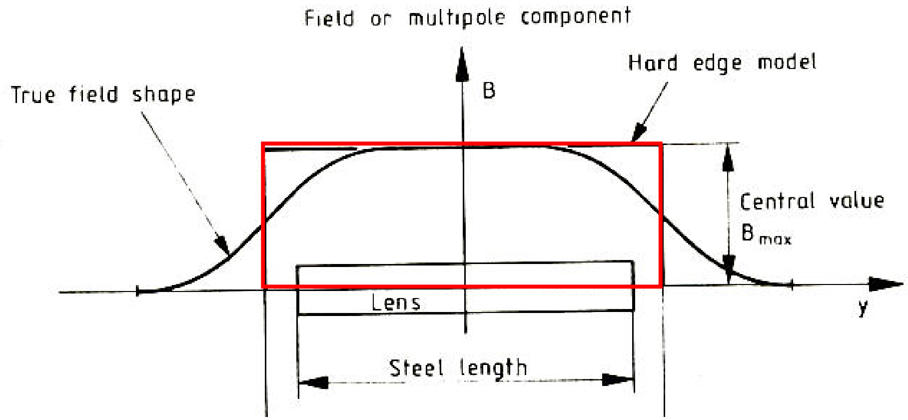
!



Inside a magnet we assume constant focusing properties !

$$\frac{1}{\rho} = const \qquad k = const$$

$$B\, l_{eff} = \int_0^{l_{mag}} B\, ds$$



4.) Solution of Trajectory Equations

Define … hor. plane: $K = 1/\rho^2 - k$

… vert. Plane: $K = k$

$$\boldsymbol{x}'' + \boldsymbol{K}\ \boldsymbol{x} = 0$$

Differential Equation of harmonic oscillator … with spring constant K

Ansatz: $x(s) = a_1 \cdot \cos(\omega s) + a_2 \cdot \sin(\omega s)$

general solution: linear combination of two independent solutions

$$x'(s) = -a_1\omega \sin(\omega\, s) + a_2\omega \cos(\omega\, s)$$

$$x''(s) = -a_1\omega^2 \cos(\omega s) - a_2\omega^2 \sin(\omega s) = -\omega^2 x(s) \longrightarrow \omega = \sqrt{K}$$

general solution:

$$x(s) = a_1 \cos(\sqrt{K} s) + a_2 \sin(\sqrt{K} s)$$

determine a_1 , a_2 by boundary conditions:

$$s = 0 \longrightarrow \begin{cases} x(0) = x_0 \quad , \quad a_1 = x_0 \\ x'(0) = x'_0 \quad , \quad a_2 = \dfrac{x'_0}{\sqrt{K}} \end{cases}$$

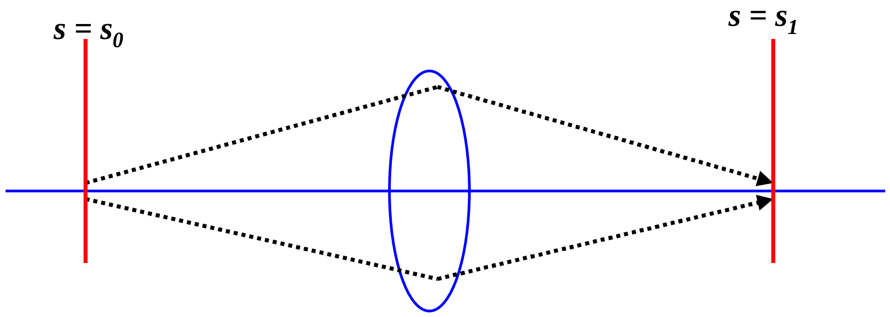
Hor. Focusing Quadrupole $K > 0$:

$$x(s) = x_0 \cdot \cos(\sqrt{|K|}s) + x'_0 \cdot \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s)$$

$$x'(s) = -x_0 \cdot \sqrt{|K|} \cdot \sin(\sqrt{|K|}s) + x'_0 \cdot \cos(\sqrt{|K|}s)$$

For convenience expressed in matrix formalism:

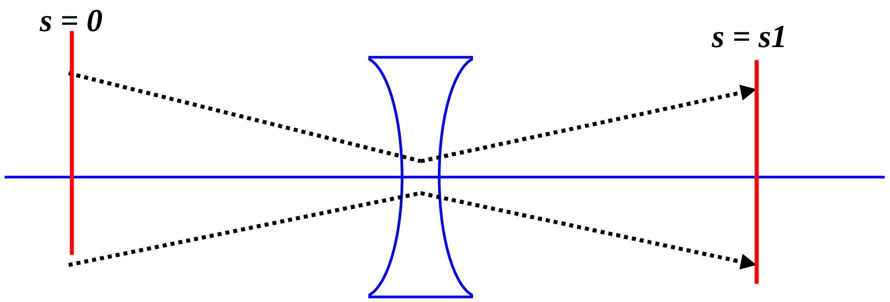
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M_{foc} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$



$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \dfrac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s \\ -\sqrt{|K|} \sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0$$

hor. defocusing quadrupole:

$$x'' - K\,x = 0$$



Remember from school:

$$f(s) = \cosh(s) \quad , \qquad f'(s) = \sinh(s)$$

Ansatz: $$x(s) = a_1 \cdot \cosh(\omega s) + a_2 \cdot \sinh(\omega s)$$

$$M_{defoc} = \begin{pmatrix} \cosh\sqrt{|K|}\,l & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}\,l \\ \sqrt{|K|}\sinh\sqrt{|K|}\,l & \cosh\sqrt{|K|}\,l \end{pmatrix}$$

drift space:

K = 0

$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

! with the assumptions made, the motion in the horizontal and vertical planes are independent „ ... the particle motion in x & y is uncoupled“

Thin Lens Approximation:

matrix of a quadrupole lens

$$M = \begin{pmatrix} \cos\sqrt{|k|}l & \frac{1}{\sqrt{|k|}}\sin\sqrt{|k|}l \\ -\sqrt{|k|}\sin\sqrt{|k|}l & \cos\sqrt{|k|}l \end{pmatrix}$$

in many practical cases we have the situation:

$$f = \frac{1}{kl_q} >> l_q$$

... focal length of the lens is much bigger than the length of the magnet

limes: $\boldsymbol{l_q} \rightarrow 0$ ***while keeping*** $\boldsymbol{k\, l_q} = \boldsymbol{const}$

$$M_x = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \qquad M_z = \begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix}$$

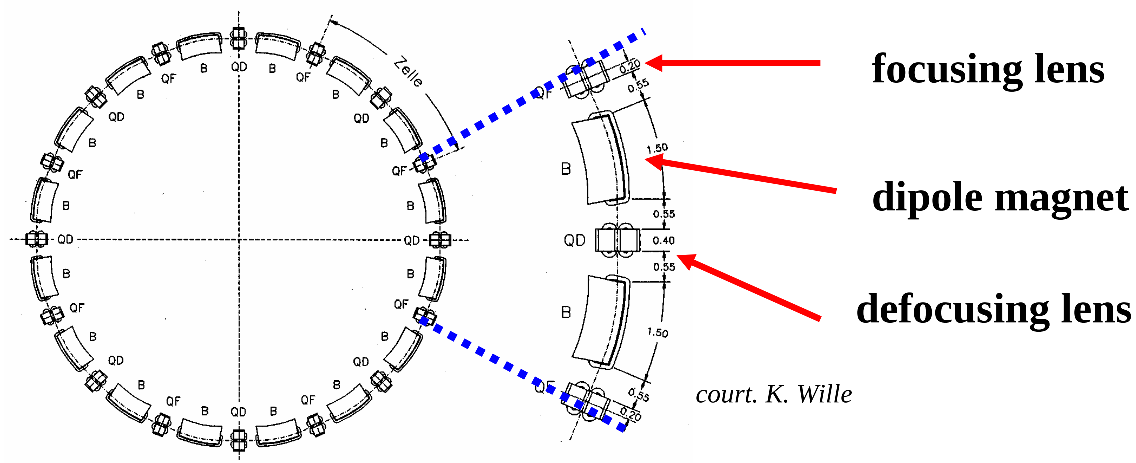
... useful for fast (and in large machines still quite accurate) „back on the envelope calculations“ ... and for the guided studies !

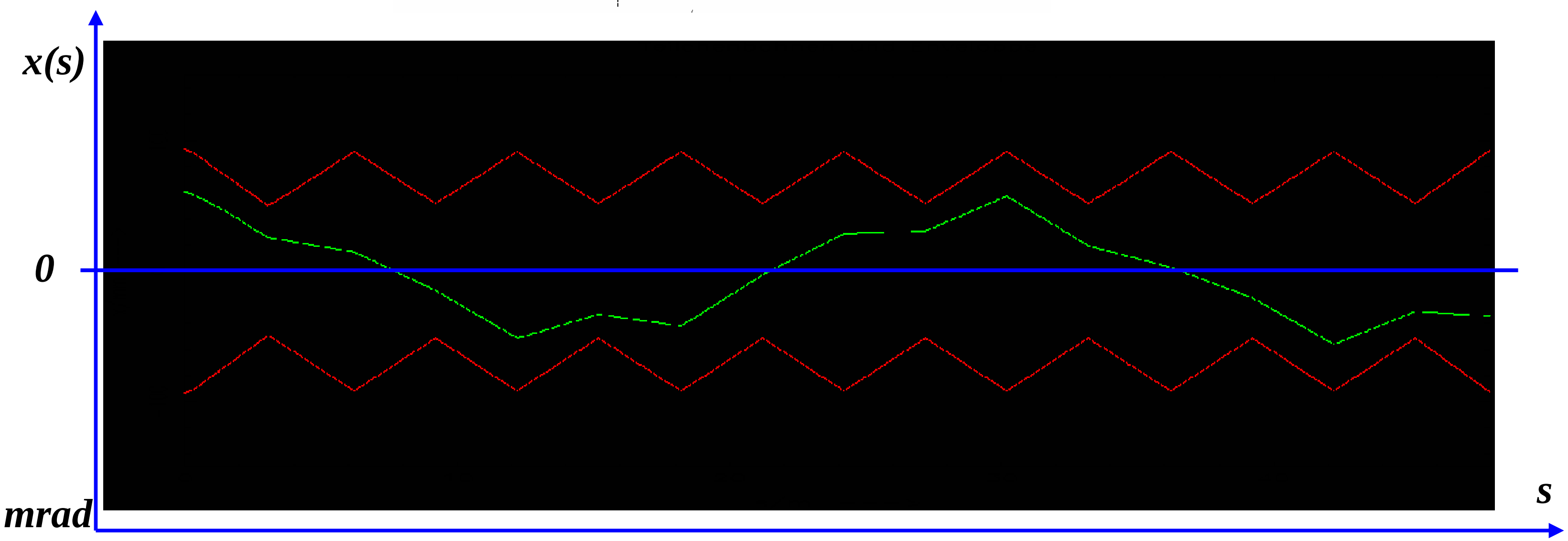
Transformation through a system of lattice elements

combine the single element solutions by multiplication of the matrices

$$M_{total} = M_{QF} * M_D * M_{QD} * M_{Bend} * M_{D^*.....}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s2,s1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$





typical values in a strong foc. machine:
$x \approx$ mm, $x' \leq$ mrad

5.) Orbit & Tune:

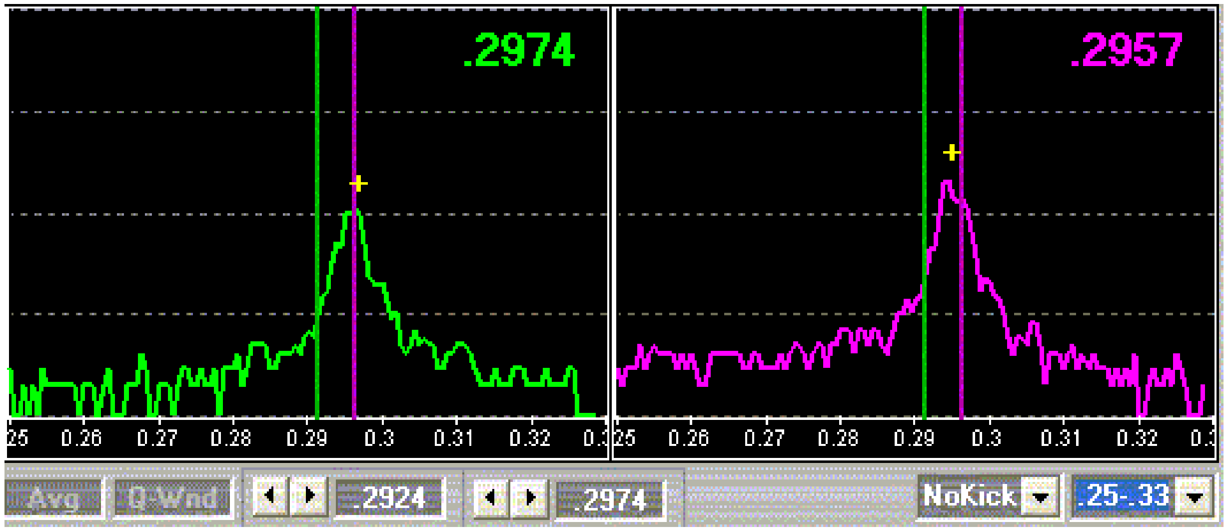
Tune: number of oscillations per turn

64.31
59.32

Relevant for beam stability:
non integer part

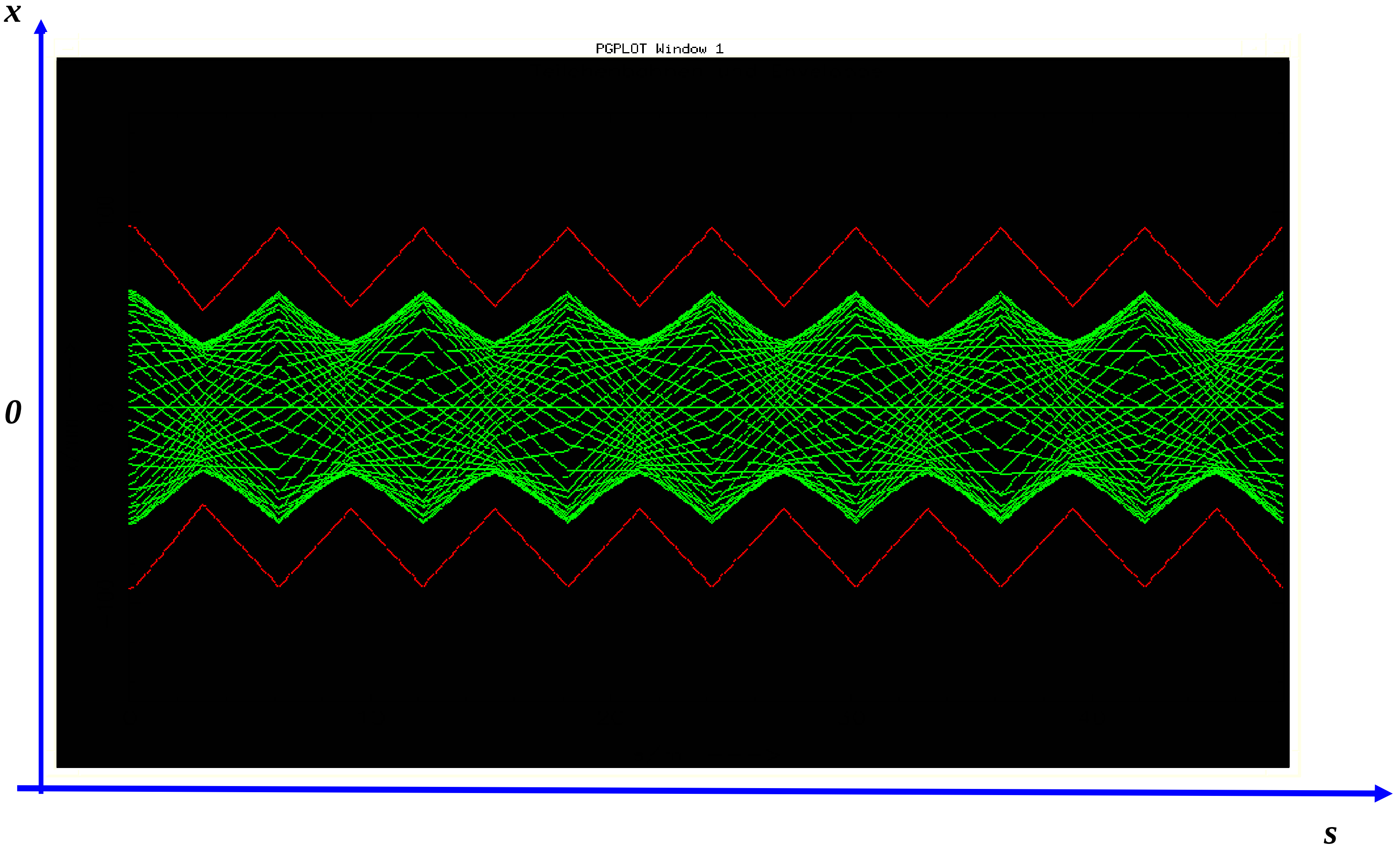
LHC revolution frequency: 11.3 kHz

$0.31 * 11.3 = 3.5\,kHz$



Question: ***what will happen, if the particle performs a second turn ?***

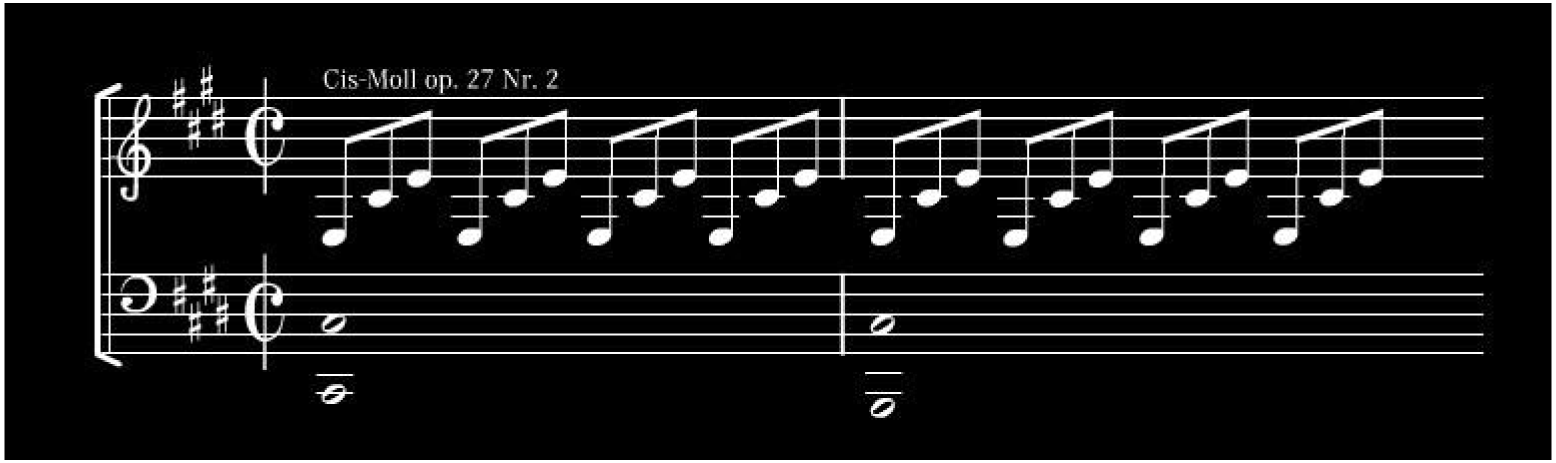
... or a third one or ... 10^{10} turns



19th century:

Ludwig van Beethoven: „Mondschein Sonate“

Sonate Nr. 14 in cis-Moll (op. 27/II,   1801)



Astronomer Hill:

differential equation for motions with periodic focusing properties
„Hill's equation"

Example: particle motion with periodic coefficient

equation of motion: $x''(s) - k(s)x(s) = 0$

restoring force $\neq$ const,
k(s) = depending on the position s
k(s+L) = k(s), periodic function

we expect a kind of quasi harmonic oscillation: amplitude & phase will depend on the position s in the ring.

6.) The Beta Function

General solution of Hill´s equation:

(i) $$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi)$$

ε, Φ = integration constants determined by initial conditions

β(s) periodic function given by focusing properties of the lattice ↔ quadrupoles

$$\beta(s+L) = \beta(s)$$

Inserting (i) into the equation of motion …

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

Ψ(s) = „phase advance“ of the oscillation between point „0“ and „s“ in the lattice.
For one complete revolution: number of oscillations per turn „Tune“

$$\boldsymbol{Q_y} = \frac{1}{2\pi}\oint \frac{\boldsymbol{ds}}{\boldsymbol{\beta(s)}}$$

7.) Beam Emittance and Phase Space Ellipse

general solution of Hill equation

$$\left\{\begin{array}{ll} (1) & x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\,\cos(\psi(s)+\phi) \\ (2) & x'(s) = -\dfrac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\,\{\alpha(s)\cos(\psi(s)+\phi)+\sin(\psi(s)+\phi)\} \end{array}\right.$$

from (1) we get

$$\cos(\psi(s)+\phi) = \frac{x(s)}{\sqrt{\varepsilon}\,\sqrt{\beta(s)}}$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

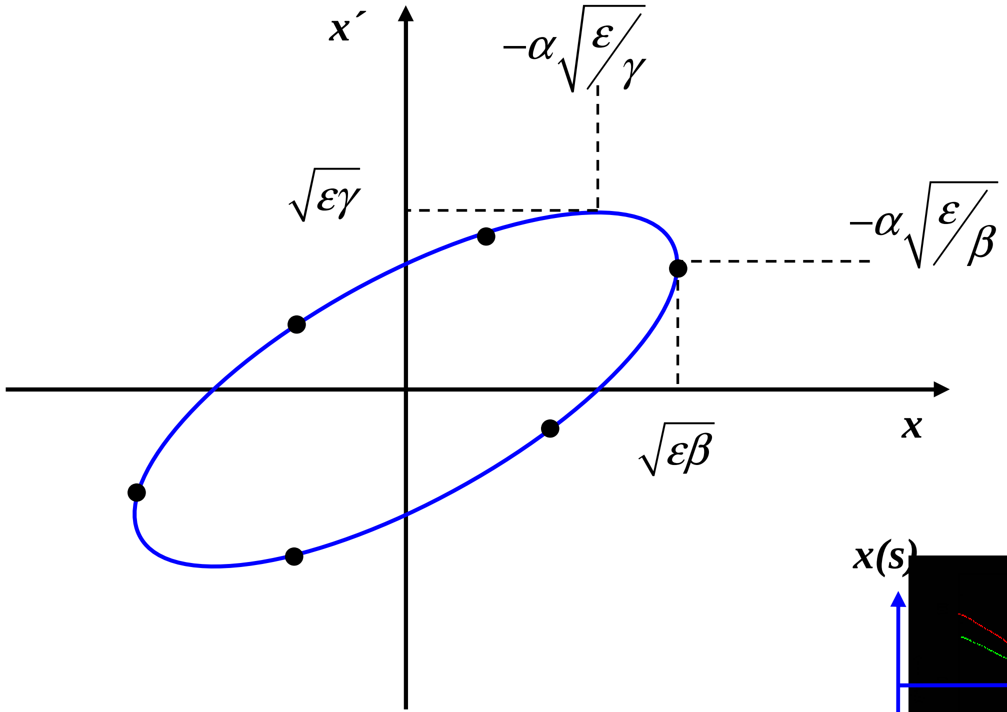
Insert into (2) and solve for ε

$$\boxed{\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)}$$

**** ε is a constant of the motion … it is independent of „s"***
**** parametric representation of an ellipse in the x x' space***
**** shape and orientation of ellipse are given by α, β, γ***

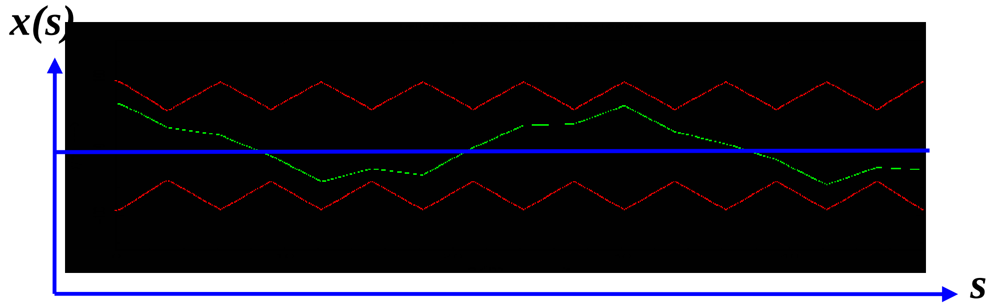
Beam Emittance and Phase Space Ellipse

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\, x'^2(s)$$



***Liouville:* in reasonable storage rings**
area in phase space is constant.

A = π*ε=const

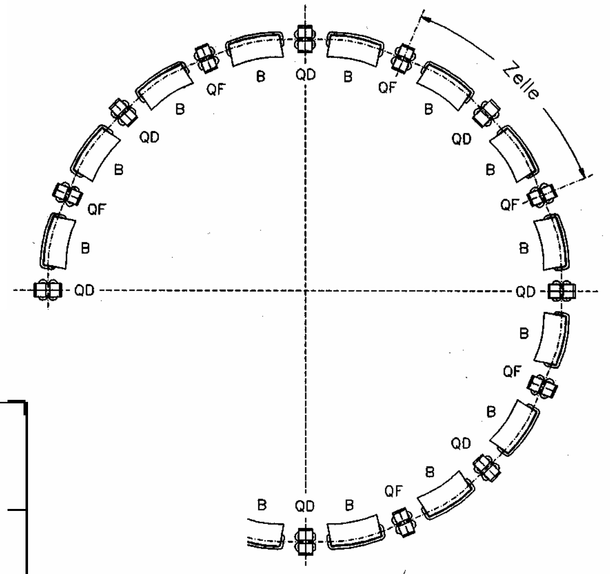


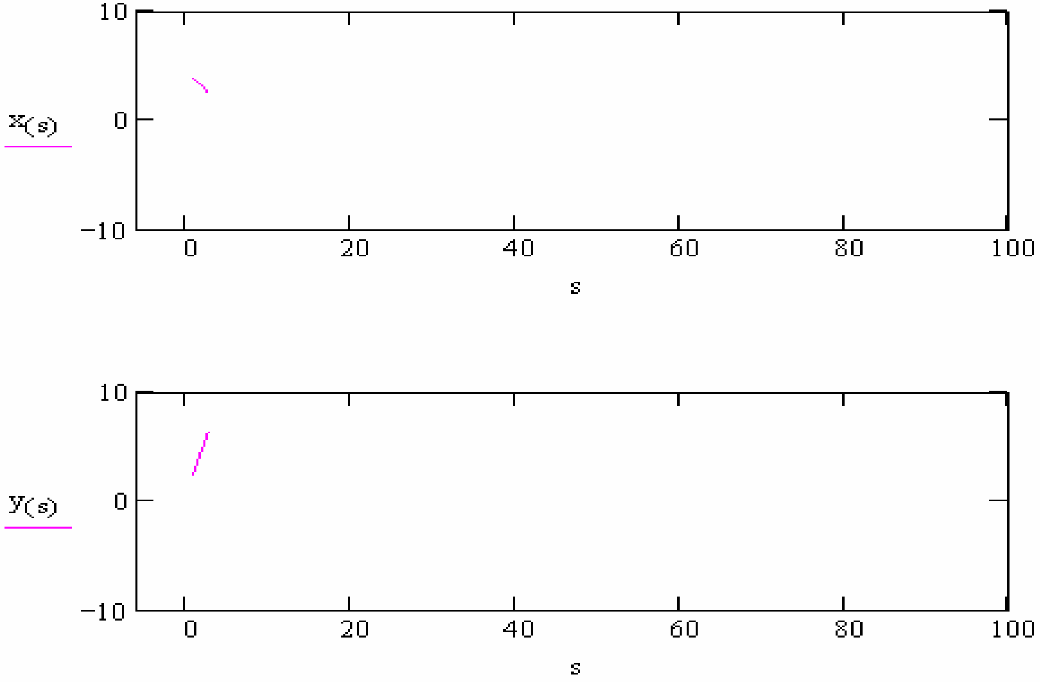
ε beam emittance = woozilycity of the particle ensemble, intrinsic beam parameter,
cannot be changed by the foc. properties.
Scientifiquely speaking: area covered in transverse x, x´ phase space … and it is constant !!!

Particle Tracking in a Storage Ring

Calculate x, x´ for each linear accelerator element according to matrix formalism

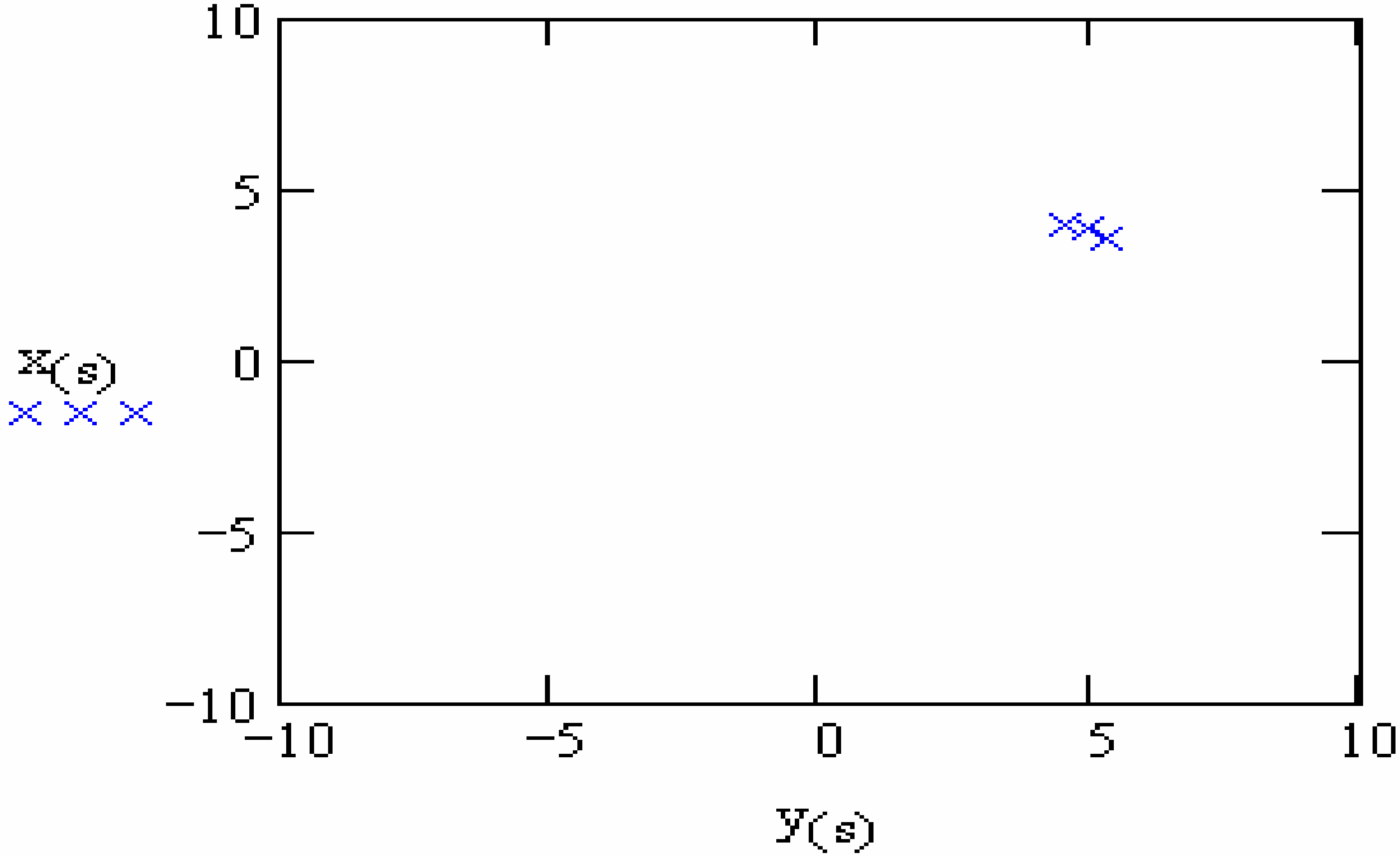
plot x, x´as a function of „s"





… and now the ellipse:

note for each turn x, x´at a given position „s_1" and plot in the phase space diagram



Résumé:

beam rigidity: $B \cdot \rho = p/q$

bending strength of a dipole: $\frac{1}{\rho}\left[m^{-1}\right] = \frac{0.2998 \cdot B_0(T)}{p(GeV/c)}$

focusing strength of a quadrupole: $k\left[m^{-2}\right] = \frac{0.2998 \cdot g}{p(GeV/c)}$

focal length of a quadrupole: $f = \frac{1}{k \cdot l_q}$

equation of motion: $x'' + Kx = \frac{1}{\rho}\frac{\Delta p}{p}$

matrix of a foc. quadrupole: $x_{s2} = M \cdot x_{s1}$

$$M = \begin{pmatrix} \cos\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sin\sqrt{|K|}l \\ -\sqrt{|K|}\sin\sqrt{|K|}l & \cos\sqrt{|K|}l \end{pmatrix} , \qquad M = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

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Accelerators, SSC Lab 1990

7.) Appendix: The equation of motion

Linear approximation:

*** ideal particle** → ***design orbit***

*** any other particle** → ***coordinates x, y*** ***small quantities***
$x,y \ll \rho$

→ ***magnetic guide field: only linear terms in x & y of B have to be taken into account***

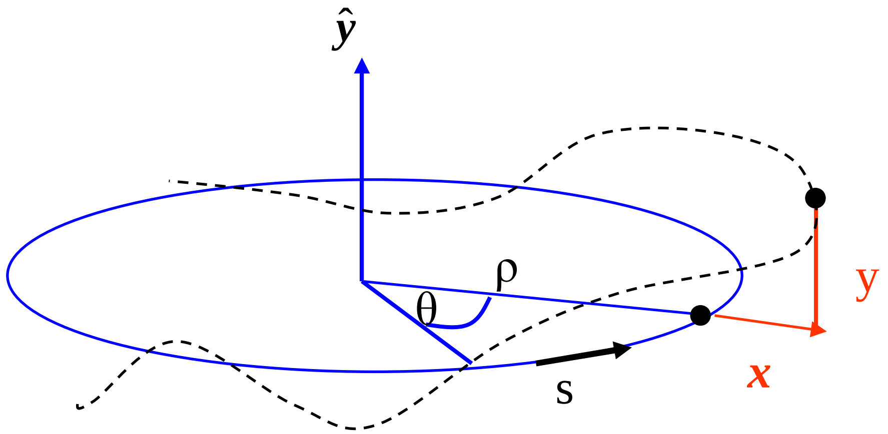
Taylor Expansion of the B field:

$$B_y(x) = B_{y0} + \frac{dB_y}{dx}x + \frac{1}{2!}\frac{d^2B_y}{dx^2}x^2 + \frac{1}{3!}\frac{eg''}{dx^3} + \ldots$$

normalise to momentum
$p/e = B\rho$

$$\frac{B(x)}{p/e} = \frac{B_0}{B_0\rho} + \frac{g * x}{p/e} + \frac{1}{2!}\frac{eg'}{p/e} + \frac{1}{3!}\frac{eg''}{p/e} + \ldots$$

Equation of Motion:



Consider local segment of a particle trajectory
... and remember the old days:
(Goldstein page 27)

radial acceleration:

$$a_r = \frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2$$

Ideal orbit: $\rho = const, \quad \frac{d\rho}{dt} = 0$

Force: $F = m\rho\left(\frac{d\theta}{dt}\right)^2 = m\rho\omega^2$

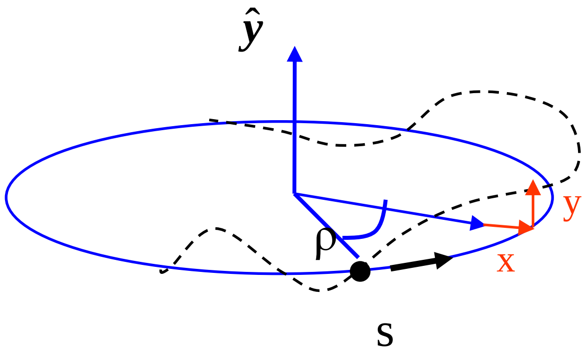
$$F = mv^2 / \rho$$

general trajectory: $\rho \rightarrow \rho + x$

$$\boldsymbol{F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v}$$

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v$$

1 2



1 $$\frac{d^2}{dt^2}(x+\rho) = \frac{d^2}{dt^2}x$$ *… as ρ = const*

2 ***remember:*** ***x ≈ mm , ρ ≈ m … → develop for small x***

$$\frac{1}{x+\rho} \approx \frac{1}{\rho}\left(1-\frac{x}{\rho}\right)$$

Taylor Expansion

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!}f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \quad .$$

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}\left(1-\frac{x}{\rho}\right) = eB_y v$$

guide field in linear approx.

$$B_y = B_0 + x\frac{\partial B_y}{\partial x}$$

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1-\frac{x}{\rho}) = ev\left\{B_0 + x\frac{\partial B_y}{\partial x}\right\} \quad \Big| \; : m$$

$$\frac{d^2x}{dt^2} - \frac{v^2}{\rho}(1-\frac{x}{\rho}) = \frac{e\,v\,B_0}{m} + \frac{e\,v\,x\,g}{m}$$

independent variable: $t \rightarrow s$

$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt}$$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{ds}\frac{ds}{dt}\right) = \frac{d}{ds}\left(\underbrace{\frac{dx}{ds}}_{x'}\underbrace{\frac{ds}{dt}}_{v}\right)\frac{ds}{dt}$$

$$\frac{d^2x}{dt^2} = x''v^2 + \cancel{\frac{dx}{ds}\frac{dv}{ds}}v$$

$$x''v^2 - \frac{v^2}{\rho}(1-\frac{x}{\rho}) = \frac{e\,v\,B_0}{m} + \frac{e\,v\,x\,g}{m} \quad \Big| \; : v^2$$

$$x'' - \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right) = \frac{e\ B_0}{mv} + \frac{e\ x\ g}{mv}$$

$m\,v = p$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = \frac{B_0}{p/e} + \frac{x\ g}{p/e}$$

normalize to momentum of particle

$$\frac{B_0}{p/e} = -\frac{1}{\rho}$$

$$\frac{g}{p/e} = k$$

$$x'' - \cancel{\frac{1}{\rho}} + \frac{x}{\rho^2} = -\cancel{\frac{1}{\rho}} + k\ x$$

$$\boxed{x'' + x\left(\frac{1}{\rho^2} - k\right) = 0}$$

* ***Equation for the vertical motion:***

$$\frac{1}{\rho^2} = 0$$ ***no dipoles … in general …***

$$k \leftrightarrow -k$$ ***quadrupole field changes sign***

$$y'' + k\ y = 0$$